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The equivalence principle of quantum mechanics: uniqueness theorem

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Abstract

Recently we showed that the postulated diffeomorphic equivalence of states implies quantum mechanics. This approach takes the canonical variables to be dependent by the relation $p = \partial_q \mathcal{S}_0$ and exploits a basic $GL(2, C)$ -symmetry which underlies the canonical formalism. In particular, we looked for the special transformations leading to the free system with vanishing energy. Furthermore, we saw that while on the one hand the equivalence principle cannot be consistently implemented in classical mechanics, on the other it naturally led to the quantum analogue of the Hamilton–Jacobi equation, thus implying the Schrödinger equation. In this letter we show that actually the principle uniquely leads to this solution. Furthermore, we find the map reducing any system to the free one with vanishing energy and derive the transformations on \mathcal{S}_0 leaving the wave function invariant. We also express the canonical and Schrödinger equations by means of the brackets recently introduced in the framework of $N = 2$ SYM. These brackets are the analogue of the Poisson brackets with the canonical variables taken as dependent. © 1998 Published by Elsevier Science B.V. All rights reserved.

It is well-known that the classical Hamilton–Jacobi (HJ) formalism stems from the problem of finding the canonical transformation yielding a vanishing Hamiltonian. In [1] we took the canonical variables q and p as dependent through the momentum generating function, that is $p = \partial_q \mathcal{S}_0$, and, according to the diffeomorphic equivalence principle [1], we looked for coordinate transformations con-

necting different physical systems including the free one with vanishing energy.

The equivalence principle was suggested by a basic $GL(2, C)$ -symmetry of the canonical equation associated to the Legendre transformation of the Hamilton’s characteristic function. This connection between the Legendre transformation and differential equations, which was used in the framework of the Schrödinger equation in [2], had been introduced in [3] for deriving the inversion formula in $N = 2$ super Yang–Mills, and had been further investigated in [4]. The formalism naturally fits with the brackets intro-

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duced in [5]. Remarkably, we can express the canonical [1] and Schrödinger equations in terms of these brackets that in our approach are analogous to the Poisson brackets with the canonical variables taken as dependent.

A basic step in the construction was the proof that the equivalence principle cannot be consistently implemented in classical mechanics. Actually, this principle leads to the quantum analogue of the HJ equation and in turn implies the Schrödinger equation [1]. We now proceed to show that the equivalence principle uniquely leads to this solution.

Let us start with a very explicit example of the transformations we will consider. Given two functions, say $f_1(x_1) = x_1^m$, $f_2(x_2) = x_2^n$, we can associate the coordinate transformation $x_1 \rightarrow x_2 = x_1^{m/n}$ which is naturally induced by the identification $f_2(x_2) = f_1(x_1)$. This is equivalent to say that given the function $f_1(x_1) = x_1^m$, the map $x_1 \rightarrow x_2 = v(x_1) = x_1^{m/n}$ induces the transformation $f_1 \rightarrow f_2$, defined by $f_2(x_2) = f_1(x_1)$. In other words, the diffeomorphism $x_1 \rightarrow x_2 = v(x_1)$ induces the functional transformation $f_1 \rightarrow f_2 = f_1 \circ v^{-1}$.

Let us now consider the case of two physical systems with Hamilton's characteristic functions³ \mathcal{S}_0 and \mathcal{S}_0^v . Let us denote the coordinates of the two systems by q and q^v respectively. Setting

$$\mathcal{S}_0^v(q^v) = \mathcal{S}_0(q), \quad (1)$$

induces the map

$$q \rightarrow q^v = v(q), \quad (2)$$

where $v = \mathcal{S}_0^{v^{-1}} \circ \mathcal{S}_0$, with $\mathcal{S}_0^{v^{-1}}$ denoting the inverse of \mathcal{S}_0^v . This construction is equivalent to say that the map (2) induces the transformation $\mathcal{S}_0 \rightarrow \mathcal{S}_0^v = \mathcal{S}_0 \circ v^{-1}$, that is $\mathcal{S}_0(q) \rightarrow \mathcal{S}_0^v(q^v) = \mathcal{S}_0(q(v^{-1}(q^v)))$. In other words, for a given v there is the induced map v^{-1*} defined by

$$v^{-1*} : \mathcal{S}_0 \mapsto v^{-1*}(\mathcal{S}_0),$$

that is $\mathcal{S}_0^v(q^v) = \mathcal{S}_0(v^{-1}(q^v))$ so that \mathcal{S}_0^v is the pullback of \mathcal{S}_0 by v^{-1*} . We will call the diffeo-

morphism (2) v -transformations. Let us consider the Legendre transformation [1]

$$\mathcal{S}_0(q) = pq - \mathcal{T}_0(p), \quad (3)$$

$$p = \frac{\partial \mathcal{S}_0}{\partial q}, \quad q = \frac{\partial \mathcal{T}_0}{\partial p}. \quad (4)$$

The second derivative of (3) with respect to $s = \mathcal{S}_0(q)$ yields the "canonical equation"

$$(\partial^{2s} + \mathcal{W}(s))q\sqrt{p} = 0 = (\partial^{2s} + \mathcal{W}(s))\sqrt{p}, \quad (5)$$

with $\mathcal{W}(s) = \{q\sqrt{p}/\sqrt{p}, s\}/2 = \{q, s\}/2$, and where $\{h(x), x\} = \frac{h''(x)}{h'(x)} - \frac{3}{2} \left(\frac{h'(x)}{h(x)} \right)^2$, denotes the Schwarzian derivative. Observe that the choice of the coordinates q and q^v , which of course does not imply any loss of generality as both q and q^v play the role of independent coordinate in their own system, allows us to look at the reduced action as a scalar function. In particular, since $\mathcal{S}_0^v(q^v) = \mathcal{S}_0(q)$, we see that the transformations (2) leave the Legendre transformation of \mathcal{T}_0 (3) unchanged. Consequently, since $\partial_{q^v} \mathcal{S}_0^v(q^v) = (\partial_q q^v)^{-1} \partial_q \mathcal{S}_0(q)$, we have

$$p \rightarrow p_v = (\partial_q q^v)^{-1} p. \quad (6)$$

However, while the Legendre transformation of \mathcal{T}_0 is invariant under arbitrary diffeomorphisms, this is not the case for the canonical potential \mathcal{W} . Nevertheless, there is an important exception as under the $GL(2, \mathcal{C})$ -transformations

$$q^v = (Aq + B)/(Cq + D),$$

$$p_v = p(Cq + D)^2 / (AD - BC), \quad (7)$$

we have $\{(Aq + B)/(Cq + D), s\}/2 = \mathcal{W}(s)$, so that we can speak of the $GL(2, \mathcal{C})$ -symmetry of the canonical equation. Involutivity of the Legendre transformation and the duality

$$\mathcal{S}_0 \leftrightarrow \mathcal{T}_0, \quad q \leftrightarrow p,$$

imply another $GL(2, \mathcal{C})$ -symmetry, with the dual versions of Eq. (5) being

$$(\partial_t^2 + \mathcal{V}(t))p\sqrt{q} = 0 = (\partial_t^2 + \mathcal{V}(t))\sqrt{q}, \quad (8)$$

where $\mathcal{V}(t) = \{p\sqrt{q}/\sqrt{q}, t\}/2 = \{p, t\}/2$, with $t = \mathcal{T}_0(p)$. We note that for $p = \gamma/q$ the solutions

