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# Equivalence principle: tunnelling, quantized spectra and trajectories from the quantum HJ equation

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## Abstract

A basic aspect of the recently proposed approach to quantum mechanics is that no use of any axiomatic interpretation of the wave function is made. In particular, the quantum potential turns out to be an intrinsic potential energy of the particle, which, similarly to the relativistic rest energy, is never vanishing. This is related to the tunnel effect, a consequence of the fact that the conjugate momentum field is real even in the classically forbidden regions. The quantum stationary Hamilton–Jacobi equation is defined only if the ratio  $\psi^D/\psi$  of two real linearly independent solutions of the Schrödinger equation, and therefore of the trivializing map, is a local homeomorphism of the extended real line into itself, a consequence of the Möbius symmetry of the Schwarzian derivative. In this respect we prove a basic theorem relating the request of continuity at spatial infinity of  $\psi^D/\psi$ , a consequence of the  $q \leftrightarrow q^{-1}$  duality of the Schwarzian derivative, to the existence of  $L^2(\mathbb{R})$  solutions of the corresponding Schrödinger equation. As a result, while in the conventional approach one needs the Schrödinger equation with the  $L^2(\mathbb{R})$  condition, consequence of the axiomatic interpretation of the wave function, the equivalence principle by itself implies a dynamical equation that does not need any assumption and reproduces both the tunnel effect and energy quantization. © 1999 Elsevier Science B.V. All rights reserved.

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Let us consider a one-dimensional stationary system of energy  $E$  and potential  $V$  and set  $\mathcal{W} \equiv V(q) - E$ . Let us denote by  $\mathcal{S}_0$  the quantum analogue of the Hamilton characteristic function, also called reduced action. This

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function satisfies the Quantum Stationary Hamilton–Jacobi Equation (QSHJE)

$$\frac{1}{2m} \left( \frac{\partial \mathcal{S}_0(q)}{\partial q} \right)^2 + V(q) - E + \frac{\hbar^2}{4m} \{ \mathcal{S}_0, q \} = 0, \quad (1)$$

that in [1] was uniquely derived from the equivalence principle. It states that

*For each pair  $\mathcal{W}^a, \mathcal{W}^b$ , there is a transformation  $q^a \rightarrow q^b = v(q^a)$ , such that*

$$\mathcal{W}^a(q^a) \rightarrow \mathcal{W}^{av}(q^b) = \mathcal{W}^b(q^b). \quad (2)$$

This principle implies the existence of the trivializing map [1]

$$q \rightarrow q^0 = \mathcal{S}_0^{0^{-1}} \circ \mathcal{S}_0(q), \quad (3)$$

reducing the system with a given  $\mathcal{W}$  to the one corresponding to  $\mathcal{W}^0 \equiv 0$ .

If  $(\psi^D, \psi)$  is a pair of real linearly independent solutions of the Schrödinger equation, then we have

$$e^{\frac{2i}{\hbar} \mathcal{S}_0} = e^{i\alpha} \frac{w + i\ell}{w - i\ell}, \quad (4)$$

where  $w = \psi^D/\psi$ , and  $\text{Re } \ell \neq 0$ .

A basic property of the conjugate momentum  $p = \partial_q \mathcal{S}_0$ , is that it is real even in the classically forbidden regions [1] [2]. This fact is an important check in considering the trajectories described by (1). To understand this, observe that in the conventional formulation of quantum mechanics the tunnel effect is a consequence of the axiomatic interpretation of the wave function  $\psi$ . Actually, the fact that it describes the probability amplitude of finding the particle in the interval  $[q, q + dq]$ , implies that the tunnel effect simply arises in the cases in which  $\psi$  is not identically zero in the classically forbidden regions. In the case at hand, there is no need for any axiomatic interpretation, it is just a consequence of reality of the conjugate momentum in the classically forbidden regions. This result suggests considering if the theory reproduces the other fundamental aspect of quantum mechanics, namely quantization of the energy spectra. This would be a basic check as also energy quantization is strictly related to the axiomatic interpretation of the wave function. We will see that this is actually the case.

