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Equivalence principle, Planck length and quantum Hamilton–Jacobi equation

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Abstract

The Quantum Stationary HJ Equation (QSHJE) that we derived from the equivalence principle, gives rise to initial conditions which cannot be seen in the Schrödinger equation. Existence of the classical limit leads to a dependence of the integration constant $\ell = \ell_1 + i\ell_2$ on the Planck length. Solutions of the QSHJE provide a trajectory representation of quantum mechanics which, unlike Bohm's theory, has a non-trivial action even for bound states and no wave guide is present. The quantum potential turns out to be an intrinsic potential energy of the particle which, similarly to the relativistic rest energy, is never vanishing.

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Let us consider a one-dimensional stationary system of energy E and potential V and set $\mathscr{W} \equiv V(q) - E$. In [1] the following equivalence principle has been formulated

For each pair $\mathscr{W}^a, \mathscr{W}^b$, there is a transformation $q^a \to q^b = v(q^a)$, such that

$$\mathscr{W}^a(q^a) \to \mathscr{W}^{av}(q^b) = \mathscr{W}^b(q^b). \qquad (1)$$

Implementation of this principle uniquely leads to the Quantum Stationary HJ Equation (QSHJE) [1]

$$\frac{1}{2m}\left(\frac{\partial \mathscr{S}_0(q)}{\partial q}\right)^2 + V(q) - E + \frac{\hbar^2}{4m}\{\mathscr{S}_0, q\} = 0, \qquad (2)$$

where $\mathscr{S}_0$ is the Hamilton's characteristic function also called reduced action. In this equation the Planck constant plays the role of covariantizing parameter. The fact that a fundamental constant follows from the equivalence principle suggests that other fundamental constants as well may be related to such a principle.

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We have seen in [1] that the implementation of the equivalence principle implied a cocycle condition which in turn determines the structure of the quantum potential. A property of the formulation is that unlike in Bohm's theory [2,3], the quantum potential

$$Q(q) = \frac{\hbar^2}{4m}\{\mathscr{S}_0, q\}, \tag{3}$$

like $\mathscr{S}_0$, is never trivial. This reflects in the fact that a general solution of the Schrödinger equation will have the form

$$\psi = \frac{1}{\sqrt{\mathscr{S}_0'}}\left(Ae^{-\frac{i}{\hbar}\mathscr{S}_0} + Be^{\frac{i}{\hbar}\mathscr{S}_0}\right). \tag{4}$$

If (ψ^D,ψ) is a pair of real linearly independent solutions of the Schrödinger equation, then we have

$$e^{\frac{2i}{\hbar}\mathscr{S}_0} = e^{i\alpha}\frac{w + i\bar{\ell}}{w - i\ell}, \tag{5}$$

where $w = \psi^D/\psi$, and $\mathrm{Re}\,\ell \neq 0$. We note that ℓ, on which the dynamics depends, does not appear in the conventional formulation of quantum mechanics.

In this Letter we show that non-triviality of the quantum potential of the free particle with vanishing energy is at the heart of the existence of a length scale. Actually, we will show the appearance of the Planck length in the complex integration constant ℓ, indicating that gravity is intrinsically and deeply connected with quantum mechanics. Therefore, there is trace of gravity in the constant ℓ whose role is that of initial condition for the dynamical Eq. (2). In this context we stress that ℓ plays the role of ''hidden'' constant as it does not appear in the Schrödinger equation. This is a consequence of the fact that whereas the Schrödinger equation is a second-order linear differential equation, the QSHJE is a third-order non-linear one, with the associated dynamics being deeply connected to the Möbius symmetry of the Schwarzian derivative.

