REND. SEM. MAT. UNIV. PADOVA, Vol. 109 (2003)

Cohomological Descent of Rigid Cohomology for Etale Coverings.

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1. Introduction.

Rigid cohomology (introduced by P. Berthelot [3]) is thought to be a good p-adic cohomology theory for schemes of positive characteristic p. It is known that rigid cohomology with trivial coefficient sheaf is of finite dimension [4] (see also [17]), and admits Poincaré duality, satisfies the Künneth formula [5]. However, in the general case of non-trivial coefficients, it is still unknown whether or not rigid cohomology satisfies the analogous good properties.

In this paper we shall discuss rigid cohomology from the point of view of cohomological descent. The theory of cohomological descent was studied by B. Saint-Donat [1, V^{bis}]. A direct translation to the case of rigid cohomology is not straightforward, so we will first develop a cohomological descent theory for the cohomology of coherent sheaves over overconvergent functions on tubular neighbourhoods and then we will study a cohomological descent theory for coherent sheaves with integrable connections. We will extend Berthelot's definition of rigid cohomology to

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Supported by a fellowship program of the ministry of education, Japan.

schemes which are separated and locally of finite type over a field of characteristic p by means of cohomological descent theory. One of the main results is that an étale hypercovering is universally de Rham descendable. This means that, for rigid cohomology, we have cohomological descent for étale hypercoverings, and there exists a spectral sequence of rigid cohomology for étale hypercoverings.

Let us now briefly explain the contents of this paper.

1.1. Let k be a field of characteristic p, which is the residue field of a complete discrete valuation ring $\mathcal{V}$ having K as fraction field. We call $(X, \overline{X})$ «a pair separated locally of finite type over Spec k» if X and $\overline{X}$ are k-schemes separated locally of finite type endowed with an open immersion $X \to \overline{X}$, and we say that $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ is «a triple separated locally of finite type over Spf $\mathcal{V}$» if $(X, \overline{X})$ is a pair locally of finite type over Spec k and $\mathcal{X}$ is a formal separated $\mathcal{V}$-scheme locally of finite type and endowed with a closed immersion $\overline{X} \to \mathcal{X}$. Let $\mathcal{X}_K^{\mathrm{an}}$ be the rigid analytic space associated to the generic fiber of $\mathcal{X}$ in the sense of M. Raynaud and let $]\overline{X}[_{\mathcal{X}}$ (resp. $]X[_{\mathcal{X}}$) be a tube of $\overline{X}$ (resp. X) in $\mathcal{X}_K^{\mathrm{an}}$. We denote by $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$ the sheaf of rings of overconvergent functions on $]\overline{X}[_{\mathcal{X}}$ along a complement of X in $\overline{X}$. A morphism $w : \mathfrak{Y} \to \mathfrak{X}$ of triples is a diagram as in 2.3.

Let $\mathfrak{X} = (X, \overline{X}, \mathfrak{X})$ be a triple separated locally of finite type over Spf $\mathcal{V}$ and let $\mathfrak{Y}_. = (Y_., \overline{Y}_., \mathcal{Y}_.)$ be a simplicial triple separated locally of finite type over $\mathfrak{X}$. Then one has a simplicial rigid analytic space $]\overline{Y}_.[_{\mathcal{Y}_.}$ over $]\overline{X}[_{\mathcal{X}}$ and a sheaf $j^\dagger \mathcal{O}_{]\overline{Y}_.[_{\mathcal{Y}_.}}$ of rings of overconvergent functions on $]\overline{Y}_.[_{\mathcal{Y}_.}$ along a complement of $Y_.$ in $\overline{Y}_.$. In this situation we construct a Čech complex $\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; w_.^\dagger E)$ and a derived Čech complex $\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; w_.^\dagger E)$ for a sheaf E of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules (4.1 and 4.2). The Čech complex is the usual complex of sheaves on $]\overline{X}[_{\mathcal{X}}$ for the simplicial triple $w_.: \mathfrak{Y}_. \to \mathfrak{X}$ and the derived Čech complex is a derived version of the Čech complex.

We say that $w_.: \mathfrak{Y}_. \to \mathfrak{X}$ is cohomologically descendable (resp. universally cohomologically descendable) if $w_.$ is exact (Definition 6.1.1) and the natural morphism

$$E \to \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; w_.^\dagger E)$$

is an isomorphism for any sheaf E of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules (resp. if $w_.$ is cohomologically descendable after any base change by $\mathfrak{Z} \to \mathfrak{X}$ of triples locally of finite type) (Definition). We also say that a morphism $w : \mathfrak{Y} \to \mathfrak{X}$ is cohomologically descendable (resp. universally cohomologically descendable) if the Čech diagram

$$\mathfrak{X} \leftarrow \mathfrak{Y} \begin{smallmatrix} \leftarrow \\ \rightarrow \\ \leftarrow \end{smallmatrix} \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y} \begin{smallmatrix} \leftarrow \\ \rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow \end{smallmatrix} \cdots$$

associated to w (Example 3.1.1 (4)) is so.

Let $w : \mathfrak{Z} \to \mathfrak{Y}$ be a morphism of triples separated locally of finite type over $\mathfrak{X}$ and suppose that w is universally cohomologically descendable. Then, $\mathfrak{Y} \to \mathfrak{X}$ is universally cohomologically descendable if and only if $\mathfrak{Z} \to \mathfrak{X}$ is so (Theorem 6.3.1). This is an expected property for cohomological descent theory. Since a strict Zariski covering of $\mathfrak{X}$ (see 2.3.3 and 2.3.4) and a covering of $\mathfrak{X}$ associated to a finite Zariski covering of X are universally cohomologically descendable (Propositions 6.2.5 and 6.2.6), the notion of universally cohomological descendability of a morphism $\mathfrak{Y} \to \mathfrak{X}$ is Zariski local both on $\mathfrak{X}$ and on $\mathfrak{Y}$. Suppose that $\mathfrak{Y}$ and $\mathfrak{Y}'$ are triples separated locally of finite type over $\mathfrak{X}$ such that $Y = Y'$, both $\overline{Y}$ and $\overline{Y}'$ are proper over $\overline{X}$, and both $\mathcal{Y}$ and $\mathcal{Y}'$ are smooth over $\mathfrak{X}$ around Y and Y'. Then $\mathfrak{Y} \to \mathfrak{X}$ is universally cohomologically descendable if and only if $\mathfrak{Y}' \to \mathfrak{X}$ is so by the fibration theorem of tubular neighbourhoods (Corollary 6.4.2).

The cohomological descent theory of rigid cohomology for étale coverings is as follows. Let $w : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of triples separated locally of finite type over Spf $\mathcal{V}$ such that $\overline{Y} \to \overline{X}$ is étale surjective, $\overline{w}^{-1}(X) = Y$ and $\mathcal{Y} \to \mathcal{X}$ is smooth around Y. Then w is universally cohomologically descendable (Theorem 7.3.1). Now we give a sketch of the proof. By the properties of cohomological descent given above, one can reduce the universally cohomological descendability of w to the case where $\widehat{w}(\overline{X}) = \overline{Y}$ (see 2.3). In this case the natural homomorphism

$$\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}.; w_{\cdot}^{\dagger} E) \to \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}.; w_{\cdot}^{\dagger} E)$$

is an isomorphism for any sheaf E of coherent $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules, as follows from a theorem on the vanishing of higher cohomology of coherent sheaves over sheaves of rings of overconvergent functions (Theorems 5.2.1 and 5.2.2). Since there exists a lattice over formal schemes for sheaves of coherent modules on affinoids, one can reduce our problem to the faithfully flat descent theorem of coherent sheaves on formal schemes (Proposition 7.1.2). We also have a version for a morphism w such that $Y \to X$ is étale surjective, $\overline{Y} \to \overline{X}$ is proper and $\mathcal{Y} \to \mathcal{X}$ is smooth around Y (Theorem 7.4.1). By using a homotopy theory of simplicial triples (Corollary 6.5.4), we obtain more generally that étale-étale and étale-proper hypercoverings (Definition 7.2.2) are universally cohomologically descendable (Corollaries 7.3.3 and 7.4.3).

Let $\mathfrak{X}$ be a triple separated locally of finite type over Spf $\mathcal{V}$ such that $\mathcal{X}$ is smooth over Spf $\mathcal{V}$ around X. Let (E, ∇) be a sheaf E of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}$-modules with an integrable connection $\nabla : E \to E \otimes_{j^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j^\dagger \Omega^1_{]\overline{X}[_\mathcal{X}/\mathrm{Spm}\ K}$. If we replace coherent sheaves by the de Rham complex associated to the integrable connection (E, ∇) in the derived Čech complex, then we can define a notion of de Rham descent and universal de Rham descent for simplicial triples over $\mathfrak{X}$ just as in the case of the cohomological descent discussed above (Definition 8.3.2).

The typical universally de Rham descendable hypercovering is a constant simplicial triple $\mathfrak{Y}^{\Delta^\circ} \to \mathfrak{X}$ associated to a separated morphism $\mathfrak{Y} \to \to \mathfrak{X}$ of finite type such that $X = Y$, $\overline{Y} \to \overline{X}$ is proper and $\mathcal{Y} \to \mathcal{X}$ is smooth around Y (Corollary 8.3.6). This example plays an important role in the proof that our definition of rigid cohomology in section 10 coincides with the original definition of Berthelot. Another important example is that $\mathfrak{Y} \to \mathfrak{X}$ is universally cohomologically descendable if and only if it is universally de Rham descendable (Corollary 8.5.2). As a consequence, étale-étale and étale-proper hypercoverings are universally de Rham descendable (Theorem 9.1.1).

1.2. In general a scheme over Spec k can not be embedded in a formal smooth scheme over Spf $\mathcal{V}$. To define a notion of universally de Rham descendable hypercoverings in general cases, we use base changes. In fact, let $(X, \overline{X})$ be a pair separated locally of finite type over Spec k and let $\mathfrak{Y}_. = (Y_., \overline{Y}_., \mathcal{Y}_.)$ be a simplicial triple separated locally of finite type over Spf $\mathcal{V}$ such that $(Y_., \overline{Y}_.)$ is a simplicial pair over $(X, \overline{X})$. We say that $\mathfrak{Y}_.$ is a universally de Rham descendable hypercovering of $(X, \overline{X})$ if, for any triple $\mathfrak{Z} = (Z, \overline{Z}, \mathcal{Z})$ locally of finite type over Spf $\mathcal{V}$, $(Y_. \times_X Z, \overline{Y}_. \times_{\overline{X}} \overline{Z}, \mathcal{Y}_. \times_{\mathrm{Spec}\ \mathcal{V}} \mathcal{Z}) \to \mathfrak{Z}$ is de Rham descendable (Definition 10.1.3).

Our definition of rigid cohomology uses universally de Rham descendable hypercoverings (10.4). Let $(X, \overline{X})$ be a pair separated locally of finite type over Spec k and let $\mathfrak{Y}_.$ be a universally de Rham descendable hypercovering of $(X, \overline{X})$. We define a rigid cohomology $H^\bullet_{\mathrm{rig}}((X, \overline{X})/K, E)$ by the cohomology of derived Čech complexes for de Rham complexes $E \otimes_{j^\dagger \mathcal{O}_{]\overline{X}[_\mathcal{X}}} j^\dagger \Omega^\bullet_{]\overline{X}[_\mathcal{X}/\mathrm{Spm}\, K}$ associated to overconvergent isocrystals (E, ∇) on $(X, \overline{X})/K$. Then this definition is independent of the choice of universally de Rham descendable hypercoverings. For example, if $\overline{X}$ is a completion of X, then our rigid cohomology is independent of the choice of completions of X and we denote it by $H^*_{\mathrm{rig}}(X/K, E)$ as in the cases considered by Berthelot. Since constant simplicial triples as above are universally de Rham descendable, our rigid cohomology coincides with that of Berthelot (Theorem 10.6.1).

Let X be a scheme separated of finite type over Spec k and let $g_{\cdot}: Y_{\cdot} \to X$ be an étale hypercovering. In this case we construct a truncated universally de Rham descendable hypercovering $\mathfrak{Z}_{\cdot} = (Z_{\cdot}, \overline{Z}_{\cdot}, \mathcal{Z}_{\cdot})$ of X such that $Z_{\cdot}$ is a refinement of $Y_{\cdot}$ over X (Definition 11.3.1) and $\overline{Z}_{\cdot}$ is proper over Spec k (11.4 and 11.5). Using an argument involving limits of truncations, we obtain a spectral sequence

$$E_1^{qr} = H^r_{\mathrm{rig}}(Y_q/K, g_q^* E) \Rightarrow H^{q+r}_{\mathrm{rig}}(X/K, E)$$

with respect to the étale hypercovering $Y_{\cdot} \to X$ for an overconvergent isocrystal E on X/K (Theorem 11.7.1).

If one consider overconvergent F-isocrystals, then our rigid cohomology has a Frobenius structure and it coincides with the Frobenius structure in Berthelot's definition. The Frobenius structure commutes with spectral sequences for étale hypercoverings (Section 12).

1.3. We fix terminology and notations.

1.3.1. All rings in this paper are commutative rings with a unity 1.

1.3.2. Throughout this paper, we fix a rational prime number p. Let us fix a multiplicative valuation $|\ |$ on the quotient field $\mathbb{Q}_p$ of the ring $\mathbb{Z}_p$ of p-adic integers. We define a category $\mathrm{CDVR}_{\mathbb{Z}_p}$ as follows.

- An object of $\mathrm{CDVR}_{\mathbb{Z}_p}$ is a complete discrete valuation ring $\mathcal{V}$ over $\mathbb{Z}_p$ with a multiplicative valuation $|\ |$ on the field $K = K(\mathcal{V})$ of fractions of $\mathcal{V}$ such that $\mathbb{Q}_p \to K$ is an isometry.
- A morphism of $\mathrm{CDVR}_{\mathbb{Z}_p}$ is an isometric ring homomorphism $\mathcal{V} \to \mathcal{W}$ over $\mathbb{Z}_p$.

We denote by $k(\mathcal{V})$ the residue field of $\mathcal{V}$ and put $\sqrt{|K|} = |K^\times| \otimes \mathbb{Q} \cup \{0\}$. We regard an object $\mathcal{V}$ of $\mathrm{CDVR}_{\mathbb{Z}_p}$ as a p-adic formal algebra over $\mathbb{Z}_p$ (See 2.2.).

1.3.3. Let I be a category. We denote by Ob(I), (resp. Mor(I), resp. $\mathrm{Mor}_I(m, n)$) the class of objects of I (resp. the class of morphisms of I, resp. the set of morphisms from m to n for $m, n \in \mathrm{Ob}(I)$). We denote by I° the dual category of I.

1.3.4. We denote by Δ the standard simplicial category:

– $\mathrm{Ob}(\Delta)$ consists of sets $\{0, 1, \ldots, n\}$, which we simply denote by n and identify with the integer n, for any nonnegative integer n;

– $\mathrm{Mor}(\Delta)$ consists of maps $\eta : m \to n$ with $\eta(k) \leqq \eta(l)$ for $k \leqq l$.

We denote by $\eta_n^l : n \to n+1$, $(0 \leqq l \leqq n+1)$ (resp. $\xi_n^l : n \to n-1, (0 \leqq l \leqq n-1)$) the monomorphism (coface map) (resp. the epimorphism (codegeneracy)) of Δ with $l \notin \eta_n^l(\{0, 1, \ldots, n\})$ (resp. $\xi_n^l(l) = \xi_n^l(l+1)$).

1.3.5. For any nonnegative integer n, we put $\Delta[n]$ to be the full subcategory of Δ whose set of objects consists of $0, 1, \ldots, n$.

1.3.6. For a complex $\mathcal{C}^{\bullet}$ of sheaves of abelian groups, we denote by $\underline{H}^q(\mathcal{C}^{\bullet})$ the q-th cohomology sheaf of $\mathcal{C}^{\bullet}$.

Acknowledgment. This work was done during the stay of second author at the University of Padova. He expresses his thanks for the hospitality of members of Department of Mathematics. The first author also thanks the University of Hiroshima for its hospitality.

2. Preliminaries.

We introduce the terminology of triples and recall the definition of tubes associated to triples, strict neighbourhoods, and sheaves of coherent $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules which were studied by Berthelot. (See [7].) In the absence of explicit mention to the contrary, $\mathcal{V}$ (resp. K, resp. k) denotes an object of $\mathrm{CDVR}_{\mathbb{Z}_p}$ (resp. the field of fractions of $\mathcal{V}$, resp. the residue field of $\mathcal{V}$). We will deal with tubes associated to $\mathcal{V}$-triples locally of finite type and prove that the functor $j^{\dagger}$ of taking sheaf of overconvergent sections commutes with localization. Hence, one can study the behavior of sheaves of coherent $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules by using localization of triples and reduce to the case of $\mathcal{V}$-triples of finite type.

2.1. We introduce the terminology of «pairs».

We call $(X, \overline{X})$ a pair of schemes if X and $\overline{X}$ are schemes over $\mathrm{Spec}\ \mathbb{F}_p$ with an open immersion $X \to \overline{X}$. A morphism $w : (Y, \overline{Y}) \to (X, \overline{X})$ of pairs is a commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\mathring{w}"'] & \overline{Y} \arrow[d, "\overline{w}"] \\ X \arrow[r] & \overline{X} \end{tikzcd}$$

of $\mathbb{F}_p$-schemes.

Let $(X, \overline{X})$ be a pair. A pair $(Y, \overline{Y})$ with a morphism $(Y, \overline{Y}) \to (X, \overline{X})$ of pairs is called a pair over $(X, \overline{X})$ or an $(X, \overline{X})$-pair. A morphism of pairs over $(X, \overline{X})$ is a morphism of pairs which commutes with structure morphisms. A k-pair is a pair over (Spec k, Spec k).

Let (P) be a property of morphisms of schemes such that (i) (P) is stable under any base change, (ii) under the assumption that f is (P), g is (P) if and only if fg is (P), and (iii) an open immersion and a closed immersion are (P). We say that a morphism $w:(Y, \overline{Y}) \to (X, \overline{X})$ of pairs is (P) if $\overline{w}$ is (P).

We define a fiber product of pairs of $(Y, \overline{Y})$ and $(Z, \overline{Z})$ over $(X, \overline{X})$ by

$$(Y, \overline{Y}) \times_{(X, \overline{X})} (Z, \overline{Z}) = (Y \times_X Z, \overline{Y} \times_{\overline{X}} \overline{Z}).$$

2.2. We recall some notions on formal schemes. In this paper, the term «formal $\mathcal{V}$-algebra» means a $\mathcal{V}$-algebra with the complete and separated p-adic topology, that is, a $\mathcal{V}$-algebra $\mathcal{A}$ is a formal $\mathcal{V}$-algebra if the canonical homomorphism $\mathcal{A} \to \varprojlim_n \mathcal{A}/p^n\mathcal{A}$ is an isomorphism. Any homomorphism between formal $\mathcal{V}$-algebras is continuous in the p-adic topology.

Let $\mathcal{A}$ be a formal $\mathcal{V}$-algebra. A formal $\mathcal{A}$-algebra $\mathcal{B}$ is topologically of finite type if there exists a surjective homomorphism $\mathcal{A}\{x_1, \ldots, x_d\} \to B$ of formal $\mathcal{A}$-algebras, where $\mathcal{A}\{x_1, \ldots, x_d\}$ is a p-adic completion of a polynomial ring over $\mathcal{A}$ with d indeterminates. Suppose that $\mathcal{A}$ is noetherian. A formal $\mathcal{A}$-algebra $\mathcal{B}$ is topologically of finite type if and only if it is noetherian and $\mathcal{B}/p\mathcal{B}$ is of finite type over $\mathcal{A}/p\mathcal{A}$ [11, Chap. 0, Proposition 7.5.3]. Hence, every ideal of a formal $\mathcal{A}$-algebra topologically of finite type is closed in the p-adic topology.

An affine formal $\mathcal{V}$-scheme is a formal spectrum $\mathrm{Spf}\,\mathcal{A}$ of a formal $\mathcal{V}$-algebra $\mathcal{A}$. A formal $\mathcal{V}$-scheme is a topological ringed space with a covering $\{\mathcal{U}_\lambda\}_\lambda$ which consists of affine formal $\mathcal{V}$-schemes. A morphism $\widehat{w}: \mathcal{Y} \to \mathcal{X}$ of formal $\mathcal{V}$-schemes is locally of finite type (resp. of finite type) if, for any open affine formal subscheme $\mathcal{U}$ of $\mathcal{X}$, there exists an affine covering (resp. a finite affine covering) $\{V_\lambda\}_\lambda$ of $\widehat{w}^{-1}(\mathcal{U})$ such that

$\Gamma(\mathcal{V}_\lambda, \mathcal{O}_{\mathcal{V}_\lambda})$ is topologically of finite type over $\Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$ for any λ. If a formal $\mathcal{V}$-scheme $\mathcal{X}$ is affine and locally of finite type, then $\mathcal{X}$ is a formal spectrum of a formal $\mathcal{V}$-algebra topologically of finite type [11, Chap. 1, Corollaire 10.6.5].

2.3. We introduce the terminology of «triples locally of finite type».

We call $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ a triple if $(X, \overline{X})$ is a pair, $\mathcal{X}$ is a formal scheme over Spf $\mathbb{Z}_p$ and $\overline{X}$ is a closed subscheme of $\mathcal{X} \times_{\mathrm{Spf}\,\mathbb{Z}_p} \mathrm{Spec}\ \mathbb{F}_p$. Here $\mathbb{F}_p$ is the field of p elements. For a triple $\mathfrak{X}$, we usually use the symbol $(X, \overline{X}, \mathcal{X})$. We put $\mathfrak{S}_{\mathcal{V}} = (\mathrm{Spec}\ k, \mathrm{Spec}\ k, \mathrm{Spf}\ \mathcal{V})$ for an object $\mathcal{V}$ in $\mathrm{CDVR}_{\mathbb{Z}_p}$.

We say that $w : \mathfrak{Y} \to \mathfrak{X}$ is a morphism of triples if it consists of morphisms $w : (Y, \overline{Y}) \to (X, \overline{X})$ of pairs, and a morphism $\widehat{w} : \mathcal{Y} \to \mathcal{X}$ of formal $\mathbb{Z}_p$-schemes such that the diagram

$$\begin{array}{ccccc} Y & \longrightarrow & \overline{Y} & \longrightarrow & \mathcal{Y} \\ {\scriptstyle \mathring{w}}\downarrow & & {\scriptstyle \overline{w}}\downarrow & & \downarrow{\scriptstyle \widehat{w}} \\ X & \longrightarrow & \overline{X} & \longrightarrow & \mathcal{X} \end{array}$$

is commutative. For a morphism w of triples, we usually use the symbol $w = (\mathring{w}, \overline{w}, \widehat{w})$.

Let $\mathfrak{X}$ be a triple. A triple $\mathfrak{Y}$ with a morphism $\mathfrak{Y} \to \mathfrak{X}$ of triples is called a triple over $\mathfrak{X}$ or an $\mathfrak{X}$-triple. A morphism of triples over $\mathfrak{X}$ is a morphism of triples which commutes with structure morphisms. A $\mathcal{V}$-triple is a triple over $\mathfrak{S}_{\mathcal{V}}$. Note that, if $\mathfrak{Y}$ is a triple over $\mathfrak{X}$, then the natural morphism $\overline{Y} \to \mathcal{Y} \times_{\mathcal{X}} \overline{X}$ is a closed immersion.

We now define several notions used throughout the sequel.

2.3.1. Let (P) be a property of morphisms of schemes and formal schemes satisfying conditions (i) (P) is stable under any base change, (ii) under the assumption that f is (P), g is (P) if and only if fg is (P), and (iii) an open immersion and a closed immersion are (P). We say that a morphism $w : \mathfrak{Y} \to \mathfrak{X}$ of triples is (P) if $\widehat{w} : \mathcal{Y} \to \mathcal{X}$ is (P). In this case we sometimes say that an $\mathfrak{X}$-triple $\mathfrak{Y}$ is (P).

For example, a morphism $w : \mathfrak{Y} \to \mathfrak{X}$ of triples is locally of finite type (resp. of finite type) if and only if $\widehat{w} : \mathcal{Y} \to \mathcal{X}$ is locally of finite type (resp. of finite type).

2.3.2. Let $\widehat{w}: \mathcal{Y} \to \mathcal{X}$ be a morphism of formal schemes and let Z be a subset of $\mathcal{Y}$. $\widehat{w}$ is smooth around Z if there exists an open formal subscheme $\mathcal{U}$ of $\mathcal{Y}$ such that $Z \subset \mathcal{U}$ and $\widehat{w}|_{\mathcal{U}}: \mathcal{U} \to \mathcal{X}$ is smooth.

2.3.3. A morphism $w: \mathfrak{Y} \to \mathfrak{X}$ of triples is strict as a morphism of triples (resp. strict as a morphism of pairs) if $\overline{Y} = \widehat{w}^{-1}(\overline{X})$ and $Y = \overline{w}^{-1}(X)$ (resp. $Y = \overline{w}^{-1}(X)$).

Let $\mathfrak{X}$ be a triple and let $\widehat{w}: \mathcal{Y} \to \mathcal{X}$ be a morphism of formal schemes. If we put $\overline{Y} = \widehat{w}^{-1}(\overline{X})$, $Y = \widehat{w}^{-1}(X)$ and $\mathfrak{Y} = (Y, \overline{Y}, \mathcal{Y})$, then the natural map $w: \mathfrak{Y} \to \mathfrak{X}$ which is induced from $\widehat{w}$ is a morphism of triples. We call $\mathfrak{Y}$ (resp. w) a triple over $\mathfrak{X}$ induced from $\mathcal{Y}$ (resp. a morphism induced from $\widehat{w}$). In particular, if $\mathfrak{X} = \mathfrak{S}_{\mathcal{V}}$, $\mathfrak{Y}$ is called a $\mathcal{V}$-triple induced from $\mathcal{Y}$.

2.3.4. A morphism $f: T \to S$ of (formal) schemes is a Zariski covering if T is a disjoint union of open (formal) subschemes and f is a natural surjective morphism. A Zariski covering $f: T \to S$ is finite if T is a finite disjoint union of open subschemes of S, and a Zariski covering $f: T \to S$ is affine if each component of T is affine.

A morphism $w: \mathfrak{Y} \to \mathfrak{X}$ of triples is a Zariski covering, (resp. a finite Zariski covering, resp. an affine Zariski covering) if (i) $\overline{w}: \overline{Y} \to \overline{X}$ is so, (ii) w is strict as a morphism of pairs, and (iii) $\widehat{w}$ is separated, locally of finite type and smooth around Y. ■

Let $\mathfrak{Y}$ and $\mathfrak{Z}$ be a triple over $\mathfrak{X}$. We define a fiber product of triples of $\mathfrak{Y}$ and $\mathfrak{Z}$ over $\mathfrak{X}$ by

$$\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z} = (Y, \overline{Y}, \mathcal{Y}) \times_{(X, \overline{X}, \mathcal{X})} (Z, \overline{Z}, \mathcal{Z}) = (Y \times_X Z, \overline{Y} \times_{\overline{X}} \overline{Z}, \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}).$$

Both projections $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z} \to \mathfrak{Y}$ and $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z} \to \mathfrak{Y}$ are morphisms of triples over $\mathfrak{X}$. It is easily seen that

(1) if $\mathfrak{Y}$ and $\mathfrak{Z}$ satisfies a property (P) as in 2.3.1, then $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z}$ is (P) over $\mathfrak{X}$;

(2) any property (P) as in 2.3.1 is stable under any base change;

(3) any base change of strict morphisms as a morphism of triples (resp. pairs) is again so.

2.3.5 Definition. (1) We define a category $\mathrm{TRI}^{\mathrm{lft}}$ as follows.

– An object is a $\mathcal{V}$-triple $\mathfrak{X}$ locally of finite type for some object $\mathcal{V}$ of $\mathrm{CDVR}_{\mathbb{Z}_p}$;

– a morphism w from a $\mathcal{V}$-triple $\mathfrak{X}$ locally of finite type to a $\mathcal{W}$-triple $\mathfrak{Y}$ locally of finite type is a commutative diagram

$$\begin{array}{ccc} \mathfrak{Y} & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \mathfrak{S}_{\mathcal{W}} & \longrightarrow & \mathfrak{S}_{\mathcal{V}} \end{array}$$

of triples. Here the bottom arrow is a morphism induced from a morphism $\mathcal{V} \to \mathcal{W}$ of $\mathrm{CDVR}_{\mathbb{Z}_p}$.

An object of $\mathrm{TRI}^{\mathrm{lft}}$ is called an lft-triple We usually denote by $\mathfrak{X}$ (resp. $w : \mathfrak{Y} \to \mathfrak{X}$) an object $\mathfrak{X} \to \mathfrak{S}_{\mathcal{V}}$ (resp. a morphism as above) of $\mathrm{TRI}^{\mathrm{lft}}$, for simplicity.

(2) Let $\mathfrak{X}$ be an lft-triple. An lft-triple $\mathfrak{Y}$ with a morphism $\mathfrak{Y} \to \mathfrak{X}$ of lft-triples is called an lft-triple over $\mathfrak{X}$. A morphism of lft-triples over $\mathfrak{X}$ is a morphism of $\mathrm{TRI}^{\mathrm{lft}}$ which commutes with structure morphisms. ■

For example, a category of $\mathcal{V}$-triples locally of finite type is naturally a subcategory of lft-triples.

In section 10 we will introduce a generalized notion «$(X, \overline{X})$-triples» over a triple $\mathfrak{T}$.