³ In literature the Hamilton's characteristic function is also called reduced action.

of (5) and (8) coincide. Therefore we have the self-dual states

$$\mathcal{S}_0 = \gamma \ln \gamma_q q, \quad \mathcal{F}_0 = \gamma \ln \gamma_p p, \quad (9)$$

where the three constants satisfy

$$\gamma_p \gamma_q \gamma = e. \quad (10)$$

Observe that

$$\mathcal{S}_0 + \mathcal{F}_0 = pq = \gamma, \quad \mathcal{U}(\mathfrak{s}) = -1/4\gamma^2 = \mathcal{V}(\mathfrak{t}). \quad (11)$$

The canonical Eq. (5) and its dual (8) correspond to two equivalent descriptions of the physical system. Remarkably, for the self-dual states the two descriptions overlap. Furthermore, we observe that the canonical equation and its dual are covariant under arbitrary transformations. Actually, under $q \rightarrow \tilde{q}$, $\mathcal{S}_0 \rightarrow \tilde{\mathcal{S}}_0(\tilde{q})$ the transformation properties of \mathcal{F}_0 are determined by the fact that $\tilde{\mathcal{F}}_0(\tilde{p})$ is the Legendre transformation of $\tilde{\mathcal{S}}_0(\tilde{q})$: $\mathcal{F}_0(p) \rightarrow \tilde{\mathcal{F}}_0(\tilde{p}) = \tilde{p}\tilde{q} - \tilde{\mathcal{S}}_0(\tilde{q})$. Repeating the above derivation one sees that the canonical equation and its dual have the same form as the original ones.

The transformations in (2) and (6) do not correspond to canonical transformations. Since p and q are considered dependent, a transformation of q induces a transformation of p and vice versa. Thus, in [1], as in the search for canonical transformations leading to a system with vanishing Hamiltonian one obtains the HJ equation, we looked for transformations on the *dependent* quantities q and $p = \partial_q \mathcal{S}_0(q)$ reducing to the free system with vanishing energy.

The answer to this basic question led to the formulation of an equivalence principle, suggested by the fact that \mathcal{U} , though invariant under Möbius transformations, changes under arbitrary diffeomorphisms. This equivalence principle led to the quantum analogue of the HJ equation [1]. Therefore, we have the following problem: given an arbitrary reduced action $\mathcal{S}_0(q)$, find the map $q \rightarrow q^0 = v_0(q)$, such that the new reduced action \mathcal{S}_0^0 , defined by

$$\mathcal{S}_0^0(q^0) = \mathcal{S}_0(q), \quad (12)$$

corresponds to the free system with vanishing energy. Observe that the structure of the states de-

scribed by \mathcal{S}_0^0 and \mathcal{S}_0 determines the “trivializing coordinate” q^0 to be

$$q \rightarrow q^0 = \mathcal{S}_0^{0^{-1}} \circ \mathcal{S}_0(q), \quad (13)$$

Let us set $\mathcal{W} \equiv V(q) - E$, where V is the potential and E is the energy. We denote by \mathcal{H} the space of all possible \mathcal{W} 's. If the above question has solution then there is the following “diffeomorphic equivalence principle” [1]

For each pair $\mathcal{W}^a, \mathcal{W}^b \in \mathcal{H}$, there is a v -transformation such that

$$\mathcal{W}^a(q) \rightarrow \mathcal{W}^{av}(q^v) = \mathcal{W}^b(q^v). \quad (14)$$

This implies that there always exists the trivializing coordinate q^0 for which $\mathcal{W}(q) \rightarrow \mathcal{W}^0(q^0) \equiv 0$. In particular, since the inverse transformation should exist as well, it is clear that the trivializing transformation should be a continuous, locally one-to-one map.

In [1] it has been shown that this principle cannot be consistently implemented in classical mechanics. Actually, note that the Classical Stationary HJ Equation (CSHJE)

$$\frac{1}{2m} (\partial_q \mathcal{S}_0^{cl}(q))^2 + \mathcal{W}(q) = 0, \quad (15)$$

provides a correspondence between \mathcal{W} and \mathcal{S}_0^{cl} . In particular, $\mathcal{S}_0^{cl\ v}(q^v)$ must satisfy the CSHJE $(\partial_{q^v} \mathcal{S}_0^{cl\ v}(q^v))^2 / 2m + \mathcal{W}^v(q^v) = 0$. Since $\mathcal{S}_0^{cl\ v}(q^v) = \mathcal{S}_0^{cl}(q)$, by (15)

$$\mathcal{W}(q) \rightarrow \mathcal{W}^v(q^v) = (\partial_{q^v})^{-2} \mathcal{W}(q). \quad (16)$$

Therefore, in classical mechanics consistency requires that $\mathcal{W}(q)$ belongs to the space \mathcal{Q} of functions transforming as quadratic differentials under v -maps.

Let us now consider the case of the state \mathcal{W}^0 . By (16) it follows that

$$\mathcal{W}^0(q^0) \rightarrow \mathcal{W}^v(q^v) = (\partial_{q^0} q^v)^{-2} \mathcal{W}^0(q^0) = 0. \quad (17)$$

Then we have [1]

In classical mechanics consistency requires that a state \mathcal{W} transforms as a quadratic differential under the v -maps. As a consequence the state \mathcal{W}^0 is a fixed point in \mathcal{H} . Equivalently, in classical mechan-

ics the space \mathcal{H} cannot be reduced to a point upon factorization by the diffeomorphisms. Hence, the equivalence principle (14) cannot be consistently implemented in classical mechanics.

It is therefore clear that in order to preserve the equivalence principle we have to deform the CSHJE. As we will see, this request will determine the equation for \mathcal{S}_0 . Let us discuss its general form. First of all observe that adding a constant to \mathcal{S}_0 does not change the dynamics. Actually, Eqs. (3)(4) are unchanged upon adding a constant to either \mathcal{S}_0 or \mathcal{S}_v . Then, the most general differential equation that \mathcal{S}_0 should satisfy has the structure

$$F(\mathcal{S}_0, \mathcal{S}_0'', \dots) = 0, \quad (18)$$

where $' \equiv \partial_q$. Let us write down Eq. (18) in the general form

$$\frac{1}{2m} (\partial_q \mathcal{S}_0(q))^2 + \mathcal{W}(q) + Q(q) = 0. \quad (19)$$

The properties of $\mathcal{W} + Q$ under the v -transformations (2) are determined by the transformed equation $(\partial_{q^v} \mathcal{S}_0^v(q^v))^2 / 2m + \mathcal{W}^v(q^v) + Q^v(q^v) = 0$ that by (1) and (19) yields

$$\mathcal{W}^v(q^v) + Q^v(q^v) = (\partial_{q^v} q^v)^{-2} (\mathcal{W}(q) + Q(q)), \quad (20)$$

that is

$$(\mathcal{W} + Q) \in \mathcal{Q}. \quad (21)$$

A basic guidance in deriving the differential equation for \mathcal{S}_0 is that in some limit it should reduce to the CSHJE. Therefore, in determining the structure of the Q term we have to take into account that in the classical limit $Q \rightarrow 0$. In doing this we need some parameter which will suitably select the classical phase.