The properties of the Schwarzian derivative imply that the QSHJE is well defined if and only if [3]

$$w \neq \text{cst}, w \in C^2(\hat{\mathbb{R}}), \text{ and } \partial_q^2 w \text{ differentiable on } \hat{\mathbb{R}}, \quad (5)$$

and  $w(q)$  is locally invertible  $\forall q \in \hat{\mathbb{R}}$ , where  $\hat{\mathbb{R}} \equiv \mathbb{R} \cup \{\infty\}$  denotes the extended real line. In particular, this implies the following joining condition at spatial infinity

$$w(-\infty) = \begin{cases} w(+\infty), & \text{for } w(-\infty) \neq \pm\infty, \\ -w(+\infty), & \text{for } w(-\infty) = \pm\infty. \end{cases} \quad (6)$$

It follows that  $w$ , and therefore the trivializing map, is a local homeomorphism of  $\hat{\mathbb{R}}$  into itself.

Let us start proving a result concerning the energy spectra. If  $V(q) > E$ ,  $\forall q \in \mathbb{R}$ , then, as we will see, there are no solutions such that the ratio of two linearly independent solutions of the Schrödinger equation corresponds to a local homeomorphism of  $\hat{\mathbb{R}}$  into itself. The fact that this is an unphysical situation can be also seen from the fact that the case  $V > E$ ,  $\forall q \in \mathbb{R}$ , has not classical limit. Therefore, if  $V(q) > E$  both at  $-\infty$  and  $+\infty$ , a physical situation requires that there are at least two points in which  $V - E = 0$ . More generally, if the potential is not continuous, we should have at least two turning points for which  $V(q) - E$  changes sign. Let us

denote by  $q_-$  ( $q_+$ ) the lowest (highest) value of the turning points. In the following we will prove the following basic fact

If

$$V(q) - E \geq \begin{cases} P_-^2 > 0, & q < q_-, \\ P_+^2 > 0, & q > q_+, \end{cases} \quad (7)$$

then the ratio  $w = \psi^D / \psi$  is a local homeomorphism of  $\hat{\mathbb{R}}$  into itself if and only if the corresponding Schrödinger equation admits an  $L^2(\mathbb{R})$  solution.

Note that by (7) we have

$$\int_{q_-}^{-\infty} dx \kappa(x) = -\infty, \quad \int_{q_+}^{+\infty} dx \kappa(x) = +\infty, \quad (8)$$

where

$$\kappa = \frac{\sqrt{2m(V-E)}}{\hbar}. \quad (9)$$

Let us first show that the request that the corresponding Schrödinger equation admits an  $L^2(\mathbb{R})$  solution is a sufficient condition for the ratio  $w = \psi^D / \psi$  to be continuous at  $\pm\infty$ . Let us denote by  $\psi$  the  $L^2(\mathbb{R})$  solution and by  $\psi^D$  a linearly independent solution. As we will see, the fact that  $\psi^D$  and  $\psi$  are linearly independent imply that if  $\psi \in L^2(\mathbb{R})$ , then  $\psi^D \notin L^2(\mathbb{R})$ , in particular  $\psi^D$  is divergent both at  $q = -\infty$  and  $q = +\infty$ . Let us consider the real ratio

$$w = \frac{A\psi^D + B\psi}{C\psi^D + D\psi}, \quad (10)$$

where  $AD - BC \neq 0$ . Since  $\psi \in L^2(\mathbb{R})$ , we have

$$\lim_{q \rightarrow \pm\infty} w = \lim_{q \rightarrow \pm\infty} \frac{A\psi^D + B\psi}{C\psi^D + D\psi} = \lim_{q \rightarrow \pm\infty} \frac{A\psi^D}{C\psi^D} = \frac{A}{C}, \quad (11)$$

that is  $w(-\infty) = w(+\infty)$ . In the case in which  $C = 0$  we have

$$\lim_{q \rightarrow \pm\infty} w = \lim_{q \rightarrow \pm\infty} \frac{A\psi^D}{D\psi} = \pm\epsilon \cdot \infty, \quad (12)$$

