Fekete points for bivariate polynomials restricted to $y = x^m$

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Abstract

We study the so-called Fekete points which maximize Vandermonde determinants of the form

$$V_\alpha(x_1, ..., x_N) = \left| x_i^{\alpha_j} \right|, \quad i, j = 1, ..., N \ ,$$

where the x_i are distinct points belonging to an interval $[a, b]$ of the real line and the α_j's are ordered integers $\alpha_1 > \alpha_2 > \cdots > \alpha_N \geq 0$ obtained as the exponents for the monomial basis of bivariate polynomials of degree n, restricted to the curve $y = x^m$. We prove that every Vandermonde determinant, so generalized, can be factored as a product of the corresponding classical Vandermonde determinant and a homogeneous symmetric function of the points, a *Schur function*, and that the resulting generalized Fekete points have the same asymptotic distribution as the classical ones.

keywords: Vandermonde determinants, Schur functions, Fekete interpolating points.

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1 Introduction

In the classical study of the convergence properties of the Lagrange interpolation the Lebesgue function

$$\lambda_n(X;x) = \sum_{i=0}^{n} |\ell_i(x)| \ , \tag{1}$$

and the corresponding Lebesgue constant

$$\Lambda_n(X) = \max_{x\in[-1,1]} \lambda_n(X;x) \tag{2}$$

play a fundamental role. As usual, $\ell_i(x)$ denotes the i-th fundamental Lagrange polynomial of degree n on the set $X = \{x_0, ..., x_n\}$ of distinct nodes in the *canonical* interval $[-1,1]$. The Lebesgue constant is also the norm of the Lagrange operator, which takes $f \in C[-1,1]$ to its interpolant (as an operator on $C[-1,1]$).

One may also consider the several variable analogues of such problems. Given a compact set $K \subset I\!R^d$, the polynomials of degree n restricted to K form a certain vector space of dimension $N = d_n(K)$, say. Suppose that $\{p_i, : 1 \le i \le N\}$ is a basis for this space. Then given N points $X = \{x_i\} \subset K$ and a function $f \in C(K)$, one may ask for the polynomial $p = \sum_k a_k p_k$, such that $p(x_i) = f(x_i)$, $1 \le i \le N$. This interpolating polynomial exists and is unique provided the Vandermonde determinant

$$\det[p_i(x_j)]_{1\le i,j\le N} \neq 0$$

If this is indeed the case then we may also form the corresponding fundamental Lagrange polynomials ℓ_i, defined by the condition that $\ell_i(x_j) = \delta_{ij}$ (the Kronecker delta). Then we may write the interpolant in the form

$$p(x) = \sum_k f(x_k)\ell_k(x).$$

Just as in the univariate case, the Lebesgue function is defined to be

$$\lambda_n(X;x) = \sum_{i=1}^{N} |\ell_i(x)|$$

and its maximum over K is the norm of the projector that takes $f \in C(K)$ to its interpolant.

In one variable, there is a very beautiful and useable characterization of the points that minimize the Lebesgue constant (see e.g. the monograph [10] Chapter 3) but in several variables this is far from the case. In fact, the multivariate theory is still in its relative infancy and not nearly complete. Moreover, it appears that multivariate analysis of the Lebesgue function is a rather imposing problem, and this has a lead to the consideration of a set of extremal points which are known to be near optimal in the univariate case, but that generalize much more readily to higher dimensions, i.e. to the so-called Fekete points. These may succintly be defined as those points in K which maximize the Vandermonde determinant, $\det[p_i(x_j)]$.

For the interval $[-1,1]$, Fejér [5] showed that the set of Fekete points, F_n say, consists of the zeros of the polynomial $(x^2-1)\,P_n'(x)$, where $P_n(x)$ is the Legendre polynomial of degree n. Moreover, he showed that $\sum_k \ell_k^2(x) \le 1$ on $[-1,1]$ from which it follows that the associated Lebesgue constant

$$\Lambda_n(F_n) \le \sqrt{n+1}\ .$$

This bound was improved by Sündermann ([9]) who showed that

$$\Lambda_n(F_n) = \mathcal{O}(\log n), \tag{3}$$

the order of the optimal points (cf. [10]). We would point out that from this explicit characterization of F_n, but also from the more abstract considerations of complex Potential Theory, it follows that the Fekete points have asymptotically the so-called arcsin or Chebyshev distribution (which is also the equilibrium distribution from Potential Theory).