2.4. Let $\mathcal{P}$ be a formal $\mathcal{V}$-scheme locally of finite type. We denote by $\mathcal{P}_K^{\mathrm{an}}$ (resp. $\mathrm{sp} : \mathcal{P}_K^{\mathrm{an}} \to \mathcal{P}$) the rigid analytic space over Spm K associated to $\mathcal{P}$ in Raynaud's sense [20] (resp. the specialization morphism). (See [3, Sect. 1].)

Suppose that $\{\mathcal{P}_\alpha\}$ is a Zariski covering of $\mathcal{P}$. W is an admissible open subset of $\mathcal{P}_K^{\mathrm{an}}$ (resp. $\{W_\beta\}_\beta$ is an admissible covering of an admissible open subset W) if and only if $W \cap \mathcal{P}^{\mathrm{an}}_{\alpha, K}$ is an admissible open subset of $\mathcal{P}^{\mathrm{an}}_{\alpha, K}$ (resp. $\{W_\beta \cap \mathcal{P}^{\mathrm{an}}_{\alpha, K}\}_\beta$ is an admissible covering of $W \cap \mathcal{P}^{\mathrm{an}}_{\alpha, K}$) for all α.

$\mathcal{P}_K^{\mathrm{an}}$ is quasi-separated, and it is separated (resp. quasi-compact) if $\mathcal{P}$ is separated (resp. of finite type) over Spf $\mathcal{V}$. (See [7, 0.1.7, 0.2.4].)

For a locally closed k-subscheme Z in $\mathcal{P} \times_{\mathrm{Spf}\ \mathcal{V}} \mathrm{Spec}\ k$,

$$]Z[_{\mathcal{P}} = \mathrm{sp}^{-1}(Z)$$

with the induced Grothendieck topology from that of $\mathcal{P}_K^{\mathrm{an}}$ is called a tube of Z in $\mathcal{P}_K^{\mathrm{an}}$. $]Z[_{\mathcal{P}}$ is an admissible open subset in $\mathcal{P}_K^{\mathrm{an}}$ [7, 1.1].

Let $w : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of lft-triples such that $\mathfrak{X}$ (resp. $\mathfrak{Y}$) is a $\mathcal{V}$-triple (resp. a $\mathcal{W}$-triple). Since the canonical morphism $\mathfrak{Y} \to \mathfrak{X} \times_{\mathfrak{S}_{\mathcal{V}}} \mathfrak{S}_{\mathcal{W}}$

is locally of finite type, w induces a commutative diagram

$$\begin{array}{ccc}]\overline{Y}[_{\mathcal{Y}} & \longrightarrow &]\overline{X}[_{\mathcal{X}} \\ \downarrow & & \downarrow \\ \operatorname{Spm} K(\mathcal{W}) & \longrightarrow & \operatorname{Spm} K(\mathcal{V}) \end{array}$$

of rigid analytic spaces. We denote by

$$\widetilde{w}:]\overline{Y}[_{\mathcal{Y}} \to]\overline{X}[_{\mathcal{X}}$$

the induced morphism of rigid analytic spaces. $\widetilde{w}$ is continuous. The correspondence from the category of lft-triples to the category of quasi-separated rigid analytic spaces is a functor.

2.5. Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ be a $\mathcal{V}$-triple locally of finite type, in such a setting we will define by ∂X any scheme-theoretic complement of X in $\overline{X}$ (and we will refer to it as the «complement» of X in $\overline{X}$, or simply the complement). We recall the definition of strict neighbourhoods of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ [7, 1.2.1 Définition]. A subset V in $]\overline{X}[_{\mathcal{X}}$ is called a strict neighbourhood of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ if $\{V,]\partial X[_{\mathcal{X}}\}$ is an admissible covering of $]\overline{X}[_{\mathcal{X}}$. Trivially, $]\overline{X}[_{\mathcal{X}}$ is a strict neighbourhood. Note that the notion of strict neighbourhood does not depend on the choice of complements ∂X and it depends only on the triple $\mathfrak{X}$.

2.5.1. Lemma ([7, 1.2.3 Remarque]). *With the notation as above, let $\coprod_\alpha \mathcal{X}_\alpha$ be a Zariski covering of $\mathcal{X}$ and let $\mathfrak{X}_\alpha = (X_\alpha, \overline{X}_\alpha, \mathcal{X}_\alpha)$ be a triple over $\mathfrak{X}$ induced by the morphism $\mathcal{X}_\alpha \to \mathcal{X}$ for each α. A subset V of $]\overline{X}[_{\mathcal{X}}$ is a strict neighbourhood of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ if and only if $V \cap]\overline{X}_\alpha[_{\mathcal{X}_\alpha}$ is a strict neighbourhood of $]X_\alpha[_{\mathcal{X}_\alpha}$ in $]\overline{X}_\alpha[_{\mathcal{X}_\alpha}$ for all α.*

Proof. The assertion follows from the fact $\{]\overline{X}_\alpha[_{\mathcal{X}_\alpha}\}_\alpha$ is an admissible covering of $]\overline{X}[_{\mathcal{X}}$. ■

For admissible open subsets $V \subset U$ of $]\overline{X}[_{\mathcal{X}}$, we denote by $j_V^U: V \to U$ the open immersion. In the case where $U =]\overline{X}[_{\mathcal{X}}$ we simply put $j_V = j_V^{]\overline{X}[_{\mathcal{X}}}$.

If V and W are strict neighbourhoods of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$, then the intersection $V \cap W$ is also so [7, 1.2.10 Proposition (i)]. Hence, the category of strict neighbourhoods of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ forms a filtered category. Let $\mathcal{F}$ be a sheaf of abelian groups on a strict neighbourhood U of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$. A sheaf of overconvergent sections of $\mathcal{F}$ on $]\overline{X}[_{\mathcal{X}}$ along ∂X is defined by

$$j_U^\dagger \mathcal{F} = \varinjlim_{V} j_{V*}(j_V^U)^{-1}\mathcal{F},$$

where V runs through all strict neighbourhoods of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ which are included in U [7, 2.1.1]. If $\mathcal{F}$ is a sheaf of rings (resp. $\mathcal{A}$-modules for a sheaf $\mathcal{A}$ of rings on U), then $j_U^\dagger\mathcal{F}$ is a sheaf of rings (resp. $j_U^\dagger\mathcal{A}$-modules). In the case where $U =]\overline{X}[_{\mathcal{X}}$ we denote by $j^\dagger$ the functor $j^\dagger_{]\overline{X}[_{\mathcal{X}}}$ as usual. Note that our definition is slightly different from the definition in [7, 2.1.1]. The $j_U^\dagger$ of [7, 2.1.1] corresponds to $j_U^{-1}j_U^\dagger$ in our notation.

We will discuss some properties of $j_U^\dagger$ in sections 2.6 and 2.7.

2.6. We will calculate the group of sections of sheaves of overconvergent sections on a quasi-compact admissible open subset. We denote by π a uniformizer of $\mathcal{V}$. The following construction of $U^{\geqq \nu}_{\mathfrak{X}, \partial X}$ and $U^{\leqq \nu}_{\mathfrak{X}, \partial X}$ is due to Berthelot, cf. [7, 1.1.8].

Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ be a $\mathcal{V}$-triple of finite type such that $\mathcal{X}$ is affine with $\mathcal{A} = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and let us fix a complement ∂X of X in $\overline{X}$. Since $\overline{X}$ is a closed subscheme of $\mathcal{X} \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}\, k$, the homomorphism $\mathcal{A} \otimes_{\mathcal{V}} k \to \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$ of k-algebras is surjective. We take lifts $g_1, g_2, \ldots, g_s \in \mathcal{A}$ of generators of the ideal of definition of ∂X in $\overline{X}$. For $\nu \in \sqrt{|K|} \cap]0,1[$, we define admissible open subsets $U_{g_i}^{\geqq \nu}$, $U_{g_i}^{\leqq \nu}$ $(1 \leqq i \leqq s)$ and $U^{\geqq \nu}_{\mathfrak{X}, \partial X}$, $U^{\leqq \nu}_{\mathfrak{X}, \partial X}$ of $]\overline{X}[_{\mathcal{X}}$ by

$$\begin{aligned}
U_{g_i}^{\geqq \nu} &= \{x \in]\overline{X}[_{\mathcal{X}} \mid |g_i(x)| \geqq \nu,\ |g_i(x)| \geqq |g_j(x)| \text{ for } 1 \leqq j \leqq s\},\\
U_{g_i}^{\leqq \nu} &= \{x \in]\overline{X}[_{\mathcal{X}} \mid |g_i(x)| \leqq \nu\},\\
U^{\geqq \nu}_{\mathfrak{X}, \partial X} &= \bigcup_{i=1}^{s} U_{g_i}^{\geqq \nu},\\
U^{\leqq \nu}_{\mathfrak{X}, \partial X} &= \bigcap_{i=1}^{s} U_{g_i}^{\leqq \nu}.
\end{aligned}$$

Indeed, the subset of $\mathcal{X}_K^{\mathrm{an}}$ which defined by the same inequalities in the definition of $U_{g_i}^{\geqq \nu}$ (resp. $U_{g_i}^{\leqq \nu}$) is an affinoid subvariety. Hence, $U_{g_i}^{\geqq \nu}$ and $U_{g_i}^{\leqq \nu}$ are admissible open subsets of $]\overline{X}[_{\mathcal{X}}$ for all i. Since $\mathcal{X}_K^{\mathrm{an}}$ is quasi-separated, both $U^{\geqq \nu}_{\mathfrak{X}, \partial X}$ and $U^{\leqq \nu}_{\mathfrak{X}, \partial X}$ are also admissible open subsets of $]\overline{X}[_{\mathcal{X}}$. Moreover, $\{U_{g_i}^{\geqq \nu}, U_{g_i}^{\leqq \nu} \mid 1 \leqq i \leqq s\}$ and $\{U^{\geqq \nu}_{\mathfrak{X}, \partial X}, U^{\leqq \nu}_{\mathfrak{X}, \partial X}\}$ are admissible coverings of $]\overline{X}[_{\mathcal{X}}$. Therefore, $U^{\geqq \nu}_{\mathfrak{X}, \partial X}$ is a strict neighbourhood of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$. Note that $U^{\leqq \nu}_{\mathfrak{X}, \partial X}$ is the closed tube $[\partial X]_{\mathcal{X}\nu}$ of radius ν and $U^{\geqq\nu}_{\mathfrak{X},\partial X}$ is $]\overline{X}[_{\mathcal{X}}\setminus]\partial X[_{\mathcal{X}\nu}$ ($]\partial X[_{\mathcal{X}\nu}$ is an open tube of radius ν) in [7, 1.1.8].

The following lemma is easy.

2.6.1. Lemma. *With the notation as above, let W be a subset in $U^{\geqq\delta}_{g_i}$ for some g_i and some $\delta\in\sqrt{|K|}\cap]0,1[$. Then, for any $\nu>\delta$, we have $W\cap U^{\geqq\nu}_{\mathfrak{X},\partial X}=W\cap U^{\geqq\nu}_{g_i}$. In particular, if W is affinoid (resp. quasi-compact), then $W\cap U^{\geqq\nu}_{\mathfrak{X},\partial X}$ is also affinoid (resp. quasi-compact).*

2.6.2. Lemma. *With the notation as above, let $g_1', g_2', \ldots, g_{s'}'\in\mathcal{A}$ be other lifts of generators of the ideal of definition of ∂X in $\overline{X}$ and let $U^{\geqq\nu}_{g_i'}$, $U^{\leqq\nu}_{g_i'}$ $(1\leqq i\leqq s')$ and $(U^{\geqq\nu}_{\mathfrak{X},\partial X})'$, $(U^{\leqq\nu}_{\mathfrak{X},\partial X})'$ be admissible open subsets of $]\overline{X}[_{\mathcal{X}}$ for $g_1', g_2', \ldots, g_{s'}'$ as above. If $\nu>|\pi|$, then we have*

$$(U^{\geqq\nu}_{\mathfrak{X},\partial X})'=U^{\geqq\nu}_{\mathfrak{X},\partial X},$$
$$(U^{\leqq\nu}_{\mathfrak{X},\partial X})'=U^{\leqq\nu}_{\mathfrak{X},\partial X}.$$

Proof. Note that

$$g_j'=\sum_{i=1}^{s}a_{ij}g_i+\pi b_j$$

with some a_{ij}, $b_j\in\mathcal{A}$. If $x\in U^{\geqq\nu}_{g_j'}$, then

$$|g_j'(x)|=\left|\sum_{i=1}^{s}a_{ij}(x)\,g_i(x)+\pi b_j(x)\right|\leqq\max\{|a_{ij}(x)\,g_i(x)|\mid 1\leqq i\leqq s\}.$$

Since $|a_{ij}(x)|\leqq 1$ $(1\leqq i\leqq s)$, there exists at least one i such that $|g_i(x)|\geqq\nu$. Hence, we have $x\in U^{\geqq\nu}_{\mathfrak{X},\partial X}$ and $(U^{\geqq\nu}_{\mathfrak{X},\partial X})'\subset U^{\geqq\nu}_{\mathfrak{X},\partial X}$. If $x\in U^{\leqq\nu}_{\mathfrak{X},\partial X}$, then

$$|g_j'(x)|=\left|\sum_{i=1}^{s}a_{ij}(x)\,g_i(x)+\pi b_j(x)\right|\leqq$$
$$\leqq\max\{|a_{ij}(x)\,g_i(x)|\mid 1\leqq i\leqq s\}\cup\{|\pi|\}\leqq\nu.$$

Hence, $U^{\leqq\nu}_{\mathfrak{X},\partial X}\subset(U^{\leqq\nu}_{\mathfrak{X},\partial X})'$. One can prove the opposite inclusions in the same way. Therefore, $(U^{\geqq\nu}_{\mathfrak{X},\partial X})'=U^{\geqq\nu}_{\mathfrak{X},\partial X}$ and $(U^{\leqq\nu}_{\mathfrak{X},\partial X})'=U^{\leqq\nu}_{\mathfrak{X},\partial X}$. ■

If $\nu>|\pi|$, the admissible open subsets $U^{\geqq\nu}_{\mathfrak{X},\partial X}$ and $U^{\leqq\nu}_{\mathfrak{X},\partial X}$ are independent of the choices of lifts $g_1,\ldots,g_s\in\mathcal{A}$ of generators of the ideal of definition of ∂X in $\overline{X}$. They depend only on the choice of complement ∂X of X in $\overline{X}$. If we chose another complement $\partial X'$, then, for any ν sufficiently close to 1, there exist λ, μ with $\lambda < \nu < \mu$ such that $U^{\geqq \mu}_{\mathfrak{X}, \partial X} \subset U^{\geqq \nu}_{\mathfrak{X}, \partial X'} \subset$ $\subset U^{\geqq \lambda}_{\mathfrak{X}, \partial X}$.

2.6.3. LEMMA. *Let $w : \mathfrak{Y} \to \mathfrak{X}$ be a morphism $\mathcal{V}$-triple of finite type such that both $\mathcal{X}$ and $\mathcal{Y}$ are affine and w is strict as a morphism of pairs, let ∂X be a complement of X in $\overline{X}$ and put $\partial Y = \overline{w}^{-1}(\partial X)$ so that ∂Y is a complement of Y in $\overline{Y}$. If $\nu > |\pi|$, then*

$$\widetilde{w}^{-1}(U^{\geqq \nu}_{\mathfrak{X}, \partial X}) = U^{\geqq \nu}_{\mathfrak{Y}, \partial Y}$$
$$\widetilde{w}^{-1}(U^{\leqq \nu}_{\mathfrak{X}, \partial X}) = U^{\leqq \nu}_{\mathfrak{Y}, \partial Y}.$$

Moreover, if $\widehat{w} : \mathcal{Y} \to \mathcal{X}$ is an open immersion and w is strict as a morphism of triples, then we have

$$U^{\geqq \nu}_{\mathfrak{X}, \partial X} \cap \,]\overline{Y}[_{\mathcal{Y}} = U^{\geqq \nu}_{\mathfrak{Y}, \partial Y}$$
$$U^{\leqq \nu}_{\mathfrak{X}, \partial X} \cap \,]\overline{Y}[_{\mathcal{Y}} = U^{\leqq \nu}_{\mathfrak{Y}, \partial Y}.$$

PROOF. Let us put $\mathcal{A} = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\mathcal{B} = \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ with a homomorphism $\theta : \mathcal{A} \to \mathcal{B}$ of formal $\mathcal{V}$-algebras. Let $g_1, \ldots, g_s \in \mathcal{A}$ be lifts of generators of the ideal of definition of ∂X in $\overline{X}$. Since $\overline{w}^{-1}(\partial X) = \partial Y$, the images of $g_1, \ldots, g_s$ in $\mathcal{B}$ are lifts of generators of the ideal of definition of ∂Y in $\overline{Y}$. The assertion follows from the fact that $\theta(g_i)(y) = g_i(\widetilde{w}(y))$ for any $y \in]\overline{Y}[_{\mathcal{Y}}$ and any i. ■

From now on, let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ be a $\mathcal{V}$-triple locally of finite type and let ∂X be a complement of X in $\overline{X}$. Let $\{\mathcal{X}_\alpha\}_{\alpha \in I}$ be an affine Zariski covering of $\mathcal{X}$, let $\mathfrak{X}_\alpha$ be the induced triple over $\mathfrak{X}$ from $\mathcal{X}_\alpha \to \mathcal{X}$ and put $\partial X_\alpha =$ $= \partial X \cap \overline{X}_\alpha$. For $\nu \in \sqrt{K} \cap]|\pi|, 1[$, we define subsets of $]\overline{X}[_{\mathcal{X}}$ as follows:

$$U^{\geqq \nu}_{\mathfrak{X}, \partial X} = \bigcup_{\alpha \in I} U^{\geqq \nu}_{\mathfrak{X}_\alpha, \partial X_\alpha}$$
$$U^{\leqq \nu}_{\mathfrak{X}, \partial X} = \bigcup_{\alpha \in I} U^{\leqq \nu}_{\mathfrak{X}_\alpha, \partial X_\alpha}.$$

2.6.4. LEMMA. *With the notation as above, suppose that $\nu > |\pi|$. Then we have*

$$U^{\geqq \nu}_{\mathfrak{X}_\alpha, \partial X_\alpha} = U^{\geqq \nu}_{\mathfrak{X}, \partial X} \cap \,]\overline{X}_\alpha[_{\mathcal{X}_\alpha}$$
$$U^{\leqq \nu}_{\mathfrak{X}_\alpha, \partial X_\alpha} = U^{\leqq \nu}_{\mathfrak{X}, \partial X} \cap \,]\overline{X}_\alpha[_{\mathcal{X}_\alpha}.$$

PROOF. Let us put $\mathfrak{X}_{\alpha\beta} = \mathfrak{X}_\alpha \times_{\mathfrak{X}} \mathfrak{X}_\beta$ for α and β. We have only to prove that $U^{\geqq \nu}_{\mathfrak{X}_{\alpha\beta},\, \partial X_{\alpha\beta}} = U^{\geqq \nu}_{\mathfrak{X}_\alpha,\, \partial X_\alpha} \cap]\overline{X}_{\alpha\beta}[_{\mathcal{X}_{\alpha\beta}}$ and $U^{\leqq \nu}_{\mathfrak{X}_{\alpha\beta},\, \partial X_{\alpha\beta}} = U^{\leqq \nu}_{\mathfrak{X}_\alpha,\, \partial X_\alpha} \cap]\overline{X}_{\alpha\beta}[_{\mathcal{X}_{\alpha\beta}}$ for any α and β.

Let $\mathcal{Y}$ be an open affine formal subscheme of $\mathcal{X}_{\alpha\beta}$ and let $\mathfrak{Y}$ be a triple over $\mathfrak{X}_{\alpha\beta}$ induced from $\mathcal{Y} \to \mathcal{X}_{\alpha\beta}$. Then we have $U^{\geqq \nu}_{\mathfrak{X},\, \partial X} \cap]\overline{Y}[_{\mathcal{Y}} = U^{\geqq \nu}_{\mathfrak{Y},\, \partial Y}$ and $U^{\leqq \nu}_{\mathfrak{X},\, \partial X} \cap]\overline{Y}[_{\mathcal{Y}} = U^{\leqq \nu}_{\mathfrak{Y},\, \partial Y}$ by Lemma 2.6.3. If $\mathcal{Y}$ varies all open affine formal subschemes of $\mathcal{X}_{\alpha\beta}$, we have the desired formulas. ■

Lemmas 2.5.1 and 2.6.4 imply the following proposition.

2.6.5. PROPOSITION. *With the notation as above, suppose that* $\nu > |\pi|$. *Then* $\{U^{\geqq \nu}_{\mathfrak{X},\, \partial X}, U^{\leqq \nu}_{\mathfrak{X},\, \partial X}\}$ *is an admissible covering of* $]\overline{X}[_{\mathcal{X}}$. *In particular,* $U^{\geqq \nu}_{\mathfrak{X},\, \partial X}$ *is a strict neighbourhood of* $]X[_{\mathcal{X}}$ *in* $]\overline{X}[_{\mathcal{X}}$. ■

The following lemma is a locally of finite type version of [7, 1.2.5 Proposition (i)]. However, [7, 1.2.5 Proposition (ii)] does not apply to an arbitrary covering locally of finite type. Let us consider the situation as in Lemma 2.5.1. Unfortunately, for an arbitrary collection $\{V_\alpha\}_\alpha$ such that each V_α is a strict neighbourhood of $]X_\alpha[_{\mathcal{X}_\alpha}$ in $]\overline{X}_\alpha[_{\mathcal{X}_\alpha}$, we can not find a strict neighbourhood V of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ with $V \cap]\overline{X}_\alpha[_{\mathcal{X}_\alpha} \subset V_\alpha$ for all α.

2.6.6. LEMMA. *With the notation as above, we have the following:*

(1) *Let* W *be a quasi-compact subset of* $]\partial X[_{\mathcal{X}}$. *Then there exists* $\delta \in \sqrt{|K|} \cap]|\pi|, 1[$ *such that* $W \subset U^{\leqq \delta}_{\mathfrak{X},\, \partial X}$.

(2) *Let* W *be a quasi-compact admissible open subset of* $]\overline{X}[_{\mathcal{X}}$. *For any strict neighbourhood* V *of* $]X[_{\mathcal{X}}$ *in* $]\overline{X}[_{\mathcal{X}}$, *there exists* $\nu \in \sqrt{|K|} \cap]|\pi|, 1[$ *such that*

$$W \cap U^{\geqq \nu}_{\mathfrak{X},\, \partial X} \subset W \cap V\,.$$

PROOF. (1) Since W is quasi-compact, there exists a finite subset I_0 of I such that $W \subset \bigcup_{\alpha \in I_0}]\overline{X}_\alpha[_{\mathcal{X}_\alpha}$. Since $W \cap]\overline{X}_\alpha[_{\mathcal{X}_\alpha}$ is quasi-compact (see 2.6.1); for any $\alpha \in I_0$, it follows from the maximum principle (Lemma 2.6.7 below) that there exists δ_α in $\sqrt{|K|} \cap]|\pi|, 1[$ such that

$$W \cap]\overline{X}_\alpha[_{\mathcal{X}_\alpha} \subset U^{\leqq \delta_\alpha}_{\mathfrak{X},\, \partial X}.$$

If we put $\delta = \max\{\delta_\alpha \,|\, \alpha \in I_0\}$, we have $W \subset U^{\leqq \delta}_{\mathfrak{X},\, \partial X}$.

(2) Since W is quasi-compact, there exists a finite subset I_0 of I such that $W \subset \bigcup_{\alpha \in I_0}]\overline{X}_\alpha[_{\mathcal{X}_\alpha}$. Let $\{W_{\alpha\beta}\}_\beta$ be a finite affinoid admissible open covering of $W \cap]\overline{X}_\alpha[_{\mathcal{X}_\alpha}$ such that $\{W_{\alpha\beta}\}_\beta$ is a refinement of the admissible covering $\{W \cap V \cap]\overline{X}_\alpha[_{\mathcal{X}_\alpha}, W \cap]\partial X_\alpha[_{\mathcal{X}_\alpha}\}$ of $W \cap]\overline{X}_\alpha[_{\mathcal{X}_\alpha}$. Since $W \cap$ $\cap]\overline{X}_\alpha[_{\mathcal{X}_\alpha}$ is quasi-compact by Lemma 2.6.1, such a covering always exists for any α. Then there exists $\delta_\alpha \in \sqrt{|K|} \cap]|\pi|, 1[$ such that

$$W_{\alpha\beta} \subset U^{\leqq \delta_\alpha}_{\mathfrak{X}_\alpha, \partial X_\alpha}$$

for all β with $W_{\alpha\beta} \subset]\partial X_\alpha[_{\mathcal{X}_\alpha}$ and any $\alpha \in I_0$ by (1). We put $\delta = \max\{\delta_\alpha \mid \alpha \in$ $\in I_0\}$. If $\nu > \delta$, then $W \cap U^{\geqq \nu}_{\mathfrak{X}_\alpha, \partial X_\alpha} \subset W \cap V \cap]\overline{X}_\alpha[_{\mathcal{X}_\alpha}$ for any $\alpha \in I_0$. Since $W \subset \bigcup_{\alpha \in I_0}]\overline{X}_\alpha[_{\mathcal{X}_\alpha}$, we have $W \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X} \subset W \cap V$. ■

2.6.7. Lemma. (The maximum principle [8, 9.1.4 Lemma 6]). *Let X be an affinoid variety over* Spm K *and let $g \in \Gamma(X, \mathcal{O}_X)$. Suppose that Y is an affinoid subvariety with $Y \subset X(|g| < 1) = \{x \in X \mid |g(x)| < 1\}$. Then there exists a real number $\nu \in \sqrt{|K|} \cap]0, 1[$ such that*

$$Y \subset X(|g| \leqq \nu) = \{x \in X \mid |g(x)| \leqq \nu\}.$$

2.6.8. Proposition. *With the notation as above, let W be a quasi-compact admissible open subset of $]\overline{X}[_{\mathcal{X}}$. For a sheaf of abelian groups $\mathcal{F}$ on a strict neighbourhood U of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$, we have*

$$H^q(W, j_U^\dagger \mathcal{F}) \cong \lim_{\nu \to 1^-} H^q(W, j_{U \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X} *}(j^U_{U \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X}})^{-1} \mathcal{F})$$

for any q, where ν runs through $\nu \in \sqrt{|K|} \cap]|\pi|, 1[$. In particular, we have

$$\Gamma(W, j_U^\dagger \mathcal{F}) = \lim_{\nu \to 1^-} \Gamma(W \cap U \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X}, \mathcal{F}).$$

Proof. Since W is quasi-compact and quasi-separated, cohomological functors and filtered direct limits commute with each other [1, VI, Corollaire 5.3]. Hence we have

$$H^q(W, j_U^\dagger \mathcal{F}) \cong \varinjlim_V H^q(W, j_{V*}(j^U_V)^{-1} \mathcal{F}),$$

where V runs through all strict neighbourhoods of $]X[_{\mathcal{X}}$ in U. Let V be a strict neighbourhood of $]X[_{\mathcal{X}}$ in U. Then there exists a real number $\nu \in \sqrt{|K|} \cap]|\pi|, 1[$ such that $W \cap U^{\geqq \nu} \subset W \cap V$ by Lemma 2.6.6. Since

$V \cap U^{\geqq \nu}$ is also a strict neighbourhood of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ which is inside U [7, Proposition 1.2.10], we have the assertion. ■

Since any rigid analytic space has an admissible covering which consists of open affinoid subvarieties, one can check the vanishing of sheaves on tubes using the proposition just proved.

2.7. We prove some properties of the functor $j^{\dagger}$. First we introduce a functor $f_!$ of extension by zero outside an open set. Let $f: S \to T$ be an open immersion of rigid analytic spaces. For a sheaf $\mathcal{F}$ of abelian groups on S, we define a sheaf $f_! \mathcal{F}$ by the sheaf abelian groups on T which is associated to the presheaf

$$W \mapsto \begin{cases} \Gamma(W, \mathcal{F}) & \text{if } W \subset S \\ 0 & \text{if } W \not\subset S \end{cases}$$

for any admissible open subset W of T. $f_! \mathcal{F}$ is a subsheaf of $\mathcal{F}$ by [8, 9.2.2 Lemma 3] and there exists a canonical homomorphism $f_! \mathcal{F} \to f_* \mathcal{F}$ such that $f^{-1} f_! \mathcal{F} \to f^{-1} f_* \mathcal{F} \cong \mathcal{F}$. If $\mathcal{F}$ is a sheaf of rings (resp. $\mathcal{A}$-modules), then $f_! \mathcal{F}$ is a sheaf of rings (resp. $f_! \mathcal{A}$-modules). The following lemma is easy.

2.7.1. Lemma. *With the notation as above, we have*
(1) $f_!$ *is exact;*
(2) $f_!$ *is a left adjoint of* f^{-1}.

By (2) of the above lemma there exists a canonical homomorphism $f_!(f^{-1}\mathcal{A}) \to \mathcal{A}$ of sheaves of rings.

Now let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ be a $\mathcal{V}$-triple locally of finite type and let us fix a complement ∂X of X in $\overline{X}$.

2.7.2. Proposition. *With the notation as above, let* $V \subset U$ *be strict neighbourhoods of* $]X[_{\mathcal{X}}$ *in* $]\overline{X}[_{\mathcal{X}}$ *and let* $\mathcal{A}$ *be a sheaf of rings on* U.