where  $\epsilon = \pm 1$ . The fact that  $\lim_{q \rightarrow \pm\infty} A\psi^D / D\psi$  is divergent follows from the mentioned properties of  $\psi^D$  and  $\psi$ . It remains to understand the fact that if the limit is  $-\infty$  at  $q = -\infty$ , then this limit is  $+\infty$  at  $q = +\infty$ , and vice versa. This can be understood by observing that

$$\frac{\psi^D(q)}{\psi(q)} = c \int_{q_0}^q dx \psi^{-2}(x) + d, \quad (13)$$

for some real constants  $c \neq 0$  and  $d$ . Now, since  $\psi \in L^2(\mathbb{R})$ , it follows that  $\psi^{-1} \notin L^2(\mathbb{R})$ . In particular,  $\int_{q_0}^{+\infty} dx \psi^{-2}(x) = +\infty$  and  $\int_{q_0}^{-\infty} dx \psi^{-2}(x) = -\infty$ , so that  $\psi^D(-\infty) / \psi(-\infty) = -\epsilon \cdot \infty = -\psi^D(\infty) / \psi(\infty)$  where  $\epsilon = \text{sgn } c$ .

We now show that the existence in the case (7) of an  $L^2(\mathbb{R})$  solution of the Schrödinger equation is a necessary condition for the ratio  $w = \psi^D / \psi$  to be continuous at  $\pm\infty$ . To this end we recall that if  $V(q) - E \geq P_+^2 > 0$ ,  $q > q_+$ , then as  $q \rightarrow +\infty$ , we have ( $P_+ > 0$ )

- There is one solution of the Schrödinger equation, defined up to a multiplicative constant, that vanishes at least as  $e^{-P+q}$ .
- Any other solution diverges at least as  $e^{P+q}$ .

The proof of this fact is based on Wronskian arguments and can be found in Messiah's book [4]. The above result extends also to the case in which  $V(q) - E \geq P_-^2 > 0$ ,  $q < q_-$ . In particular, as  $q \rightarrow -\infty$ , we have ( $P_- > 0$ )

- There is one solution of the Schrödinger equation, defined up to a multiplicative constant, that vanishes at least as  $e^{P-q}$ .
- Any other solution diverges at least as  $e^{-P-q}$ .

These properties imply that if there is a solution of the Schrödinger equation in  $L^2(\mathbb{R})$ , then any solution is either in  $L^2(\mathbb{R})$  or diverges both at  $-\infty$  and  $+\infty$ . Let us show that the possibility that a solution vanishes only at one of the two spatial infinities is excluded. Suppose that, besides the  $L^2(\mathbb{R})$  solution, which we denote by  $\psi_1$ , there is a solution  $\psi_2$  which is divergent only at  $+\infty$ . On the other hand, the above properties show that there exists also a solution which is divergent at  $-\infty$ . Let us denote by  $\psi_3$  this solution. Since the number of linearly independent solutions of the Schrödinger equation is two, we have

$$\psi_3 = A\psi_1 + B\psi_2, \quad (14)$$

for some constants  $A$  and  $B$ . However, since  $\psi_1$  vanishes both at  $-\infty$  and  $+\infty$ , we have that (14) cannot be satisfied unless  $\psi_2$  and  $\psi_3$  are divergent both at  $-\infty$  and  $+\infty$ . This fact and the above properties imply the following

*If the Schrödinger equation has an  $L^2(\mathbb{R})$  solution, then any solution has one of the following two possible asymptotic behaviors*

- Vanishes both at  $-\infty$  and  $+\infty$  at least as  $e^{P-q}$  and  $e^{-P+q}$  respectively.
- Diverges both at  $-\infty$  and  $+\infty$  at least as  $e^{-P-q}$  and  $e^{P+q}$  respectively.