According to the equivalence principle, all the \mathcal{W} 's are connected by a v -transformation. On the other hand, we have seen that if $\mathcal{W} \in \mathcal{Q}$, then \mathcal{W}^0 would be a fixed point in the \mathcal{H} space. This remark and Eq. (21) imply

$$\mathcal{W} \notin \mathcal{Q}, \quad Q \notin \mathcal{Q}. \quad (22)$$

Therefore, the only possible way to reach $\mathcal{W}^v \neq 0$ from \mathcal{W}^0 , is that it transforms with an inhomoge-

neous term. In particular, by (21)(22) it follows that for an arbitrary state \mathcal{W}^a we have

$$\mathcal{W}^v(q^v) = (\partial_{q^a} q^v)^{-2} \mathcal{W}^a(q^a) + (q^a; q^v), \quad (23)$$

and

$$Q^v(q^v) = (\partial_{q^a} q^v)^{-2} Q^a(q^a) - (q^a; q^v). \quad (24)$$

Setting $\mathcal{W}^a = \mathcal{W}^0$ in Eq. (23) yields

$$\mathcal{W}^v(q^v) = (q^0; q^v), \quad (25)$$

so that, according to the equivalence principle (14), all the states correspond to the inhomogeneous part in the transformation of the state \mathcal{W}^0 induced by some diffeomorphism.

Let us denote by a, b, c, \dots different v -transformations. Comparing

$$\mathcal{W}^b(q^b) = (\partial_{q^b} q^a)^2 \mathcal{W}^a(q^a) + (q^a; q^b) = (q^0; q^b), \quad (26)$$

with the same formula with q^a and q^b interchanged we have

$$(q^b; q^a) = -(\partial_{q^a} q^b)^2 (q^a; q^b), \quad (27)$$

and in particular $(q; q) = 0$. More generally, comparing

$$\begin{aligned} \mathcal{W}^b(q^b) &= (\partial_{q^b} q^c)^2 \mathcal{W}^c(q^c) + (q^c; q^b) \\ &= (\partial_{q^b} q^a)^2 \mathcal{W}^a(q^a) + (\partial_{q^b} q^c)^2 (q^a; q^c) \\ &\quad + (q^c; q^b), \end{aligned}$$

with (26) we obtain

$$(q^a; q^c) = (\partial_{q^c} q^b)^2 (q^a; q^b) - (\partial_{q^c} q^b)^2 (q^c; q^b), \quad (28)$$

which is a direct consequence of the equivalence principle.

Thus, we see that the choice of representing the state transformations by the pullback of \mathcal{S}_0 by v^{-1*} is the simplest one. In particular, under the v -transformations \mathcal{W} , Q and $(q^a; q^b)$ transform as projective connections. We will see that Eq. (28), that can be seen as a cocycle condition, implies $(q; \gamma(q)) = 0 = (\gamma(q); q)$, with γ a Möbius transformation. As this is a crucial step in the formulation

we will analyze it in detail. Actually, it is remarkable that besides the translations and dilatations there appears a highly non trivial symmetry such as the inversion.

Let us first evaluate $(Aq; q)$ with A a non vanishing constant. Since $(q; q) = 0$ we have

$$(Aq; q) = \sum_{n=1}^{\infty} a_n(q)(A-1)^n. \tag{29}$$

To evaluate the q -dependent coefficients $a_k(q)$'s we first observe that

$$\begin{aligned} (q; A^{-1}q) &= (AA^{-1}q; A^{-1}q) \\ &= \sum_{n=1}^{\infty} a_n(A^{-1}q)(A-1)^n, \end{aligned} \tag{30}$$

which can be also evaluated by first using (27) and then the expansion (29)

$$\begin{aligned} (q; A^{-1}q) &= -A^2(A^{-1}q; q) \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} a_n(q) A^{2-n} (A-1)^n. \end{aligned} \tag{31}$$

Comparing (30) with (31) yields $a_n(A^{-1}q) = (-1)^{n+1} A^{2-n} a_n(q)$, that is $a_n(q) = \alpha_n q^{n-2}$ where $\alpha_{2n} = 0, n \in \mathbf{Z}_+$; moreover, since $(q; q) = 0$, we have that $(Aq; q)$ is vanishing at $q = 0$, so that $\alpha_1 = 0$. Therefore (29) becomes

$$(Aq; q) = \sum_{n=0}^{\infty} \alpha_{2n+3} (A-1)^{2n+3} q^{2n+1}. \tag{32}$$

To fix the α_k 's we first consider $(q+B, q)$ with B an arbitrary constant. We have

$$(q+B; q) = \sum_{n=1}^{\infty} b_n(q) B^n, \tag{33}$$

which follows by $(q; q) = 0$. By (27)(33) we have $(q; q+B) = -(q+B; q) = -\sum_{n=1}^{\infty} b_n(q) B^n$, that compared with $(q; q+B) = (q+B-B; q+B) = \sum_{n=1}^{\infty} b_n(q+B)(-B)^n$ yields $b_n(q+B) = (-1)^{n+1} b_n(q)$, that is $b_{2n-1}(q) = \beta_{2n-1}$, where $b_{2n} = 0, n \in \mathbf{Z}_+$. Therefore (33) becomes

$$(q+B; q) = \sum_{n=0}^{\infty} \beta_{2n+1} B^{2n+1}, \tag{34}$$

Subtracting $(-q; -q+B) = (-q; q) - (-q+B; q)$ from $(q; q-B) = (q; -q+B) - (q-B; -q+B)$

gives $(-q; -q+B) - (q; q-B) - (-q, q) - (q-B; -q+B) = 0$, that by (32) and (34) becomes

$$\begin{aligned} &2 \sum_{n=0}^{\infty} \beta_{2n+1} B^{2n+1} \\ &- \sum_{n=0}^{\infty} \alpha_{2n+3} 2^{2n+3} (q^{2n+1} + (B-q)^{2n+1}) \\ &= 0. \end{aligned} \tag{35}$$