(1) [7, 2.1.1]. *Let* $\mathcal{F}$ *(resp.* $\mathcal{G}$*) be a sheaf of* $\mathcal{A}$*-modules (resp.* $(j_V^U)^{-1}\mathcal{A}$*-modules). Then the natural homomorphism* $j_V^{\dagger}((j_V^U)^{-1}\mathcal{F}) \to j_U^{\dagger}\mathcal{F}$ *(resp.* $j_V^{\dagger}\mathcal{G} \to j_U^{\dagger}(j_{V*}^U \mathcal{G})$*) is an isomorphism. The canonical homomorphisms* $j_{V!}^U \mathcal{G} \to \mathcal{A} \otimes_{j_{V!}^U((j_V^U)^{-1}\mathcal{A})} j_{V!}^U \mathcal{G} \to j_{V*}^U \mathcal{G}$ *induce isomorphisms* $j_U^{\dagger}(j_{V!}^U \mathcal{G}) \xrightarrow{\cong} j_U^{\dagger}(\mathcal{A} \otimes_{j_{V!}^U((j_V^U)^{-1}\mathcal{A})} j_{V!}^U \mathcal{G}) \xrightarrow{\cong} j_U^{\dagger}(j_{V*}^U \mathcal{G})$.

(2) [7, 2.1.1]. *If* $\mathcal{F}$ *is a sheaf of* $\mathcal{A}$*-modules, then we have* $j_{]\partial X[_{\mathcal{X}}}^{-1}(j_U^{\dagger}\mathcal{F}) = 0$.

(3) [7, 2.1.3 Proposition (i)]. *Let $\mathcal{F}$ be a sheaf of $\mathcal{A}$-modules. The natural $\mathcal{A}$-homomorphism $j_{U!}\mathcal{F} \to j_U^\dagger \mathcal{F}$ induced by the natural transform $j_{U!} \to j_{U*}$ is surjective. In particular, if $(j^U_{]\partial X[_{\mathcal{X}} \cap U})^{-1}\mathcal{F} = 0$, then the natural homomorphism $j_{U!}\mathcal{F} \to j_U^\dagger \mathcal{F}$ is an isomorphism.*

(4) *Let $\mathcal{F}$ be a sheaf of $\mathcal{A}$-modules and let $\mathcal{G}$ be a sheaf of $j_U^\dagger \mathcal{A}$-modules. Then the map*

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, j_U^{-1}\mathcal{G}) \xrightarrow{j_U^\dagger} \mathrm{Hom}_{j_U^\dagger \mathcal{A}}(j_U^\dagger \mathcal{F}, \mathcal{G})$$

induced by the natural transform $j_U^\dagger j_U^{-1} \to \mathrm{id}$ is bijective. In other words, $j_U^\dagger$ is a left adjoint of the functor j_U^{-1} from the category of sheaves of $j^\dagger \mathcal{A}$-modules to that of sheaves of $\mathcal{A}$-modules.

(5) *For sheaves $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{A}$-modules, the homomorphism*

$$j_U^\dagger(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}) \to j_U^\dagger \mathcal{F} \otimes_{j_U^\dagger \mathcal{A}} j_U^\dagger \mathcal{G}$$

which is induced from the composition $j_{U!}(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}) \xrightarrow{\cong} j_{U!}\mathcal{F} \otimes_{j_{U!}\mathcal{A}} j_{U!}\mathcal{G} \to$ $\to j_U^\dagger \mathcal{F} \otimes_{j_U^\dagger \mathcal{A}} j_U^\dagger \mathcal{G}$ of homomorphisms ((3) and Lemma 2.7.1 (2)) is an isomorphism.

(6) [7, 2.1.3 Proposition (iii)]. *The functor $j_U^\dagger$ is exact from the category of sheaves of $\mathcal{A}$-modules to the category of sheaves of $j_U^\dagger \mathcal{A}$-modules.*

PROOF. (1) Since a finite intersection of strict neighbourhoods is also a strict neighbourhood, the assertion is easy by definition.

(2) Let W be a quasi-compact admissible open subset of $]\overline{X}[_{\mathcal{X}}$ such that W is included in $]\partial X[_{\mathcal{X}}$. Then there exists a strict neighbourhood V of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ such that $V \subset U$ and $V \cap W = \emptyset$ by Proposition 2.6.5 and Lemma 2.6.6. Hence the assertion follows from the definition of $j_U^\dagger$.

(3) Let W be a quasi-compact admissible open subset of $]\overline{X}[_{\mathcal{X}}$. A section s of $j_U^\dagger \mathcal{F}$ on W is represented by a section $t \in \Gamma(W \cap U^{\geqq \nu}_{\mathcal{X}, \partial X}, \mathcal{F})$ for some ν sufficiently close to 1 with $W \cap U^{\geqq \nu}_{\mathcal{X}, \partial X} \subset W \cap U$ by Proposition 2.6.8 and Lemma 2.6.6. Since $t \in \Gamma(W \cap U^{\geqq \nu}_{\mathcal{X}, \partial X}, j_{U!}\mathcal{F})$ goes to $s|_{W \cap U^{\geqq \nu}_{\mathcal{X}, \partial X}}$ and $s|_{W \cap U^{\leqq \nu}_{\mathcal{X}, \partial X}} = 0$ by (2), $j_{U!}\mathcal{F} \to j_U^\dagger \mathcal{F}$ is surjective by Proposition 2.6.5.

Suppose that $(j^U_{]\partial X[_{\mathcal{X}} \cap U})^{-1}\mathcal{F} = 0$. Both sides are 0 on $]\partial X[_{\mathcal{X}}$. Let W be a quasi-compact admissible open subset of U. Since $\Gamma(W \cap U^{\leqq \nu}_{\mathcal{X}, \partial X}, \mathcal{F}) =$ $= 0$, the restriction map $\Gamma(W, \mathcal{F}) \to \Gamma(W \cap U^{\geqq \nu}_{\mathcal{X}, \partial X}, \mathcal{F})$ induces an isomorphism by Proposition 2.6.5. Hence the natural homomorphism $j_{U!}\mathcal{F} \to$ $\to j_U^\dagger \mathcal{F}$ is an isomorphism by Proposition 2.6.8.

(4) If $\mathcal{G}$ is a sheaf of $j_U^\dagger\mathcal{A}$-modules, then any homomorphism $j_{U!}\mathcal{F}\to\mathcal{G}$ uniquely factors through $j_U^\dagger\mathcal{F}\to\mathcal{G}$ via the surjection $j_{U!}\mathcal{F}\to j_U^\dagger\mathcal{F}$ by (1), (2) and (3). The assertion easily follows from Lemma 2.7.1.

(5) We check the universality of the functor $j_U^\dagger$ in (4). Let $\mathcal{H}$ be a sheaf of $j_U^\dagger\mathcal{A}$-modules. We also regard $\mathcal{H}$ as a sheaf of $j_{U!}\mathcal{A}$-modules through the natural homomorphism $j_{U!}\mathcal{A}\to j_U^\dagger\mathcal{A}$. Since $j_U^\dagger\mathcal{H}=\mathcal{H}$, the set of $j_U^\dagger\mathcal{A}$-bilinear homomorphisms from $j_U^\dagger\mathcal{F}\times j_U^\dagger\mathcal{G}$ to $\mathcal{H}$ is naturally bijective to the set of $j_{U!}\mathcal{A}$-bilinear homomorphisms from $j_{U!}\mathcal{F}\times j_{U!}\mathcal{G}$ to $\mathcal{H}$ by (2) and (3). Hence we have a bijection $\mathrm{Hom}_{j^\dagger\mathcal{A}}(j_U^\dagger\mathcal{F}\otimes_{j^\dagger\mathcal{A}}j_U^\dagger\mathcal{G},\mathcal{H})\xrightarrow{\cong}$ $\xrightarrow{\cong}\mathrm{Hom}_{j_{U!}\mathcal{A}}(j_{U!}(\mathcal{F}\otimes_{\mathcal{A}}\mathcal{G}),\mathcal{H})$ by the universality of tensor products. Then the assertion follows from (4).

(6) Since the exactness is stable under taking direct limits, we have only to prove that the surjectivity of $\mathcal{F}\to\mathcal{G}$ of sheaves of $\mathcal{A}$-modules implies the surjectivity of $j_U^\dagger\mathcal{F}\to j_U^\dagger\mathcal{G}$. The assertion follows from (3) and Lemma 2.7.1 (1). ■

Let $w:\mathfrak{Y}\to\mathfrak{X}$ be a morphism of $\mathcal{V}$-triples locally of finite type such that w is strict as a morphism of triples and $\widehat{w}:\mathcal{Y}\to\mathcal{X}$ is an open immersion. If V is a strict neighbourhood of $]X[_{\mathcal{X}}$ over $]\overline{X}[_{\mathcal{X}}$, then $V\cap]\overline{Y}[_{\mathcal{Y}}$ is a strict neighbourhood of $]Y[_{\mathcal{Y}}$ over $]\overline{Y}[_{\mathcal{Y}}$ (see 2.4). Hence, for any sheaf $\mathcal{F}$ of abelian groups on $]\overline{X}[_{\mathcal{X}}$, there exists a natural homomorphism

$$\widetilde{w}^{-1}(j^\dagger\mathcal{F})\to j^\dagger(\widetilde{w}^{-1}\mathcal{F}).$$

2.7.3. Proposition. *With the notation as above, the morphism* $\widetilde{w}^{-1}(j^\dagger\mathcal{F})\to j^\dagger(\widetilde{w}^{-1}\mathcal{F})$ *above is an isomorphism.*

Proof. Let W be a quasi-compact admissible open subset of $]\overline{Y}[_{\mathcal{Y}}$. For a strict neighbourhood V of $]Y[_{\mathcal{Y}}$ in $]\overline{Y}[_{\mathcal{Y}}$, there exists a strict neighbourhood U of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ such that $U\cap W\subset V\cap W$ by Proposition 2.6.5 and Lemma 2.6.6 (2). Since $U\cap]\overline{Y}[_{\mathcal{Y}}$ is a strict neighbourhood of $]Y[_{\mathcal{Y}}$ in $]\overline{Y}[_{\mathcal{Y}}$, the assertion follows from Proposition 2.6.8. ■

The proposition above means that the functor $j^\dagger$ commutes with localizations on $\mathcal{X}$. By Proposition 2.7.3 and Lemma 2.5.1 we can reduce many problems regrading sheaves of overconvergent sections on tubes to those of the case where $\mathfrak{X}$ is a $\mathcal{V}$-triple of finite type.

The following proposition was proved in Berthelot's unpublished note [6].

2.7.4. PROPOSITION. *Let $\mathfrak{X}$ be a $\mathcal{V}$-triple locally of finite type with a complement ∂X of X in $\overline{X}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $]\overline{X}[_{\mathcal{X}}$ such that $j_{]\partial X[_{\mathcal{X}}}^{-1}\mathcal{F}=0$. Then, for any strict neighbourhood V of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$, we have*

$$\mathbb{R}^q j_{V*}(j_V^{-1}\mathcal{F}) = \begin{cases} \mathcal{F} & \text{if } q=0 \\ 0 & \text{if } q \neq 0\,. \end{cases}$$

PROOF. Since $\{V,]\partial X[_{\mathcal{X}}\}$ is an admissible covering of $]\overline{X}[_{\mathcal{X}}$, it is sufficient to calculate the cohomology sheaves both on V and on $]\partial X[_{\mathcal{X}}$. Then it is clear on V since j_V is the identity. Both sides are zero on $]\partial X[_{\mathcal{X}}$ by the hypothesis on $\mathcal{F}$. Hence, we have proved the assertion. ■

2.8. Let $w : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of lft-triples, let ∂X and ∂Y be complements of X (resp. Y) in $\overline{X}$ (resp. $\overline{Y}$) with $\overline{w}^{-1}(\partial X) \subset \partial Y$, and let $\mathcal{A}$ (resp. $\mathcal{B}$) be a sheaf of rings on $]\overline{X}[_{\mathcal{X}}$ (resp. $]\overline{Y}[_{\mathcal{Y}}$) with a homomorphism $\widetilde{w}^{-1}(\mathcal{A}) \to \mathcal{B}$. For a strict neighbourhood V of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$, the inverse image $\widetilde{w}^{-1}(V)$ is a strict neighbourhood of $]Y[_{\mathcal{Y}}$ in $]\overline{Y}[_{\mathcal{Y}}$ since $\widetilde{w}$ is continuous [7, 1.2.7 Proposition]. Hence, we have a natural homomorphism

$$\widetilde{w}^{-1}(j^\dagger \mathcal{A}) \to j^\dagger \mathcal{B}$$

of sheaves of rings.

Let $\mathcal{F}$ be a sheaf of $\mathcal{A}$-modules. The inverse image functor $w^\dagger$ as a sheaf of overconvergent sections along ∂Y with respect to $\widetilde{w}^{-1}(j^\dagger \mathcal{A}) \to$ $\to j^\dagger \mathcal{B}$ is defined by

$$w^\dagger \mathcal{F} = j^\dagger (\mathcal{B} \otimes_{\widetilde{w}^{-1}(\mathcal{A})} \widetilde{w}^{-1}\mathcal{F})$$

[7, 2.1.4 Proposition]. Then, there exists a natural homomorphism

$$j^\dagger \mathcal{B} \otimes_{\widetilde{w}^{-1}(j^\dagger \mathcal{A})} \widetilde{w}^{-1}(j^\dagger \mathcal{F}) \to w^\dagger \mathcal{F}$$

of sheaves of $j^\dagger \mathcal{B}$-modules.

2.8.1. PROPOSITION [7, 2.1.4 Proposition]. *With the notation as above, suppose that w is strict as a morphism of pairs. If $\mathcal{F}$ is a sheaf of $\mathcal{A}$-modules, then the homomorphism $j^\dagger \mathcal{B} \otimes_{\widetilde{w}^{-1}(j^\dagger \mathcal{A})} \widetilde{w}^{-1}(j^\dagger \mathcal{F}) \to w^\dagger \mathcal{F}$ is an isomorphism.*

PROOF. First we prove that the natural homomorphism $\widetilde{w}^{-1}(j^\dagger \mathcal{F}) \to j^\dagger(\widetilde{w}^{-1}\mathcal{F})$ is an isomorphism. Suppose that $\mathcal{G}$ is a sheaf of

$j^\dagger(\widetilde{w}^{-1}\mathcal{A})$-modules. Then $\widetilde{w}_*\mathcal{G}$ is a sheaf of $j^\dagger\mathcal{A}$-modules by the canonical homomorphism $j^\dagger\mathcal{A}\to\widetilde{w}_*j^\dagger(\widetilde{w}^{-1}\mathcal{A})$ and we have

$$\mathrm{Hom}_{\widetilde{w}^{-1}(j^\dagger\mathcal{A})}(\widetilde{w}^{-1}(j^\dagger\mathcal{F}),\mathcal{G})=\mathrm{Hom}_{j^\dagger\mathcal{A}}(j^\dagger\mathcal{F},\widetilde{w}_*\mathcal{G})=\mathrm{Hom}_{\mathcal{A}}(\mathcal{F},\widetilde{w}_*\mathcal{G})=$$

$$=\mathrm{Hom}_{\widetilde{w}^{-1}\mathcal{A}}(\widetilde{w}^{-1}\mathcal{F},\mathcal{G})=\mathrm{Hom}_{j^\dagger(\widetilde{w}^{-1}\mathcal{A})}(j^\dagger(\widetilde{w}^{-1}\mathcal{F}),\mathcal{G})$$

by adjoints and Proposition 2.7.2 (4). Hence, $\widetilde{w}^{-1}(j^\dagger\mathcal{F})\to j^\dagger(\widetilde{w}^{-1}\mathcal{F})$ is an isomorphism.

We put $(\mathcal{A},\mathcal{F})=(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}},\mathcal{A})$. Since w is strict as a morphism of pairs, we have $j_{]\partial Y[_{\mathcal{Y}}}^{-1}(\widetilde{w}^{-1}(j^\dagger\mathcal{A}))=0$. Hence the universality of $j^\dagger$ in Proposition 2.7.2 (4) implies that the natural homomorphism $\widetilde{w}^{-1}(j^\dagger\mathcal{A})\to j^\dagger(\widetilde{w}^{-1}\mathcal{A})$ is an isomorphism.

Finally we prove the general cases. Since there exists natural isomorphisms $\widetilde{w}^{-1}(j^\dagger\mathcal{A})\cong j^\dagger(\widetilde{w}^{-1}\mathcal{A})$ and $\widetilde{w}^{-1}(j^\dagger\mathcal{F})\cong j^\dagger(\widetilde{w}^{-1}\mathcal{F})$, we have isomorphisms

$$j^\dagger\mathcal{B}\otimes_{\widetilde{w}^{-1}(j^\dagger\mathcal{A})}\widetilde{w}^{-1}(j^\dagger\mathcal{F})\cong j^\dagger\mathcal{B}\otimes_{j^\dagger(\widetilde{w}^{-1}\mathcal{A})}j^\dagger(\widetilde{w}^{-1}\mathcal{F})\cong j^\dagger(\mathcal{B}\otimes_{\widetilde{w}^{-1}\mathcal{A}}\widetilde{w}^{-1}\mathcal{F})$$

by Proposition 2.7.2 (5). This completes the proof. ■

2.9. Let $\mathfrak{X}$ be a $\mathcal{V}$-triple locally of finite type and let $\mathcal{A}$ be a sheaf of rings on $]\overline{X}[_{\mathcal{X}}$. We denote by $\mathrm{Coh}(\mathcal{A})$ the category of coherent sheaves of $\mathcal{A}$-modules.

We say that $\mathcal{A}$ is a sheaf of coherent rings if $\mathcal{A}$ is a sheaf of coherent $\mathcal{A}$-modules. For example, $j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$ is a sheaf of coherent rings by Proposition 2.7.3 and [7, 2.1.9 Proposition (i)]. Suppose that $\mathcal{A}$ is a sheaf of coherent rings. Then E is a sheaf of coherent $\mathcal{A}$-modules if and only if E is a sheaf of $\mathcal{A}$-modules of finite presentation, that is, there exists an admissible covering $\{U_\lambda\}_\lambda$ of $]\overline{X}[_{\mathcal{X}}$ such that $E|_{U_\lambda}$ is a cokernel of some homomorphism $\mathcal{A}^m|_{U_\lambda}\to\mathcal{A}^n|_{U_\lambda}$.

Let E be a sheaf of $j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules. E is coherent if and only if, for any strict morphism of triples $w:\mathfrak{Y}\to\mathfrak{X}$ and such that $\mathcal{Y}$ is an open formal subscheme of $\mathcal{X}$ of finite type over Spf $\mathcal{V}$, there exists a strict neighbourhood U of $]Y[_{\mathcal{Y}}$ in $]\overline{Y}[_{\mathcal{Y}}$ and a sheaf $\mathcal{E}$ of coherent $\mathcal{O}_U$-modules with $\widetilde{w}^{-1}E\cong j_U^\dagger\mathcal{E}$. If $\varphi:E\to F$ is a homomorphism of sheaves of coherent $j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules, then, for any strict morphism $w:\mathfrak{Y}\to\mathfrak{X}$ as a morphism of triples such that $\mathcal{Y}$ is an open formal subscheme of $\mathcal{X}$ which is of finite type over Spf $\mathcal{V}$, there exist a strict neighbourhood U of $]Y[_{\mathcal{Y}}$ in $]\overline{Y}[_{\mathcal{Y}}$ and a homomorphism $\psi:\mathcal{E}\to\mathcal{F}$ of sheaves of coherent $\mathcal{O}_U$-modules with $\widetilde{w}^{-1}(\varphi)=j_U^\dagger(\psi)$ [7, 2.1.10 Proposition].

2.9.1. LEMMA ([7, 2.1.11 Corollaire]). *Let $\mathcal{X}$ be a $\mathcal{V}$-triple locally of finite type. Then the restriction functor*

$$j^*(=j_{]\overline{X}[_{\mathcal{X}}}^{-1}) : \mathrm{Coh}\,(j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}) \to \mathrm{Coh}\,(\mathcal{O}_{]X[_{\mathcal{X}}})$$

is exact and faithful.

2.10. Let $w : \mathcal{Y} \to \mathcal{X}$ be a morphism of lft-triples and let $\mathcal{A}$ (resp. $\mathcal{B}$) be a sheaf of rings on $]\overline{X}[_{\mathcal{X}}$ (resp. $]\overline{Y}[_{\mathcal{Y}}$) with a homomorphism $\widetilde{w}^{-1}(\mathcal{A}) \to \to \mathcal{B}$. For a sheaf $\mathcal{F}$ of $\mathcal{A}$-modules, we define the inverse image sheaf $\widetilde{w}^* \mathcal{F}$ with respect to $\widetilde{w}^{-1}(\mathcal{A}) \to \mathcal{B}$ by

$$\widetilde{w}^* \mathcal{F} = \mathcal{B} \otimes_{\widetilde{w}^{-1}\mathcal{A}} \widetilde{w}^{-1} \mathcal{F}.$$

Suppose that $\mathcal{B}$ is a sheaf of coherent rings. Then, if $\mathcal{F}$ is a sheaf of coherent $\mathcal{A}$-modules, $\widetilde{w}^* \mathcal{F}$ is a sheaf of coherent $\mathcal{B}$-modules.

Now we consider the case where $\mathcal{A} = j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$ and $\mathcal{B} = j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}$. Let E be a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules and let $\mathcal{E}$ be a sheaf of coherent $\mathcal{O}_U$-modules for a strict neighbourhood U of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ such that $E \cong \cong j_U^\dagger \mathcal{E} \cong j^\dagger (j_{U!} \mathcal{E}) \cong j^\dagger (\mathcal{O}_{]\overline{X}[_{\mathcal{X}}} \otimes_{j_{U!}\mathcal{O}_U} j_{U!} \mathcal{E})$ (locally on $\mathcal{X}$ such U and $\mathcal{E}$ always exist). Applying $j^\dagger \widetilde{w}^*$ to the natural homomorphism $\mathcal{O}_{]\overline{X}[_{\mathcal{X}}} \otimes_{j_{U!}\mathcal{O}_U} j_{U!} \mathcal{E} \to E$, we obtain a homomorphism

$$w^\dagger (\mathcal{O}_{]\overline{X}[_{\mathcal{X}}} \otimes_{j_{U!}\mathcal{O}_U} j_{U!} \mathcal{E}) \to \widetilde{w}^* E$$

of $j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}$-modules by Proposition 2.7.2 (3). Note that $\widetilde{w}^* (\mathcal{O}_{]\overline{X}[_{\mathcal{X}}} \otimes \otimes_{j_{U!}\mathcal{O}_U} j_{U!} \mathcal{E}) = \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}} \otimes_{\widetilde{w}^{-1}(j_{U!}\mathcal{O}_U)} \widetilde{w}^{-1}(j_{U!} \mathcal{E})$ is coherent on the strict neighbourhood $\widetilde{w}^{-1}(U)$ of $]Y[_{\mathcal{Y}}$ in $]\overline{Y}[_{\mathcal{Y}}$. If $\mathcal{X}$ is a $\mathcal{V}$-triple of finite type for some object $\mathcal{V}$ of $\mathrm{CDVR}_{\mathbb{Z}_p}$, the left-hand side does not depend on the choice of U and $\mathcal{E}$.

2.10.1. PROPOSITION. *With the notation as above, the homomorphism above is an isomorphism.*

PROOF. Since both $\widetilde{w}^\dagger j_{U!}$ and $\widetilde{w}^*$ are right exact by Proposition 2.7.2 (6) and Lemma 2.7.1 (2), both sides are sheaves of coherent $j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}$-modules. The homomorphism is an isomorphism on $]Y[_{\mathcal{Y}}$. Hence the assertion follows from Lemma 2.9.1. ■

In this paper we use the notation $w^\dagger E$ for the inverse image $\widetilde{w}^* E$ for a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules because we want to stress that the inverse image is a sheaf of overconvergent sections.

2.10.2. Proposition. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be $\mathcal{V}$-triples locally of finite type and let $w : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of finite type such that*

(i) $\widehat{w} : \mathcal{Y} \to \mathcal{X}$ *is étale around* Y;

(ii) $\overline{w} : \overline{Y} \to \overline{X}$ *is proper*;

(iii) $\overset{\circ}{w} : Y \to X$ *is an isomorphism.*

Then we have the following:

(1) *The inverse image functor $w^\dagger$ gives an equivalence*

$$\mathrm{Coh}\,(j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}) \xrightarrow{\cong} \mathrm{Coh}\,(j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}})$$

of categories and the direct image functor $\widetilde{w}_*$ is a quasi-inverse of $w^\dagger$.

(2) *Let $\mathcal{G}$ be a sheaf of abelian groups on $]\overline{Y}[_{\mathcal{Y}}$ such that $j^{-1}_{]\partial Y[_{\mathcal{Y}}} \mathcal{G} = 0$, where ∂Y is a complement of Y in $\overline{Y}$. Then we have $\mathbb{R}^q \widetilde{w}_* \mathcal{G} = 0$ for any $q > 0$.*

Proof. Since the problem is local on $\mathcal{X}$, we may assume that both $\mathfrak{X}$ and $\mathfrak{Y}$ are $\mathcal{V}$-triples of finite type by Proposition 2.7.3. Then there exist a strict neighbourhood U of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ and a strict neighbourhood V of $]Y[_{\mathcal{Y}}$ in $]\overline{Y}[_{\mathcal{Y}}$ such that $\widetilde{w}$ induces an isomorphism from V to U [7, 1.3.5 Théorème].

(1) We prove that $\widetilde{w}_*$ is a quasi-inverse of $w^\dagger$. Let E be a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules such that $E \cong j^\dagger_W \mathcal{E}$ for a strict neighbourhood W of $]X[_{\mathcal{X}}$ in U and a sheaf $\mathcal{E}$ of coherent $\mathcal{O}_W$-modules. In this situation $w^\dagger E \cong j^\dagger_W \mathcal{E}$, where we identify W with $\widetilde{w}^{-1}(W) \cap V$ through $\widetilde{w}$. On the other hand, if F is a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}$-modules such that $F \cong j^\dagger_W \mathcal{F}$ for a strict neighbourhood W of $]Y[_{\mathcal{Y}}$ in V and a sheaf $\mathcal{F}$ of coherent $\mathcal{O}_W$-modules, then we have $\widetilde{w}_* F \cong j^\dagger_W \mathcal{F}$. Here we identify W with $\widetilde{w}^{-1}(W) \cap V$ through $\widetilde{w}$. Hence, $\widetilde{w}_* F$ is coherent and $\widetilde{w}_*$ is a quasi-inverse of $w^\dagger$.

(2) Since the natural homomorphism $\mathcal{G} \to \mathbb{R} j_{V*}(j_V^{-1} \mathcal{G})$ is an isomorphism by Proposition 2.7.4, we have

$$\mathbb{R}^q \widetilde{w}_* \mathcal{G} \cong \mathbb{R}^q (\widetilde{w} j_V)_* (j_V^{-1} \mathcal{G})$$

for any q. It vanishes on U for any $q > 0$ since $\widetilde{w} j_V$ is an isomorphism from V to U and it vanishes on $]\partial X[_{\mathcal{X}}$ for any q since $j^{-1}_{]\partial Y[_{\mathcal{Y}}} \mathcal{G} = 0$. Hence, $\mathbb{R}^q \widetilde{w}_* \mathcal{G}$ vanishes for any $q > 0$. ■

2.11. We give a resolution of sheaves on a rigid analytic space using the Čech complex for sheaves. We will construct a Čech complex for general simplicial tubes in 3.9. First we fix our notation.

Let $\coprod_{\alpha\in I} w_\alpha \colon \coprod_{\alpha} X_\alpha \to X$ be a covering of schemes (resp. rigid analytic spaces, resp. triples). We denote by

$$w_{\alpha_0\dots\alpha_r}\colon X_{\alpha n_0\dots\alpha_r} \to X$$

the open immersion from the intersection of X_{α_0}, X_{α_1}, ..., and X_{α_r} to X. Then we have a Čech diagram

$$X \leftarrow \coprod_{\alpha_0} X_{\alpha_0} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \coprod_{\alpha_0,\,\alpha_1} X_{\alpha_0\alpha_1} \;\substack{\longleftarrow \\ \rightarrow \\ \longleftarrow \\ \rightarrow \\ \longleftarrow}\; \cdots,$$

for the morphism $\coprod_{\alpha} w_\alpha \colon \coprod_{\alpha} X_\alpha \to X$. Let $\mathcal{A}$ be a sheaf of rings on X. For any sheaf $\mathcal{F}$ of $\mathcal{A}$-modules on X, we will denote by $\mathcal{C}^{\cdot}(\{X_\alpha\}_\alpha, \mathcal{F})$ the Čech complex of sheaves with respect to $\{X_\alpha\}_\alpha$:

$$0 \to \prod_{\alpha_0} w_{\alpha_0 *} w_{\alpha_0}^{-1}\mathcal{F} \to \prod_{\alpha_0,\,\alpha_1} w_{\alpha_0\alpha_1 *} w_{\alpha_0\alpha_1}^{-1}\mathcal{F} \to \prod_{\alpha_0,\,\alpha_1,\,\alpha_2} w_{\alpha_0\alpha_1\alpha_2 *} w_{\alpha_0\alpha_1\alpha_2}^{-1}\mathcal{F} \to \dots .$$

Here $\prod_{\alpha_0} w_{\alpha_0 *} w_{\alpha_0}^{-1}\mathcal{F}$ is located at degree 0 and each coboundary map is an alternating sum which is induced by the Čech diagram above as usual (see 3.9). Then, there is a natural morphism

$$\mathcal{F} \to \mathcal{C}^{\cdot}(\{X_\alpha\}_\alpha, \mathcal{F})$$

of complexes of sheaves of $\mathcal{A}$-modules, where we regard $\mathcal{F}$ a complex which is concentrated at degree 0.