Before going further it is worth stressing that the basic difference between our $\mathscr{S}_0$ and the one in Bohm's theory [2,3] arises for bound states. In this case the wave function ψ is proportional to a real function. This implies that with Bohm's identification $\psi(q) = R(q)\exp(i\mathscr{S}_0/\hbar)$, one would have $\mathscr{S}_0 = \mathrm{cnst}$. Therefore, all bound states, like in the case of the harmonic oscillator, would have $\mathscr{S}_0 = \mathrm{cnst}$ (for which the Schwarzian derivative is not defined). Besides the difficulties in getting a non-trivial classical limit for $p = \partial_q\mathscr{S}_0$, this seems an unsatisfactory feature of Bohm's theory which completely disappears if one uses Eq. (4). This solution directly follows from the QSHJE: reality of ψ simply implies that $|A|^2 = |B|^2$ and there is no trace of the solution $\mathscr{S}_0 = \mathrm{cnst}$ of Bohm theory. Furthermore, we would like to remark that in Bohm's approach some interpretational aspects are related to the concept of a pilot-wave guide. There is no need for this in the present formulation. This aspect and related ones have been investigated also by Floyd [4]. Nevertheless, there are some similarities between the approach and Bohm's interpretation of quantum mechanics. In particular, solutions of the QSHJE provide a trajectory representation of quantum mechanics. That is for a given quantum mechanical system, by solving the QSHJE for $\mathscr{S}_0(q)$, we can evaluate $p = \partial_q\mathscr{S}_0(q)$ as a function of the initial conditions. Thus, for a given set of initial conditions we have a predetermined orbit in phase space. The solutions to the third-order non-linear QSHJE are obtained by utilizing the two linearly independent solutions of the corresponding Schrödinger equation.

In the case of the free particle with vanishing energy, i.e. with $\mathscr{W}(q) = \mathscr{W}^0(q^0) \equiv 0$, we have that two real linearly independent solutions of the Schrödinger equation are $\psi^{D^0} = q^0$ and $\psi^0 = 1$. As the different dimensional properties of p and q led to introduce the Planck constant in the QSHJE [1], in the case of ψ^{D^0} and ψ^0 we have to introduce a length constant.

Let us derive the quantum potential in the case of the state $\mathscr{W}^0$. By (3) and (5) we have

$$Q^0 = -\frac{p_0^2}{2m} = -\frac{\hbar^2\left(\ell_0 + \bar{\ell}_0\right)^2}{8m|q^0 - i\ell_0|^4}, \tag{6}$$

where $p_0 = \partial_{q^0}\mathscr{S}_0^0(q^0)$. Therefore, we see that the quantum potential is an intrinsic property of the particle as even in the case of the free particle of vanishing energy one has $Q_0 \not\equiv 0$. This is strictly related to the local homeomorphicity properties that the ratio of any real pair of linearly independent solutions of the Schrödinger equation should satisfy, a consequence of the Möbius symmetry of the Schwarzian derivative [5]. In this context let us recall that $\{\mathscr{S}_0, q\}$ is not defined for $\mathscr{S}_0 = \text{cnst}$. More generally, the QSHJE is well defined if and only if the corresponding $w \neq \text{cnst}$ is of class $C^2(\hat{\mathbb{R}})$ with $\partial_q^2 w$ differentiable on the extended real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ [5].

We stress that Q does not correspond to the Bohm's quantum potential which, in a different context, was considered as internal potential in [6]. Here we have the basic fact that the quantum potential is never trivial, an aspect which is strictly related to p–q duality and to the existence of the Legendre transformation of $\mathscr{S}_0$ for any $\mathscr{W}$ [1,5].

Besides the different dimensionality of ψ^{D^0} and ψ^0, we also note that similarity between the equivalence principle we formulated and the one at the heart of general relativity would suggest the appearance of some other fundamental constants besides the Planck constant. We now show that non-triviality of Q^0 or, equivalently, of p_0, has an important consequence in considering the classical and $E \to 0$ limits in the case of the free particle. In doing this, we will see the appearance of the Planck length in the complex integration constant ℓ of the QSHJE.

Let us consider the conjugate momentum in the case of the free particle of energy E. We have [5]

$$p_E = \pm \frac{\hbar\left(\ell_E + \bar{\ell}_E\right)}{2|k^{-1}\sin kq - i\ell_E \cos kq|^2}, \tag{7}$$

where $k = \sqrt{2mE}/\hbar$. The first condition is that in the $\hbar \to 0$ limit the conjugate momentum reduces to the classical one

$$\lim_{\hbar \to 0} p_E = \pm\sqrt{2mE}. \tag{8}$$

On the other hand, we should also have

$$\lim_{E \to 0} p_E = p_0 = \pm \frac{\hbar\left(\ell_0 + \bar{\ell}_0\right)}{2|q - i\ell_0|^2}. \tag{9}$$