In general, we have only that

$$\max_{x\in K} |\ell_k(x)| = 1, \quad 1 \le k \le N$$

from which follows the estimate

$$\Lambda_n(X) \le N.$$

This implies already that the Fekete points are always quite good interpolation points, but the upper bound N is almost certainly rather pessimistic. From numerical experiments, one may expect that in fact the Fekete points are very close to optimal (see e.g. [7]).

We hope that the reader is convinced by our preamble, or by other means, that the Fekete points for a given set $K \subset \mathbb{R}^d$, are interesting and important for the study of multivariate polynomial interpolation. An especially intriguing problem is to determine their asymptotic distribution. Likely, this several variable problem is intimately related to complex Pluripotential Theory (see e.g. [6]) but to date very little is known.

In this paper we begin by considering the case when $K \subset \mathbb{R}^2$ is a piece of an algebraic curve of the form

$$K = \{(x,y)\ :\ y = x^m,\ a \le x \le b\}.$$

We show that a basis for the polynomials restricted to such K are a certain collection of monomials (Proposition 1 below) and then consider a more general class of Vandermonde type determinants, involving the so-called Schur functions, and show by means of direct, elementary calculations that the associated Fekete points always have the same asymptotic distribution as the classical ones.

2 The polynomials restricted to the curve $y = x^m$

Proposition 1 *Bivariate polynomials of degree $n \geq m$ restricted to $y = x^m$ are the*

$$span \left\{ \begin{array}{lll} x^k & k \equiv 0 \; mod \; m & 0 \leq k \leq mn \\ x^k & k \equiv 1 \; mod \; m & 0 \leq k \leq m(n-1)+1 \\ x^k & k \equiv 2 \; mod \; m & 0 \leq k \leq m(n-2)+2 \\ \vdots & \vdots & \vdots \\ x^k & k \equiv (m-1) \; mod \; m & 0 \leq k \leq m(n-(m-1))+m-1 \end{array} \right\} .$$

Proof. Any integer $k \geq 0$ is congruent to one of $0, 1, ..., m-1$ mod m. Indeed suppose $k \equiv i \; mod \; m, \quad 0 \leq i \leq m-1$, then $k = jm + i$ for some j and x^k is the restriction of $y^j x^i$. In fact, this is the monomial of lowest degree which restricts to x^k.

For $y^{j'} x^{i'}_{|y=x^m} = x^k$ if and only if $mj' + i' = k = mj + i$.

If $j' > j$ then $mj' + i' \geq m(j+1) + i' = mj + m + i' > mj + i + i' > k$ which is a contradiction. Therefore, $j' \leq j$.

If $j' = j$ then $i' = i$ and there is nothing more to prove. Suppose then that $j' < j$. For the sake of simplicity let $j' = j - t, \quad t \geq 1$. Then $m(j-t) + i' = mj + i$, that is $i' = i + mt$ and then $i' + j' = i + mt + j - t = i + j + (m-1)t > i + j$. Hence

$$x^k, \quad k \equiv \; i \bmod m \,, \tag{4}$$

is the restriction of $x^i y^j, \quad i + j \leq n$ `if and only if`

$$k = mj + i \leq m(n-1) + i \; . \tag{5}$$

This concludes the proof. ■

Remark. The space of bivariate polynomials restricted to $y = x^m$ consists of univariate polynomials of degree at most mn. However, the dimension of this space is

$$N = \binom{n+2}{2} - \binom{n+2-m}{2} = mn - \frac{m(m-3)}{2} \leq mn + 1 \tag{6}$$

with *strict* inequality for $m > 2$. Hence, for $m > 2$, the basis consists of a strict subset of the monomials of degree at most mn, i.e. there are "missing" powers or gaps.

It does however include all univariate polynomials of degree

$$m(n-(m-1)) + 2(m-1) = mn - (m-1)(m-2) \; .$$

In fact, we may describe the basis more precisely as

$$\cup_{k=0}^{n-(m-1)} \cup_{j=0}^{m-1} \{x^{km+j}\} \bigcup \cup_{k=0}^{m-2} \cup_{j=0}^{k} \{x^{(n-k)m+j}\} \tag{7}$$

so that the missing powers are precisely

$$\cup_{k=1}^{m-2} \cup_{j=k+1}^{m-1} \{x^{(n-k)m+j}\}.$$

Let then

$$\{x^{\alpha_1}, x^{\alpha_2}, \cdots, x^{\alpha_N}\}$$

with $\alpha_1 > \alpha_2 > \cdots > \alpha_N \geq 0$ be the basis described in Proposition 1, and consider, for given N distinct points $x_1 < x_2 < \cdots < x_N$, the associated Vandermonde determinant

$$V_\alpha(x_1, \cdots, x_N) := \det[x_i^{\alpha_j}]_{1 \leq i,j \leq N}. \tag{8}$$

Such determinants are intimately connected to the so-called Schur functions (cf. [8]), defined as follows.