If the index set I is well-ordered, we denote by $\mathcal{C}^{\cdot}_{\mathrm{alt}}(\{X_\alpha\}_\alpha, \mathcal{F})$ the alternating Čech complex of sheaves with respect to $\{X_\alpha\}_\alpha$:

$$0 \to \prod_{\alpha_0} w_{\alpha_0 *} w_{\alpha_0}^{-1}\mathcal{F} \to \prod_{\alpha_0<\alpha_1} w_{\alpha_0\alpha_1 *} w_{\alpha_0\alpha_1}^{-1}\mathcal{F} \to \prod_{\alpha_0<\alpha_1<\alpha_2} w_{\alpha_0\alpha_1\alpha_2 *} w_{\alpha_0\alpha_1\alpha_2}^{-1}\mathcal{F} \to \dots .$$

Then, there also exists a natural morphism

$$\mathcal{F} \to \mathcal{C}^{\cdot}_{\mathrm{alt}}(\{X_\alpha\}_\alpha, \mathcal{F})$$

of complexes of sheaves of $\mathcal{A}$-modules.

2.11.1. LEMMA. *With the notation as above, the natural morphism*

$$\mathcal{F} \to \mathcal{C}^{\cdot}(\{X_\alpha\}_\alpha, \mathcal{F})$$

is a quasi-isomorphism. The same result holds for alternating Čech complexes.

PROOF. The q-th cohomology sheaf $\underline{H}^q(\mathcal{C}^{\cdot}(\{X_\alpha\}_\alpha, \mathcal{F}))$ of the Čech complex is the sheaf associated to the presheaf

$$U \mapsto H^q(\Gamma(U, \mathcal{C}^{\cdot}(\{X_\alpha\}_\alpha, \mathcal{F})))$$

for any admissible open subset U of X. So it is sufficient to prove that

$$H^q(\Gamma(U, \mathcal{C}^{\cdot}(\{X_\alpha\}_\alpha, \mathcal{F}))) = \begin{cases} \Gamma(U, \mathcal{F}) & \text{if } q = 0 \\ 0 & \text{if } q > 0 . \end{cases}$$

for any $\beta \in I$ and any admissible open subset U of X_β. In the case where $q = 0$ the required identity follows from the axioms defining sheaves. Let us fix an element β of I and an admissible open subset U of X_β: the proof is then analogous to [14], III, 4.2. ■

2.12. From Proposition 2.7.3 and Lemma 2.11.1 we have

2.12.1. PROPOSITION. *Let $\mathcal{X}$ be a $\mathcal{V}$-triple locally of finite type, let $\{\mathcal{X}_\alpha\}_\alpha$ be a Zariski covering of $\mathcal{X}$ and let $\coprod_\alpha w_\alpha : \coprod_\alpha \mathfrak{X}_\alpha \to \mathfrak{X}$ be a strict morphism of triples which is induced from $\{\mathcal{X}_\alpha\}_\alpha$. Let $\mathcal{A}$ be a sheaf of rings on $]\overline{X}[_{\mathcal{X}}$ and let $\mathcal{F}$ be a sheaf of $\mathcal{A}$-modules. Then the sequence*

$$0 \to j^\dagger \mathcal{F} \to \prod_{\alpha_0} \widetilde{w}_{\alpha_0 *} w_{\alpha_0}^\dagger \mathcal{F} \to$$
$$\to \prod_{\alpha_0, \alpha_1} \widetilde{w}_{\alpha_0\alpha_1 *} w_{\alpha_0\alpha_1}^\dagger \mathcal{F} \to \prod_{\alpha_0, \alpha_1, \alpha_2} \widetilde{w}_{\alpha_0\alpha_1\alpha_2 *} w_{\alpha_0\alpha_1\alpha_2}^\dagger \mathcal{F} \to \dots$$

of $j^\dagger\mathcal{A}$-modules which is induced from the Čech diagram for $\coprod_\alpha w_\alpha : \coprod_\alpha \mathfrak{X}_\alpha \to \mathfrak{X}$ is exact. The same holds for alternating complexes.

Let $\mathcal{X}$ be a $\mathcal{V}$-triple locally of finite type and let $\{X_\alpha\}$ be a finite Zariski covering of X. We put $\mathfrak{X}_\alpha = (X_\alpha, \overline{X}, \mathcal{X})$ and denote by $w_\alpha : \mathfrak{X}_\alpha \to \mathfrak{X}$

the structure morphism. Let $\mathcal{A}$ be a sheaf of rings on $]\overline{X}[_{\mathcal{X}}$ and let $\mathcal{F}$ be a sheaf of $\mathcal{A}$-modules. Then we have a sequence

$$(*) \qquad 0 \to j^\dagger \mathcal{F} \to \prod_{\alpha_0} w_{\alpha_0}^\dagger \mathcal{F} \to \prod_{\alpha_0, \alpha_1} w_{\alpha_0\alpha_1}^\dagger \mathcal{F} \to \prod_{\alpha_0, \alpha_1, \alpha_2} w_{\alpha_0\alpha_1\alpha_2}^\dagger \mathcal{F} \to \dots$$

of sheaves of $j^\dagger \mathcal{A}$-modules which is induced from the Čech diagram for $\coprod_\alpha w_\alpha : \coprod_\alpha \mathfrak{X}_\alpha \to \mathfrak{X}$.

2.12.2. PROPOSITION ([7, 2.1.8 Proposition, Remarque]). *With the notation as above, the sequence* $(*)$ *is exact. The same holds in the alternating case.*

PROOF. Since the problem is local on $\mathcal{X}$ by Proposition 2.7.3, we may assume that $\mathcal{X}$ is affine. Since $j^\dagger(j^\dagger\mathcal{F}) = j^\dagger\mathcal{F}$ and $w_{\alpha_0\dots\alpha_r}^\dagger(j^\dagger\mathcal{F}) = \widetilde{w}_{\alpha_0\dots\alpha_r}^{-1} j^\dagger(j^\dagger\mathcal{F}) = \widetilde{w}_{\alpha_0\dots\alpha_r}^{-1} j^\dagger\mathcal{F} = w_{\alpha_0\dots\alpha_r}^\dagger \mathcal{F}$ by Propositions 2.7.2 (3) and 2.8.1, we may assume that $j^\dagger\mathcal{F} = \mathcal{F}$.

Let us fix a complement ∂X_α of X_α in $\overline{X}$, put $\partial X = \bigcap_\alpha \partial X_\alpha$ so that it is a complement of X in $\overline{X}$ and define a complement $\partial X_{\alpha_0\dots\alpha_r} = \bigcup_{l=0}^r \partial X_{\alpha_l}$ of $X_{\alpha_0\dots\alpha_r}$ in $\overline{X}$.

We let $U_{\mathfrak{X},\partial X}^{\geqq\nu}$ (resp. $U_{\alpha_0\dots\alpha_r}^{\geqq\nu}$) be the strict neighbourhood of $]X[_{\mathcal{X}}$ (resp. $]X_{\alpha_0\dots\alpha_r}[_{\mathcal{X}}$) in $]\overline{X}[_{\mathcal{X}}$ with respect to ∂X (resp. $\partial X_{\alpha_0\dots\alpha_r}$) as in 2.6. We denote by $j^\nu : U_{\mathfrak{X},\partial X}^{\geqq\nu} \to]\overline{X}[_{\mathcal{X}}$ (resp. $j_{\alpha_0\dots\alpha_r}^\nu : \bigcap_{l=0}^r U_{\alpha_l}^{\geqq\nu} \to]\overline{X}[_{\mathcal{X}}$) the open immersion of rigid analytic spaces.

Suppose that $g_{\alpha,1}, \dots, g_{\alpha,s_\alpha} \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ are lifts of generators of the ideal of definition of ∂X_α in $\overline{X}$, then $\bigcap_\alpha \{g_{\alpha,1}, \dots, g_{\alpha,s_\alpha}\}$ is a set of lifts of generators of the ideal of definition of ∂X in $\overline{X}$ and $\{g_{\alpha_0,i_0} g_{\alpha_1,i_1} \cdots g_{\alpha_r,i_r} \mid 1 \leqq i_l \leqq s_l \text{ for } 0 \leqq l \leqq r\}$ is a set of lifts of generators of the ideal of definition of $\partial X_{\alpha_0\dots\alpha_r}$ in $\overline{X}$. Hence we have

$$U_{\mathfrak{X},\partial X}^{\geqq\nu} = \bigcup_\alpha U_\alpha^{\geqq\nu}$$

$$\bigcap_{l=0}^r U_{\alpha_l}^{\geqq\nu} \subset U_{\alpha_0\dots\alpha_r}^{\geqq\nu^r} \subset \bigcap_{l=0}^r U_{\alpha_l}^{\geqq\nu^r}$$

for any ν sufficiently close to 1 by definition.

Note that $j_{]\partial X[_{\mathcal{X}}}^{-1}\mathcal{F} = 0$ because $j^\dagger\mathcal{F} = \mathcal{F}$ by our assumption. Since $U_{\mathfrak{X},\partial X}^{\geqq\nu}$

is a strict neighbourhood of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$, we have an exact sequence

$$0 \to j^{\nu}_{*}(j^{\nu})^{-1}\mathcal{F} \to \prod_{\alpha_0} j^{\nu}_{\alpha_0 *}(j^{\nu}_{\alpha_0})^{-1}\mathcal{F} \to$$

$$\to \prod_{\alpha_0,\,\alpha_1} j^{\nu}_{\alpha_0\alpha_1 *}(j^{\nu}_{\alpha_0\alpha_1})^{-1}\mathcal{F} \to \prod_{\alpha_0,\,\alpha_1,\,\alpha_2} j^{\nu}_{\alpha_0\alpha_1\alpha_2 *}(j^{\nu}_{\alpha_0\alpha_1\alpha_2})^{-1}\mathcal{F} \to \dots$$

of sheaves of $\mathcal{A}$-modules.

Since $U^{\geqq \nu}_{\mathfrak{X},\partial X}$ is a strict neighbourhood of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ (Proposition 2.6.5), we have

$$j^{\dagger}\mathcal{F} \cong \lim_{\nu \to 1^-} j^{\nu}_{*}(j^{\nu})^{-1}\mathcal{F}$$

by Proposition 2.6.8 and Lemma 2.6.6 (2). We also have

$$w^{\dagger}_{\alpha_0 \dots \alpha_r}\mathcal{F} \cong \lim_{\nu \to 1^-} j^{\nu}_{\alpha_0 \dots \alpha_r *}(j^{\nu}_{\alpha_0 \dots \alpha_r})^{-1}\mathcal{F}$$

by the inclusion relations above. Since a filtered inductive limit preserves exactness, we have the desired exactness. ■

3. Diagrams of triples.

In this section we will deal with diagrams of triples and associated diagrams of tubes following [1, VI, 7], [15, VI] and [2, V, 3.4]. In 3.10 we discuss a homotopy theory for simplicial triples and simplicial tubes (see [1, $\mathrm{V}^{\mathrm{bis}}$, 3.0]).

3.1. Let I be a small category and let $\mathfrak{S}$ be an lft-triple. We say that a covariant functor

$$\mathfrak{X}_{\textstyle\cdot} : I \to \mathrm{TRI}^{\mathrm{lft}}$$

is a diagram of lft-triples over $\mathfrak{S}$ indexed by I, which we usually denote by $(\mathfrak{X}_{\textstyle\cdot}, I)$. We denote by $\mathfrak{X}_n = (X_n, \overline{X}_n, \mathcal{X}_n)$ the corresponding lft-triple for an object n of I, and by $\eta_{\mathfrak{X}_{\cdot}} = (\mathring{\eta}_{\mathfrak{X}_{\cdot}}, \overline{\eta}_{\mathfrak{X}_{\cdot}}, \widehat{\eta}_{\mathfrak{X}_{\cdot}}) : \mathfrak{X}_m \to \mathfrak{X}_n$ the corresponding morphisms of lft-triples for a morphism $\eta : m \to n$ of I. We fix a complement ∂X_n of X_n in $\overline{X}_n$ for each n. Note that «$\partial X_{\textstyle\cdot}$» is not a diagram of schemes in general.

Let $\mathfrak{X}_{\textstyle\cdot}$ and $\mathfrak{Y}_{\textstyle\cdot}$ be diagrams of lft-triples over $\mathfrak{S}$ indexed by small categories I and J, respectively. We say that $(f_{\textstyle\cdot}, t) : (\mathfrak{Y}_{\textstyle\cdot}, J) \to (\mathfrak{X}_{\textstyle\cdot}, I)$ is a morphism of diagrams of triples over $\mathfrak{S}$ if it consists of a functor $t : J \to I$

of categories and a morphism $f_n\colon \mathfrak{Y}_n \to \mathfrak{X}_{t(n)}$ of lft-triples over $\mathfrak{S}$ for any object n of J such that the diagram

$$\begin{tikzcd} \mathfrak{Y}_m \arrow[r, "f_m"] \arrow[d, "\eta_{\mathfrak{Y}.}"'] & \mathfrak{X}_{t(m)} \arrow[d, "t(\eta)_{\mathfrak{X}.}"] \\ \mathfrak{Y}_n \arrow[r, "f_n"'] & \mathfrak{X}_{t(n)} \end{tikzcd}$$

is commutative for any morphism $\eta : m \to n$ of J.

Let (P) be a property of morphisms of schemes and formal schemes such that (i) (P) is stable under any base change, (ii) under the condition f is (P), g is (P) if and only if fg is (P), and (iii) an open immersion and a closed immersion are (P). We say that a morphism $(f_., t)\colon (\mathfrak{Y}_., J) \to \to (\mathfrak{X}_., I)$ of diagrams of triples locally of finite type is (P) if $f_n\colon \mathfrak{Y}_n \to \mathfrak{X}_{t(n)}$ is (P) for any object n of I.

If there is no confusion, we simply denote diagrams of lft-triples and their morphisms without the index categories. If the target category is a category of $\mathfrak{S}$-triples locally of finite type, we say that $\mathfrak{X}_.$ is a diagram of $\mathfrak{S}$-triples locally of finite type.

3.1.1. Example. (1) Let $\underline{0}$ be a category whose set of objects consists of a unique set 0 and whose set of morphisms consists of the identity id_0. Then the category of diagrams of lft-triples over $\mathfrak{S}$ indexed by $\underline{0}$ is that of lft-triples over $\mathfrak{S}$.

(2) Let I be a small category and let $\mathfrak{X}$ be an lft-triple over $\mathfrak{S}$. We denote by $\mathfrak{X}^I_.$ the constant diagram of lft-triples over $\mathfrak{S}$ for $\mathfrak{X}$ indexed by I: $\mathfrak{X}^I_n = \mathfrak{X}$ for any object n of I and $\eta_{\mathfrak{X}^I_.} = \mathrm{id}_{\mathfrak{X}}$ for any morphism η of I. If we regard $\mathfrak{X}$ as a diagram of lft-triples over $\mathfrak{S}$ indexed by $\underline{0}$ as in (1), then there exists a unique morphism

$$\varepsilon_.\colon \mathfrak{X}^I_. \to \mathfrak{X}$$

of diagrams of lft-triples over $\mathfrak{S}$ such that ε_n is the identity for each n.

Let $(\mathfrak{Y}_., I)$ be a diagram of lft-triples over $\mathfrak{X}$. If $w_n\colon \mathfrak{Y}_n \to \mathfrak{X}$ is a structure morphism, then the set $\{w_n\}$ induces a morphism

$$w_.\colon \mathfrak{Y}_. \to \mathfrak{X}$$

of diagrams of lft-triples over $\mathfrak{S}$. We define a morphism

$$w^I_.\colon \mathfrak{Y}_. \to \mathfrak{X}^I_.$$

of diagrams of triples indexed by I by $w_n^I = w_n$ for an object n. Then $w_{.} = \varepsilon_{.} w_{.}^I$.

(3) Suppose that $\mathfrak{S}$ is a $\mathcal{V}$-triple locally of finite type. Let $(\mathfrak{X}_{.}, I)$ and $(\mathfrak{Y}_{.}, J)$ be diagrams of $\mathfrak{S}$-triples locally of finite type. We define the fiber product of $(\mathfrak{X}_{.}, I)$ and $(\mathfrak{Y}_{.}, J)$ over $\mathfrak{S}$ by

$$(\mathfrak{X}_{.}, I) \times_{\mathfrak{S}} (\mathfrak{Y}_{.}, J) = (\mathfrak{X}_{.} \times_{\mathfrak{S}} \mathfrak{Y}_{.}, I \times J).$$

One can easily check that each of the two projections

$$\mathrm{pr}_1 = (\mathrm{pr}_1, \mathrm{pr}_1) : (\mathfrak{X}_{.}, I) \times_{\mathfrak{S}} (\mathfrak{Y}_{.}, J) \to (\mathfrak{X}_{.}, I)$$

$$\mathrm{pr}_2 = (\mathrm{pr}_2, \mathrm{pr}_2) : (\mathfrak{X}_{.}, I) \times_{\mathfrak{S}} (\mathfrak{Y}_{.}, J) \to (\mathfrak{Y}_{.}, J)$$

is a morphism of diagrams of $\mathfrak{S}$-triples locally of finite type.

Suppose that $I = J$. We define a diagonal diagram $(\mathrm{diag}(\mathfrak{X}_{.} \times_{\mathfrak{S}} \mathfrak{Y}_{.}), I)$ inside $(\mathfrak{X}_{.} \times_{\mathfrak{S}} \mathfrak{Y}_{.}, I \times I)$ by

- $\mathrm{diag}(\mathfrak{X}_{.} \times_{\mathfrak{S}} \mathfrak{Y}_{.})_n = \mathfrak{X}_n \times_{\mathfrak{S}} \mathfrak{Y}_n$ for any object n of I;
- $\eta_{\,\mathrm{diag}} = \eta_{\,\mathfrak{X}_{.}} \times \eta_{\,\mathfrak{Y}_{.}}$ for any morphism η of I.

Then the natural morphism $\mathrm{diag}(\mathfrak{X}_{.} \times_{\mathfrak{S}} \mathfrak{Y}_{.}) \to \mathfrak{X}_{.} \times_{\mathfrak{S}} \mathfrak{Y}_{.}$ induced by a diagonal morphism $I \to I \times I$ is a morphism of diagrams of $\mathfrak{S}$-triples locally of finite type.

(4) We say that a diagram $\mathfrak{X}_{.}$ of lft-triples over $\mathfrak{S}$ indexed by the dual category Δ° of the standard simplicial category Δ (see 1.3.4) is a simplicial lft-triple over $\mathfrak{S}$. We also say that a diagram $\mathfrak{X}_{.}$ of lft-triples over $\mathfrak{S}$ indexed by $(\Delta^r)^{\circ}$ is an r-simplicial lft-triple over $\mathfrak{S}$, where Δ^r is a product of r-copies of Δ. If $r = 0$, we put $\Delta^0 = \underline{0}$.

(5) Let $w : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of $\mathfrak{S}$-triples locally of finite type. We define $\mathfrak{Y}_n$ to be the fiber product of $n+1$ copies of $\mathfrak{Y}$ over $\mathfrak{X}$, and put $w_n : \mathfrak{Y}_n \to \mathfrak{X}$ the structure morphism.

We define a functor $\mathfrak{Y}_{.}$ from Δ° to the category of $\mathfrak{S}$-triples locally of finite type as follows: $\mathfrak{Y}_n$ is as above for any nonnegative integer n, and $\eta_{\,\mathfrak{Y}_{.}} : \mathfrak{Y}_m \to \mathfrak{Y}_n$ by $(y_0, y_1, \ldots, y_m) \mapsto (y_{\eta(0)}, y_{\eta(1)}, \ldots, y_{\eta(n)})$ for any morphism $\eta : n \to m$ in Δ. Then $\mathfrak{Y}_{.}$ is a simplicial $\mathfrak{S}$-triple locally of finite type over $\mathfrak{X}$ and we call it the Čech diagram for $w : \mathfrak{Y} \to \mathfrak{X}$.

(6) Let n be a nonnegative integer. We say that a covariant functor from the dual category $\Delta[n]^{\circ}$ of $\Delta[n]$ (see 1.3.5) to the category of lft-triples is an n-truncated simplicial lft-triple.

The forgetful functor

$$\begin{array}{ccc} \left(\begin{array}{c}\text{simplicial } \mathfrak{S}\text{-triples locally} \\ \text{of finite type}\end{array}\right) & \to & \left(\begin{array}{c}n\text{-truncated simplicial } \mathfrak{S}\text{-triples locally} \\ \text{of finite type}\end{array}\right) \\ \mathfrak{X}_{\bullet} & \mapsto & \mathfrak{X}_{\bullet}^{(n)}, \end{array}$$

which is called the n-skeleton functor, has a right adjoint $\mathrm{cosk}_n^{\mathfrak{S}}$ which is called the coskeleton functor [13]. Indeed, the category of simplicial $\mathfrak{S}$-triples locally of finite type is closed under finite and nonempty inverse limits. Then $\mathrm{cosk}_n^{\mathfrak{S}}(\mathfrak{Y}_{\bullet})_l = \mathfrak{Y}_l$ for $l \leqq n$, and $\mathrm{cosk}_n^{\mathfrak{S}}(\mathfrak{Y}_{\bullet})_l$ is obtained by fiber products whose entries are $\mathfrak{S}$, $\mathfrak{Y}_0, \ldots, \mathfrak{Y}_n$ for any l. (See an explicit formula in the proof of Lemma 3.10.2.)

If $n = -1$, we put $\mathfrak{X}_{\bullet}^{(-1)} = \mathfrak{S}$ and define $\mathrm{cosk}_{-1}^{\mathfrak{S}}(\mathfrak{Y})$ to be the constant simplicial lft-triple $\mathfrak{S}_{\bullet}^{\Delta^{\circ}}$. If $n = 0$, then $\mathrm{cosk}_0^{\mathfrak{S}}(\mathfrak{Y})$ is the Čech diagram for $\mathfrak{Y} \to \mathfrak{S}$. ■

We give an example of diagrams of triples which will often be used later.

3.1.2. Example. Let

$$\begin{array}{ccc} (\mathfrak{Y}_{\bullet}, I) & \leftarrow & (\mathfrak{Z}_{\bullet}, I) \\ \downarrow & & \downarrow \\ (\mathfrak{W}, \underline{0}) & \leftarrow & (\mathfrak{X}, \underline{0}) \end{array}$$

be a commutative diagram of diagrams of $\mathfrak{S}$-triples locally of finite type and let us denote by $\mathfrak{X}_{\bullet} \to \mathfrak{W}$ the Čech diagram for $\mathfrak{X} \to \mathfrak{W}$. We define a diagram $\mathfrak{U}_{\bullet}$ of triples over $\mathfrak{S}$ indexed by $\Delta^{\circ} \times I$ as follows:

– for an object (m, n) in $\Delta^{\circ} \times I$, $\mathfrak{U}_{(m,n)}$ is a fiber product of $m+1$-copies of $\mathfrak{Z}_n$ over $\mathfrak{Y}_n$;

– $(\xi, \eta)_{\mathfrak{U}_{\bullet}} : \mathfrak{U}_{(m,n)} \to \mathfrak{U}_{(k,l)}$ is the morphism defined by

$$(z_0, z_1, \ldots, z_m) \mapsto (\eta_{\mathfrak{Z}_{\bullet}}(z_{\xi(0)}), \eta_{\mathfrak{Z}_{\bullet}}(z_{\xi(1)}), \ldots, \eta_{\mathfrak{Z}_{\bullet}}(z_{\xi(k)}))$$

for morphisms $\xi : k \to m$ in Δ and $\eta : n \to l$ in I.

One can easily check that $\mathfrak{U}_{\bullet}$ is a diagram of $\mathfrak{S}$-triple locally of finite type indexed by $\Delta^{\circ} \times I$ over $\mathfrak{S}$ and $\mathfrak{U}_{(0,\bullet)} = \mathfrak{Z}_{\bullet}$. If we define a morphism $\mathfrak{U}_{(m,n)} \to \mathfrak{X}_m$ (resp. $\mathfrak{U}_{(m,n)} \to \mathfrak{Y}_n$) by the $m+1$-copies of the structure morphism $\mathfrak{Z}_n \to \mathfrak{X}$ (resp. by the natural morphism), then the collection of morphisms determines a morphism $\mathfrak{U}_{\bullet} \to \mathfrak{X}_{\bullet}$ (resp. $\mathfrak{U}_{\bullet} \to \mathfrak{Y}_{\bullet}$) of diagrams of triples over $\mathfrak{S}$. Moreover, the diagram

$$\begin{array}{ccc} (\mathfrak{Y}_., I) & \leftarrow & (\mathfrak{U}_., \Delta^{\circ} \times I) \\ \downarrow & & \downarrow \\ (\mathfrak{W}, \underline{0}) & \leftarrow & (\mathfrak{X}_., \Delta^{\circ}) \end{array}$$

is commutative. We call $\mathfrak{U}_.$ a Čech diagram of $\mathfrak{Z}_.$ over $\mathfrak{Y}_.$ and denote it by $\mathrm{cosk}_0^{\mathfrak{Y}_.}(\mathfrak{Z}_.)$. ■

3.2. Let $\mathfrak{S}$ be an lft-triple and let $(\mathfrak{X}_., I)$ be a diagram of lft-triples over $\mathfrak{S}$. By the functoriality of the construction of tubes associated to lft-triples we obtain a diagram $]\overline{X}_.[_{\mathcal{X}_.}$ of rigid analytic spaces over $]\overline{S}[_{\mathcal{S}}$ indexed by I: for each object n of I, $n \mapsto]\overline{X}_n[_{\mathcal{X}_n}$, and for each morphism $\eta : m \mapsto n$ of I, we have $\tilde{\eta}_{\mathfrak{X}_.} :]\overline{X}_m[_{\mathcal{X}_m} \to]\overline{X}_n[_{\mathcal{X}_n}$ which is the morphism of rigid analytic spaces induced by the morphism $\eta_{\mathfrak{X}_.}$ of triples. We call $]\overline{X}_.[_{\mathcal{X}_.}$ the diagram of tubes associated to $\mathfrak{X}_.$. The correspondence we have constructed between diagrams of lft-triples and the associated diagrams of tubes is functorial.

3.3. Let $\mathfrak{X}_.$ be a diagram of lft-triples indexed by I and let $]\overline{X}_.[_{\mathcal{X}_.}$ be a diagram of tubes associated to $\mathfrak{X}_.$.

A sheaf $\mathcal{F}_.$ of abelian groups (resp. rings, resp. $\mathcal{A}_.$-modules for a sheaf $\mathcal{A}_.$ of rings) on $]\overline{X}_.[_{\mathcal{X}_.}$ consists of the following data:

– for each object n of I, $\mathcal{F}_n$ is a sheaf of abelian groups (resp. rings, resp. $\mathcal{A}_n$-modules) on $]\overline{X}_n[_{\mathcal{X}_n}$;

– for each morphism $\eta : m \to n$ of I, $\mathcal{F}_.(\eta) : \tilde{\eta}_{\mathfrak{X}_.}^{-1} \mathcal{F}_n \to \mathcal{F}_m$ is a homomorphism of sheaves of abelian groups (resp. rings, resp. $\tilde{\eta}_{\mathfrak{X}_.}^{-1} \mathcal{A}_n$-modules) such that the diagram

$$\begin{array}{ccc} \widetilde{\eta\xi}_{\mathfrak{X}_.}^{-1} \mathcal{F}_n & \xrightarrow{\tilde{\xi}_{\mathfrak{X}_.}^{-1}(\mathcal{F}_.(\eta))} & \tilde{\xi}_{\mathfrak{X}_.}^{-1} \mathcal{F}_m \\ & {\scriptstyle \mathcal{F}_.(\eta\xi)} \searrow \quad \swarrow {\scriptstyle \mathcal{F}_.(\xi)} & \\ & \mathcal{F}_l & \end{array}$$

is commutative for each pair of morphisms $\xi : l \to m$ and $\eta : m \to n$ of I.

A homomorphism $\varphi_. : \mathcal{F}_. \to \mathcal{G}_.$ of sheaves of abelian groups (resp. rings, resp. $\mathcal{A}_.$-modules) on $]\overline{X}_.[_{\mathcal{X}_.}$ consists of the following data:

(i) for each object n of I, $\varphi_n : \mathcal{F}_n \to \mathcal{G}_n$ is a homomorphism of sheaves of abelian groups (resp. rings, resp. $\mathcal{A}_n$-modules);

(ii) for each morphism $\eta : m \to n$ of I, the diagram

$$\begin{array}{ccc} \tilde{\eta}_{\mathfrak{X}.}^{-1}\mathcal{F}_n & \xrightarrow{\tilde{\eta}_{\mathfrak{X}.}^{-1}(\varphi_n)} & \tilde{\eta}_{\mathfrak{X}.}^{-1}\mathcal{G}_n \\ {\scriptstyle \mathcal{F}.(\eta)}\downarrow & & \downarrow{\scriptstyle \mathcal{G}.(\eta)} \\ \mathcal{F}_m & \xrightarrow[\varphi_m]{} & \mathcal{G}_m \end{array}$$

is commutative.

3.3.1. Example. We give some examples of sheaves of rings on $]\overline{X}.[_{\mathcal{X}.}$. (In fact they are all coherent, see 2.9 and Definition 3.3.3.)

(1) $\mathbb{Z}_{]\overline{X}.[_{\mathcal{X}.}}$: the sheaf of rings associated to the presheaf $U \mapsto \mathbb{Z}$ for any object n of I and for any admissible open subset U in $]\overline{X}_n[_{\mathcal{X}_n}$.

(2) The structure sheaf $\mathcal{O}_{]\overline{X}.[_{\mathcal{X}.}}$: for each object n of I, n corresponds to the structure sheaf $\mathcal{O}_{]\overline{X}_n[_{\mathcal{X}_n}}$ of the rigid analytic space $]\overline{X}_n[_{\mathcal{X}_n}$.