Let us first consider the limit (8). By (7) we see that in order to reach the classical value $\sqrt{2mE}$ in the $\hbar \to 0$ limit, the quantity ℓ_E should depend on E. Let us set

$$\ell_E = k^{-1} f(E,\hbar) + \lambda_E, \tag{10}$$

where f is dimensionless. Since λ_E is still arbitrary, we can choose f to be real. By (7) we have

$$p_E = \pm \frac{\sqrt{2mE}\, f(E,\hbar) + mE\left(\lambda_E + \bar{\lambda}_E\right)/\hbar}{\left|e^{ikq} + \left(f(E,\hbar) - 1 + \lambda_E k\right)\cos kq\right|^2}. \tag{11}$$

Observe that if one ignores λ_E and sets $\lambda_E = 0$, then by (8) we have

$$\lim_{\hbar \to 0} f(E,\hbar) = 1. \tag{12}$$

We now consider the properties that λ_E and f should have in order that (12) be satisfied in the physical case in which λ_E is arbitrary but for the condition $\mathrm{Re}\,\ell_E \neq 0$, as required by the existence of the QSHJE. First of all note that cancellation of the divergent term $E^{-1/2}$ in

$$p_E \underset{E \to 0}{\sim} \pm \frac{2\hbar^2(2mE)^{-1/2} f(E,\hbar) + \hbar\left(\lambda_E + \bar{\lambda}_E\right)}{2|q - i\hbar(2mE)^{-1/2} f(E,\hbar) - i\lambda_E|^2}, \tag{13}$$

yields

$$\lim_{E \to 0} E^{-1/2} f(E,\hbar) = 0. \tag{14}$$

The limit (9) can be seen as the limit in which the trivializing map reduces to the identity. Actually, the trivializing map, introduced in [1] and further investigated in [5], which connects the state $\mathscr{W} = -E$ with the state $\mathscr{W}^0$, reduces to the identity map in the $E \to 0$ limit. In the above investigation we considered q as independent variable, however one can also consider $q_E(q^0)$ so that $\lim_{E \to 0} q_E = q^0$ and in the above formulas one can replace q with q_E.

We know from (14) that k must enter in the expression of $f(E,\hbar)$. Since f is a dimensionless constant, we need at least one more constant with the dimension of a length. Two fundamental lengths one can consider are the Compton length

$$\lambda_c = \frac{\hbar}{mc}, \tag{15}$$

and the Planck length

$$\lambda_p = \sqrt{\frac{\hbar G}{c^3}}. \tag{16}$$

Two dimensionless quantities depending on E are

$$x_c = k\lambda_c = \sqrt{\frac{2E}{mc^2}}\,, \tag{17}$$

and

$$x_p = k\lambda_p = \sqrt{\frac{2mEG}{\hbar c^3}}\,. \tag{18}$$

On the other hand, concerning x_c we see that it does not depend on $\hbar$ so that it cannot be used to satify (12). Therefore, we see that a natural expression for f is a function of the Planck length times k. Let us set

$$f(E,\hbar) = e^{-\alpha(x_p^{-1})}, \tag{19}$$

where

$$\alpha\left(x_p^{-1}\right) = \sum_{k\geq 1} \alpha_k x_p^{-k}. \tag{20}$$

The conditions (12), (14) correspond to conditions on the coefficients α_k. For example, in the case in which one considers α to be the function

$$\alpha\left(x_p^{-1}\right) = \alpha_1 x_p^{-1}, \tag{21}$$

then by (14) we have

$$\alpha_1 > 0. \tag{22}$$

In order to consider the structure of λ_E, we note that although $e^{-\alpha(x_p^{-1})}$ cancelled the $E^{-1/2}$ divergent term, we still have some conditions to be satisfied. To see this note that

$$p_E = \pm \frac{\sqrt{2mE}\, e^{-\alpha(x_p^{-1})} + mE\left(\lambda_E + \bar{\lambda}_E\right)/\hbar}{\left| e^{ikq} + \left(e^{-\alpha(x_p^{-1})} - 1 + k\lambda_E\right)\cos kq\right|^2}\,, \tag{23}$$

so that the condition (8) implies

$$\lim_{\hbar\to 0} \frac{\lambda_E}{\hbar} = 0. \tag{24}$$

To discuss this limit, we first note that

$$p_E = \pm \frac{2\hbar k^{-1} e^{-\alpha(x_p^{-1})} + \hbar\left(\lambda_E + \bar{\lambda}_E\right)}{2\left| k^{-1}\sin kq - i\left(k^{-1}e^{-\alpha(x_p^{-1})} + \lambda_E\right)\cos kq\right|^2}. \tag{25}$$