Similarly, we have

*If the Schrödinger equation does not admit an  $L^2(\mathbb{R})$  solution, then any solution has one of the following three possible asymptotic behaviors*

- Diverges both at  $-\infty$  and  $+\infty$  at least as  $e^{-P-q}$  and  $e^{P+q}$  respectively.
- Diverges at  $-\infty$  at least as  $e^{-P-q}$  and vanishes at  $+\infty$  at least as  $e^{-P+q}$ .
- Vanishes at  $-\infty$  at least as  $e^{P-q}$  and diverges at  $+\infty$  at least as  $e^{P+q}$ .

Let us consider the ratio  $w = \psi^D/\psi$  in the latter case. Since any different choice of linearly independent solutions of the Schrödinger equation corresponds to a Möbius transformation of  $w$ , we can choose<sup>3</sup>

$$\psi^D \underset{q \rightarrow -\infty}{\sim} a_- e^{P-q}, \quad \psi^D \underset{q \rightarrow +\infty}{\sim} a_+ e^{P+q}, \quad (15)$$

and

$$\psi \underset{q \rightarrow -\infty}{\sim} b_- e^{-P-q}, \quad \psi \underset{q \rightarrow +\infty}{\sim} b_+ e^{-P+q}. \quad (16)$$

Their ratio has the asymptotics

$$\frac{\psi^D}{\psi} \underset{q \rightarrow -\infty}{\sim} c_- e^{2P-q} \rightarrow 0, \quad \frac{\psi^D}{\psi} \underset{q \rightarrow +\infty}{\sim} c_+ e^{2P+q} \rightarrow \pm\infty, \quad (17)$$

so that  $w$  cannot satisfy the continuity condition (6) at  $\pm\infty$ .

The above results imply that the quantized spectrum one obtains from the conventional approach to quantum mechanics arises as a consequence of the equivalence principle. Actually, even in the conventional approach the

<sup>3</sup> Here by  $\sim$  we mean that  $\psi^D$  and  $\psi$  either diverge or vanish “at least as”.

quantized spectrum and its structure arose by the condition that the values of  $E$  satisfying (7) should correspond to a Schrödinger equation having an  $L^2(\mathbb{R})$  solution. Then, all the standard results on the quantized spectrum are reproduced in our formulation.

Let  $n$  be the index  $I[q^0]$  of the trivializing map. This is the number of times  $q^0$  spans  $\hat{\mathbb{R}}$  while  $q$  spans  $\hat{\mathbb{R}}$ . In other words,  $n$  is the index of the covering associated to the trivializing map. Since  $q^0$  and  $w$  are related by a Möbius transformation [3], we have that the index of  $q$  and  $w$  coincide

$$I[q^0] = I[w]. \quad (18)$$

Another property of the trivializing map is that its index depends on  $\mathcal{W}$  but not on the specific Möbius state. The Möbius states are the states with the same  $\mathcal{W}$  but with different values of the constant  $\ell$  (see [3] for related aspects). Observe that since  $p = \partial_q \mathcal{S}_0$  does not vanish for finite values of  $w$ , it follows that  $I[q^0]$  coincides with the number of zeroes of  $w$ . This aspect may be also understood by recalling a Sturm theorem about second-order differential equations. The theorem states that given two linearly independent solutions  $\psi^D$  and  $\psi$  of the equation  $\psi''(q) = K(q)\psi(q)$ , between any two zeroes of  $\psi^D$  there is one zero of  $\psi$ .<sup>4</sup> This theorem, and the condition that the values of  $E$  satisfying (7) should correspond to a Schrödinger equation having an  $L^2(\mathbb{R})$  solution guarantees local homeomorphicity of the trivializing map. It remains to understand the case in which  $V(q) > E$ ,  $\forall q \in \mathbb{R}$ . We already noticed that, since there is not the classical limit in this case, these solutions are not admissible ones. We now show that these solutions do not satisfy the continuity condition for  $w$  at  $\pm\infty$ . To see this it is sufficient to note that if  $\psi$  decreases as  $q \rightarrow -\infty$ , then by  $\psi''/\psi = 4m\mathcal{W}/\hbar^2 > 0$ ,  $\forall q \in \mathbb{R}$ , it follows that  $\psi$  is always convex,  $\psi \notin L^2(\mathbb{R})$ . The absence of turning points does not modify the essence of the above conclusions and if  $V(q) > E$ ,  $\forall q \in \mathbb{R}$ , then the ratio  $\psi^D/\psi$  is discontinuous at  $\pm\infty$ . As an example, let us consider the equation