(3) The sheaf $j^\dagger\mathcal{O}_{]\overline{X}.[_{\mathcal{X}.}}$ of overconvergent functions: for each object n of I, n corresponds to the sheaf $j^\dagger\mathcal{O}_{]\overline{X}_n[_{\mathcal{X}_n}}$ of overconvergent functions on $]\overline{X}_n[_{\mathcal{X}_n}$ along a complement ∂X_n of X_n in $\overline{X}_n$. (See the definition of sheaves of overconvergent sections on a diagrams of tubes in 3.5.) ■

Let $\mathcal{A}.$ be a sheaf of rings on $]\overline{X}.[_{\mathcal{X}.}$. The category of sheaves of $\mathcal{A}.$-modules is an abelian category which is closed under the direct sum and the direct product for any family of objects. A sequence $\mathcal{F}^\bullet_.$ of sheaves of $\mathcal{A}.$-modules is exact if and only if $\mathcal{F}_n$ is exact for each object n of I. Hence, for any filtered inductive system of exact sequences of sheaves of $\mathcal{A}.$-modules, the limit is also exact. The following Proposition is in [15, VI, 5.3]. (See also [2, V, Proposition 3.4.4].) We will give a sketch of the proof in 3.8.

3.3.2. Proposition. *The category of sheaves of $\mathcal{A}.$-modules has enough injectives.*

We introduce the notion of coherent sheaf on diagrams of tubes.

3.3.3. Definition. Let $\mathcal{A}.$ be a sheaf of rings on $]\overline{X}.[_{\mathcal{X}.}$.

(1) We say that a sheaf $\mathcal{E}.$ is a sheaf of coherent $\mathcal{A}.$-modules if it satisfies the following conditions:

(i) $\mathcal{E}_n$ is a sheaf of coherent $\mathcal{A}_n$-modules for any n;

(ii) $\mathcal{E}_{\cdot}(\eta)$ induces an isomorphism

$$\mathcal{A}_m \otimes_{\tilde{\eta}_{\mathfrak{X}_{\cdot}}^{-1}(\mathcal{A}_n)} \tilde{\eta}_{\mathfrak{X}_{\cdot}}^{-1} \mathcal{E}_n \to \mathcal{E}_m$$

for any morphism $\eta : m \to n$ of I.

We denote by $\mathrm{Coh}(\mathcal{A}_{\cdot})$ the category of sheaves of coherent $\mathcal{A}_{\cdot}$-modules.

(2) $\mathcal{A}_{\cdot}$ is a sheaf of coherent rings if $\mathcal{A}_n$ is coherent as an $\mathcal{A}_n$-module for any object n. ■

If $\mathcal{A}_{\cdot}$ is a sheaf of coherent rings, the category of sheaves of coherent $\mathcal{A}_{\cdot}$-modules is a full subcategory of that of sheaves of $\mathcal{A}_{\cdot}$-modules. It is abelian and closed under tensor products and internal homs.

3.4. Let $(f_{\cdot}, t) : (\mathfrak{Y}_{\cdot}, J) \to (\mathfrak{X}_{\cdot}, I)$ be a morphism of diagrams of lft-triples.

For a sheaf $\mathcal{F}_{\cdot}$ of abelian groups on $]\overline{X}_{\cdot}[_{\mathcal{X}_{\cdot}}$, we define an inverse image $\tilde{f}_{\cdot}^{-1}\mathcal{F}_{\cdot}$ as follows. We put

$$(\tilde{f}_{\cdot}^{-1}\mathcal{F}_{\cdot})_n = \tilde{f}_n^{-1}\mathcal{F}_{t(n)}$$

for an object n of J and put

$$(\tilde{f}_{\cdot}^{-1}\mathcal{F}_{\cdot})(\eta) = \tilde{f}_m^{-1}(\mathcal{F}_{\cdot}(t(\eta)))$$

for a morphism $\eta : m \to n$ in J. One can easily see that $\tilde{f}_{\cdot}^{-1}\mathcal{F}_{\cdot}$ satisfies the condition of sheaves on $]\overline{Y}_{\cdot}[_{\mathcal{Y}_{\cdot}}$. If $\mathcal{F}_{\cdot}$ is a sheaf of rings (resp. $\mathcal{A}_{\cdot}$-modules for a sheaf $\mathcal{A}_{\cdot}$ of rings on $]\overline{X}[_{\mathcal{X}}$), then $\tilde{f}^{-1}\mathcal{A}_{\cdot}$ is as well.

The following Proposition follows easily from the definition.

3.4.1. Proposition. *With the notation as above, we have*

(1) $\tilde{f}_{\cdot}^{-1}$ *is exact*;

(2) *if* $g_{\cdot} : \mathfrak{Z}_{\cdot} \to \mathfrak{Y}_{\cdot}$ *is a morphism of diagrams of lft-triples, then* $(\widetilde{fg})_{\cdot}^{-1} = \tilde{g}_{\cdot}^{-1}\tilde{f}_{\cdot}^{-1}$.

Let $\mathcal{A}_{\cdot}$ (resp. $\mathcal{B}_{\cdot}$) be a sheaf of rings on $]\overline{X}_{\cdot}[_{\mathcal{X}_{\cdot}}$ (resp. $]\overline{Y}_{\cdot}[_{\mathcal{Y}_{\cdot}}$) with a homomorphism $\tilde{f}_{\cdot}^{-1}\mathcal{A}_{\cdot} \to \mathcal{B}_{\cdot}$ of sheaves of rings. We define a functor $\tilde{f}_{\cdot}^*$ from the category of sheaves of $\mathcal{A}_{\cdot}$-modules to that of $\mathcal{B}_{\cdot}$-modules by

$$(\tilde{f}_{\cdot}^*\mathcal{F}_{\cdot})_n = \mathcal{B}_n \otimes_{\tilde{f}_n^{-1}(\mathcal{A}_{t(n)})} \tilde{f}_n^{-1}\mathcal{F}_{t(n)}$$

for an object n of J and by

$$(\tilde{f}^*_{\cdot}\mathcal{F}_{\cdot})(\eta) = \mathcal{B}_{\cdot}(\eta)\otimes \tilde{f}_m^{-1}(\mathcal{F}_{\cdot}(t(\eta)))$$

for a morphism $\eta : m \to n$ of J. One can easily check that $\tilde{f}^*_{\cdot}$ is a functor. We also call $\tilde{f}^*_{\cdot}$ an inverse image functor if there is no confusion.

3.4.2. Proposition. *With the notation as above, we have the following.*

(1) *$\tilde{f}^*_{\cdot}$ is right exact.*

(2) *Let $g_{\cdot} : \mathfrak{Z}_{\cdot} \to \mathfrak{Y}_{\cdot}$ be a morphism of diagrams of lft-triples and let $\mathcal{C}_{\cdot}$ be a sheaf of rings on $]\overline{Z}_{\cdot}[_{\mathcal{Z}_{\cdot}}$ with a homomorphism $\tilde{g}^{-1}_{\cdot}\mathcal{B}_{\cdot} \to \mathcal{C}_{\cdot}$ of sheaves of rings. Then $(\widetilde{fg})^*_{\cdot} = \tilde{g}^*_{\cdot}\tilde{f}^*_{\cdot}$.*

The following proposition follows easy from the definition.

3.4.3. Proposition. *With the notation as above, assume furthermore that $\mathcal{B}_{\cdot}$ is coherent. Then the inverse image $\tilde{f}^*_{\cdot}\mathcal{E}_{\cdot}$ of a sheaf $\mathcal{E}_{\cdot}$ of coherent $\mathcal{A}_{\cdot}$-modules is a sheaf of coherent $\mathcal{B}_{\cdot}$-modules.*

3.5. Let $(\mathfrak{X}_{\cdot}, I)$ be a diagram of lft-triples and let $\mathcal{A}_{\cdot}$ be a sheaf of rings on $]\overline{X}[_{\mathcal{X}}$. For a sheaf $\mathcal{F}_{\cdot}$ of $\mathcal{A}_{\cdot}$-modules, we define a sheaf $j^\dagger\mathcal{F}_{\cdot}$ of overconvergent sections along $\partial X_{\cdot}$ by

$$\begin{aligned} j^\dagger\mathcal{F}_m &= \varinjlim_V j_{V*}j_V^{-1}\mathcal{F}_m \\ (j^\dagger\mathcal{F}_{\cdot})(\eta) &= \varinjlim_V j_{V*}j_V^{-1}\mathcal{F}_{\cdot}(\eta) \end{aligned}$$

for an object m and a morphism $\eta : m \to n$ of I, where V runs through all strict neighbourhoods of $]X_m[_{\mathcal{X}_m}$ in $]\overline{X}_m[_{\mathcal{X}_m}$. Then, $j^\dagger\mathcal{F}_{\cdot}$ is a sheaf of $j^\dagger\mathcal{A}_{\cdot}$-modules.

Now let $(f_{\cdot}, t) : (\mathfrak{Y}_{\cdot}, J) \to (\mathfrak{X}_{\cdot}, I)$ be a morphism of diagrams of lft-triples and let $\mathcal{B}_{\cdot}$ be a sheaf of rings on $]\overline{Y}_{\cdot}[_{\mathcal{Y}_{\cdot}}$ with a morphism $\tilde{f}^{-1}_{\cdot}\mathcal{A}_{\cdot} \to \mathcal{B}_{\cdot}$ of sheaves of rings. For a sheaf $\mathcal{F}_{\cdot}$ of $\mathcal{A}_{\cdot}$-modules, we define a functor $f^\dagger_{\cdot}$ by

$$f^\dagger_{\cdot}\mathcal{F}_{\cdot} = j^\dagger(\tilde{f}^*_{\cdot}\mathcal{F}_{\cdot}).$$

$f^\dagger_{\cdot}\mathcal{F}_{\cdot}$ is a sheaf of $j^\dagger\mathcal{B}_{\cdot}$-modules. One can easily see that there is a natural transform $\tilde{f}^*_{\cdot}j^\dagger \to f^\dagger_{\cdot}$.

Let $E_{\cdot}$ be a sheaf of coherent $j^\dagger\mathcal{O}_{]\overline{X}_{\cdot}[_{\mathcal{X}_{\cdot}}}$-modules and suppose that $\mathcal{E}_m$ is a sheaf of coherent $\mathcal{O}_{U_m}$-modules for a strict neighborhood U_m of $]X_m[_{\mathcal{X}_m}$

in $]\overline{X}_m[_{\mathcal{X}_m}$ with an isomorphism $E_m \cong j^\dagger_{U_m}\mathcal{E}_m$. In this situation there is a canonical isomorphism

$$(\tilde{f}^*_{\cdot} E_{\cdot})_n \cong f_n^\dagger(\mathcal{O}_{]\overline{X}_{t(n)}[_{\mathcal{X}_{t(n)}}} \otimes_{j_{U_{t(n)}}!\mathcal{O}_{U_{t(n)}}} j_{U_{t(n)}}!\,\mathcal{E}_{t(n)})$$

for any object n of J by Proposition 2.10.1.

Thus $\tilde{f}^*_{\cdot} E_{\cdot}$ is a sheaf of overconvergent sections: it coincides with $f^\dagger_{\cdot} E_{\cdot}$.

3.6. Let $t : J \to I$ be a covariant functor of categories. For an object m of I, we define a category J/m as follows:

– an object is a data (n, ξ) such that $\xi : t(n) \to m$ is a morphism of I;

– a morphism $\zeta : (n_1, \xi_1) \to (n_2, \xi_2)$ is a morphism $\zeta : n_1 \to n_2$ of J with $\xi_1 = \xi_2 t(\zeta)$ in I.

Let $(f_{\cdot}, t) : (\mathcal{Y}_{\cdot}, J) \to (\mathfrak{X}_{\cdot}, I)$ be a morphism of diagrams of lft-triples. For a sheaf $\mathcal{G}_{\cdot}$ of abelian groups on $]\overline{Y}_{\cdot}[_{\mathcal{Y}_{\cdot}}$, we define a direct image $\tilde{f}_{\cdot *}\mathcal{G}_{\cdot}$ as follows. For an object m of I,

$$(\tilde{f}_{\cdot *}\mathcal{G}_{\cdot})_m = \varprojlim_{(n,\xi) \in \mathrm{Ob}(J/m)} \tilde{\xi}_{\mathfrak{X}_{\cdot} *}\tilde{f}_{n*}\mathcal{G}_n$$

For a morphism $\eta : m_1 \to m_2$ in I, a homomorphism

$$(\tilde{f}_{\cdot *}\mathcal{G}_{\cdot})(\eta) : \tilde{\eta}^{-1}_{\mathfrak{X}_{\cdot}}(\tilde{f}_{\cdot *}\mathcal{G}_{\cdot})_{m_2} \to (\tilde{f}_{\cdot *}\mathcal{G}_{\cdot})_{m_1}$$

is defined by the adjoint

$$\tilde{\eta}^{-1}_{\mathfrak{X}_{\cdot}}(\widetilde{\eta\xi})_{\mathfrak{X}_{\cdot}*}\tilde{f}_{n*}\mathcal{G}_n \to \tilde{\xi}_{\mathfrak{X}_{\cdot}*}\tilde{f}_{n*}\mathcal{G}_n$$

for an object (n, ξ) of J/m_1. Since the diagram

$$\begin{array}{ccc}
\tilde{\eta}^{-1}_{\mathfrak{X}_{\cdot}}(\widetilde{\eta\xi_2})_{\mathfrak{X}_{\cdot}*}\tilde{f}_{n_2*}\mathcal{G}_{n_2} & \to & \tilde{\xi}_{2\mathfrak{X}_{\cdot}*}\tilde{f}_{n_2*}\mathcal{G}_{n_2} \\
\downarrow & & \downarrow \\
\tilde{\eta}^{-1}_{\mathfrak{X}_{\cdot}}(\widetilde{\eta\xi_2 t(\zeta)})_{\mathfrak{X}_{\cdot}*}\tilde{f}_{n_1*}\tilde{\zeta}^{-1}_{\mathcal{Y}_{\cdot}}\mathcal{G}_{n_2} & \to & (\widetilde{\xi_2 t(\zeta)})_{\mathfrak{X}_{\cdot}*}\tilde{f}_{n_1*}\tilde{\zeta}^{-1}_{\mathcal{Y}_{\cdot}}\mathcal{G}_{n_2} \\
{\scriptstyle \tilde{\eta}^{-1}_{\mathfrak{X}_{\cdot}}(\widetilde{\eta\xi_1})_{\mathfrak{X}_{\cdot}*}\tilde{f}_{n_1*}(\mathcal{G}_{\cdot}(\zeta))}\downarrow & & \downarrow{\scriptstyle \tilde{\xi}_{1\mathfrak{X}_{\cdot}*}\tilde{f}_{n_1*}(\mathcal{G}_{\cdot}(\zeta))} \\
\tilde{\eta}^{-1}_{\mathfrak{X}_{\cdot}}(\widetilde{\eta\xi_1})_{\mathfrak{X}_{\cdot}*}\tilde{f}_{n_1*}\mathcal{G}_{n_1} & \to & \tilde{\xi}_{1\mathfrak{X}_{\cdot}*}\tilde{f}_{n_1*}\mathcal{G}_{n_1}
\end{array}$$

is commutative for a morphism $\eta : m_1 \to m_2$ in I and a morphism $\zeta : (n_1, \xi_1) \to (n_2, \xi_2)$ of J/m_1; one sees that $\tilde{f}_{\cdot *}\mathcal{G}_{\cdot}$ satisfies the conditions for a sheaf on $]\overline{X}_{\cdot}[_{\mathcal{X}_{\cdot}}$.

We recall some notions on categories. A category C is connected if, for any two objects m and n, there exists a sequence $m \to l_1 \leftarrow l_2 \to \to \ldots \leftarrow n$ (possibly infinite) of morphisms of C. Then a small category is a disjoint union of connected full subcategories. We call each connected full subcategory a connected component of C. A category is discrete if the only morphisms are the identities.

A category is filtered if it satisfies the conditions: (i) it is connected, (ii) for any two morphisms $\xi_i : l \to m_i$ $(i = 1, 2)$, there exist an object n and morphisms $\eta_i : m_i \to n$ $(i = 1, 2)$ such that $\eta_1 \xi_1 = \eta_2 \xi_2$, and (iii) for any two morphisms $\xi_i : l \to m$ $(i = 1, 2)$, there exists an object n and a morphism $\eta : m \to n$ such that $\eta \xi_1 = \eta \xi_2$. A category is cofiltered if and only if the dual category is filtered. If a category has a final object (resp. an initial object), then it is filtered (resp. cofiltered).

If $J/m = \coprod_\lambda (J/m)_\lambda$ is the decomposition into connected components, then

$$(\tilde{f}_{.*} \mathcal{G}_.)_m = \prod_\lambda \varprojlim_{(n, \xi) \in \mathrm{Ob}((J/m)_\lambda)} \tilde{\xi}_{\mathfrak{X}.*} \tilde{f}_{n*} \mathcal{G}_n .$$

If each connected component $(J/m)_\lambda$ has an initial object (n_λ, ξ_λ), then

$$(\tilde{f}_{.*} \mathcal{G}_.)_m = \prod_\lambda \tilde{\xi}_{\lambda \mathfrak{X}.*} \tilde{f}_{n_\lambda *} \mathcal{G}_{n_\lambda} .$$

If $I = J$ and t is the identity functor, then, for any object n of I, I/n is connected and (n, id_n) is a final object of the category I/n. Hence, we have

$$(\tilde{f}_{.*} \mathcal{G}_.)_n \cong \tilde{f}_{n*} \mathcal{G}_n$$

for any object n of I. In particular, if $I = J = \underline{0}$, the direct image functor $\tilde{f}_{.*}$ is a usual direct image functor of sheaves on rigid analytic spaces.

3.7. Let $(f_., t) : (\mathfrak{Y}_., J) \to (\mathfrak{X}_., I)$ be a morphism of diagrams of lft-triples and let $\mathcal{A}_.$ (resp. $\mathcal{B}_.$) be a sheaf of rings on $]\overline{X}_.[_{\mathcal{X}_.}$ (resp. $]\overline{Y}_.[_{\mathcal{Y}_.}$) with a homomorphism $\mathcal{A}_. \to \tilde{f}_{.*} \mathcal{B}_.$ of sheaves of rings on $]\overline{X}_.[_{\mathcal{X}_.}$. If $\mathcal{G}_.$ is a sheaf of $\mathcal{B}_.$-modules, then $\tilde{f}_{.*} \mathcal{G}_.$ is a sheaf of $\mathcal{A}_.$-modules.

For a sheaf $\mathcal{F}_.$ of $\mathcal{A}_.$-modules, we define a homomorphism

$$\mathcal{F}_. \to \tilde{f}_{.*} \tilde{f}_.^* \mathcal{F}_.$$

of sheaves of $\mathcal{A}_{\bullet}$-modules by the diagonal homomorphism

$$\mathcal{F}_m \to \varprojlim_{(n,\,\xi)\,\in\,\mathrm{Ob}\,(J/m)} \tilde{\xi}_{\mathfrak{X}.*}\tilde{f}_{n*}(\tilde{f}_n^*\mathcal{F}_{t(n)})$$

whose (n, ξ)-component is the adjunction of the composition $\tilde{\xi}^*_{\mathfrak{X}.}\mathcal{F}_m \xrightarrow{\mathcal{F}.(\xi)} \mathcal{F}_{t(n)} \to \tilde{f}_{n*}\tilde{f}_n^*\mathcal{F}_{t(n)}$. Since the diagram

$$\begin{array}{ccc} \tilde{\eta}^*_{\mathfrak{X}.}\mathcal{F}_{m_2} & \xrightarrow{\mathcal{F}.(\eta\xi)} & \tilde{\eta}^*_{\mathfrak{X}.}(\widetilde{\eta\xi})_{\mathfrak{X}.*}\tilde{f}_{n*}(\tilde{f}_n^*\mathcal{F}_{t(n)}) \\ {\scriptstyle \mathcal{F}.(\eta)}\downarrow & & \downarrow \\ \mathcal{F}_{m_1} & \xrightarrow[\mathcal{F}(\xi)]{} & \tilde{\xi}_{\mathfrak{X}.*}\tilde{f}_{n*}(\tilde{f}_n^*\mathcal{F}_{t(n)}) \end{array}$$

is commutative for a morphism $\eta : m_1 \to m_2$ of I and an object (n, ξ) of J/m_1, the map $\mathcal{F}. \to \tilde{f}_{.*}\tilde{f}^*_{.}\mathcal{F}.$ is a homomorphism of sheaves of $\mathcal{A}_{\bullet}$-modules.

For a sheaf $\mathcal{G}.$ of $\mathcal{B}_{\bullet}$-modules, we define a homomorphism

$$\tilde{f}^*_{.}\tilde{f}_{.*}\mathcal{G}. \to \mathcal{G}.$$

of sheaves of $\mathcal{B}_{\bullet}$-modules by the composition

$$\tilde{f}_l^*\left(\varprojlim_{(n,\,\xi)\,\in\,\mathrm{Ob}\,(J/t(l))} \tilde{\xi}_{\mathfrak{X}.*}\tilde{f}_{n*}\mathcal{G}_n\right) \to \tilde{f}_l^*\tilde{f}_{l*}\mathcal{G}_l \to \mathcal{G}_l$$

for any object l of J. Here the first arrow is the projection of the (l, id)-component and the second arrow follows from adjonction. One can easily see that the map $\tilde{f}^*_{.}\tilde{f}_{.*}\mathcal{G}. \to \mathcal{G}.$ is a homomorphism of sheaves of $\mathcal{B}_{\bullet}$-modules.

By the homomorphisms given above we have

3.7.1. PROPOSITION. *$\tilde{f}_{.*}$ is a right adjoint of $\tilde{f}^*_{.}$.*

3.7.2. COROLLARY. *Let $(f., t) : (\mathfrak{Y}., J) \to (\mathfrak{X}., I)$ and $(g., u) : (\mathfrak{Z}., K) \to (\mathfrak{Y}., J)$ be morphisms of diagrams of lft-triples. Then we have $(\widetilde{fg})_{.*} = \tilde{f}_{.*}\tilde{g}_{.*}$.*

PROOF. Both $(\widetilde{fg})_{.*}$ and $\tilde{f}_{.*}\tilde{g}_{.*}$ are right adjoints of $(\widetilde{fg})^*_{.} = \tilde{g}^*_{.}\tilde{f}^*_{.}$. ■

3.7.3. COROLLARY. *Let*

$$\begin{array}{ccc} \mathfrak{Y}_{\cdot} & \xleftarrow{g_{\cdot}} & \mathfrak{Y}'_{\cdot} \\ {\scriptstyle w_{\cdot}}\downarrow & & \downarrow{\scriptstyle w'_{\cdot}} \\ \mathfrak{X}_{\cdot} & \xleftarrow[f_{\cdot}]{} & \mathfrak{X}'_{\cdot} \end{array}$$

be a commutative diagram of lft-triples and let $\mathcal{A}_{\cdot}$ (*resp.* $\mathcal{A}'_{\cdot}$, *resp.* $\mathcal{B}_{\cdot}$, *resp.* $\mathcal{B}'_{\cdot}$) *be a ring of sheaves on* $]\overline{X}_{\cdot}[_{\mathcal{X}_{\cdot}}$ (*resp.* $]\overline{X}'_{\cdot}[_{\mathcal{X}'_{\cdot}}$, *resp.* $]\overline{Y}_{\cdot}[_{\mathcal{Y}_{\cdot}}$, *resp.* $]\overline{Y}'_{\cdot}[_{\mathcal{Y}'_{\cdot}}$) *with commutative morphisms corresponding to the diagram above. Then, for a sheaf* $\mathcal{G}_{\cdot}$ *of* $\mathcal{B}_{\cdot}$*-modules, the adjoints induce a functorial morphism*

$$\tilde{f}^*_{\cdot}\widetilde{w}_{\cdot *}\mathcal{G}_{\cdot} \to \widetilde{w}'_{\cdot *}\tilde{g}^*_{\cdot}\mathcal{G}_{\cdot}$$

of sheaves of $\mathcal{A}'_{\cdot}$*-modules.*

We call the homomorphism of sheaves given above a base change homomorphism.

3.7.4. PROPOSITION. *Let*

$$\begin{array}{ccc} (\mathfrak{Y}_{\cdot}, J) & \xleftarrow{(g_{\cdot}, \mathrm{id}_J)} & (\mathfrak{Y}'_{\cdot}, J) \\ {\scriptstyle (w_{\cdot}, t)}\downarrow & & \downarrow{\scriptstyle (w'_{\cdot} t)} \\ (\mathfrak{X}_{\cdot}, I) & \xleftarrow[(f_{\cdot}, \mathrm{id}_I)]{} & (\mathfrak{X}'_{\cdot}, I) \end{array}$$

be a commutative diagram of lft-triples such that

(i) $\widehat{f}_m\colon \mathcal{X}'_m \to \mathcal{X}_m$ *is an open immersion and* $\widehat{f}_m^{-1}(\overline{X}_m) = \overline{X}'_m$ *for any object* m *of* I;

(ii) $\widehat{w}_n^{-1}(\mathcal{X}'_{t(n)}) = \mathcal{Y}'_n$ *and* $\overline{w}_n^{-1}(\overline{X}'_{t(n)}) = \overline{Y}'_n$ *via* g_n *for any object* n *of* J.

Let $\mathcal{A}_{\cdot}$ (*resp.* $\mathcal{B}_{\cdot}$) *be a ring of sheaves on* $]\overline{X}_{\cdot}[_{\mathcal{X}_{\cdot}}$ (*resp.* $]\overline{Y}_{\cdot}[_{\mathcal{Y}_{\cdot}}$) *with a homomorphism* $\widetilde{w}_{\cdot}^{-1}(\mathcal{A}_{\cdot}) \to \mathcal{B}_{\cdot}$ *of sheaves of rings. Then, for a sheaf* $\mathcal{G}_{\cdot}$ *of* $\mathcal{B}_{\cdot}$*-modules, the base change homomorphism*

$$\tilde{f}_{\cdot}^{-1}\widetilde{w}_{\cdot *}\mathcal{G}_{\cdot} \to \widetilde{w}'_{\cdot *}\tilde{g}_{\cdot}^{-1}\mathcal{G}_{\cdot}$$

is an isomorphism.

PROOF. The assertion follows from the fact that the base change homomorphism $\tilde{f}_{t(n)}^{-1}\widetilde{w}_{n*}\mathcal{G}_n \to \widetilde{w}'_{n*}\tilde{g}_n^{-1}\mathcal{G}_n$ is an isomorphism. ■

3.8. Let I be a small category and let J be a subcategory of I. For an object m of I, we define a category $J(m)$ as follows:

– an object of $J(m)$ is a data (n, ξ) such that n is an object in J and $\xi : m \rightarrow n$ is a morphism of I;

– a morphism $\zeta : (n_1, \xi_1) \rightarrow (n_2, \xi_2)$ of $J(m)$ is a morphism $\zeta : n_1 \rightarrow n_2$ of J with $\xi_2 = \zeta\xi_1$ in I.

If J is discrete, $J(m)$ is discrete. By definition we have

3.8.1. Lemma. *With the notation as above, if we regard J° as a subcategory of I°, then $J(m)$ is the dual category of J°/m (see the definition in* 3.6).

3.8.2. Example. Let r and s be nonnegative integers with $r \leqq s$. We define $|m| = m_1 + \ldots + m_r$ for any object $m = (m_1, \ldots, m_r)$ in Δ^r. For an integer q, we define a subcategory $\Delta_q^{r, s}$ of Δ^s as follows:

– objects have the form (m, n) as an object of $\Delta^s = \Delta^r \times \Delta^{s-r}$ where $|m| = q$;

– $\mathrm{Mor}_{\Delta_q^{r,s}}((m, n_1), (m, n_2)) = \{(\mathrm{id}_m, \xi) \mid \xi \in \mathrm{Mor}_{\Delta^{s-r}}(n_1, n_2)\}$, and $\mathrm{Mor}_{\Delta_q^{r,s}}((m_1, n_1), (m_2, n_2)) = \emptyset$ if $m_1 \neq m_2$.

For an object m of Δ^r with $|m| = q$, we define a full subcategory $m \times \times \Delta^{s-r}$ of $\Delta_q^{r,s}$ whose objects consist of (m, n) for some object n of Δ^{s-r}. The category $m \times \Delta^{s-r}$ is equivalent to Δ^{s-r} by the functor $(m, n) \mapsto n$. One can easily see that

$$\Delta_q^{r,s} = \coprod_{|m| = q} m \times \Delta^{s-r}$$

is a decomposition of connected components of categories.

Let m be an object of Δ^r and let (l, n) be an object of $\Delta^s = \Delta^r \times \times \Delta^{s-r}$.

(1) If we denote by $(m \times \Delta^{s-r}/(l, n))_{((m, n), (\eta, \mathrm{id}_n))}$ the connected component of the object $((m, n), (\eta, \mathrm{id}_n))$ in the category $m \times \Delta^{s-r}/(l, n)$ (see the definition in 3.6) for some morphism $\eta : m \rightarrow l$ of Δ^r, then $((m, n), (\eta, \mathrm{id}_n))$ is a final object of $(m \times \Delta^{s-r}/(l, n))_{((m, n), (\eta, \mathrm{id}_n))}$ and

$$m \times \Delta^{s-r}/(l, n) = \coprod_{\eta \in \mathrm{Mor}_{\Delta^r}(m, l)} (m \times \Delta^{s-r}/(l, n))_{((m, n), (\eta, \mathrm{id}_n))}$$

is the decomposition into connected components.