Definition 1 *Given a partition $\lambda = (\lambda_1, ..., \lambda_N) \in I\!N^N$, and N distinct points $x_1, ..., x_N$, the associated Schur function, s_λ, defined on $I\!R^N$, is the ratio*

$$s_\lambda(x_1, ..., x_N) = \frac{det(x_i^{\lambda_j+N-j})}{det(x_i^{N-j})}, \quad 1 \leq i, j \leq N. \tag{9}$$

Note that in this definition the denominator is the classical Vandermonde determinant

$$VDM(x_1, \cdots, x_N) = \det(x_i^{N-j}).$$

Since the α_j are positive integers we must have $\alpha_j \geq N - j$ and thus by taking $\lambda_j = \alpha_j - (N - j)$ we have

$$V_\alpha(x_1, \cdots, x_N) = \det(x_i^{\lambda_j+N-j}) = VDM(x_1, ..., x_N)\ s_\lambda(x_1, \cdots, x_N)\ . \tag{10}$$

Of importance in what follows is the fact that λ depends only on m, i.e., is constant with respect to N. More precisely we have:

Lemma 1 *If λ is defined as above, then*

$$\lambda = (\underbrace{\sum_{k=1}^{m-2} k}_{1}, \underbrace{\sum_{k=1}^{m-3} k, \sum_{k=1}^{m-3} k}_{2}, \underbrace{\sum_{k=1}^{m-4} k, \sum_{k=1}^{m-4} k, \sum_{k=1}^{m-4} k}_{3}, \cdots, \underbrace{\sum_{k=1}^{1} k, \cdots, \sum_{k=1}^{1} k}_{m-2}, 0, \cdots, 0).$$

Proof. This is just a technical fact that follows easily (albeit with some tedium) from the representataion of the basis (7). ■

Some simple consequences are that

$$|\lambda| = \sum_{j=1}^{m-2} \left(j \sum_{k=1}^{m-1-j} k \right) = \binom{m+1}{4}. \tag{11}$$

We may also compute $\ell(\lambda)$, the length of λ, i.e., the index of the last non-zero term, to be

$$\ell(\lambda) = \sum_{j=1}^{m-2} j = \binom{m-1}{2}. \tag{12}$$

3 The Main Theorem

Consider now the Vandermonde type determinant $V_\alpha(x_1, \cdots, x_N)$ defined by (8) for points

$$a \le x_1 < x_2 < \cdots < x_N \le b.$$

The points x_j which maximize $|V_\alpha|$ we will refer to as Fekete points for the piece of the curve $K = \{(x, x^m) : a \le x \le b\}$. The points f_j in $[a, b]$ which maximize the classical Vandermonde determinant $|VDM(f_1, \cdots, f_N)|$ are the *classical* Fekete points. Our main Theorem is as follows.

Theorem 1 *The Fekete points for the piece of the curve K have the same asymptotic distribution as do the classical Fekete points for $[a, b]$.*

Proof. By Theorem 1.5 of [1] it suffices to show that

$$\lim_{N\to\infty} |VDM(x_1, \cdots, x_N)|^{1/\binom{N}{2}} = \lim_{N\to\infty} |VDM(f_1, \cdots, f_N)|^{1/\binom{N}{2}}. \tag{13}$$

We will show that this is indeed the case, by means of a sequence of lemmas. To begin, we will need some of the standard symmetric polynomials. Let $p_r = p_r(t_1, \cdots, t_N)$ denote the rth *power sum* of N variables, i.e.,

$$p_r(t_1, \cdots, t_N) := \sum_{k=1}^{N} t_k^r.$$

Then, since the classical Fekete points have asymptotically the arcsin distribution, we have

$$\begin{aligned} \lim_{N\to\infty} \frac{p_r(f_1, \cdots, f_N)}{N} &= \frac{1}{\pi} \int_a^b \frac{x^r}{\sqrt{(x-a)(b-x)}} dx \\ &= \frac{1}{\pi} \int_0^\pi \left(\frac{a+b}{2} + \frac{b-a}{2} \cos(\theta) \right)^r d\theta. \end{aligned} \tag{14}$$