Since this equation must be satisfied for any B and q , we have

$$\beta_1 = 4\alpha_3, \quad \alpha_{k>3} = 0, \quad \beta_{k>1} = 0. \tag{36}$$

Note that by (28) $(Aq+B; Aq) = A^{-2}(Aq+B; q) - A^{-2}(Aq; q)$. On the other hand, (36) implies $(Aq+B; q) = \beta_1 B$ and $(Aq; q) = \alpha_3(A-1)^3 q$, so that, as $\beta_1 = 4\alpha_3$, we have

$$(Aq+B; q) = \alpha_3 [4A^2 B + (A-1)^3 q]. \tag{37}$$

Now observe that

$$\begin{aligned} (Aq+B; q) &= -A^{-2}(q; Aq+B) \\ &= -A^{-2}(A^{-1}Q - A^{-1}B; Q), \end{aligned} \tag{38}$$

where $Q = Aq+B$. By (37) we have $(A^{-1}Q - A^{-1}B; Q) = \alpha_3 [4A^{-2}(-A^{-1}B) + (A^{-1}-1)^3 Q]$, so that (38) becomes $(Aq+B; q) = -\alpha_3 A^2 [4A^{-2}(-A^{-1}B) + (A^{-1}-1)^3 (Aq+B)]$, that compared with (37) yields $\beta_1 = \alpha_3 = 0$. Therefore

$$(Aq; q) = 0 = (q; Aq), \tag{39}$$

and

$$(q+B; q) = 0 = (q; q+B). \tag{40}$$

Eq. (28) implies $(q^a; Aq^b) = A^{-2}((q^a; q^b) - (Aq^b; q^b))$ so that by (39)

$$(q^a; Aq^b) = A^{-2}(q^a; q^b). \tag{41}$$

By (27) and (41) we have $(Aq^a; q^b) = -(\partial_{q^b} q^a)^2 (q^b; q^a)$, then using again (27)

$$(Aq^a; q^b) = (q^a; q^b). \tag{42}$$

Likewise, by (40)

$$(q^a+B; q^b) = (q^a; q^b) = (q^a; q^b+B). \tag{43}$$

Let us set $f(q) = q^{-2}(q; q^{-1})$. Eqs. (27)(42) imply $f(Aq) = -f(q^{-1})$, so that $(q; q^{-1}) = 0 = (q^{-1}; q)$

that together to (28) yields $(q^a; q^{b^{-1}}) = q^{b^4}(q^a; q^b)$. It follows that

$$\begin{aligned} (q^{a^{-1}}; q^b) &= -(\partial_{q^b} q^{a^{-1}})^2 (q^b; q^{a^{-1}}) \\ &= -(\partial_{q^b} q^a)^2 (q^b; q^a) = (q^a; q^b). \end{aligned}$$

Therefore

$$(q^{a^{-1}}; q^b) = (q^a; q^b) = q^{b^{-4}}(q^a; q^{b^{-1}}). \quad (44)$$

Since dilatations, translations and inversion generate the Möbius group, we have by (41)–(44)

$$(\gamma(q^a); q^b) = (q^a; q^b), \quad (45)$$

and

$$(q^a; \gamma(q^b)) = (Cq^b + D)^4 (q^a; q^b), \quad (46)$$

where $\gamma(q) = \frac{Aq+B}{Cq+D}$, with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2, C)$. We also have

$$(\gamma(q); q) = 0 = (q; \gamma(q)). \quad (47)$$

The above investigation implies that $(q^a; q^b)$ is proportional to $\{q^a; q^b\}$. To show this, we first note that (28) is the transformation rule of the Schwarzian derivative. Then note that the identities $\partial_x h^{1/2} h^{-1/2} = 0 = \partial_x h^{-1/2} h^{1/2} h^{-1/2} h$, imply that the second-order operator

$$h^{1/2} \frac{\partial}{\partial x} \frac{1}{h} \frac{\partial}{\partial x} h^{1/2} = \frac{\partial^2}{\partial x^2} + \frac{1}{2} \{h, x\}, \quad (48)$$

has solutions

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} + \frac{1}{2} \{h, x\} \right) h^{-1/2} (Ah + B) &= 0 \\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{1}{2} \{h, x\} \right) h^{-1/2} (Ch + D). \end{aligned} \quad (49)$$

Therefore, the Schwarzian derivative of the ratio of two linearly independent solutions of $(\partial_x^2 + V(x))f = 0$, is twice $V(x)$. Noticing that for any A and B , not simultaneously vanishing, $(\partial_x^2 + V(x))f_k(x) = 0$, $k = 1, 2$, is equivalent to $V = -(Af_1'' + Bf_2'') / (Af_1 + Bf_2)$, we have $\{\gamma(h), x\} = \{h, x\}$, which implies $\{\gamma(x), x\} = \{x, x\} = 0$. Conversely, if $\{h, x\} = 0$, then solving $(\ln h'(x))'' - \frac{1}{2} [(\ln h'(x))']^2 / 2 = 0$, gives $h(x) = \gamma(x)$. Summarizing, we have

$$\{f, x\} = \{h, x\}, \quad (50)$$

if and only if $f = \gamma(h)$. Let us now solve the Schwarzian equation for $f(q^a)$

$$\{f(q^a), q^b\} = -\frac{4m}{\beta^2} (q^a; q^b), \quad (51)$$

where β is a constant that (51) fixes to have the dimension of an action.

Eq. (47), which represents the core of the properties of $(q^a; q^b)$ derived from (28), is crucial to derive $f(q^a)$. Actually $(q^a; \gamma(q^a)) = 0$ implies $\{f(q^a), \gamma(q^a)\} = 0$. On the other hand, by (50) $f(q^a) = (A'q^a + B') / (C'q^a + D')$. Therefore, we can state the central result

$$(q^a; q^b) = -\frac{\beta^2}{4m} \{q^a, q^b\}, \quad (52)$$

which, as we have seen, uniquely follows from the equivalence principle (14). Remarkably, (52) also naturally selects the parameter leading to the classical phase. Actually, comparing $\mathscr{W}^v(q^v) = (\partial_q q^v)^{-2} \mathscr{W}(q) + (q; q^v)$, with the classical version $\mathscr{W}^v(q^v) = (\partial_q q^v)^{-2} \mathscr{W}(q)$ one sees that in the classical limit $\beta^2 \{q, q^v\} / 4m \rightarrow 0$. Thus β is precisely the parameter we are looking for. In particular

$$\lim_{\beta \rightarrow 0} Q = 0, \quad (53)$$

and $\lim_{\beta \rightarrow 0} \mathcal{S}_0 = \mathcal{S}_0^{cl}$. Eqs. (25)(52) imply that \mathscr{W} itself is a Schwarzian derivative $\mathscr{W}^v(q^v) = -\frac{\beta^2}{4m} \{q^0, q^v\}$. We will see that the unique possible Q in (19) is

$$Q = \frac{\beta^2}{4m} \{\mathcal{S}_0, q\}, \quad (54)$$

that by (19) and the basic identity

$$(\partial_q \mathcal{S}_0)^2 = \beta^2 \left(\left\{ e^{\frac{2i}{\beta} \mathcal{S}_0}, q \right\} - \{\mathcal{S}_0, q\} \right) / 2, \quad (55)$$

is equivalent to

$$\mathscr{W} = -\frac{\beta^2}{4m} \left\{ e^{\frac{2i}{\beta} \mathcal{S}_0}, q \right\}. \quad (56)$$

By (48) and (49) it follows that [1]

$$e^{\frac{2i}{\beta} \mathcal{S}_0} = \frac{A\psi^D / \psi + B}{C\psi^D / \psi + D}, \quad (57)$$

where ψ^D and ψ are linearly independent solutions of the stationary Schrödinger equation

$$\left(-\frac{\beta^2}{2m} \frac{\partial^2}{\partial q^2} + V(q) \right) \psi = E\psi. \tag{58}$$