(2) If we denote by $m \times \Delta^{s-r}(l, n)_{((m, n), (\eta, \mathrm{id}_n))}$ the connected component of the object $((l, n), (\eta, \mathrm{id}_n))$ in the category $m \times \Delta^{s-r}(l, n)$ for a morphism $\eta : l \to m$ of Δ^r, then $((m, n), (\eta, \mathrm{id}_n))$ is an initial object of $m \times \Delta^{s-r}(l, n)_{((m, n), (\eta, \mathrm{id}_n))}$ and

$$m \times \Delta^{s-r}(l, n) = \coprod_{\eta \in \mathrm{Mor}_{\Delta^r}(l, m)} (m \times \Delta^{s-r}/(l, n))_{((m, n), (\eta, \mathrm{id}_n))}$$

is the decomposition into connected components. ■

Let $(\mathfrak{X}_., I)$ be a diagram of lft-triples and let J be a subcategory of I. As a restriction of the functor $\mathfrak{X}_.$, we obtain a diagram $\mathfrak{Y}_.$ of lft-triples indexed by J and denote by $i_. : (\mathfrak{Y}_., J) \to (\mathfrak{X}_., I)$ the induced transform. Let $\mathcal{A}_.$ be a sheaf of rings on $]\overline{X}_.[_{\mathcal{X}_.}$

We define a functor $\tilde{i}_{.!}$ from the category of sheaves of $\tilde{i}_.^{-1}\mathcal{A}_.$-modules to that of $\mathcal{A}_.$-modules as follows. Let $\mathcal{F}_.$ be a sheaf of $\tilde{i}_.^{-1}\mathcal{A}_.$-modules. For an object m of I, we put

$$(\tilde{i}_{.!}\mathcal{F}_.)_m = \varinjlim_{(n, \xi) \in \mathrm{Ob}(J(m))} \tilde{\xi}^*_{\mathfrak{X}_.}\mathcal{F}_n$$

and, for a morphism $\eta : m_1 \to m_2$ of I, we define a morphism

$$\tilde{i}_{.!}\mathcal{F}_.(\eta) : \tilde{\eta}^*_{\mathfrak{X}_.}(i_{.!}\mathcal{F}_.)_{m_2} \to (i_{.!}\mathcal{F}_.)_{m_1}$$

by the induced homomorphism from the natural identity

$$\tilde{\eta}^*_{\mathfrak{X}_.}\tilde{\xi}^*_{\mathfrak{X}_.}\mathcal{F}_n \to (\widetilde{\xi\eta})^*_{\mathfrak{X}_.}\mathcal{F}_n$$

for $(n, \xi) \in J(m_2)$. Since the diagram

$$\begin{array}{ccc}
\tilde{\eta}^*_{\mathfrak{X}_.}\tilde{\xi}^*_{1\mathfrak{X}_.}(\tilde{\zeta}^*_{\mathfrak{Y}_.}\mathcal{F}_{n_2}) & \xrightarrow{\cong} & (\widetilde{\xi_1\eta})^*_{\mathfrak{X}_.}(\tilde{\zeta}^*_{\mathfrak{Y}_.}\mathcal{F}_{n_2}) \\
\tilde{\eta}^*_{\mathfrak{X}_.}\tilde{\xi}^*_{1\mathfrak{X}_.}(\mathcal{F}_.(\zeta)) \downarrow & & \downarrow \widetilde{\xi_1\eta}^*_{\mathfrak{X}_.}(\mathcal{F}_.(\zeta)) \\
\tilde{\eta}^*_{\mathfrak{X}_.}\tilde{\xi}^*_{1\mathfrak{X}_.}\mathcal{F}_{n_1} & \xrightarrow{\cong} & (\widetilde{\xi_1\eta})^*_{\mathfrak{X}_.}\mathcal{F}_{n_1}
\end{array}$$

is commutative for a morphism $\eta : m_1 \to m_2$ of I and a morphism $\zeta : (n_1, \xi_1) \to (n_2, \xi_2)$ of $J(m_2)$, we conclude that $\tilde{i}_{.!}\mathcal{F}_.$ is a sheaf of $\mathcal{A}_.$-modules on $]\overline{X}_.[_{\mathcal{X}_.}$. One can easily see that $\tilde{i}_{.!}$ is functorial in $\mathcal{F}_.$.

If $J(m) = \coprod_\lambda J(m)_\lambda$ is a decomposition into connected components, then

$$(\tilde{i}_{.!}\mathcal{F}_.)_m = \oplus_\lambda \varinjlim_{(n, \xi) \in \mathrm{Ob}(J(m)_\lambda)} \tilde{\xi}^*_{\mathfrak{X}_.}\mathcal{F}_n.$$

If each connected component $J(m)_\lambda$ has an initial object (n_λ, ξ_λ), then

$$(\tilde{i}_{.!}\mathcal{F}_.)_m = \oplus_\lambda \widetilde{\xi}_{\lambda\mathfrak{X}^*_{.}\mathcal{F}_{n_\lambda}}.$$

3.8.3. Proposition. *Suppose that $\tilde{\eta}^*_{\mathfrak{X}.}$ is exact for every morphism η of I and that, for each object m of I, each connected component of $J(m)$ is cofiltered. Then $\tilde{i}_{.!}$ is exact.*

Let $\mathcal{F}_.$ be a sheaf of $\mathcal{A}_.$-modules. We define a homomorphism $\tilde{i}_{.!}\tilde{i}^*_.\mathcal{F}_. \to \mathcal{F}_.$ by the map

$$\varinjlim_{(n,\,\xi)\,\in\,\mathrm{Ob}(J(m))} \tilde{\xi}^*_{\mathfrak{X}.}\mathcal{F}_n \to \mathcal{F}_m$$

induced by $\varinjlim_{(n,\,\xi)\,\in\,\mathrm{Ob}(J(m))} \mathcal{F}_.(\xi)$ for an object m of I. One can easily see that it is a homomorphism of sheaves of $\mathcal{A}_.$-modules.

Let $\mathcal{G}_.$ be a sheaf of $\tilde{i}^{-1}_.\mathcal{A}_.$-modules. We define a homomorphism $\mathcal{G}_. \to \tilde{i}^*_.\tilde{i}_{.!}\mathcal{G}_.$ by the map

$$\mathcal{G}_m \to \varinjlim_{(n,\,\xi)\,\in\,\mathrm{Ob}(J(m))} \tilde{\xi}^*_{\mathfrak{X}.}\mathcal{G}_n$$

which goes to the $(m$, id)-component by the identity for an object m of J. One can easily see that it is a homomorphism of sheaves of $\tilde{i}^{-1}_.\mathcal{A}_{]\overline{X}.[_{\mathfrak{X}.}}$-modules.

By the homomorphisms given above we have

3.8.4. Proposition. *$\tilde{i}_{.!}$ is a left adjoint of $\tilde{i}^*_.$.*

3.8.5. Corollary. *If K is a subcategory of J and if $j_. : (\mathfrak{Z}_., K) \to \to (\mathfrak{Y}_., J)$ is the induced morphism of diagrams of lft-triples, then $(\widetilde{ij})_{.!} = \tilde{i}_{.!}\tilde{j}_{.!}$.*

3.8.6. Corollary. *Let $(f_., t) : (\mathfrak{Y}_., J) \to (\mathfrak{X}_., I)$ be a diagram of lft-triples and let $\mathcal{A}_.$ (resp. $\mathcal{B}_.$) be a sheaf of rings on $]\overline{X}_.[_{\mathfrak{X}.}$ (resp. $]\overline{Y}_.[_{\mathfrak{Y}.}$) with a homomorphism $\tilde{f}^{-1}_.\mathcal{A}_. \to \mathcal{B}_.$ of sheaves of rings. Suppose that I' (resp. J') is a subcategory of I (resp. J) such that $t(J')$ is a subcategory* *of I'. If*

$$\begin{array}{ccc} (\mathfrak{Y}_{\bullet}, J) & \overset{j_{\bullet}}{\leftarrow} & (\mathfrak{Y}'_{\bullet}, J') \\ f_{\bullet} \downarrow & & \downarrow f'_{\bullet} \\ (\mathfrak{X}_{\bullet}, I) & \underset{i_{\bullet}}{\leftarrow} & (\mathfrak{X}'_{\bullet}, I') \end{array}$$

is the induced commutative diagram of lft-triples, then, for any sheaf $\mathcal{F}_{\bullet}$ of $(\tilde{i}_{\bullet})^{-1}\mathcal{A}_{\bullet}$-modules, the adjoints induce a functorial homomorphism

$$\tilde{j}_{\bullet !}(\tilde{f}'_{\bullet})^* \mathcal{F}_{\bullet} \to \tilde{f}^*_{\bullet} \tilde{i}_{\bullet !} \mathcal{F}_{\bullet}$$

of sheaves of $\mathcal{B}_{\bullet}$-modules.

Now we give a sketch of a proof of Proposition 3.3.2 following [15, VI, 5.3]. It is sufficient to see that the category of sheaves of $\mathcal{A}_{\bullet}$-modules has a family of generators [9, Théorème 1.10.1]. For an object n of I, we denote by $i_{\underline{n}}: \mathfrak{X}_n \to \mathfrak{X}_{\bullet}$ a morphism of diagrams of lft-triples which corresponds to the discrete subcategory $\underline{n}$ of I (see Example 3.1.1 (1)). In this case $\tilde{i}^*_{\underline{n}}$ is a restriction functor from the category of sheaves of $\mathcal{A}_{\bullet}$-modules to that of $\mathcal{A}_n$-modules. Then, for any sheaf $\mathcal{F}_n$ of $\mathcal{A}_n$-modules, $\mathcal{F}_n \to \tilde{i}^*_{\underline{n}} \tilde{i}_{\underline{n}!} \mathcal{F}_n$ is a monomorphism. Hence, the set

$$\left\{ \tilde{i}_{\underline{n}!}(j_{U!}\mathcal{A}_n) \;\middle|\; \begin{array}{l} n \text{ is an object of } I, \\ U \text{ is an admissible open subset of }]\overline{X}_n[_{\mathcal{X}_n} \end{array} \right\}.$$

is a family of generators by Proposition 3.8.4. Here $j_U: U \to]\overline{X}_n[_{\mathcal{X}_n}$ is an open immersion and $j_{U!}$ is a left adjoint of the inverse image functor j_U^{-1} which was introduced in 2.7.

Since $\tilde{i}^*_{\bullet}$ is a right adjoint of the exact functor $\tilde{i}_{\bullet !}$ by Propositions 3.8.3 and 3.8.4, we have the following Corollary. (See [2, V, Proposition 3.4.4].)

3.8.7. Corollary. *Suppose that $\tilde{\eta}^*_{\mathfrak{X}_{\bullet}}$ is exact for any morphism η of I and that, for each object m of I, each connected component of $J(m)$ is cofiltered. Then, for any injective sheaf $\mathfrak{I}_{\bullet}$ of $\mathcal{A}_{\bullet}$-modules, the inverse image $\tilde{i}^*_{\bullet}\mathfrak{I}_{\bullet}$ is an injective sheaf of $\tilde{i}^{-1}_{\bullet}\mathcal{A}_{\bullet}$-modules.*

3.8.8. Example. The first assumption in Corollary 3.8.7 is satisfied in the case where $\mathcal{B}_{\bullet} = \tilde{w}^{-1}_{\bullet}(\mathcal{A}_{\bullet})$ for a morphism $w_{\bullet}: \mathfrak{Y}_{\bullet} \to \mathfrak{X}$ of diagrams of lft-triples (Proposition 3.4.1). In particular, $\mathcal{A}_{\bullet} = \mathbb{Z}_{]\overline{X}_{\bullet}[_{\mathcal{X}_{\bullet}}}$ for any diagram $\mathcal{X}_{\bullet}$ of lft-triples. The second assumption is satisfied if the subcategory J is discrete. ■

3.9. Let $\mathfrak{X}$ be an lft-triple, let $\mathfrak{X}_{\bullet}^{(\Delta^r)^\circ}$ be the constant diagram of triples indexed by $(\Delta^r)^\circ$ and let $\varepsilon_{\bullet}^r: \mathfrak{X}^{(\Delta^r)^\circ} \to \mathfrak{X}$ be a morphism of triples as in Example 3.1.1 (2). We define a morphism

$$\eta^{i,l}_{n_1,\dots,n_r} =$$

$$= \mathrm{id}_{n_1\times\dots\times n_{i-1}} \times \eta^l_{n_i} \times \mathrm{id}_{n_{i+1}\times\dots\times n_r}: (n_1,\dots,n_r) \to (n_1,\dots,n_i+1,\dots,n_r)$$

of Δ^r for $1 \leqq i \leqq r$ and $0 \leqq l \leqq n_i+1$, where $\eta^l_{n_i}$ is as in 1.3.4.

Let $\mathcal{A}$ be a sheaf of rings on $]\overline{X}[_{\mathcal{X}}$. Starting from a complex $(\mathcal{F}_{\bullet}^{\bullet}, \delta_{\bullet}^{\bullet})$ of sheaves of $(\tilde{\varepsilon}_{\bullet}^r)^{-1}\mathcal{A}$-modules, we define an $r+1$-multi complex

$$(\mathcal{C}(\mathcal{F}_{\bullet}^{\bullet}), \partial_0^{\bullet}, \partial_1^{\bullet}, \dots, \partial_r^{\bullet})$$

of sheaves of $\mathcal{A}$-modules as follows. For $(n_0, n_1, \dots, n_r) \in \mathbb{Z}^{r+1}$, we put

$$\mathcal{C}(\mathcal{F}_{\bullet}^{\bullet})^{n_0, n_1, \dots, n_r} = \begin{cases} \mathcal{F}^{n_0}_{n_1,\dots,n_r} & \text{if } n_1,\dots,n_r \geqq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We define a derivation

$$\partial_i^{n_0, n_1, \dots, n_r}: \mathcal{C}(\mathcal{F}_{\bullet}^{\bullet})^{n_0, n_1, \dots, n_r} \to \mathcal{C}(\mathcal{F}_{\bullet}^{\bullet})^{n_0, n_1, \dots, n_{i-1}, n_i+1, n_{i+1}, \dots, n_r}$$

by

$$\partial_i^{n_0, n_1, \dots, n_r} = \begin{cases} \delta^{n_0}_{n_1,\dots,n_r} & \text{if } i=0 \text{ and } n_1,\dots,n_r \geqq 0 \\ \sum_{l=0}^{n_i+1} (-1)^l \mathcal{F}_{\bullet}^{n_0}(\eta^{i,l}_{n_1,\dots,n_r}) & \text{if } i \neq 0 \text{ and } n_1,\dots,n_r \geqq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, this is an $r+1$-multi complex since $\mathcal{F}_{\bullet}^{\bullet}$ is a complex of sheaves on $]\overline{X}_{\bullet}^{(\Delta^\circ)^r}[_{\mathcal{X}_{\bullet}^{(\Delta^\circ)^r}}$. The total complex $(\mathrm{tot}(\mathcal{C}(\mathcal{F}_{\bullet}^{\bullet})), d^{\bullet})$ of $(\mathcal{C}(\mathcal{F}_{\bullet}^{\bullet}), \partial_0^{\bullet}, \partial_1^{\bullet}, \dots, \partial_r^{\bullet})$ is as follows:

$$\mathrm{tot}(\mathcal{C}(\mathcal{F}_{\bullet}^{\bullet}))^n = \prod_{n_0+\dots+n_r=n} \mathcal{C}(\mathcal{F}_{\bullet}^{\bullet})^{n_0, n_1, \dots, n_r}$$

$$d^n = \sum_{n_0+\dots+n_r=n} \sum_{i=0}^{r} (-1)^{n_0+\dots+n_{i-1}} \partial_i^{n_0,\dots,n_r}.$$

Here the sum is well-defined since it is a finite sum on each component, and the sign is the plus if $i=0$. The $r+1$-multi complex $(\mathcal{C}(\mathcal{F}^{\cdot}_{\cdot}), \partial^{\cdot}_0, \partial^{\cdot}_1, \dots, \partial^{\cdot}_r)$ is functorial in $\mathcal{F}^{\cdot}_{\cdot}$ by construction.

3.10. We now introduce a homotopy theory of simplicial triples and tubes. (See [1, $\mathrm{V}^{\mathrm{bis}}$, 3.0].)

Let $\mathfrak{X}$ be a $\mathcal{V}$-triple locally of finite type and let $\mathfrak{Z}_{\cdot}$ and $\mathfrak{Y}_{\cdot}$ be simplicial $\mathfrak{X}$-triples locally of finite type with two commutative diagrams

$$\begin{array}{ccc} \mathfrak{Z}_{\cdot} & \xrightarrow{w_{\cdot}^{(i)}} & \mathfrak{Y}_{\cdot} \\ & v_{\cdot} \searrow \quad \swarrow u_{\cdot} & \\ & \mathfrak{X} & \end{array}$$

for $i=0,1$. We say that a collection of morphisms

$$h_n(\chi)\colon \mathfrak{Z}_n \to \mathfrak{Y}_n$$

over $\mathfrak{X}$ for any object n of Δ and any morphism $\chi: n \to 1$ of Δ is a homotopy from $w_{\cdot}^{(0)}$ to $w_{\cdot}^{(1)}$ if it satisfies the following conditions:

(i) $h_n(\partial_n^{(i)}) = w_n^{(i)}$ for any n and any $i=0,1$, where $\partial_n^{(i)}\colon n \to 1$ is a constant map onto i;

(ii) for any morphism $\eta: m \to n$ of Δ and any map $\chi: n \to 1$ of Δ, the diagram

$$\begin{array}{ccc} \mathfrak{Z}_n & \xrightarrow{h_n(\chi)} & \mathfrak{Y}_n \\ {\scriptstyle \eta_{\mathfrak{Z}_{\cdot}}}\downarrow & & \downarrow{\scriptstyle \eta_{\mathfrak{Y}_{\cdot}}} \\ \mathfrak{Z}_m & \xrightarrow[h_m(\chi\eta)]{} & \mathfrak{Y}_m \end{array}$$

is commutative.

3.10.1. Example. Let $w^{(i)}\colon \mathfrak{Z} \to \mathfrak{Y}$ be a morphism of triples over $\mathfrak{X}$ for $i=0,1$ and let $w_{\cdot}^{(i)}\colon \mathfrak{Z}_{\cdot} \to \mathfrak{Y}_{\cdot}$ be the morphism of the associated Čech diagrams over $\mathfrak{X}$. We define a morphism

$$h_n(\chi)\colon \mathfrak{Z}_n \to \mathfrak{Y}_n$$

over $\mathfrak{X}$ by $(z_0, z_1, \dots, z_n) \mapsto (w_n^{(\chi(0))}(z_0), \dots, w_n^{(\chi(n))}(z_n))$ for an object n and a map $\chi: n \to 1$. Then $\{h_n(\xi)\}_{n,\chi}$ is a homotopy from $w_{\cdot}^{(0)}$ to $w_{\cdot}^{(1)}$. ■

We generalize this example (in the case where $n=0$) to general simplicial triples. We explicitly give the entire proof of the following lemma, a hint of which was given in [1, $\mathrm{V}^{\mathrm{bis}}$, Lemme 3.0.2.4].

3.10.2. LEMMA. *Let n be a nonnegative integer and let $w_{\cdot}^{(i)}: \mathfrak{Z}_{\cdot} \to \mathfrak{Y}_{\cdot}$ $(i=0,1)$ be a morphism of simplicial triples over $\mathfrak{X}$ such that* $\mathrm{cosk}_n^{\mathfrak{X}}(\mathfrak{Y}_{\cdot}^{(n)}) = \mathfrak{Y}_{\cdot}$ *and* $\mathrm{cosk}_n^{\mathfrak{X}}(\mathfrak{Z}_{\cdot}^{(n)}) = \mathfrak{Z}_{\cdot}$ *(see the definition of* $\mathrm{cosk}_n^{\mathfrak{X}}$ *in Example* 3.1.1 (6)). *Suppose that $w_l^{(0)} = w_l^{(1)}$ for any $l < n$ and* $\mathrm{cosk}_n^{\mathfrak{X}}((w_{\cdot}^{(i)})^{(n)}) = w_{\cdot}^{(i)}$ *for $i = 0, 1$. Then there exists a homotopy from $w_{\cdot}^{(0)}$ to $w_{\cdot}^{(1)}$.*

PROOF. Let $\Delta[n]$ be as in 1.3.4, let $\Delta[n]^{\mathrm{mon}}$ be a subcategory of $\Delta[n]$ whose set of morphism consists of monomorphisms, and let $\Delta[n]/l$ (resp. $\Delta[n]^{\mathrm{mon}}/l$) be the category for $\Delta[n] \subset \Delta$ (resp. $\Delta[n]^{\mathrm{mon}} \subset \Delta^{\mathrm{mon}}$) as defined in 3.6. We put $\mathfrak{Y}_{\zeta} = \mathfrak{Y}_a$ and $\mathfrak{Y}'_{\zeta} = \mathfrak{Y}_b$ for a morphism $\zeta : a \to b$ of Δ. We also put

$$\mathfrak{T} = \prod_{(c,\zeta) \in \mathrm{Ob}(\Delta[n]^{\mathrm{mon}}/l)} \mathfrak{Y}_{\zeta}$$

$$\mathfrak{S} = \prod_{\lambda \in \mathrm{Mor}(\Delta[n]^{\mathrm{mon}}/l)} \mathfrak{Y}'_{\lambda},$$

where all the fiber products are over $\mathfrak{X}$, and define morphisms $f^{(i)}: \mathfrak{T} \to \to \mathfrak{S}$ $(i = 1, 2)$ over $\mathfrak{X}$ as follows: if we put $(y^{(i)}_{\lambda:(a,\xi)\to(b,\eta)}) = f^{(i)}((x_{(c,\zeta)}))$, then

$$y^{(1)}_{\lambda:(a,\xi)\to(b,\eta)} = x_{(a,\xi)}$$
$$y^{(2)}_{\lambda:(a,\xi)\to(b,\eta)} = \lambda_{\mathfrak{Y}_{\cdot}}(x_{(b,\eta)}).$$

Then we have an explicit formula

$$\mathfrak{Y}_l = \mathrm{cosk}_n^{\mathfrak{X}}(\mathfrak{Y}_{\cdot}^{(n)})_l = \varprojlim_{(a,\zeta) \in \mathrm{Ob}(\Delta[n]/l)} \mathfrak{Y}_{(a,\zeta)} = \varprojlim_{(a,\zeta) \in \mathrm{Ob}(\Delta[n]^{\mathrm{mon}}/l)} \mathfrak{Y}_{(a,\zeta)} =$$

$$= \text{the fiber product of } f^{(1)}: \mathfrak{T} \to \mathfrak{S} \textit{ and } f^{(2)}: \mathfrak{T} \to \mathfrak{S} \text{ over } \mathfrak{S}$$

of the coskeleton functor in the category of triples over $\mathfrak{X}$. (See [11, Chap. 0, Remarque 1.4.12] in general situations.)

For a morphism $\eta : l \to l'$ of Δ, the map $\eta_{\mathfrak{Y}_{\cdot}}: \mathfrak{Y}_{l'} \to \mathfrak{Y}_l$ is given by $\mathfrak{Y}_{\eta\zeta} \xrightarrow{\mathrm{id}_{\mathfrak{Y}_a}} \mathfrak{Y}_{\zeta}$ for an object (a, ζ) of $\Delta[n]/l$ in each component.

We define a morphism $h_l(\chi)\colon \mathfrak{Z}_l \to \mathfrak{Y}_l$ by

$$h_l(\chi) = \begin{cases} w_l^{(0)} & \text{if } \chi = \partial_l^{(0)} \\ w_l^{(1)} & \text{if } \chi \neq \partial_l^{(0)} \end{cases}$$

for any $l \leqq n$ and $\chi : l \to 1$ and by

$$h_l(\chi) = \varprojlim_{(a,\,\zeta) \in \mathrm{Ob}(\Delta[n]/l)} h_a(\chi\zeta)$$

for any $l > n$ and $\chi : l \to 1$. This inverse limit exists in the category of triples over $\mathfrak{X}$ by the explicit formula. By the assumption, $h_l(\partial_l^{(i)}) = \varprojlim_{(a,\,\zeta)} w_a^{(i)} = w_l^{(i)}$ for $i = 0, 1$ and the condition $\eta_{\mathfrak{Y}.}\, h_l(\chi) = h_{l'}(\chi\eta)\, \eta_{\mathfrak{Z}.}$ follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{Z}_{\eta\zeta} & \xrightarrow{h_a(\chi(\eta\zeta))} & \mathfrak{Y}_{\eta\zeta} \\ {\scriptstyle \mathrm{id}_{\mathfrak{Z}_m}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathfrak{Y}_a}} \\ \mathfrak{Z}_\zeta & \xrightarrow[h_m((\chi\eta)\,\zeta)]{} & \mathfrak{Y}_\zeta \end{array}$$

for an object $(a,\,\zeta)$ of $\Delta[n]/l$ and a morphism $\eta : l \to l'$ of Δ. Hence, $\{h_l(\chi)\}_{l,\,\chi}$ forms a homotopy from $w_.^{(0)}$ to $w_.^{(1)}$. ■

We define a category $H(\Delta)$ as follows:

$\mathrm{Ob}(H(\Delta)) = \mathrm{Ob}(\Delta) \times \{0,\,1\}$;

$$\mathrm{Mor}_{H(\Delta)}((m,i),(n,j)) = \begin{cases} \mathrm{Mor}_\Delta(m,n) & \text{if } i = j \\ \mathrm{Mor}_\Delta(m,n) \times \mathrm{Mor}_\Delta(m,1) & \text{if } (i,j) = (0,1) \\ \emptyset & \text{if } (i,j) = (1,0) \end{cases}.$$

The composition law is given by

$$\zeta(\eta,\chi)\,\xi = (\zeta\eta\xi,\,\chi\xi)$$

for $\xi : (k,\,0) \to (l,0)$, $(\eta,\chi) : (l,\,0) \to (m,1)$ and $\zeta : (m,1) \to (n,1)$, and is given as usual on $\mathrm{Mor}((m,\,i),(n,\,j))$ for $i = j$.

One can easily see that a homotopy induces a diagram of triples indexed by $H(\Delta)^{\circ}$ as follows.

3.10.3. Proposition. *Let $\{h_n(\chi)\}_{n,\,\chi}$ be a homotopy from $w_.^{(0)}\colon \mathfrak{Z}. \to \mathfrak{Y}.$ to $w_.^{(1)}\colon \mathfrak{Z}. \to \mathfrak{Y}.$ over $\mathfrak{X}$. If one defines a contravariant functor*

$$H(\mathfrak{Z}. \rightrightarrows \mathfrak{Y}.)\colon H(\Delta) \to (\text{triples over } \mathfrak{X})$$

by $(m,0)\mapsto \mathfrak{Y}_m$, $(n,1)\mapsto \mathfrak{Z}_n$, $(\xi:(k,0)\to(l,0))\mapsto \xi_{\mathfrak{Y}.}$, $((\eta,\chi):(l,0)\to$ $\to(m,1))\mapsto h_l(\chi)\,\eta_{\mathfrak{Z}.}$ *and* $(\zeta:(m,1)\to(n,1))\mapsto \zeta_{\mathfrak{Z}.}$, *then* $H(\mathfrak{Z}.\rightrightarrows\mathfrak{Y}.)$ *is a diagram of triples over* $\mathfrak{X}$ *indexed by* $H(\varDelta)^0$ *and* $((\mathrm{id}_n,\partial_n^{(i)}):(n,1)\to$ $\to(n,0))\mapsto w_n^{(i)}$.

Now we are given a homotopy $\{h_n(\chi)\}_{n,\chi}$ from $w_.^{(0)}$ to $w_.^{(1)}$. Let $\mathcal{A}$, $\mathcal{B}.$ and $\mathcal{C}.$ be sheaves of rings on $]\overline{X}[_{\mathcal{X}}$, $]\overline{Y}.[_{\mathcal{Y}.}$ and $]\overline{Z}.[_{\mathcal{Z}.}$, respectively, and let compatible homomorphisms of sheaves of rings for $u.$, $v.$ and $w_.^{(i)}$ ($i=$ $=0,1$) (see the notation at the begining of 3.10) also be given.

Let $\mathcal{F}_.^{\cdot}$ (resp. $\mathcal{G}_.^{\cdot}$) be a complexes of sheaves of $\mathcal{B}.$-modules (resp. $\mathcal{C}.$-modules) and let

$$\varphi_.^{(i)^{\cdot}}:(\tilde{w}_.^{(i)})^*\mathcal{F}_.^{\cdot}\to\mathcal{G}_.^{\cdot}$$

be homomorphisms of complexes of sheaves of $\mathcal{C}.$-modules for $i=0,1$. We say that a collection of homomorphisms

$$\theta_n(\chi):\widetilde{h_n(\chi)}^*\mathcal{F}_n^{\cdot}\to\mathcal{G}_n^{\cdot}$$

of $\mathcal{C}_n$-modules for any object n and any morphism $\chi:n\to 1$ of $\varDelta$ is a homotopy from $\varphi_.^{(1)^{\cdot}}$ to $\varphi_.^{(0)^{\cdot}}$ if it satisfies the following conditions:

(i) $\theta_n(\partial_n^{(i)})=\varphi_n^{(i)^{\cdot}}$;

(ii) for any morphisms $\eta:m\to n$ and $\chi:n\to 1$ of $\varDelta$, the diagram

$$\begin{array}{ccc}\tilde{\eta}_{\mathfrak{Z}.}^*\widetilde{h_n(\chi\eta)}^*\mathcal{F}_m\cong\widetilde{h_n(\chi)}^*\tilde{\eta}_{\mathfrak{Y}.}^*\mathcal{F}_m & \xrightarrow{\widetilde{h_n(\chi)}^*(\mathcal{F}.(\eta))} & \widetilde{h_n(\chi)}^*\mathcal{F}_n\\ \tilde{\eta}_{\mathfrak{Z}.}^*(\theta_m(\chi\eta))\downarrow & & \downarrow\theta_n(\chi)\\ \tilde{\eta}_{\mathfrak{Z}.}^*\mathcal{G}_m & \xrightarrow[\mathcal{G}.(\eta)]{} & \mathcal{G}_n\end{array}$$

is commutative.