Also, let $h_r = h_r(t_1, \cdots, t_N)$ denote the rth so-called *complete symmetric function* of N variables. It is defined as the sum of all the monomials of exact degree r, in the variables $t_1, \cdots, t_N$, but may also be computed from the Newton like formula (see [8, formula (2.11)])

$$kh_k = \sum_{r=1}^{k} p_r h_{k-r}. \tag{15}$$

Lemma 2

$$\lim_{N\to\infty} \frac{h_k(f_1, \cdots, f_N)}{N^k} = \frac{1}{k!}\left(\frac{a+b}{2}\right)^k$$

Proof. We proceed by induction. If $k = 1$, then $h_1 = p_1$ and the result follows easily from (14) with $r = 1$. Hence, suppose that the Lemma is true up to $k-1$ and consider

$$\begin{aligned}
\frac{h_k}{N^k} &= \frac{1}{k}\sum_{r=1}^{k} \frac{p_r}{N}\frac{h_{k-r}}{N^{k-r}}\frac{1}{N^{r-1}} \\
&= \frac{1}{k}\left(\frac{p_1}{N}\frac{h_{k-1}}{N^{k-1}} + \sum_{r=2}^{k} \frac{p_r}{N}\frac{h_{k-r}}{N^{k-r}}\frac{1}{N^{r-1}}\right).
\end{aligned}$$

Because of the $N^{-(r-1)}$ factor, the terms in the summation tend to zero, and hence,

$$\begin{aligned}
\lim_{N\to\infty} \frac{h_k}{N^k} &= \frac{1}{k}\lim_{N\to\infty}\frac{p_1}{N}\lim_{N\to\infty}\frac{h_{k-1}}{N^{k-1}} \\
&= \frac{1}{k}\left(\frac{a+b}{2}\right)\frac{1}{(k-1)!}\left(\frac{a+b}{2}\right)^{k-1} \\
&= \frac{1}{k!}\left(\frac{a+b}{2}\right)^{k}.
\end{aligned}$$

This concludes the proof. ■

Lemma 3 *For any fixed* λ,

$$\lim_{N\to\infty} \frac{s_\lambda(f_1, \cdots, f_N)}{N^{|\lambda|}} = \left(\frac{a+b}{2}\right)^{|\lambda|} \sum_{\substack{\omega \in S_\ell \\ \lambda+\delta-\omega(\delta) \geq 0}} \epsilon(\omega)\frac{1}{(\lambda+\delta-\omega(\delta))!} \tag{16}$$

where $\delta = (\ell-1, \ell-2, ..., 1, 0)$ and $\ell = \ell(\lambda)$ *and* S_ℓ *denotes the group of permutations of* ℓ *objects.*

Proof. By formula [8, (3.4)′, p. 42] with

$$s_\lambda = \sum_{\omega\in S_\ell} \epsilon(\omega) h_{\lambda+\delta-\omega(\delta)} = \sum_{\substack{\omega \in S_\ell \\ \lambda+\delta-\omega(\delta)\geq 0}} \epsilon(\omega) h_{\lambda+\delta-\omega(\delta)}$$

since $h_r = 0$ for $r < 0$.

Hence,

$$\begin{aligned} \frac{s_\lambda}{N^{|\lambda|}} &= \sum_{\substack{\omega \in S_\ell \\ \lambda+\delta-\omega(\delta)\geq 0}} \epsilon(\omega) \frac{h_{\lambda+\delta-\omega(\delta)}}{N^{|\lambda|}} \\ &= \sum_{\substack{\omega \in S_\ell \\ \lambda+\delta-\omega(\delta)\geq 0}} \epsilon(\omega) \frac{h_{\lambda_1+\delta_1-\omega(\delta)_1}}{N^{\lambda_1+\delta_1-\omega(\delta)_1}} \cdots \frac{h_{\lambda_\ell+\delta_\ell-\omega(\delta)_\ell}}{N^{\lambda_\ell+\delta_\ell-\omega(\delta)_\ell}} \end{aligned}$$

Now, letting $N \to \infty$, by Lemma 2, the right side tends to

$$\begin{aligned} &\sum_{\substack{\omega \in S_\ell \\ \lambda+\delta-\omega(\delta)\geq 0}} \epsilon(\omega) \frac{\left(\frac{a+b}{2}\right)^{\lambda_1+\delta_1-\omega(\delta)_1}}{(\lambda_1+\delta_1-\omega(\delta)_1)!} \cdots \frac{\left(\frac{a+b}{2}\right)^{\lambda_\ell+\delta_\ell-\omega(\delta)_\ell}}{(\lambda_\ell+\delta_\ell-\omega(\delta)_\ell)!} \\ &= \left(\frac{a+b}{2}\right)^{|\lambda|} \sum_{\substack{\omega \in S_\ell \\ \lambda+\delta-\omega(\delta)\geq 0}} \epsilon(\omega) \frac{1}{(\lambda+\delta-\omega(\delta))!} \ . \end{aligned}$$