$$\frac{\hbar^2}{2m} \partial_q^2 \psi = a^2 \psi. \quad (19)$$

Any pair of linearly independent solutions has the form

$$\psi^D = Ae^{aq} + Be^{-aq}, \quad \psi = Ce^{aq} + De^{-aq}, \quad (20)$$

where

$$AD - BC \neq 0. \quad (21)$$

Their ratio

$$\frac{\psi^D}{\psi} = \frac{Ae^{2aq} + B}{Ce^{2aq} + D}, \quad (22)$$

has the asymptotics

$$\lim_{q \rightarrow -\infty} \frac{\psi^D}{\psi} = \frac{B}{D}, \quad \lim_{q \rightarrow +\infty} \frac{\psi^D}{\psi} = \frac{A}{C}, \quad (23)$$

so that by (21) neither the case  $w(-\infty) = \text{finite} = w(+\infty)$ , nor  $w(-\infty) = -w(+\infty) = \pm\infty$  can occur.

We now consider the cases of the potential well and of the simple and double harmonic oscillators. We will explicitly see that in the case one considers the energy values for which the corresponding Schrödinger equation has not  $L^2(\mathbb{R})$  solutions, the ratio  $\psi^D/\psi$  has a discontinuity at spatial infinity.

<sup>4</sup> Observe that this can be seen as a sort of duality between  $\psi^D$  and  $\psi$ . In this context we note that, while in the conventional approach one usually selects the wave function which is a particular solution of the Schrödinger equation,  $\mathcal{S}_0$  and  $p$  contain both  $\psi^D$  and  $\psi$ .

Let us consider the potential well

$$V(q) = \begin{cases} 0, & |q| \leq L, \\ V_0, & |q| > L, \end{cases} \quad (24)$$

and set

$$k = \frac{\sqrt{2mE}}{\hbar}, \quad \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}. \quad (25)$$

According to (4), in order to determine  $\mathcal{S}_0$ , and therefore to solve the dynamical problem, we have to find two real linearly independent solutions of the Schrödinger equation. Since the potential is even, we can choose solutions of definite parity. We have

$$\psi = k^{-1} \cdot \begin{cases} -\alpha \exp[\kappa(q+L)] - \beta \exp[-\kappa(q+L)], & q < -L, \\ \sin(kq), & |q| \leq L, \\ \alpha \exp[-\kappa(q-L)] + \beta \exp[\kappa(q-L)], & q > L, \end{cases} \quad (26)$$

where for any  $E \geq 0$

$$\alpha = \frac{1}{2} \sin(kL) - \frac{k}{2\kappa} \cos(kL), \quad \beta = \frac{1}{2} \sin(kL) + \frac{k}{2\kappa} \cos(kL). \quad (27)$$

For the dual solution, we have

$$\psi^D = \begin{cases} \gamma \exp[\kappa(q+L)] + \delta \exp[-\kappa(q+L)], & q < -L, \\ \cos(kq), & |q| \leq L, \\ \gamma \exp[-\kappa(q-L)] + \delta \exp[\kappa(q-L)], & q > L, \end{cases} \quad (28)$$

where

$$\gamma = \frac{1}{2} \cos(kL) + \frac{k}{2\kappa} \sin(kL), \quad \delta = \frac{1}{2} \cos(kL) - \frac{k}{2\kappa} \sin(kL). \quad (29)$$

The ratio of the solutions is given by

$$\frac{\psi^D}{\psi} = k \cdot \begin{cases} -(\gamma \exp[\kappa(q+L)] + \delta \exp[-\kappa(q+L)]) / (\alpha \exp[\kappa(q+L)] + \beta \exp[-\kappa(q+L)]), & q < -L, \\ \cot(kq), & |q| \leq L, \\ (\gamma \exp[-\kappa(q-L)] + \delta \exp[\kappa(q-L)]) / (\alpha \exp[-\kappa(q-L)] + \beta \exp[\kappa(q-L)]), & q > L, \end{cases} \quad (30)$$

whose asymptotic behavior is

$$\lim_{q \rightarrow \pm\infty} \frac{\psi^D}{\psi} = \pm \frac{\delta}{\beta} k. \quad (31)$$