3.10.4. LEMMA. *With the notation as above, let* $\mathcal{E}^{\cdot}$ *be a complex of sheaves of* $\mathcal{A}$*-modules. If one gives a homomorphism* $\theta_n(\chi):\widetilde{h_n(\chi)}^*(\tilde{u}_n^*\mathcal{E}^{\cdot})\to\tilde{v}_n^*\mathcal{E}^{\cdot}$ *by the induced map from the identity* $u_nh_n(\chi)=v_n$, *then the collection* $\{\theta_n(\chi)\}_{n,\chi}$ *gives a homotopy from* $\varphi_.^{(1)^{\cdot}}=\theta.(\partial_.^{(1)})$ *to* $\varphi_.^{(0)^{\cdot}}=\theta.(\partial_.^{(0)})$.

3.10.5. PROPOSITION. (1) *Let* $\mathcal{F}_.^{\cdot}$ *(resp.* $\mathcal{G}_.^{\cdot}$*) be a complex of sheaves of abelian groups on* $]\overline{Y}[_{\mathcal{Y}}$ *(resp.* $]\overline{Z}[_{\mathcal{Z}}$*) and let* $\{\theta_n(\chi)\}_{n,\chi}$ *be a homotopy* *from* $\varphi^{(1)^{\cdot}}_{\cdot} : (\tilde{w}^{(1)}_{\cdot})^* \mathcal{F}^{\cdot}_{\cdot} \to \mathcal{G}^{\cdot}_{\cdot}$ *to* $\varphi^{(0)^{\cdot}}_{\cdot} : (\tilde{w}^{(0)}_{\cdot})^* \mathcal{F}^{\cdot}_{\cdot} \to \mathcal{G}^{\cdot}_{\cdot}$. *If we define a homomorphism*

$$\tilde{\eta}^{-1}_{\mathfrak{Z}_{\cdot}} \widetilde{h_m(\chi)}^{-1} \mathcal{F}^{\cdot}_m \to \mathcal{G}^{\cdot}_n$$

by the homomorphisms induced from the composition $\mathcal{G}^{\cdot}_n(\eta)\, \tilde{\eta}^*_{\mathfrak{Z}_{\cdot}}(\theta_m(\chi))$ *for* $(\eta : m \to n,\ \chi : m \to 1)$ *of* $H(\Delta)$, *then it determines a sheaf* $H(\mathcal{F}_{\cdot} \rightrightarrows \mathcal{G}_{\cdot})$ *of abelian groups on the tube associated to* $H(\mathfrak{Z}_{\cdot} \rightrightarrows \mathfrak{Y}_{\cdot})$. *In particular, if* $\mathcal{F}_{\cdot}$ *and* $\mathcal{G}_{\cdot}$ *is a sheaf of* $\tilde{u}^{-1}_{\cdot}(\mathcal{A})$*-rings (resp.* $\tilde{v}^{-1}_{\cdot}(\mathcal{A})$*-rings), then* $H(\mathcal{F}_{\cdot} \rightrightarrows \mathcal{G}_{\cdot})$ *is a a sheaf of* $H(\tilde{v}_{\cdot} \rightrightarrows \tilde{u}_{\cdot})^{-1}(\mathcal{A})$*-rings. Here* $H(\tilde{v}_{\cdot} \rightrightarrows \tilde{u}_{\cdot})$ *is the induced structure morphism between tubes.*

(2) *Let* $\mathcal{F}^{\cdot}_{\cdot}$ *(resp.* $\mathcal{G}^{\cdot}_{\cdot}$*) be a complex of sheaves of* $\mathcal{B}_{\cdot}$*-modules (resp.* $\mathcal{C}_{\cdot}$*-modules) and let* $\{\theta_n(\eta)\}_{n,\eta}$ *be a homotopy from* $\varphi^{(1)^{\cdot}}_{\cdot} : (\tilde{w}^{(1)}_{\cdot})^* \mathcal{F}^{\cdot}_{\cdot} \to \mathcal{G}^{\cdot}_{\cdot}$ *to* $\varphi^{(0)^{\cdot}}_{\cdot} : (\tilde{w}^{(0)}_{\cdot})^* \mathcal{F}^{\cdot}_{\cdot} \to \mathcal{G}^{\cdot}_{\cdot}$. *Then* $H(\mathcal{F}^{\cdot}_{\cdot} \rightrightarrows \mathcal{G}^{\cdot}_{\cdot})$ *is a complex of sheaves of* $H(\mathcal{B}_{\cdot} \rightrightarrows \mathcal{C}_{\cdot})$*-modules on the tubes associated to* $H(\mathfrak{Z}_{\cdot} \rightrightarrows \mathfrak{Y}_{\cdot})$.

Assume furthermore that $\mathcal{B}_{\cdot}$ *and* $\mathcal{C}_{\cdot}$ *are coherent. If* $\mathcal{F}_{\cdot}$ *and* $\mathcal{G}_{\cdot}$ *are coherent and* $\theta_n(\chi)$ *is an isomorphism for any* n *and* χ, *then* $H(\mathcal{F}_{\cdot} \rightrightarrows \mathcal{G}_{\cdot})$ *is a sheaf of coherent* $H(\mathcal{B}_{\cdot} \rightrightarrows \mathcal{C}_{\cdot})$*-modules.*

(3) *The converses of* (1) *and* (2) *hold.*

The following Proposition follows from Corollary 3.8.7 and the lemma below.

3.10.6. Proposition. *With the notation as above, let* $\mathcal{J}_{\cdot}$ *be an injective sheaf of* $H(\mathcal{B}_{\cdot} \rightrightarrows \mathcal{C}_{\cdot})$*-modules on the tube associated to the triple* $H(\mathfrak{Z}_{\cdot} \rightrightarrows \mathfrak{Y}_{\cdot})$. *Suppose that* $\tilde{\eta}^*_{H_{\cdot}}$ *is exact for any morphism* η *of* $H(\Delta)$. *(For example,* $\mathcal{B}_{\cdot} = \tilde{u}^{-1}_{\cdot}(\mathcal{A})$ *and* $\mathcal{C}_{\cdot} = \tilde{v}^{-1}_{\cdot}(\mathcal{A})$ *(Example* 3.8.8)).

(1) $\mathcal{J}_{(m,0)}$ *is injective for any object* m.

(2) *The restriction* $\mathcal{J}_{(\cdot,1)}$ *of* $\mathcal{J}_{\cdot}$ *on* $]\overline{Z}_{\cdot}[_{\mathcal{Z}_{\cdot}}$ *is injective.*

3.10.7. Lemma. (1) *Let* $(m, 0)$ *be an object of* $H(\Delta)$. *Then the category* $\Delta \times \{1\}/(m, 0)$ *is empty.*

(2) *Let* $(m, 1)$ *be an object of* $H(\Delta)$. *Then the object* $((m, 1), \mathrm{id}_{(m,1)})$ *is a final object of the category* $\Delta \times \{1\}/(m, 1)$.

Let $\mathcal{F}^{\cdot}_{\cdot}$ (resp. $\mathcal{G}^{\cdot}_{\cdot}$) be complexes of sheaves of $\mathcal{B}_{\cdot}$-modules (resp. $\mathcal{C}_{\cdot}$-modules) and let $\{\theta_n(\eta)\}_{n,\eta}$ be a homotopy from $\varphi^{(1)^{\cdot}}_{\cdot}$ to $\varphi^{(0)^{\cdot}}_{\cdot}$. We construct a homotopy H^n: $\mathrm{tot}(\mathcal{C}(\tilde{u}^{\Delta^\circ}_{\cdot *} \mathcal{F}^{\cdot}_{\cdot}))^n \to \mathrm{tot}(\mathcal{C}(\tilde{v}^{\Delta^\circ}_{\cdot *} \mathcal{G}^{\cdot}_{\cdot}))^{n-1}$ $(n \in \mathbb{Z})$ from $\mathrm{tot}(\mathcal{C}(\tilde{v}^{\Delta^\circ}_{\cdot *}(\varphi^{(1)^{\cdot}}_{\cdot})))$ to $\mathrm{tot}(\mathcal{C}(\tilde{v}^{\Delta^\circ}_{\cdot *}(\varphi^{(0)^{\cdot}}_{\cdot}))$ (see the notation in 3.9). Let ξ^l_n: $n \to n-1$ $(0 \leqq l \leqq n-1)$ be as in 1.3.4 and let χ^l_n: $n \to 1$ $(-1 \leqq l \leqq n)$

be a morphism of $\varDelta$ with $\chi^l_n(l) = 0$ and $\chi^l_n(l+1) = 1$. We define an $\mathcal{A}$-homomorphism

$$(\psi^l_n)^{\cdot} : \tilde{u}_{n*}\mathcal{F}^{\cdot}_n \to \tilde{v}_{n-1*}\mathcal{G}^{\cdot}_{n-1}$$

by the composition

$$\tilde{u}_{n*}\mathcal{F}^{\cdot}_n \to \tilde{v}_{n*}\widetilde{h_n(\chi^l_n)}^*\mathcal{F}^{\cdot}_n \xrightarrow{\tilde{v}_{n*}(\theta_n(\chi^l_n))} \tilde{v}_{n*}\mathcal{G}^{\cdot}_n \xrightarrow{\tilde{\xi}^l_{n*}} \tilde{v}_{n-1*}\mathcal{G}^{\cdot}_{n-1}$$

for $0 \leqq l \leqq n-1$, where the first map is induced by the adjoint and the third map is induced by $v_{n-1} = v_{n_1}\xi^l_n$, and put

$$H^n = \sum_{n_0+n_1=n} \sum_{l=0}^{n_1-1} (-1)^{l+n_0}(\psi^l_{n_1})^{n_0}.$$

Note that $(\psi^l_n)^{\cdot}$ commutes with $\partial^{\cdot}_0$ as defined in 3.9.

3.10.8. Proposition. *With the notation as above, we have*

$$\mathrm{tot}\,(\mathcal{C}(\tilde{v}^{\varDelta^\circ}_{\cdot *}(\varphi^{(1)^{\cdot}}_{\cdot})))^n - \mathrm{tot}\,(\mathcal{C}(\tilde{v}^{\varDelta^\circ}_{\cdot *}(\varphi^{(0)^{\cdot}}_{\cdot})))^n = d^{n-1}H^n + H^{n+1}d^n.$$

4. Čech complexes.

In this section we introduce a Čech complex and a derived Čech complex and discuss several of their properties.

4.1. Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ be an lft-triple, let $\mathfrak{Y}_{\cdot} = (Y_{\cdot}, \overline{Y}_{\cdot}, \mathcal{Y}_{\cdot})$ be an r-simplicial lft-triple over $\mathfrak{X}$, and let us denote by

$$w_{\cdot} : \mathfrak{Y}_{\cdot} \to \mathfrak{X}$$

the structure morphism. We put $w^{(\varDelta^r)^\circ}_{\cdot} : \mathfrak{Y}_{\cdot} \to \mathfrak{X}^{(\varDelta^r)^\circ}_{\cdot}$ and $\varepsilon^r_{\cdot} : \mathfrak{X}^{(\varDelta^r)^\circ}_{\cdot} \to \mathfrak{X}$ to be morphisms of lft-triples as in Example 3.1.1 (2). Then $w_{\cdot} = \varepsilon^r_{\cdot} w^{(\varDelta^r)^\circ}_{\cdot}$.

Let $\mathcal{F}^{\cdot}_{\cdot}$ be a complex of sheaves of $\tilde{w}^{-1}_{\cdot}(j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$-modules and let $\mathcal{C}(\tilde{w}^{(\varDelta^r)^\circ}_{\cdot *}\mathcal{F}^{\cdot}_{\cdot})$ be an $(r+1)$-complex of sheaves of $(\tilde{\varepsilon}^r_{\cdot})^{-1}(j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$-modules defined in 3.9. We define a Čech complex of sheaves for $\mathcal{F}^{\cdot}_{\cdot}$ with respect to $w_{\cdot} : \mathfrak{Y}_{\cdot} \to \mathfrak{X}$ by the total complex

$$\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_{\cdot}; \mathcal{F}^{\cdot}_{\cdot}) = \mathrm{tot}\,(\mathcal{C}(\tilde{w}^{(\varDelta^r)^\circ}_{\cdot *}\mathcal{F}^{\cdot}_{\cdot})).$$

By the definition we have

$$\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\cdot}; \mathcal{F}^{\cdot}_{\cdot})^{n} = \prod_{n_0 + n_1 + \ldots + n_r = n} \widetilde{w}_{n_1, \ldots, n_r *} \mathcal{F}^{n_0}_{n_1, \ldots, n_r}$$

and the coboundary map d^{n} is defined by

$$d^{n} = \sum_{n_0 + \ldots + n_r = n} \sum_{i = 0}^{r} (-1)^{n_0 + \ldots + n_i} \partial_i^{n_0, \ldots, n_r},$$

where $\partial_i^{n_0, \ldots, n_r}$ is as in 3.9. The Čech complex of sheaves is a complex of sheaves of $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules. The functor $\mathcal{C}^{\dagger}$ is functorial in $\mathcal{F}^{\cdot}_{\cdot}$.

4.2. We maintain the notation of 4.1. For a complex $\mathcal{F}^{\cdot}_{\cdot}$ of sheaves of $\widetilde{w}_{\cdot}^{-1}(j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$-modules bounded below, we will define a complex $\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\cdot}; \mathcal{F}^{\cdot}_{\cdot})$, which we call a derived Čech complex of sheaves of $\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}}$-modules for $\mathcal{F}^{\cdot}_{\cdot}$ with respect to $w_{\cdot}: \mathfrak{Y}_{\cdot} \to \mathfrak{X}$, in the derived category $D^{+}(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$ of complexes of sheaves of $\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}}$ bounded below.

First we give a general assertion. Using a standard argument, we have the following propositions by Proposition 3.3.2, Corollary 3.8.7 and Example 3.8.8.

4.2.1. Proposition. *Let $(\mathfrak{Y}_{\cdot}, I)$ be a diagram of lft-triples. and let $\mathcal{A}$ (resp. $\mathcal{B}_{\cdot}$) be a sheaf of rings on $]\overline{X}[_{\mathcal{X}}$ (resp. $]\overline{Y}_{\cdot}[_{\mathcal{Y}_{\cdot}}$) with a homomorphism $\widetilde{w}_{\cdot}^{-1}(\mathcal{A}) \to \mathcal{B}_{\cdot}$ of sheaves of rings.*

(1) *If $\mathcal{F}^{\cdot}_{\cdot}$ is a complex of sheaves of $\mathcal{B}_{\cdot}$-modules bounded below, then there exists an injective resolution*

$$\mathcal{F}^{\cdot}_{\cdot} \to \mathcal{I}^{\cdot}_{\cdot}$$

as sheaves of $\widetilde{w}_{\cdot}^{-1}(\mathcal{A})$-modules (we insist that $\mathcal{I}^{\cdot}_{\cdot}$ be bounded below). Moreover, $\mathcal{F}^{\cdot}_{n} \to \mathcal{I}^{\cdot}_{n}$ is an injective resolution as sheaves of $\widetilde{w}_{n}^{-1}(\mathcal{A})$-modules for any object n of I.

(2) *Let $\varphi^{\cdot}_{\cdot}: \mathcal{F}^{\cdot}_{\cdot} \to \mathcal{G}^{\cdot}_{\cdot}$ be a homomorphism of complexes of sheaves of $\mathcal{B}_{\cdot}$-modules bounded below and let $\mathcal{F}^{\cdot}_{\cdot} \to \mathcal{I}^{\cdot}_{\cdot}$ (resp. $\mathcal{G}^{\cdot}_{\cdot} \to \mathcal{J}^{\cdot}_{\cdot}$) be a quasi-isomorphism (resp. an injective resolution) as complexes of sheaves of $\widetilde{w}_{\cdot}^{-1}(\mathcal{A})$-modules bounded below. Then there exists a homomorphism*

$$\psi^{\cdot}_{\cdot}: \mathcal{I}^{\cdot}_{\cdot} \to \mathcal{J}^{\cdot}_{\cdot}$$

of complexes of sheaves of $\widetilde{w}_{\bullet}^{-1}(\mathcal{A})$*-modules such that the diagram*

$$\begin{array}{ccc} \mathcal{F}_{\bullet}^{\cdot} & \to & \mathcal{I}_{\bullet}^{\cdot} \\ {\scriptstyle \varphi_{\bullet}^{\cdot}}\downarrow & & \downarrow{\scriptstyle \psi_{\bullet}^{\cdot}} \\ \mathcal{G}_{\bullet}^{\cdot} & \to & \mathcal{J}_{\bullet}^{\cdot} \end{array}$$

is commutative. Moreover, if $\psi_{\bullet}^{\cdot} : (\mathcal{I}_{\bullet}^{\cdot}, \partial_{\mathcal{I}}^{\cdot}) \to (\mathcal{J}_{\bullet}^{\cdot}, \partial_{\mathcal{J}}^{\cdot})$ *and* $(\psi')_{\bullet}^{\cdot} : (\mathcal{I}_{\bullet}^{\cdot}, \partial_{\mathcal{I}}^{\cdot}) \to (\mathcal{J}_{\bullet}^{\cdot}, \partial_{\mathcal{J}}^{\cdot})$ *are homomorphisms of complexes of sheaves of* $\widetilde{w}_{\bullet}^{-1}(\mathcal{A})$*-modules which satisfy the commutativity given above, then there exists a homomorphism*

$$H_{\bullet}^{n} : \mathcal{I}_{\bullet}^{n} \to \mathcal{J}_{\bullet}^{n-1}$$

of sheaves of $\widetilde{w}_{\bullet}^{-1}(\mathcal{A})$*-modules for* $n \in \mathbb{Z}$ *with*

$$\psi_{\bullet}^{n} - (\psi')_{\bullet}^{n} = H_{\bullet}^{n+1} \partial_{\mathcal{I}}^{n} + \partial_{\mathcal{J}}^{n-1} H_{\bullet}^{n}.$$

(3) *Let*

$$0 \to \mathcal{F}_{\bullet}^{\cdot} \to \mathcal{G}_{\bullet}^{\cdot} \to \mathcal{H}_{\bullet}^{\cdot} \to 0$$

be an exact sequence of complexes of sheaves of $\mathcal{B}_{\bullet}$*-modules bounded below (for each degree, the short exact sequence is exact). Then there exist injective resolutions of* $\mathcal{F}_{\bullet}^{\cdot} \to \mathcal{I}_{\bullet}^{\cdot}$, $\mathcal{G}_{\bullet}^{\cdot} \to \mathcal{J}_{\bullet}^{\cdot}$ *and* $\mathcal{H}_{\bullet}^{\cdot} \to \mathcal{K}_{\bullet}^{\cdot}$ *as complexes of sheaves of* $\widetilde{w}_{\bullet}^{-1}(\mathcal{A})$*-modules with a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{F}_{\bullet}^{\cdot} & \to & \mathcal{G}_{\bullet}^{\cdot} & \to & \mathcal{H}_{\bullet}^{\cdot} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{I}_{\bullet}^{\cdot} & \to & \mathcal{J}_{\bullet}^{\cdot} & \to & \mathcal{K}_{\bullet}^{\cdot} & \to & 0 \end{array}$$

of $\widetilde{w}_{\bullet}^{-1}(\mathcal{A})$*-modules such that the bottom sequence is exact. Moreover, one can take* $\mathcal{I}_{\bullet}^{\cdot}, \mathcal{J}_{\bullet}^{\cdot}$ *and* $\mathcal{K}_{\bullet}^{\cdot}$ *functorially in given short exact sequences.*

4.2.2. Proposition. *Under the assumption of Proposition* 4.2.1 (2), *assume furthermore that* $I = (\Delta^r)^{\circ}$, *that is,* $\mathcal{Y}_{\bullet}$ *is an r-simplicial triple. Then the homomorphism* $\psi_{\bullet}^{\cdot} : \mathcal{I}_{\bullet}^{\cdot} \to \mathcal{J}_{\bullet}^{\cdot}$ *induces a homomorphism*

$$\mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^{\circ}}(\psi_{\bullet}^{\cdot})) : \mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^{\circ}}\mathcal{I}_{\bullet}^{\cdot}) \to \mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^{\circ}}\mathcal{J}_{\bullet}^{\cdot})$$

of $r+1$*-multi complexes of sheaves of* $\mathcal{A}$*-modules. Moreover, if* $\psi_{\bullet}^{\cdot} : \mathcal{I}_{\bullet}^{\cdot} \to \mathcal{J}_{\bullet}^{\cdot}$ *and* $(\psi')_{\bullet}^{\cdot} : \mathcal{I}_{\bullet}^{\cdot} \to \mathcal{J}_{\bullet}^{\cdot}$ *are homomorphisms of complexes as in Proposi-*

tion 4.2.1 (2), *then there exists a homomorphism*

$$H^n : \mathrm{tot}\,(\mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^\circ}\mathfrak{I}^\bullet_\bullet))^n \to \mathrm{tot}\,(\mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^\circ}\mathfrak{J}^\bullet_\bullet))^{n-1}$$

of sheaves of $\mathcal{A}$*-modules for* $n \in \mathbb{Z}$ *which satisfies the relation*

$$\mathrm{tot}\,(\mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^\circ}(\psi^\bullet_\bullet)))^n - \mathrm{tot}\,(\mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^\circ}((\psi')^\bullet_\bullet)))^n = H^{n+1}d^n_{\mathfrak{I}} + d^{n-1}_{\mathfrak{J}}H^n .$$

Here $d^\bullet_{\mathfrak{I}}$ *(resp.* $d^\bullet_{\mathfrak{J}}$*) is a coboundary map of the total complex* $\mathrm{tot}\,(\mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^\circ}\mathfrak{I}^\bullet_\bullet))$ *(resp.* $\mathrm{tot}\,(\mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^\circ}\mathfrak{J}^\bullet_\bullet))$*) in* 3.9.

PROOF. The first part is clear. Let $H^\bullet_\bullet : \mathfrak{I}^\bullet_\bullet \to \mathfrak{J}^{\bullet -1}_\bullet$ be a homomorphism as in Proposition 4.2.1 (2). We put

$$H^n = \sum_{n_0 + \ldots + n_r = n} \widetilde{w}_{n_1, \ldots, n_r *}(H^{n_0}_{n_1, \ldots, n_r}) .$$

Then one can easily see that H^n $(n \in \mathbb{Z})$ satisfies the desired formula since $H^{n_0}_{n_1, \ldots, n_r}\mathfrak{I}^{n_0}_\bullet(\eta^{i,l}_{n_1, \ldots, n_r}) = \mathfrak{J}^{n_0 - 1}_\bullet(\eta^{i,l}_{n_1, \ldots, n_r})\, H^{n_0}_{n_1, \ldots, n_r}$. ■

4.2.3. PROPOSITION. *Under the assumption of Proposition* 4.2.1 (3), *assume furthermore that* $I = (\Delta^r)^\circ$, *that is,* $\mathfrak{Y}_\bullet$ *is an* r*-simplicial triple. Then the short exact sequence* $0 \to \mathfrak{I}^\bullet_\bullet \to \mathfrak{J}^\bullet_\bullet \to \mathfrak{K}^\bullet_\bullet \to 0$ *induces a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^\circ}\mathcal{F}^\bullet_\bullet) & \to & \mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^\circ}\mathcal{G}^\bullet_\bullet) & \to & \mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^\circ}\mathcal{H}^\bullet_\bullet) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^\circ}\mathfrak{I}^\bullet_\bullet) & \to & \mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^\circ}\mathfrak{J}^\bullet_\bullet) & \to & \mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^\circ}\mathfrak{K}^\bullet_\bullet) & \to & 0
\end{array}$$

of $r+1$*-multi complexes of sheaves of* $\mathcal{A}$*-modules such that the bottom sequence is exact. Moreover, if one can take* $\mathfrak{I}^\bullet_\bullet$, $\mathfrak{J}^\bullet_\bullet$ *and* $\mathfrak{K}^\bullet_\bullet$ *functorially in given short exact sequences, the commutative diagram is also functorial.*

PROOF. Since $\mathfrak{I}^{n_0}_{n_1, \ldots, n_r}$ is injective for any $n_0, \ldots, n_r$ by Proposition 4.2.1, we have $\mathbb{R}^q \widetilde{w}^{n_0}_{n_1, \ldots, n_r *}\mathfrak{I}^{n_0}_{n_1, \ldots, n_r} = 0$ for any $q > 0$. Now the assertion is easy. ■

Now we return to the situation at the beginning of 4.2. Let $\mathcal{F}^\bullet_\bullet$ be a complex of sheaves of $\widetilde{w}^{-1}_\bullet(j^\dagger \mathcal{O}_{]\overline{X}[_{\mathfrak{X}}})$-modules bounded below and let $\mathcal{F}^\bullet_\bullet \to \mathfrak{I}^\bullet_\bullet$ be an injective resolution as sheaves of abelian groups on $]\overline{Y}_\bullet[_{\mathfrak{Y}_\bullet}$. We will define a derived Čech complex $\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\bullet; \mathcal{F}^\bullet_\bullet)$ for $\mathcal{F}^\bullet_\bullet$ with respect to $w_\bullet : \mathfrak{Y}_\bullet \to \mathfrak{X}$ by

$$\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\bullet; \mathcal{F}^\bullet_\bullet) = \mathrm{tot}\,(\mathcal{C}(\widetilde{w}_{\bullet *}^{(\Delta^r)^\circ}\mathfrak{I}^\bullet_\bullet))$$

in the derived category $D^+(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$. The derived Čech complex $\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathcal{F}^\cdot_.)$ is independent of the choices of injective resolutions and functorial in $\mathcal{F}^\cdot_.$ by Proposition 4.2.2.

The homomorphism $\mathcal{F}^\cdot_. \to \mathcal{I}^\cdot_.$ induces a homomorphism

$$\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathcal{F}^\cdot_.) \to \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathcal{F}^\cdot_.)$$

which we call the canonical homomorphism. This homomorphism does not depend on the choice of injective resolutions and it is functorial in $\mathcal{F}^\cdot_.$ in the derived category $D^+(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$.

Though we use an injective resolution as sheaves of abelian groups in the definition of derived Čech complexes, we can take an injective resolution as sheaves of $\tilde{w}^{-1}(j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$-modules by Propositions 4.2.1, 4.2.2 and 4.2.3. Hence we have

4.2.4. Proposition. *With the notation as above, the q-th cohomology sheaf* $\underline{H}^q(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathcal{F}^\cdot_.))$ *is a sheaf of* $j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$*-modules for any q. It is functorial in* $\mathcal{F}^\cdot_.$ *as sheaves of* $j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$*-modules. Functorially for the short exact sequence* $0 \to \mathcal{F}^\cdot_. \to \mathcal{G}^\cdot_. \to \mathcal{H}^\cdot_. \to 0$ *of sheaves of* $\tilde{w}^{-1}(j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$*-modules, one can associate a long exact sequence*

$$\begin{aligned}
0 &\to \underline{H}^0(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathcal{F}^\cdot_.)) \to \underline{H}^0(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathcal{G}^\cdot_.)) \to \underline{H}^0(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathcal{H}^\cdot_.)) \\
&\to \underline{H}^1(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathcal{F}^\cdot_.)) \to \qquad \cdots \\
&\to \underline{H}^q(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathcal{F}^\cdot_.)) \to \underline{H}^q(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathcal{G}^\cdot_.)) \to \underline{H}^q(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathcal{H}^\cdot_.)) \\
&\to
\end{aligned}$$

of sheaves of $j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$*-modules.*

4.3. Let r and s be nonnegative integers with $r \leqq s$ and let

$$(*) \qquad \begin{tikzcd} (\mathfrak{Y}_., (\Delta^r)^\circ) \arrow[d, "v."'] & (\mathfrak{Z}_., (\Delta^s)^\circ) \arrow[l, "{(g_., t)}"'] \arrow[d, "w."] \\ (\mathfrak{W}, \underline{0}) & (\mathfrak{X}, \underline{0}) \arrow[l, "f"] \end{tikzcd}$$

be a commutative diagram of diagrams of lft-triples such that t is a projection to the first r components. We denote by $f_.: \mathfrak{X}^{(\Delta^s)^\circ}_. \to \mathfrak{W}^{(\Delta^r)^\circ}_.$ the morphism of constant simplicial lft-triples induced by f and t.