So that, since

$$\lim_{E\to 0} k^{-1} e^{-\alpha(x_p^{-1})} = 0, \tag{26}$$

we have by (9) and (25) that

$$\lambda_0 = \lim_{E\to 0} \lambda_E = \lim_{E\to 0} \ell_E = \ell_0. \tag{27}$$

Let us now consider the limit

$$\lim_{\hbar\to 0} p_0 = 0. \tag{28}$$

First of all note that, since

$$p_0 = \pm \frac{\hbar\left(\ell_0 + \bar{\ell}_0\right)}{2|q^0 - i\ell_0|^2}\,, \tag{29}$$

we have that the effect on p_0 of a shift of $\mathrm{Im}\,\ell_0$ is equivalent to a shift of the coordinate. Therefore, in considering the limit (28) we can set $\mathrm{Im}\,\ell_0 = 0$ and distinguish the cases $q^0 \neq 0$ and $q^0 = 0$. Observe that as we always have $\mathrm{Re}\,\ell_0 \neq 0$, it follows that the denominator in the right hand side of (29) is never vanishing. Let us define γ by

$$\mathrm{Re}\,\ell_0 \underset{\hbar\to 0}{\sim} \hbar^\gamma. \tag{30}$$

We have

$$p_0 \underset{\hbar\to 0}{\sim} \begin{cases} \hbar^{\gamma+1}, & q_0 \neq 0, \\ \hbar^{1-\gamma}, & q_0 = 0, \end{cases} \tag{31}$$

and by (28)

$$-1 < \gamma < 1. \tag{32}$$

A constant length having powers of $\hbar$ can be constructed by means of λ_c and λ_p. We also note that a constant length independent from $\hbar$ is provided by $\lambda_e = e^2/mc^2$ where e is the electric charge. Then ℓ_0 can be considered as a suitable function of λ_c, λ_p and λ_e satisfying the constraint (32).

The above investigation indicates that a natural way to express λ_E is given by

$$\lambda_E = e^{-\beta(x_p)}\lambda_0, \tag{33}$$

where

$$\beta(x_p) = \sum_{k\geq 1} \beta_k x_p^k. \tag{34}$$

Any possible choice of $\beta(x_p)$ should satisfy the conditions (24) and (27). For example, for the modulus ℓ_E built with $\beta(x_p) = \beta_1 x_p$, one should have $\beta_1 > 0$.

Summarizing, by (10), (19), (27) and (33) we have

$$\ell_E = k^{-1} e^{-\alpha(x_p^{-1})} + e^{-\beta(x_p)} \ell_0 , \qquad (35)$$

where $\ell_0 = \ell_0(\lambda_c, \lambda_p, \lambda_e)$, and for the conjugate momentum of the state $\mathscr{W} = -E$ we have

$$p_E = \pm \frac{2k^{-1}\hbar e^{-\alpha(x_p^{-1})} + \hbar e^{-\beta(x_p)}\left(\ell_0 + \bar{\ell}_0\right)}{2\left|k^{-1}\sin kq - i\left(k^{-1}e^{-\alpha(x_p^{-1})} + e^{-\beta(x_p)}\ell_0\right)\cos kq\right|^2} . \qquad (36)$$

It is worth recalling that the conjugate momentum does not correspond to the mechanical momentum $m\dot{q}$ [4,5]. However, the velocity itself is strictly related to it. In particular, following the suggestion by Floyd [4] of using Jacobi's theorem to define time parametrization, we have that velocity and conjugate momentum satisfy the relation [5]

$$\dot{q} = \frac{1}{\partial_E p} , \qquad (37)$$

which holds also classically.

We stress that the appearance of the Planck length is strictly related to p–q duality and to the existence of the Legendre transformation of $\mathscr{S}_0$ for any state. This p–q duality has a counterpart in the ψ^D–ψ duality [1,5] which sets a length scale which already appears in considering linear combinations of $\psi^{D^0} = q^0$ and $\psi^0 = 1$. This aspect is related to the fact that we always have $\mathscr{S}_0 \neq \text{cnst}$ and $\mathscr{S}_0 \not\propto q + \text{cnst}$, so that also for the states $\mathscr{W}^0$ and $\mathscr{W} = -E$ one has a non-constant conjugate momentum. In particular, the Planck length naturally emerges in considering $\lim_{E\to 0} p_E = p_0$, together with the analysis of the $\hbar \to 0$ limit of both p_E and p_0.

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