Continuity at  $\pm\infty$  implies that either

$$\beta = 0, \quad (32)$$

so that  $w(-\infty) = -\operatorname{sgn} \delta \cdot \infty = -w(+\infty)$ , or

$$\delta = 0, \quad (33)$$

so that  $w(-\infty) = 0 = w(+\infty)$ . It is easy to see that (27), (29), (32) and (33) identify the usual quantized spectrum.

Let us now consider the double and simple harmonic oscillators. The Hamiltonian describing the relative motion of the reduced mass of the double harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(|q| - q_0)^2. \quad (34)$$

The reduced action for this system is given by (4) with  $\psi^D$  and  $\psi$  real linearly independent solutions of the Schrödinger equation

$$\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} + \frac{1}{2}m\omega^2(|q| - q_0)^2 \right) \psi = E\psi. \quad (35)$$

Let us set

$$E = \left( \mu + \frac{1}{2} \right) \hbar \omega, \quad (36)$$

and

$$z' = \sqrt{\frac{2m\omega}{\hbar}} (q + q_0), \quad q \leq 0, \quad z = \sqrt{\frac{2m\omega}{\hbar}} (q - q_0), \quad q \geq 0. \quad (37)$$

Observe that  $z'(-q) = -z(q)$  and that for  $q_0 = 0$ ,  $z' = z = \sqrt{\frac{2m\omega}{\hbar}} q$ , so that the system reduces to the simple harmonic oscillator.

We stress that since we are considering Eq. (35) for arbitrary real  $E$ , we have that at this stage  $\mu$  is an arbitrary real number.

The Schrödinger equation (35) is equivalent to

$$\frac{\partial^2 \psi}{\partial z'^2} + \left( \mu + \frac{1}{2} - \frac{z'^2}{4} \right) \psi = 0, \quad q \leq 0, \quad (38)$$

and

$$\frac{\partial^2 \psi}{\partial z^2} + \left( \mu + \frac{1}{2} - \frac{z^2}{4} \right) \psi = 0, \quad q \geq 0. \quad (39)$$

For any  $\mu$  we have that a solution of (39) is given by the parabolic cylinder function (see for example [5])

$$D_\mu(z) = 2^{\mu/2} e^{-z^2/4} \left[ \frac{\Gamma(1/2)}{\Gamma[(1-\mu)/2]} {}_1F_1(-\mu/2; 1/2; z^2/2) + \frac{z}{\sqrt{2}} \frac{\Gamma(-1/2)}{\Gamma(-\mu/2)} {}_1F_1[(1-\mu)/2; 3/2; z^2/2] \right], \quad (40)$$

where  ${}_1F_1$  is the confluent hypergeometric function

$${}_1F_1(a; c; z) = 1 + \frac{a}{c} \frac{z}{1!} + \frac{a(a+1)}{c(c+1)} \frac{z^2}{2!} + \dots \quad (41)$$

We are interested in considering the continuity of the ratio  $w = \psi^D/\psi$  at  $\pm\infty$ . That is, if  $w(-\infty) \neq \pm\infty$  we have to impose  $w(-\infty) = w(+\infty)$ , while if  $w(-\infty) = \pm\infty$ , then we should have  $w(-\infty) = -w(+\infty)$ . To study

the continuity at  $\pm\infty$  we need the behavior of  $D_\mu$  for  $|z| \gg 1$ ,  $|z| \gg |\mu|$ . In the case  $\pi/4 < \arg z < 5\pi/4$  we have