Thus, for the ‘‘covariantizing parameter’’ we have

$$\beta = \hbar. \tag{59}$$

To show the uniqueness of the solution (56) we first set $Q = \frac{\beta^2}{4m} \{ \mathcal{S}_0, q \} - g(q)$, so that by (19) and (55) $\mathscr{W} = -\frac{\beta^2}{4m} \{ e^{\frac{2i}{\hbar} \mathcal{S}_0}, q \} + g(q)$. Since \mathcal{S}_0^{cl} does not depend on β , we have

$$\begin{aligned} & \lim_{\beta \rightarrow 0} \left(\frac{\beta^2}{4m} \{ \mathcal{S}_0, q \} - g(q) \right) \\ &= \lim_{\beta \rightarrow 0} \frac{\beta^2}{4m} \{ \mathcal{S}_0^{cl}, q \} - g^{cl}(q) \\ &= -g^{cl}(q), \end{aligned}$$

and by (53)

$$g^{cl} = 0. \tag{60}$$

Note that we used $\lim_{\beta \rightarrow 0} \{ \mathcal{S}_0, q \} = \{ \lim_{\beta \rightarrow 0} \mathcal{S}_0, q \} = \{ \mathcal{S}_0^{cl}, q \}$. However, it may happen that $\{ \mathcal{S}_0^{cl}, q \}$ is not defined, e.g. in the case of the state \mathscr{W}^0 the associated classical reduced action is a constant. In these cases one has to consider $\lim_{\beta \rightarrow 0} \beta^2 \{ \mathcal{S}_0, q \}$. Let us then first consider an arbitrary state \mathscr{W} for which $\{ \mathcal{S}_0^{cl}, q \}$ can be defined. Observe that by (24) $g(q) \in \mathcal{E}$. On the other hand, the only possible elements in \mathcal{E} that can be built by means of \mathcal{S}_0 have the form

$$g(q) = \frac{1}{4m} (\partial_q \mathcal{S}_0)^2 G(\mathcal{S}_0), \tag{61}$$

with $G(\mathcal{S}_0)$ an arbitrary function of \mathcal{S}_0 . In other words, there is no way to construct an element of \mathcal{E} by means of higher order derivatives of \mathcal{S}_0 , because these terms would break the covariance properties of g (note that these arise as consistency conditions). Furthermore, (18) implies $G(\mathcal{S}_0) = c$, where c is a constant. On the other hand, as by (61) this constant is dimensionless, it follows by (60) that $c = 0$. Hence

$$g = 0. \tag{62}$$

The extension to an arbitrary state \mathscr{W} follows from the observation that since $g \in \mathcal{E}$, we have that it is sufficient that $g = 0$ for some \mathscr{W} in order that (62) holds for all $\mathscr{W} \in \mathscr{H}$.

Therefore, we have seen that the equivalence principle actually uniquely leads to the quantum analogue of the HJ equation which in turn implies the Schrödinger equation.

Let us now derive the v -transformation $q \rightarrow q^0$ such that $\mathscr{W} \rightarrow \mathscr{W}^0$. Note that, under the transformation $q \rightarrow q^b = v^b(q)$, $\mathcal{S}_0(q) \rightarrow \mathcal{S}_0^b(q^b) = \mathcal{S}_0(q)$, we have

$$\begin{aligned} \left\{ e^{\frac{2i}{\hbar} \mathcal{S}_0^b(q^b)}, q^b \right\} &= \left\{ e^{\frac{2i}{\hbar} \mathcal{S}_0(q)}, q^b \right\} \\ &= (\partial_q q^b)^{-2} \left\{ e^{\frac{2i}{\hbar} \mathcal{S}_0(q)}, q \right\} \\ &\quad - (\partial_q q^b)^{-2} \{ q^b, q \}, \end{aligned} \tag{63}$$

that is $\mathscr{W}^b(q^b) = (\partial_q q^b)^{-2} \mathscr{W}(q) + \frac{\hbar^2}{4m} \times (\partial_q q^b)^{-2} \{ q^b, q \}$. Therefore, if

$$q^b = \frac{Ae^{\frac{2i}{\hbar} \mathcal{S}_0(q)} + B}{Ce^{\frac{2i}{\hbar} \mathcal{S}_0(q)} + D}, \tag{64}$$

then, according to (63), we have $\{ e^{\frac{2i}{\hbar} \mathcal{S}_0^b(q^b)}, q^b \} = 0$. Therefore, if (64) is satisfied, then $\mathscr{W}^b(q^b) = -\hbar^2 \times \{ e^{\frac{2i}{\hbar} \mathcal{S}_0^b(q^b)}, q^b \} / 4m$ coincides with the state \mathscr{W}^0 . This implies that q^b is a Möbius transformation of q^0 . It follows that the solution of the inversion problem (13) is

$$q \rightarrow q^0 = \frac{A'e^{\frac{2i}{\hbar} \mathcal{S}_0(q)} + B'}{C'e^{\frac{2i}{\hbar} \mathcal{S}_0(q)} + D'}. \tag{65}$$

The particular choice of the coefficients in (57) depends on the initial conditions of the Quantum Stationary HJ Equation (QSHJE) (19), which is a third-order differential equation. Since \mathcal{S}_0 should be a real function, and since it is always possible to choose ψ^D and ψ to be real linearly independent

solutions of the Schrödinger equation, we have ($w = \psi^D/\psi$)

$$e^{\frac{2i}{\hbar}\mathcal{S}_0\{\delta\}} = e^{i\alpha} \frac{w + i\bar{\ell}}{w - i\ell}, \quad (66)$$

where $\delta = \{\alpha, \ell\}$, with $\alpha \in \mathbf{R}$ and ℓ integration constants. Observe that $\mathcal{S}_0 \neq \text{const}$ is equivalent to non-degeneracy of the Möbius transformation (66), i.e. $\text{Re } \ell \neq 0$. We note that Eq. (19) and its solution (66) have been investigated by Floyd [6].

Let us denote by ψ_E the Schrödinger wave function associated to a given state of energy E . For any fixed set of integration constants δ , there are coefficients A and B such that

$$\psi_E\{\delta\} = \frac{1}{\sqrt{\mathcal{S}'_0\{\delta\}}} \left(A e^{-\frac{i}{\hbar}\mathcal{S}_0\{\delta\}} + B e^{\frac{i}{\hbar}\mathcal{S}_0\{\delta\}} \right). \quad (67)$$

Let us define

$$\phi = \sqrt{2} \frac{e^{-i\frac{\alpha}{2}} (\psi^D - i\ell\psi)}{\hbar^{1/2} |W(\ell + \bar{\ell})|^{1/2}}, \quad (68)$$

where the Wronskian $W = \psi'\psi^D - \psi^{D'}\psi$ is a real non vanishing constant. We have

$$e^{\frac{2i}{\hbar}\mathcal{S}_0\{\delta\}} = \frac{\bar{\phi}}{\phi}, \quad (69)$$

and since $\phi\bar{\phi} - \phi\bar{\phi}' = -2iW(\ell + \bar{\ell})/\hbar|W(\ell + \bar{\ell})|$, we obtain

$$p = \epsilon |\phi|^{-2}, \quad (70)$$

where the value of $\epsilon = W(\ell + \bar{\ell})/|W(\ell + \bar{\ell})| = \text{sgn}[W(\ell + \bar{\ell})]$ fixes the direction of motion. We observe that a basic property of p is that, due to the Q term, it takes real values and is not vanishing even in the classically forbidden regions. Exceptions concern the cases in which there is some space region where $\psi_E = 0$, such as in the case the infinitely deep potential well. In this case any linearly independent solution is infinite in this region and we have $p = 0$ (this situation arises by considering a suitable limiting procedure).