This concludes the proof. ■

Lemma 4 *Letting $k! = 0$ for $k < 0$, then*

$$\sum_{\substack{\omega \in S_\ell \\ \lambda+\delta-\omega(\delta)\geq 0}} \epsilon(\omega) \frac{1}{(\lambda+\delta-\omega(\delta))!} = \frac{1}{(\lambda+\delta)!} VDM(\lambda+\delta) \ , \tag{17}$$

where $VDM(\lambda+\delta)$ is the classical Vandermonde determinant of the points $\lambda_1+\delta_1, ..., \lambda_\ell+\delta_\ell$.

Proof.

$$(18)\qquad \sum_{\substack{\omega\in S_\ell\\ \lambda+\delta-\omega(\delta)\geq 0}} \epsilon(\omega)\frac{1}{(\lambda+\delta-\omega(\delta))!} = \det\left(\frac{1}{(\lambda_i-i+j)!}\right)_{1\leq i,j\leq\ell}\ ,$$

since $s_\lambda = \det\left(h_{\lambda_i-i+j}\right)$, formula [8, (3.4),p. 41] (this is just an expression of the same determinant in two different manners). Now,

$$\begin{aligned}
\det(\frac{1}{(\lambda_i-i+j)!}) &= \det\begin{pmatrix} \frac{1}{(\lambda_1)!} & \frac{1}{(\lambda_1+1)!} & \cdots & \frac{1}{(\lambda_1+\ell-1)!}\\ \frac{1}{(\lambda_2-1)!} & \frac{1}{(\lambda_2)!} & \cdots & \frac{1}{(\lambda_2+\ell-2)!}\\ \vdots & & \ddots & \vdots\\ \frac{1}{(\lambda_\ell-\ell+1)!} & \frac{1}{(\lambda_\ell-\ell+2)!} & \cdots & \frac{1}{(\lambda_\ell)!}\end{pmatrix}\\
&= \Lambda_\ell\ \det\begin{pmatrix} \frac{(\lambda_1+\ell-1)!}{(\lambda_1)!} & \frac{(\lambda_1+\ell-1)!}{(\lambda_1+1)!} & \cdots & \frac{(\lambda_1+\ell-1)!}{(\lambda_1+\ell-2)!} & 1\\ \frac{(\lambda_2+\ell-2)!}{(\lambda_2-1)!} & \frac{(\lambda_2+\ell-2)!}{(\lambda_2)!} & \cdots & \frac{(\lambda_2+\ell-2)!}{(\lambda_2+\ell-3)!} & 1\\ \vdots & \vdots & & \ddots & \vdots\\ \frac{(\lambda_\ell)!}{(\lambda_\ell-\ell+1)!} & \frac{(\lambda_\ell)!}{(\lambda_\ell-\ell+2)!} & \cdots & \frac{(\lambda_\ell)!}{(\lambda_\ell-1)!} & 1\end{pmatrix}\\
&= \Lambda_\ell \det\begin{pmatrix} [\lambda_1+\ell-1]_{\ell-1} & [\lambda_1+\ell-1]_{\ell-2} & \cdots & [\lambda_1+\ell-1]_1 & 1\\ [\lambda_2+\ell-2]_{m-1} & [\lambda_2+\ell-2]_{m-2} & \cdots & [\lambda_2+\ell-2]_1 & 1\\ \vdots & \vdots & & \ddots & \vdots\\ [\lambda_\ell]_{\ell-1} & [\lambda_\ell]_{\ell-2} & \cdots & [\lambda_\ell]_1 & 1\end{pmatrix}
\end{aligned}$$

where

$$\Lambda_\ell = \frac{1}{(\lambda_1+\ell-1)!}\frac{1}{(\lambda_2+\ell-2)!}\cdots\frac{1}{(\lambda_\ell)!}$$

and $[x]_k = x(x-1)(x-2)\cdots(x-k+1)$ is the *Pochammer symbol.*
Hence we have

$$\det(\frac{1}{(\lambda_i-i+j)!}) = \frac{1}{(\lambda+\delta)!}\det\left([\lambda_i+\ell-i]_{\ell-j}\right)_{1\leq i,j\leq\ell}$$