$$D_\mu(z) \underset{|z| \gg 1}{\sim} -\frac{\sqrt{2\pi}}{\Gamma(-\mu)} e^{\mu\pi i} e^{z^2/4} z^{-\mu-1} \left[ 1 + \frac{(\mu+1)(\mu+2)}{2z^2} + \frac{(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{2 \cdot 4 z^4} + \dots \right] + e^{-z^2/4} z^\mu \left[ 1 - \frac{\mu(\mu-1)}{2z^2} + \frac{\mu(\mu-1)(\mu-2)(\mu-3)}{2 \cdot 4 z^4} - \dots \right], \quad (42)$$

while for  $|\arg z| < 3\pi/4$

$$D_\mu(z) \underset{|z| \gg 1}{\sim} e^{-z^2/4} z^\mu \left[ 1 - \frac{\mu(\mu-1)}{2z^2} + \frac{\mu(\mu-1)(\mu-2)(\mu-3)}{2 \cdot 4 z^4} - \dots \right]. \quad (43)$$

A property of the cylinder parabolic function is that even  $D_\mu(-z)$  is a solution of (39). In particular, if  $\mu$  is a non-negative integer, then  $D_\mu(z)$  and  $D_\mu(-z)$  coincide.<sup>5</sup> Let us then consider the values of  $\mu$  different from a non-negative integer. We have

$$\psi = \begin{cases} D_\mu(-z') + cD_\mu(z'), & q \leq 0, \\ D_\mu(z) + cD_\mu(-z), & q \geq 0, \end{cases} \quad (44)$$

where

$$c = \frac{D'_\mu(-\alpha q_0)}{D'_\mu(\alpha q_0)}. \quad (45)$$

For any given  $\mu$ , a linearly independent solution is given by

$$\psi^D = \begin{cases} -D_\mu(-z') - dD_\mu(z'), & q \leq 0, \\ D_\mu(z) + dD_\mu(-z), & q \geq 0, \end{cases} \quad (46)$$

where

$$d = -\frac{D_\mu(-\alpha q_0)}{D_\mu(\alpha q_0)}. \quad (47)$$

The ratio

$$\frac{\psi_+^D}{\psi_-} = \begin{cases} -(D_\mu(-z') + dD_\mu(z')) / (D_\mu(-z') + cD_\mu(z')), & q \leq 0, \\ (D_\mu(z) + dD_\mu(-z)) / (D_\mu(z) + cD_\mu(-z)), & q \geq 0, \end{cases} \quad (48)$$

has the asymptotics behavior

$$\lim_{q \rightarrow \pm\infty} \frac{\psi^D}{\psi} = \pm \frac{d}{c}. \quad (49)$$

This shows that continuity at  $\pm\infty$  is satisfied in the case in which either  $c = 0$  or  $d = 0$ , which fix the standard energy quantized spectra (see also [3]).

<sup>5</sup> Note that in the case in which  $\mu$  is a non-negative integer we have  $\Gamma^{-1}(-\mu) = 0$ , so that in this case the first term in (42) cancels.



Observe that  $\psi$  in (44) and the dual solution (46) are not linearly independent in the case in which  $\mu = 0, 1, 2, \dots$ . In this case there is always a solution vanishing both at  $-\infty$  and  $+\infty$ . In the case  $q_0 = 0$ , this situation corresponds to the harmonic oscillator. Generally, for an arbitrary  $q_0$  and  $\mu = 0, 1, 2, \dots$ , the solution  $\psi^D$  in (46) is replaced by

$$\psi^D = \begin{cases} -D_n(-z') - d'D_{-n-1}(-iz'), & q \leq 0, \\ D_n(z) + d'D_{-n-1}(iz), & q \geq 0, \end{cases} \quad (50)$$

where now

$$d' = -\frac{D_n(-\alpha q_0)}{D_n(-i\alpha q_0)}. \quad (51)$$

We note that above we used the parabolic cylinder functions with real argument. Then, the fact that for  $\mu = 0, 1, 2, \dots$ ,  $D_\mu(z)$  and  $D_\mu(-z)$  are not linearly independent forced us to use  $D_\mu(-iz)$ . In this context we observe that  $D_\mu(z)$  and  $D_\mu(-iz)$  are always linearly independent so that the dual solution (50) can be extended to arbitrary values of  $\mu$ .

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