We now consider the effect of the Möbius transformations on the wave function together with the properties of the trivializing map. Then we will

discuss the case of the harmonic oscillator. In particular, we first consider the important point of determining the transformations $\delta \rightarrow \delta'$ leaving the state described by ψ_E invariant. To this end it is useful to write $e^{\frac{2i}{\hbar}\mathcal{S}_0}$ in a different form. First of all observe that the expression of $e^{\frac{2i}{\hbar}\mathcal{S}_0}$ in (66) can be seen as the composition of two maps. The first one is the Cayley transformation $w \rightarrow z = (w + i)/(w - i) \in S^1$. Then $e^{\frac{2i}{\hbar}\mathcal{S}_0}$ is obtained as the Möbius transformation

$$e^{\frac{2i}{\hbar}\mathcal{S}_0} = \gamma_{\mathcal{S}_0}(z) = \frac{az + b}{cz + c}, \quad (71)$$

where, by (66), the entries $a = \bar{d}$, $b = \bar{c}$ of the matrix $\gamma_{\mathcal{S}_0}$ are

$$\gamma_{\mathcal{S}_0} = \begin{pmatrix} e^{\frac{i}{2}\alpha} (1 + \bar{\ell}) & e^{\frac{i}{2}\alpha} (1 - \bar{\ell}) \\ e^{-\frac{i}{2}\alpha} (1 - \ell) & e^{-\frac{i}{2}\alpha} (1 + \ell) \end{pmatrix}. \quad (72)$$

Let us now consider the moduli transformation $\delta \rightarrow \delta' = \{\alpha', \ell'\}$, so that

$$e^{\frac{2i}{\hbar}\mathcal{S}_0\{\delta\}} \rightarrow e^{\frac{2i}{\hbar}\mathcal{S}_0\{\delta'\}} = \gamma'_{\mathcal{S}_0}(z) = e^{i\alpha'} \frac{w + i\bar{\ell}'}{w - i\ell'}, \quad (73)$$

where $\gamma'_{\mathcal{S}_0}$ is the matrix (72) with α and ℓ replaced by α' and ℓ' respectively. Note that $e^{\frac{2i}{\hbar}\mathcal{S}_0\{\delta'\}} = \tilde{\gamma}_{\mathcal{S}_0}(\gamma_{\mathcal{S}_0}(z))$, where $\tilde{\gamma}_{\mathcal{S}_0} = \gamma'_{\mathcal{S}_0} \gamma_{\mathcal{S}_0}^{-1}$. We can now determine the possible transformations $\delta \rightarrow \delta'$ such that $\psi_E\{\delta'\}$ describes the same state described by $\psi_E\{\delta\}$. That is, we consider the transformations of the integration constants of the QSHJE, corresponding to real p , such that ψ_E remains unchanged up to some multiplicative constant c , that is

$$\psi_E\{\delta\} \rightarrow \psi_E\{\delta'\} = c \psi_E\{\delta\}. \quad (74)$$

Observe that $\psi_E\{\delta'\} = (\hbar \partial_q \gamma'_{\mathcal{S}_0} / 2i)^{-1/2} (A + B \gamma'_{\mathcal{S}_0})$. Since $\tilde{\gamma}_{\mathcal{S}_0} = \gamma'_{\mathcal{S}_0} \gamma_{\mathcal{S}_0}^{-1}$, we have

$$\begin{aligned} \partial_q \gamma'_{\mathcal{S}_0} &= \partial_q \tilde{\gamma}_{\mathcal{S}_0} (\gamma_{\mathcal{S}_0}(z)) = \frac{\partial \tilde{\gamma}_{\mathcal{S}_0}}{\partial \gamma_{\mathcal{S}_0}} \partial_q \gamma_{\mathcal{S}_0} \\ &= \frac{\partial_q \gamma_{\mathcal{S}_0}}{(\tilde{c} \gamma_{\mathcal{S}_0} + \tilde{d})^2}, \end{aligned} \quad (75)$$

where we used $\tilde{\gamma}_{\mathcal{S}_0}(\gamma_{\mathcal{S}_0}) = (\tilde{a}\gamma_{\mathcal{S}_0} + \tilde{b})/(\tilde{c}\gamma_{\mathcal{S}_0} + \tilde{d})$, with $\tilde{a} = \tilde{d}$, $\tilde{b} = \tilde{c}$, denoting the elements of $\tilde{\gamma}_{\mathcal{S}_0}$ given by (72) with the δ -moduli replaced by $\tilde{\delta} = \{\tilde{\alpha}, \tilde{\ell}\}$. Therefore

$$\psi_E(\delta') = (\hbar \partial_q \gamma_{\mathcal{S}_0} / 2i)^{-1/2} \times \left[A\tilde{d} + B\tilde{b} + (A\tilde{c} + B\tilde{a})\gamma_{\mathcal{S}_0} \right], \quad (76)$$

and Eq. (74) is solved by

$$A^2\tilde{b} + AB\tilde{a} = AB\tilde{a} + B^2\tilde{b}, \quad (77)$$

which explicitly shows that there are transformations of the δ -moduli, and therefore of \mathcal{S}_0 and $p = \partial_q \mathcal{S}_0$, such that the unit ray Ψ_E associated to ψ_E remains invariant. In the case in which either A or B vanish, one has by (77) that the transformations of \mathcal{S}_0 leaving Ψ_E unchanged, correspond to a phase change. However, as we will see, non-trivial transformations arise for bounded states.

Since $e^{\frac{2i}{\hbar}\mathcal{S}_0}$ takes values in S^1 , reality of the trivializing map and Eq. (65) imply that $q^0 = l(A\psi^D/\psi + B)/(C\psi^D/\psi + D)$, where l is a real constant with the dimension of a length which can be determined together with the real coefficients A, B, C, D . Let us denote by $\delta_0 = \{\alpha_0, \ell_0\}$ the moduli associated to the state \mathcal{W}^0 . In this case we can choose $\psi^{D^0} = q^0$ and $\psi^0 = 1$. Since the trivializing map is defined by $\mathcal{S}_0^0(q^0) = \mathcal{S}_0(q)$, by (66) we have

$$e^{i\alpha_0} \frac{q^0 + i\bar{\ell}_0}{q^0 - i\ell_0} = e^{i\alpha} \frac{w + i\bar{\ell}}{w - i\ell}. \quad (78)$$

Therefore, the trivializing map transforming the state \mathcal{W} with moduli $\delta = \{\alpha, \ell\}$, to the state \mathcal{W}^0 with moduli $\delta_0 = \{\alpha_0, \ell_0\}$ is given by the real map

$$q^0 = \frac{(\ell_0 e^{i\beta} + \bar{\ell}_0 e^{-i\beta})w + i\ell_0 \bar{\ell} e^{i\beta} - i\bar{\ell}_0 \ell e^{-i\beta}}{2(\sin \beta)w + \ell e^{-i\beta} + \bar{\ell} e^{-i\beta}}, \quad (79)$$

where $\beta = (\alpha - \alpha_0)/2$. Let us consider the case in which the functional structure of two reduced actions differs for a constant only, that is $\mathcal{S}_0^a(q^a) = \mathcal{S}_0^b(q^a) + \hbar(\alpha_a - \alpha_b)/2$. Since $\mathcal{S}_0^a(q^a) = \mathcal{S}_0^b(q^b)$, it follows that $p_a(q_a) = p_b(q_a)$, that is the functional structure of p_a and p_b coincide. This is already clear from the fact that \mathcal{S}_0^a and \mathcal{S}_0^b define the same

system. Therefore, we can set $\alpha = \alpha_0 + 2k\pi$ and (79) becomes

$$q^0 = \frac{(\ell_0 + \bar{\ell}_0)w + i\ell_0 \bar{\ell} - i\bar{\ell}_0 \ell}{\ell + \bar{\ell}}. \quad (80)$$

We will call Möbius states the states parameterized by ℓ associated to a given \mathcal{W} .