But, since the basis $x(x-1)\cdots(x-\ell+2), x(x-1)\cdots(x-\ell+3), \ldots, x, 1$ is equivalent to the basis

$$x^{\ell-1}, x^{\ell-2}, \ldots, x, 1$$

then we can conclude that

$$\begin{aligned}
\det(\frac{1}{(\lambda_i - i + j)!}) &= \frac{1}{(\lambda+\delta)!}\det\left((\lambda_i + \ell - i)^{\ell-j}\right) \\
&= \frac{1}{(\lambda+\delta)!}\prod_{j>i}((\lambda_i + \ell - i) - (\lambda_j + \ell - j)) \\
&= \frac{1}{(\lambda+\delta)!}\prod_{j>i}((\lambda_i - \lambda_j) + (j - i)))
\end{aligned}$$

This latter is > 0 since $\lambda_i \geq \lambda_j$ and $j - i > 0$. ■

We now return to the proof of the main theorem. Our goal is to establish (13) and we will first assume that $a + b \neq 0$.

Now, since the classical Fekete points maximize $|VDM|$, we have

$$|VDM(x_1, \cdots, x_N)| \leq |VDM(f_1, \cdots, f_N)|.$$

But also, the generalized Fekete points maximize $|V_\alpha|$ and so

$$|V_\alpha(f_1, \cdots, f_N)| \leq |V_\alpha(x_1, \cdots, x_N)|,$$

i.e.,

$$|s_\lambda(f_1, \cdots, f_N)|\,|VDM(f_1, \cdots, f_N)| \leq |s_\lambda(x_1, \cdots, x_N)|\,|VDM(x_1, \cdots, x_N)|. \tag{19}$$

Thus

$$\left|\frac{s_\lambda(f_1, \cdots, f_N)}{s_\lambda(x_1, \cdots, x_N)}\right| |VDM(f_1, \cdots, f_N)| \leq |VDM(x_1, \cdots, x_N)|.$$

Hence our result follows once we have established that

$$\lim_{N\to\infty}\left|\frac{s_\lambda(f_1, \cdots, f_N)}{s_\lambda(x_1, \cdots, x_N)}\right|^{1/\binom{N}{2}} = 1.$$

But from Lemma 4, (provided $a + b \neq 0$) we have

$$\lim_{N\to\infty} |s_\lambda(f_1, \cdots, f_N)|^{1/\binom{N}{2}} = 1$$

and so are left with the problem of showing that

$$\lim_{N\to\infty} |s_\lambda(x_1, \cdots, x_N)|^{1/\binom{N}{2}} = 1. \tag{20}$$

Now, from (19), since $|VDM(x_1, \cdots, x_N)| \le |VDM(f_1, \cdots, f_N)|$ we must have

$$|s_\lambda(x_1, \cdots, x_N)| \ge |s_\lambda(f_1, \cdots, f_N)|$$

and so (20) follows from

Lemma 5 *There are constants $C = C(\lambda)$ and $k = k(\lambda)$ such that for all $a \le x_1 \le \cdots \le x_N \le b$,*

$$|s_\lambda(x_1, \cdots, x_N)| \le CN^k.$$

Proof. By Hadamard's determinant inequality applied to the identity $s_\lambda = \det(h_{\lambda_i - i + j})$ [8, (3.4),p. 41], we need only provide a polynomial bound for $h_{\lambda_i - i + j}$. But, $r = \lambda_i - i + j \le \lambda_1 + \ell(\lambda) - 1$ and recall that h_r is defined to the sum of all the monomials of *exact* degree r. Hence

$$|h_r(x_1, \cdots, x_N)| \le \binom{N + r - 1}{r} \max\{|a|, |b|\}^r.$$

■

The case $a + b = 0$ can be handled by replacing,e.g., b by $-a + \epsilon$ and then letting $\epsilon \to 0+$. The details are not instructive and so we do not include them here.

The proof of Theorem 1 is now complete. ■

Remark. Although we have stated our main theorem in terms of polynomial bases obtained by restricting bivariate polynomials to the curves $y = x^m$, the proof is valid as long as the associated λ is constant, i.e., independent of N. In particular the conclusion remains valid for a basis of monomials with any fixed number of gaps in the sequence of powers (beginning with the highest power).

Acknowledgements The first author was supported by NSERC operating grant number 8389. The second author was partially supported by the CNR of Italy "short-term mobility" program 1999.

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