Let us consider some properties of the QSHJE. First of all note that since it contains the Schwarzian derivative of \mathcal{S}_0 , it follows that in order to be defined, the reduced action should be of class C^2 with \mathcal{S}_0' differentiable. By (66) this is equivalent to require that ψ and ψ^D be of class C^1 with ψ' and $\psi^{D'}$ differentiable. Possible discontinuities of ψ' and $\psi^{D'}$, which may arise for example in the case of the infinitely deep potential well, should be studied as limit cases. The above properties was already expected as we required continuity and local univalence of the trivializing map. However, the Möbius symmetry of the Schwarzian derivative implies that the continuity properties should be extended to $\pm\infty$. In other words, the v -maps are local homeomorphisms of the extended real line $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ into itself. This means that as q varies in $\hat{\mathbf{R}}$, q^0 spans the extended real line a countable number of times. In [7] we will see that these conditions on the trivializing map, a direct consequence of the equivalence principle, actually imply the standard results about quantized energy spectra which follow from the conventional approach. Since the Möbius transformations, such as (79) and (80), are globally univalent transformations of the extended real line into itself, we have that the index of the trivializing map coincides with the index of w

$$I[q^0] = I[w] = k. \quad (81)$$

Since according to Sturm theorem, between any two consecutive zeroes of ψ there exists a zero of ψ^D , we have that k is strictly related to the number of zeroes, including the vanishing at infinity, of ψ^D (or ψ) [7].⁴

⁴ This theorem can be seen as a duality between ψ^D and ψ . In this context we note that, while in the standard approach one selects a solution of (58) as the Schrödinger wave function, with the dual one being usually ignored, we have that \mathcal{S}_0 and $p = \partial_q \mathcal{S}_0$ contain both ψ^D and ψ .

As an example we consider the case of the harmonic oscillator. In [7] we will show that the only possible solutions for the QSHJE which are consistent with the equivalence principle are those corresponding to the well-known spectrum $E_n = (n + 1/2)\hbar\omega$, $n = 0, 1, \dots$. To derive the trivializing map, we note that by Wronskian arguments, it follows that if ψ is a solution of the Schrödinger equation, then a linearly independent solution is given by ⁵ $\psi^D(q) = -W\psi(q) \int_{q_0}^q dx \psi^{-2}(x)$ so that $w = -W \int_{q_0}^q dx \psi^{-2}(x)$. We can choose ψ to be the normalized Hamiltonian eigenfunction $\psi_n = c_n e^{-\frac{m\omega}{2\hbar}q^2} H_n((m\omega/\hbar)^{1/2}q)$, with H_n denoting the n^{th} Hermite polynomial and $c_n = (m\omega/\pi\hbar)^{1/4} / \sqrt{2^n n!}$. For the dual solution we have $\psi_n^D = -W\psi_n \int_{q_0}^q dx \psi_n^{-2}(x)$. Note that replacing ψ_n^D and ψ_n with two real linearly independent combinations, which is equivalent to perform a real Möbius transformation of $w_n = \psi_n^D/\psi_n$, corresponds changing α and ℓ in \mathcal{S}_0 .

Since ψ_n has n -zeroes at finite q and vanishes at ∞ , we have that the trivializing map for the n^{th} -state of the harmonic oscillator has covering index $k = I[w] = n + 1$. As an example, note that $w_0 = -c_0^{-2} W \int_{q_0}^q dx e^{-\frac{m\omega}{\hbar}x^2}$ is univalent, i.e. $w'_0 \neq 0$ in \mathbf{R} , and vanishes at $q = q_0$.

Observe that by (67)–(70) we have $\phi = \epsilon^{1/2} e^{-\frac{i}{\hbar}\mathcal{S}_0} / \sqrt{p}$ and $\psi_E\{\delta\} = A\epsilon^{-1/2}\phi + B\epsilon^{1/2}\bar{\phi}$ (note that $\epsilon^{-1/2} = \epsilon^{1/2}$). Let us determine A and B for bounded states (e.g. the harmonic oscillator). By (68) we have

$$\begin{aligned} \psi_n &= \frac{\sqrt{2}}{\hbar^{1/2} |W(\ell + \bar{\ell})|^{1/2}} \\ &\times \left(A\epsilon^{-1/2} e^{-i\alpha/2} (\psi_n^D - i\ell\psi_n) \right. \\ &\left. + B\epsilon^{1/2} e^{i\alpha/2} (\psi_n^D + i\bar{\ell}\psi_n) \right), \end{aligned} \tag{82}$$

so that

$$A = i \left[\frac{e^{i\alpha}\hbar W}{2(\ell + \bar{\ell})} \right]^{1/2}, \quad B = -\epsilon e^{-i\alpha} A = \bar{A}. \tag{83}$$

This allows us to find the transformations of \mathcal{S}_0

⁵ Note that both ψ^D and ψ can be written in terms of hypergeometric functions.

leaving invariant the unit ray. Actually, according to (77) we have that \mathcal{S}_0 and $\tilde{\mathcal{S}}_0$, with

$$e^{\frac{2i}{\hbar}\tilde{\mathcal{S}}_0} = \frac{ae^{\frac{2i}{\hbar}\mathcal{S}_0} + b}{\bar{b}e^{\frac{2i}{\hbar}\mathcal{S}_0} + \bar{a}}, \tag{84}$$

where

$$\text{Im } a = -\epsilon \text{Im}(e^{-i\alpha}b), \tag{85}$$

correspond to the same unit ray defined by ψ_n .

Let us now consider the expectation value

$$\langle \hat{\mathcal{C}} \rangle = \int_{-\infty}^{\infty} dq \bar{\psi} \hat{\mathcal{C}} \psi, \tag{86}$$

where ψ is some superposition of eigenstates of the harmonic oscillator Hamiltonian. Since $\psi = \sum_n a_n \psi_n$ and $\hat{\mathcal{C}}\psi = \sum_n b_n \psi_n$, we have that to consider the effect of the trivializing map on the integrand of (86) reduces to the problem of considering its action on $|\psi_n|^2$ for any n . By (67) we have that ψ_n transforms as a $-1/2$ -differential under v -maps. In this context we note that since there is a trivializing map for any (Möbius) state it should be possible to consider different coordinates for each ψ_n . This is just the case as we can replace $\psi(q) = \sum_n a_n \psi_n(q)$ in (86) with $\sum_n a_n \psi_n(q_n)$ and simultaneously changing the measure, that is

$$\langle \hat{\mathcal{C}} \rangle = \int_{-\infty}^{\infty} \prod_{k=0}^{\infty} dq_k \sum_{m=0}^{\infty} \bar{a}_m \psi_m(q_m) \sum_{n=0}^{\infty} b_n \psi_n(q_n). \tag{87}$$

It is interesting that considering the trivializing map for an arbitrary state leads to consider the measure $\prod_{k=0}^{\infty} dq_k$. In this context we note that very recently the transformations leading to the free system with vanishing energy have been considered by Periwal in the path-integral framework [8].

Observe that by (66) it follows that the two self-dual states (9) with $\gamma = \pm \frac{i}{2}\hbar$ correspond to complex Möbius states of \mathcal{W}^0 . Actually, the difference between $\mathcal{S}_0^0(q^0)$ and $\mathcal{S}_0^{sd} = \pm \frac{i}{2}\hbar \ln \gamma_q q$ amount to a complex Möbius transformation which has no effect on $\mathcal{W}^0 = -\hbar^2 \{e^{\frac{2i}{\hbar}\mathcal{S}_0^0(q^0)}, q^0\} / 4m = 0$.

While in the standard approach the solution corresponding to the state \mathcal{W}^0 coincides with the classical one, here we have a basic difference as the

equivalence principle cannot be implemented if one considers the solution $\mathcal{S}_0^0 = \text{const}$. In particular, the Schwarzian derivative of \mathcal{S}_0^0 is not defined for $\mathcal{S}_0^0 = \text{const}$. This aspect is strictly related to the existence of the Legendre transformation of \mathcal{S}_0 for any state [1]. Using the identity $\{q, \mathcal{S}_0\} = -(\partial_q \mathcal{S}_0)^{-2} \{\mathcal{S}_0, q\}$, one sees that the quantum correction to the CSHJE corresponds to the conformal rescaling of the conjugate momentum [1]

$$\frac{1}{2m} \left(\frac{\partial \mathcal{S}_0}{\partial q} \right)^2 [1 - \hbar^2 \mathcal{U}(\mathcal{S}_0)] + \mathcal{W} = 0. \quad (88)$$

This shows the basic role of the purely quantum mechanical state \mathcal{W}^0 as in this case the QSHJE is solved by the overlooked zero-modes of the conformal factor, that is

$$1 - \hbar^2 \mathcal{U} \left(\frac{\hbar}{2i} \ln \frac{q^0 + i\ell_0}{q^0 - i\ell_0} \right) = 0. \quad (89)$$

Our formalism naturally fits with the brackets

$$\begin{aligned} \{X, Y\}_{(\beta)} &\equiv \frac{\partial}{\partial a^i} X (\partial_\beta \tau)^{-1ij} \partial_\beta \frac{\partial}{\partial a^j} Y \\ &\quad - \frac{\partial}{\partial a^i} Y (\partial_\beta \tau)^{-1ij} \partial_\beta \frac{\partial}{\partial a^j} X, \end{aligned} \quad (90)$$

introduced in [5] in the framework of $N = 2$ SYM [9]. In particular

$$\{a^i, a^j\}_{(\beta)} = 0 = \{a_i^D, a_j^D\}_{(\beta)}, \quad \{a^i, a_j^D\}_{(\beta)} = \delta_j^i. \quad (91)$$

We refer to [5] for notation in (90)(91). In the one-dimensional case, setting

$$a = \sqrt{p}, \quad a^D = q\sqrt{p}, \quad \tau = \partial_a a^D, \quad \partial_\beta = \partial_s,$$

the canonical Eq. (5), whose canonical potential essentially coincides with the quantum potential, has the bracket representation

$$\{\sqrt{p}, q\sqrt{p}\}_{(s)} = 1. \quad (92)$$

Similarly, setting $a = \psi$, $a^D = \psi^D$, $\partial_\beta = \partial_q$, we see that the Schrödinger Eq. (58) is equivalent to the bracket

$$\{\psi, \psi^D\}_{(q)} = 1, \quad (93)$$

which matches with the formalism in [2,4]. These brackets, which according to (90) and (91) can be extended to higher dimensions, can be seen as the analogue of the Poisson brackets in the case in which p and q are dependent. In this context, we also observe that the inversion formula in [3], including its higher dimensional extension [10]

$$u = \frac{i}{4\pi b_1} \left(\mathcal{F} - \sum_i \frac{a^i}{2} a_i^D \right), \quad (94)$$

satisfies the equation [5]

$$\mathcal{L}_\beta u = u, \quad (95)$$

where \mathcal{L}_β is a second-order modular invariant operator. In our approach, Eq. (94) corresponds to the higher dimensional analogue of the Legendre transformation of \mathcal{S}_0 . The generalization of the above $GL(2, \mathcal{C})$ -symmetry is just the symplectic group.

We observe that in [11] Gozzi showed, in the framework of the HJ theory, that the classical ‘‘symmetry’’ associated to the Lagrangian rescaling is broken by quantum effects with the corresponding ‘‘anomalous’’ conservation law leading to the Schrödinger equation.

In conclusion, we note that the equivalence principle suggests a new view of quantum mechanics and the reexamination of its basic foundation. In this context we observe that some aspects of the investigation are reminiscent of Bohm theory [12]. However, there are basic differences, some of which have been emphasized by Floyd who investigated the QSHJE in [6], and which we derived from the equivalence principle [1]. As stressed in [6], there is no wave guide in this approach. Another feature is that by (70) p is a real quantity also in classically forbidden regions. Furthermore, unlike in Bohm theory, the classical limit arises in a natural way. As we will show in [7], the simple but basic difference with respect to Bohm theory is that while he identified the physical wave function with $Re e^{\frac{i}{\hbar} \mathcal{S}_0}$, we in general have that ψ_E is a linear combination of $Re e^{\frac{i}{\hbar} \mathcal{S}_0}$ and $Re^{-\frac{i}{\hbar} \mathcal{S}_0}$ (reality of Eq. (58) implies that if $Re e^{\frac{i}{\hbar} \mathcal{S}_0}$ is a solution, then also $Re^{-\frac{i}{\hbar} \mathcal{S}_0}$ is a solution). The consequence is that while in Bohm theory \mathcal{S}_0 vanishes for physical wave functions which are real up to a possible complex constant, e.g. in the case of the harmonic oscillator, this is never the case in our

approach. Actually, as we have seen, this is strictly related to the equivalence principle as \mathcal{S}_0 is never a constant. Then in this approach we never have $\mathcal{S}_0 = 0$. Thus, while in Bohm theory one has to consider the question of recovering \mathcal{S}_0^{cl} in the $\hbar \rightarrow 0$ limit, this problem is completely under control and natural in this approach.

Our approach has a wide range of consequences, some of which will be considered in [7,13]. Besides the existence of trajectories in the classical forbidden regions [6,7], a remarkable aspect concerns the appearance of the quantized spectra from the properties of the trivializing map [7].

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