Let $\mathcal{F}^\cdot_.$ (resp. $\mathcal{G}^\cdot_.$) be a complex of sheaves of $\tilde{v}^{-1}_.(j^\dagger\mathcal{O}_{]\overline{W}[_{\mathcal{W}}})$-modules (resp. $\tilde{w}^{-1}_.(j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$-modules) bounded below and let $\varphi : \tilde{g}^{-1}_.\mathcal{F}^\cdot_. \to \mathcal{G}^\cdot_.$ be a homomorphism of complexes of sheaves of $\tilde{w}^{-1}_.(j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$-modules. Since there exists a natural transform

$$\tilde{f}_{\cdot}^{-1}\tilde{v}_{\cdot *}^{(\Delta^r)^\circ} \to \tilde{w}_{\cdot *}^{(\Delta^s)^\circ}\tilde{g}_{\cdot}^{-1}$$

by Corollary 3.7.3, φ induces a homomorphism

$$\tilde{f}_{\cdot}^{-1}\tilde{v}_{\cdot *}^{(\Delta^r)^\circ}(\mathcal{F}_{\cdot}^{\cdot}) \to (\tilde{w}_{\cdot *})^{(\Delta^s)^\circ}(\tilde{g}_{\cdot}^{-1}\mathcal{F}_{\cdot}^{\cdot}) \xrightarrow{\tilde{w}_{\cdot *}^{(\Delta^s)^\circ}(\varphi)} \tilde{w}_{\cdot *}^{(\Delta^s)^\circ}\mathcal{G}_{\cdot}^{\cdot}$$

of complexes. Hence φ induces a canonical homomorphism

$$\tilde{f}^{-1}\mathcal{C}^\dagger(\mathfrak{W}, \mathfrak{Y}_{\cdot}; \mathcal{F}_{\cdot}^{\cdot}) \to \mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Z}_{\cdot}; \mathcal{G}_{\cdot}^{\cdot})$$

of complexes of $\tilde{f}^{-1}(j^\dagger \mathcal{O}_{]\overline{W}[_{\mathfrak{W}}})$-modules. This canonical homomorphism is functorial in $\mathcal{F}_{\cdot}^{\cdot}$ and $\mathcal{G}_{\cdot}^{\cdot}$. In the case $r = 0$ and $\mathfrak{W} = \mathfrak{X}$ we have a canonical homomorphism

$$\mathcal{F}_{\cdot}^{\cdot} \to \mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Z}_{\cdot}; \mathcal{G}_{\cdot}^{\cdot}).$$

Let $\mathcal{F}_{\cdot}^{\cdot} \to \mathfrak{I}_{\cdot}^{\cdot}$ (resp. $\mathcal{G}_{\cdot}^{\cdot} \to \mathfrak{J}_{\cdot}^{\cdot}$) be an injective resolution as sheaves of abelian groups on $]\overline{Y}_{\cdot}[_{\mathcal{Y}_{\cdot}}$ (resp. $]\overline{Z}_{\cdot}[_{\mathcal{Z}_{\cdot}}$). Since $\tilde{g}_{\cdot}^{-1}$ is exact, there is a homomorphism

$$\tilde{g}_{\cdot}^{-1}\mathfrak{I}_{\cdot}^{\cdot} \to \mathfrak{J}_{\cdot}^{\cdot}$$

of complexes such that the diagram

$$\begin{array}{ccc} \tilde{g}_{\cdot}^{-1}\mathcal{F}_{\cdot}^{\cdot} & \to & \tilde{g}_{\cdot}^{-1}\mathfrak{I}_{\cdot}^{\cdot} \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ \mathcal{G}_{\cdot}^{\cdot} & \to & \mathfrak{J}_{\cdot}^{\cdot} \end{array}$$

is commutative. This homomorphism induces a canonical homomorphism

$$\tilde{f}^{-1}\mathbb{R}\mathcal{C}^\dagger(\mathfrak{W}, \mathfrak{Y}_{\cdot}; \mathcal{F}_{\cdot}^{\cdot}) \to \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Z}_{\cdot}; \mathcal{G}_{\cdot}^{\cdot})$$

in the derived Čech complex $D^+(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$. Indeed, the homomorphism between derived Čech complexes is independent of the choices of injective resolutions, and depends only on φ. It is functorial in $\mathcal{F}_{\cdot}^{\cdot}$ and $\mathcal{G}_{\cdot}^{\cdot}$. The canonical diagram

$$(*\,*)\qquad \begin{array}{ccc} \tilde{f}^{-1}\mathcal{C}^{\dagger}(\mathfrak{W},\,\mathfrak{Y}_{.};\,\mathcal{F}^{\cdot}_{.}) & \rightarrow & \tilde{f}^{-1}\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{W},\,\mathfrak{Y}_{.};\,\mathcal{F}^{\cdot}_{.}) \\ \downarrow & & \downarrow \\ \mathcal{C}^{\dagger}(\mathfrak{X},\,\mathfrak{Z}_{.};\,\mathcal{G}^{\cdot}_{.}) & \rightarrow & \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X},\,\mathfrak{Z}_{.};\,\mathcal{G}^{\cdot}_{.}) \end{array}$$

is commutative.

If we replace $\mathcal{F}^{\cdot}_{.}$ (resp. $\mathcal{G}^{\cdot}_{.}$) with an injective resolution as sheaves of $\widetilde{w}^{-1}(j^{\dagger}\mathcal{O}_{]\overline{W}[_{\mathfrak{W}}})$-modules (resp. as sheaves of $\widetilde{w}^{-1}(j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathfrak{X}}})$-modules), then the diagram $(*\,*)$ remains commutative in the derived category $D^{+}(j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathfrak{X}}})$ of complexes of sheaves of $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathfrak{X}}}$-modules.

4.3.1. Proposition. *Let q, r and s be nonnegative integers with $q \leqq r \leqq s$ and let*

$$\begin{array}{ccccc} (\mathfrak{T}_{.},(\Delta^{q})^{\mathrm{o}}) & \xleftarrow{(g_{.},\,t)} & (\mathfrak{Y}_{.},(\Delta^{r})^{\mathrm{o}}) & \xleftarrow{(g'_{.},\,t')} & (\mathfrak{Z}_{.},(\Delta^{s})^{\mathrm{o}}) \\ {\scriptstyle u_{.}}\downarrow & & \downarrow{\scriptstyle v_{.}} & & \downarrow{\scriptstyle w_{.}} \\ (\mathfrak{S},\underline{0}) & \xleftarrow{f} & (\mathfrak{W},\underline{0}) & \xleftarrow{f'} & (\mathfrak{X},\underline{0}) \end{array}$$

be a commutative diagram of diagrams of lft-triples such that t (resp. t') is a projection to the first q (resp. r) components. Let $\mathcal{F}^{\cdot}_{.}$ (resp. $\mathcal{G}^{\cdot}_{.}$, resp. $\mathcal{H}^{\cdot}_{.}$) be a complex of sheaves of $\tilde{u}_{.}^{-1}(j^{\dagger}\mathcal{O}_{]\overline{S}[_{\mathcal{S}}})$-modules (resp. $\tilde{v}_{.}^{-1}(j^{\dagger}\mathcal{O}_{]\overline{W}[_{\mathcal{X}}})$-modules, resp. $\widetilde{w}_{.}^{-1}(j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$-modules) bounded below, with a homomorphism $\tilde{g}_{.}^{-1}\mathcal{F}^{\cdot}_{.} \rightarrow \mathcal{G}^{\cdot}_{.}$ (resp. $(\tilde{g}'_{.})^{-1}\mathcal{G}^{\cdot}_{.} \rightarrow \mathcal{H}^{\cdot}_{.}$) of complexes of sheaves of $\tilde{v}_{.}^{-1}(j^{\dagger}\mathcal{O}_{]\overline{W}[_{\mathfrak{W}}})$-modules (resp. $\widetilde{w}_{.}^{-1}(j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$-modules). Then the natural diagrams

$$\begin{array}{ccc} (\widetilde{ff'})^{-1}\mathcal{C}^{\dagger}(\mathfrak{S},\,\mathfrak{T}_{.};\,\mathcal{F}^{\cdot}_{.}) & \rightarrow & \tilde{f}'^{-1}\mathcal{C}^{\dagger}(\mathfrak{W},\,\mathfrak{Y}_{.};\,\mathcal{G}^{\cdot}_{.}) \\ \searrow & & \swarrow \\ & \mathcal{C}^{\dagger}(\mathfrak{X},\,\mathfrak{Z}_{.};\,\mathcal{H}^{\cdot}_{.}) & \end{array}$$

$$\begin{array}{ccc} (\widetilde{ff'})^{-1}\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{S},\,\mathfrak{T}_{.};\,\mathcal{F}^{\cdot}_{.}) & \rightarrow & \tilde{f}'^{-1}\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{W},\,\mathfrak{Y}_{.};\,\mathcal{G}^{\cdot}_{.}) \\ \searrow & & \swarrow \\ & \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X},\,\mathfrak{Z}_{.};\,\mathcal{H}^{\cdot}_{.}) & \end{array}$$

are commutative.

4.3.2. LEMMA. *Let*

$$\begin{array}{ccc} \mathfrak{Y}_{\cdot} & \overset{w'_{\cdot}}{\longleftarrow} & \mathfrak{Y}'_{\cdot} \\ {\scriptstyle u_{\cdot}}\downarrow & & \downarrow{\scriptstyle u'_{\cdot}} \\ \mathfrak{X} & \overset{w_{\cdot}}{\longrightarrow} & \mathfrak{X}' \end{array}$$

be a commutative diagram of diagrams of lft-triples such that $\mathfrak{Y}_{\cdot}$ *and* $\mathfrak{Y}'_{\cdot}$ *are r-simplicial triples over* $\mathfrak{X}$ *and* $\mathfrak{X}'$, *respectively. Suppose that* $\widehat{w}_n : \mathcal{X}' \to \mathcal{X}$ (*resp.* $\widehat{w}'_n : \mathcal{Y}'_n \to \mathcal{Y}_n$) *is an open immersion and both* $\overset{\circ}{w}_n$ *and* $\overline{w}_n$ (*resp. both* $\overset{\circ}{w}{}'_n$ *and* $\overline{w}'_n$) *are isomorphisms* (*resp. for any n*). *Then, for any complex* $\mathcal{F}^{\cdot}_{\cdot}$ *of sheaves of* $u_{\cdot}^{-1}(j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$*-modules bounded below, the identification* $]\overline{X}'[_{\mathcal{X}'} =]\overline{X}[_{\mathcal{X}}$ *and* $]\overline{Y}'_n[_{\mathcal{Y}'_n} =]\overline{Y}_n[_{\mathcal{Y}_n}$ *for any n induce an isomorphism*

$$\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\cdot}; \mathcal{F}^{\cdot}_{\cdot}) \xrightarrow{\cong} \mathcal{C}^{\dagger}(\mathfrak{X}', \mathfrak{Y}'_{\cdot}; \mathcal{F}^{\cdot}_{\cdot})$$

of complexes of sheaves of $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$*-modules and an isomorphism*

$$\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\cdot}; \mathcal{F}^{\cdot}_{\cdot}) \xrightarrow{\cong} \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}', \mathfrak{Y}'_{\cdot}; \mathcal{F}^{\cdot}_{\cdot})$$

in $D^{+}(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$.

4.3.3. LEMMA. *Let* $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ *be an lft-triple, let* X' *be an open k-subscheme of* X *and let us put* $\mathfrak{X}' = (X', \overline{X}, \mathcal{X})$. *Let* $\mathfrak{Y}_{\cdot}$ *be an r-simplicial lft-triple over* $\mathfrak{X}'$ *with the structure morphism* $w'_{\cdot} : \mathfrak{Y}_{\cdot} \to \mathfrak{X}'$ *and let* $w_{\cdot} : \mathfrak{Y}_{\cdot} \to \mathfrak{X}$ *be the natural structure morphism. Let* $\mathcal{F}^{\cdot}_{\cdot}$ *be a complex of sheaves of* $(\widetilde{w}'_{\cdot})^{-1}(j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$*-modules (here* $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$ *is a sheaf of rings of overconvergent sections along* $\partial X'$). *We regard* $\mathcal{F}^{\cdot}_{\cdot}$ *as a complex of sheaves of* $\widetilde{w}_{\cdot}^{-1}(j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$*-modules (here* $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$ *is a sheaf of rings of overconvergent sections along* ∂X) *via the natural morphism* $\mathfrak{X}' \to \mathfrak{X}$.

(1) *The natural homomorphism*

$$\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\cdot}; \mathcal{F}^{\cdot}_{\cdot}) \to \mathcal{C}^{\dagger}(\mathfrak{X}', \mathfrak{Y}_{\cdot}; \mathcal{F}^{\cdot}_{\cdot})$$

is an isomorphism of complexes of sheaves of $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$*-modules* (*here* $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$ *is a sheaf of rings of overconvergent sections along* ∂X).

(2) *The natural morphism*

$$\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\cdot}; \mathcal{F}^{\cdot}_{\cdot}) \to \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}', \mathfrak{Y}_{\cdot}; \mathcal{F}^{\cdot}_{\cdot})$$

is an isomorphism in $D^{+}(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$.

PROOF. Note that the tube $]\overline{X}[_{\mathcal{X}}$ with respect to $\mathfrak{X}$ and the tube $]\overline{X}[_{\mathcal{X}}$ with respect to $\mathfrak{X}'$ are isomorphic to each other. The assertion follows from this fact. ■

We give a base change theorem of (derived) Čech complexes for strict open immersions. We often use it in the case where $\mathcal{F}^{\cdot}_{\cdot}$ is a complex of sheaves of $j^{\dagger}\mathcal{O}_{]\overline{Y}.[_{\mathcal{Y}.}}$-modules. Note that $\tilde{g}^{*}_{\cdot} = \tilde{g}^{-1}_{\cdot}$ in the following case.

4.3.4. PROPOSITION. *With the notation as in* (*) *of* 4.3, *assume furthermore that* $\widehat{f}: \mathcal{X} \to \mathcal{W}$ *is an open immersion and* $f: \mathfrak{X} \to \mathfrak{W}$ *is strict as a morphism of triples* (*see* 2.3.3), $\mathfrak{Z}_{\cdot}$ *is a cartesian product of* $\mathfrak{Y}_{\cdot}$ *and* $\mathfrak{X}$ *over* $\mathfrak{W}$, *and* $g_{\cdot}$ *and* $w_{\cdot}$ *are projections* (*hence* $r = s$). *Let* $\mathcal{F}^{\cdot}_{\cdot}$ *be a complex of sheaves of* $\tilde{v}^{-1}_{\cdot}(j^{\dagger}\mathcal{O}_{]\overline{W}[_{\mathcal{W}}})$-*modules bounded below. Then* $\tilde{g}^{-1}_{\cdot}$ *induces a canonical isomorphism*

$$\tilde{f}^{-1}\mathcal{C}^{\dagger}(\mathfrak{W}, \mathfrak{Y}_{\cdot}; \mathcal{F}^{\cdot}_{\cdot}) \xrightarrow{\cong} \mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Z}_{\cdot}; \tilde{g}^{-1}_{\cdot}\mathcal{F}^{\cdot}_{\cdot})$$

as complexes of sheaves of $f^{-1}(j^{\dagger}\mathcal{O}_{]\overline{W}[_{\mathcal{W}}})$-*modules and a canonical isomorphism*

$$\tilde{f}^{-1}\mathbb{R}\,\mathcal{C}^{\dagger}(\mathfrak{W}, \mathfrak{Y}_{\cdot}; \mathcal{F}^{\cdot}_{\cdot}) \xrightarrow{\cong} \mathbb{R}\,\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Z}_{\cdot}; \tilde{g}^{-1}_{\cdot}\mathcal{F}^{\cdot}_{\cdot})$$

in $D^{+}(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$.

PROOF. Since f is a strict open immersion, $g_{\cdot}$ is also. Hence the former homomorphism is an isomorphism by Propositions 2.8.1 and 3.7.4. Since $\tilde{g}_n$ is an open immersion of rigid analytic spaces and the left adjoint $\tilde{g}_{n!}$ of $\tilde{g}_n^{-1}$ is exact for any object n of $(\Delta^{r})^{\circ}$ (see 3.8), the inverse image of injective sheaves by $\tilde{g}^{-1}_{\cdot}$ is injective. Hence the later homomorphism is an isomorphism in $D^{+}(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$ by Proposition 3.7.4. ■

4.4. We define a decreasing filtration for (derived) Čech complexes. Let $\mathfrak{Y}_{\cdot}$ be an s-simplicial lft-triple over an lft-triple $\mathfrak{X}$ with the structure morphism $w_{\cdot}: \mathfrak{Y}_{\cdot} \to \mathfrak{X}$.

Let s' be a nonnegative integer with $s' \leqq s$. Let $\mathcal{F}^{\cdot}_{\cdot}$ be a complex of sheaves of $\tilde{w}^{-1}_{\cdot}(j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$-modules bounded below. We define a decreasing filtration of the $(s+1)$-complex $\mathcal{C}(\tilde{w}_{\cdot *}^{(\Delta^{s})^{\circ}}\mathcal{F}^{\cdot}_{\cdot})$ by

$$(\mathrm{fil}^{q}_{s'}\,\mathcal{C}(\tilde{w}_{\cdot *}^{(\Delta^{s})^{\circ}}\mathcal{F}^{\cdot}_{\cdot}))^{n_0, n_1, \ldots, n_s} = \begin{cases} \mathcal{C}(\tilde{w}_{\cdot *}^{(\Delta^{s})^{\circ}}\mathcal{F}^{\cdot}_{\cdot})^{n_0, n_1, \ldots, n_s} & \text{if } n_1 + \ldots + n_{s'} \geqq q \\ 0 & \text{if } n_1 + \ldots + n_{s'} < q \end{cases}$$

and define a decreasing filtration of $\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\cdot; \mathcal{F}^\cdot_\cdot)$ by

$$\mathrm{fil}^q_{s'} \mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\cdot; \mathcal{F}^\cdot_\cdot) = \mathrm{tot}(\mathrm{fil}^q_{s'} \mathcal{C}(\widetilde{w}^{(\Delta^s)^\circ}_{\cdot *} \mathcal{F}^\cdot_\cdot)).$$

Now let $\mathcal{F}^\cdot_\cdot \to \mathcal{I}^\cdot_\cdot$ be an injective resolution as sheaves of abelian groups on $]\overline{Y}_\cdot[_{\mathcal{Y}_\cdot}$. We define a decreasing filtration of $\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\cdot; \mathcal{F}^\cdot_\cdot)$ by

$$\mathrm{fil}^q_{s'} \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\cdot; \mathcal{F}^\cdot_\cdot) = \mathrm{tot}(\mathrm{fil}^q_{s'} \mathcal{C}(\widetilde{w}^{(\Delta^s)^\circ}_{\cdot *} \mathcal{I}^\cdot_\cdot))$$

in $D^+(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$. The filtration of $\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\cdot; \mathcal{F}^\cdot_\cdot)$ is independent of the choice of injective resolutions by Proposition 4.2.2. These filtrations are functorial in $\mathcal{F}^\cdot_\cdot$, $\mathfrak{X}$ and $\mathfrak{Y}_\cdot$ by 4.3 and 4.3. By Example 3.8.2 (1) and Corollary 3.8.7 the induced homomorphism $\mathcal{F}^\cdot_{(m,\cdot)} \to \mathcal{I}^\cdot_{(m,\cdot)}$ is an injective resolution for all $m \in \mathrm{Ob}((\Delta^{s'})^\circ)$. Hence we have

4.4.1. Lemma. *With the notation as above, there exist a canonical isomorphism*

$$\mathrm{gr}^q_{\mathrm{fil}_{s'}} \mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\cdot; \mathcal{F}^\cdot_\cdot) \cong \prod_{m \in \mathrm{Ob}((\Delta^{s'})^\circ), |m| = q} \mathcal{C}^\dagger(\mathcal{X}, \mathfrak{Y}_{(m,\cdot)}; \mathcal{F}^\cdot_{(m,\cdot)}[-q]$$

of complexes of sheaves of $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules and a canonical isomorphism

$$\mathrm{gr}^q_{\mathrm{fil}_{s'}} \mathbb{R}\mathcal{C}^\dagger(\mathcal{X}, \mathfrak{Y}_\cdot; \mathcal{F}^\cdot_\cdot) \cong \prod_{m \in \mathrm{Ob}((\Delta^{s'})^\circ), |m| = q} \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_{(m,\cdot)}; \mathcal{F}^\cdot_{(m,\cdot)})[-q]$$

in $D^+(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$ for any integer q. Here $|m| = m_1 + \ldots + m_{s'}$. The canonical homomorphism $\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\cdot; \mathcal{F}^\cdot_\cdot) \to \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\cdot; \mathcal{F}^\cdot_\cdot)$ is compatible with the filtration and the q-th graduation of the canonical homomorphism is given by the canonical homomorphism $\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_{(m,\cdot)}; \mathcal{F}^\cdot_{(m,\cdot)})[-q] \to$ $\to \mathbb{R}\mathcal{C}^\dagger(\mathcal{X}, \mathfrak{Y}_{(m,\cdot)}; \mathcal{F}^\cdot_{(m,\cdot)})[-q]$ for each object m of $(\Delta^{s'})^\circ$ with $|m| = q$.

Suppose that $\mathcal{F}^{n_0}_\cdot = 0$ for $n_0 \leqq -r$. Then, for any integer q, we have $\mathrm{fil}^{q+r}_{s'} \mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\cdot; \mathcal{F}^\cdot_\cdot)^q = 0$ (resp. $\mathrm{fil}^{q+r}_{s'} \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\cdot; \mathcal{F}^\cdot_\cdot)^q = 0$). Since the filtration is of finite length at each degree, we have the following convergence of spectral sequences. Here we use injective resolutions as complexes of sheaves of $\widetilde{w}^{-1}_\cdot(j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$-modules.

4.4.2. Lemma. *With the notation as above, there exists a commutative diagram*

$$\begin{array}{ccc}
\underline{E}_1^{qr} = \prod_{m \in \mathrm{Ob}((\Delta^{s'})^{\circ}), |m| = q} \underline{H}^r(\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_{(m,\cdot)}; \mathcal{F}^\cdot_{(m,\cdot)})) & \Rightarrow & \underline{H}^{q+r}(\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\cdot; \mathcal{F}^\cdot_\cdot)) \\
\downarrow & & \downarrow \\
\underline{E}_1^{qr} = \prod_{m \in \mathrm{Ob}((\Delta^{s'})^{\circ}), |m| = q} \underline{H}^r(\mathbb{R}\mathcal{C}^\dagger(\mathcal{X}, \mathfrak{Y}_{(m,\cdot)}; \mathcal{F}^\cdot_{(m,\cdot)})) & \Rightarrow & \underline{H}^{q+r}(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\cdot; \mathcal{F}^\cdot_\cdot))
\end{array}$$

of spectral sequences of sheaves of $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$*-modules. These spectral sequences are functorial in* $\mathcal{F}^\cdot_\cdot$, $\mathfrak{X}$ *and* $\mathfrak{Y}_\cdot$.

4.4.3. Corollary. *With the notation as above, assume furthermore that the natural homomorphism* $\widetilde{w}_{n*}\mathcal{F}^\cdot_n \to \mathbb{R}\widetilde{w}_{n*}\mathcal{F}^\cdot_n$ *is an isomorphism for all* n. *Then the canonical homomorphism*

$$\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\cdot; \mathcal{F}^\cdot_\cdot) \to \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\cdot; \mathcal{F}^\cdot_\cdot)$$

is an isomorphism.

Proof. Since each homomorphism of E_1-terms of spectral sequences in Lemma 4.4.2 (the case $s = s'$) is an isomorphism, the canonical homomorphism is also an isomorphism. ∎

Now we return to the situation in 4.3 and we consider the filtration in 4.4. Let $\mathcal{F}^\cdot_\cdot$ (resp. $\mathcal{G}^\cdot_\cdot$) be a complex of sheaves of $\tilde{v}_\cdot^{-1}(j^\dagger \mathcal{O}_{]\overline{W}[_{\mathfrak{W}}})$-modules (resp. $\widetilde{w}_\cdot^{-1}(j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$-modules) with a homomorphism $g_\cdot^{-1}\mathcal{F}^\cdot_\cdot \to \mathcal{G}^\cdot_\cdot$ of complexes of sheaves of $\widetilde{w}_\cdot^{-1}(j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$-modules. Then the commutative diagram (* *) of 4.3 is compatible with the filtrations $\mathrm{fil}^\cdot_{r'}$ $(r' \leqq r)$ and $\mathrm{fil}^\cdot_{s'}$ $s' \leqq s$ with $r' \leqq s'$. The q-th graduation is given by

(* * *)

$$\begin{array}{ccc}
\prod_{m \in \mathrm{Ob}((\Delta^{r'})^{\circ}), |m| = q} \tilde{f}^{-1}\mathcal{C}^\dagger(\mathfrak{W}, \mathfrak{Y}_{(m,\cdot)}; \mathcal{F}^\cdot_{(m,\cdot)})[-q] & \to & \prod_{m \in \mathrm{Ob}((\Delta^{r'})^{\circ}), |m| = q} \tilde{f}^{-1}\mathbb{R}\mathcal{C}^\dagger(\mathfrak{W}, \mathfrak{Y}_{(m,\cdot)}; \mathcal{F}^\cdot_{(m,\cdot)})[-q] \\
\downarrow & & \downarrow \\
\prod_{n \in \mathrm{Ob}((\Delta^{s'})^{\circ}), |n| = q} \mathcal{C}^\dagger(\mathcal{X}, \mathfrak{Z}_{(n,\cdot)}; \mathcal{G}^\cdot_{(n,\cdot)})[-q] & \to & \prod_{n \in \mathrm{Ob}((\Delta^{s'})^{\circ}), |n| = q} \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Z}_{(n,\cdot)}; \mathcal{G}^\cdot_{(n,\cdot)})[-q],
\end{array}$$

where each homomorphism is the canonical homomorphism. These filtrations induce a commutative diagram of spectral sequences in Lemma 4.4.2.

4.4.4. PROPOSITION. *With the notation as above, assume furthermore that* $\mathfrak{W} = \mathfrak{X}$ *and* $r = r' = s'$. *If the canonical homomorphism* $\mathcal{F}_m^{\bullet} \to \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{Y}_m, \mathfrak{Z}_{(m,\bullet)}; \mathcal{G}_{(m,\bullet)}^{\bullet})$ *is an isomorphism in* $D^+(\mathbb{Z}_{]\overline{Y}_m[\mathcal{Y}_m})$ *for every object* m *of* $(\Delta^r)^{\circ}$, *then the canonical homomorphism*

$$\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\bullet}; \mathcal{F}_{\bullet}^{\bullet}) \to \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Z}_{\bullet}; \mathcal{G}_{\bullet}^{\bullet})$$

is an isomorphism in $D^+(\mathbb{Z}_{]\overline{X}[\mathcal{X}})$.

PROOF. Note that $\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{W}, \mathfrak{Y}_{(m,\bullet)}; \mathcal{F}_m^{\bullet}) = \mathbb{R}\tilde{v}_{m*}\mathcal{F}_m^{\bullet}$. The assertion easily follows from the fact that the right vertical arrow in the diagram $(***)$ above arises from the canonical isomorphism $\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Z}_{(m,\bullet)}; \mathcal{G}_{(m,\bullet)}^{\bullet}) \cong \mathbb{R}\tilde{v}_{m*}\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{Y}_m, \mathfrak{Z}_{(m,\bullet)}; \mathcal{G}_{(m,\bullet)}^{\bullet})$. ■

4.4.5. PROPOSITION. *Let* $\mathfrak{X}$ *be a* $\mathcal{V}$*-triple locally of finite type and let* $w_{\bullet}: \mathfrak{Z}_{\bullet} \to \mathfrak{Y}_{\bullet}$ *be a morphism of r-simplicial lft-triples over* $\mathfrak{X}$ *such that, for each* m,

(i) $\mathring{w}_m: \coprod_{\alpha} Z_{m,\alpha} \to Y_m$ *is a finite Zariski covering;*

(ii) $\mathfrak{Z}_m = \coprod_{\alpha} (Z_m, \alpha, \overline{Z}_m, \mathcal{Z}_m)$, *and* $\overline{w}_m$ *and* $\widehat{w}_m$ *are the identities on each component.*

Let us denote by $\mathfrak{U}_{\bullet}$ (*resp.* $v_{\bullet}: \mathfrak{U}_{\bullet} \to \mathfrak{Y}_{\bullet}$) *the Čech diagram for* $w_{\bullet}$ *which is an* $(r+1)$*-simplicial triple defined in Example* 3.1.2 (*resp. the induced natural morphism of diagrams of triples*). *Then, for any complex* $\mathcal{F}_{\bullet}^{\bullet}$ *of sheaves of* $\mathcal{O}_{]\overline{Y}_{\bullet}[\mathcal{Y}_{\bullet}}$*-modules, the canonical homomorphism*

$$\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\bullet}; j^{\dagger}\mathcal{F}_{\bullet}^{\bullet}) \to \mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{U}_{\bullet}; v_{\bullet}^{\dagger}\mathcal{F}_{\bullet}^{\bullet})$$

induced by $v_{\bullet}$ *is a quasi-isomorphism and the canonical homomorphism*

$$\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\bullet}; j^{\dagger}\mathcal{F}_{\bullet}^{\bullet}) \to \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{U}_{\bullet}; v_{\bullet}^{\dagger}\mathcal{F}_{\bullet}^{\bullet})$$

induced by $v_{\bullet}$ *is an isomorphism in* $D^+(\mathbb{Z}_{]\overline{X}[\mathcal{X}})$.

PROOF. The assertion easily follows from Proposition and Proposition 4.4.4. ■

4.5. Let $\mathfrak{X}$ be an lft-triple and let $\mathfrak{Y}_{\bullet}$ be an s-simplicial lft-triple over $\mathfrak{X}$ with a structure morphism $w_{\bullet}$. Let $\mathcal{F}_{\bullet}^{\bullet}$ be a complex of sheaves of $\widetilde{w}_{\bullet}^{-1}(j^{\dagger}\mathcal{O}_{]\overline{X}[\mathcal{X}})$-modules bounded below with a decreasing filtration

$\{\mathrm{Fil}^q \mathscr{F}^\cdot_\cdot\}_q$ which satisfies the conditions: (i) $\mathrm{Fil}^q \mathscr{F}^\cdot_\cdot = \mathscr{F}^\cdot_\cdot$ for $q \ll 0$ and (ii) there exists an integer s such that $\mathrm{Fil}^q \mathscr{F}^r_\cdot = 0$ for all $r - q < s$.

4.5.1. LEMMA. *With the notation as above, there exists an injective resolution*

$$\mathscr{F}^\cdot_\cdot \to \mathfrak{I}^\cdot_\cdot$$

as decreasing filtered complexes of sheaves of abelian groups on $]\overline{Y}_\cdot[_{\mathcal{Y}_\cdot}$ *which satisfies the following conditions:*

(i) $\mathrm{Fil}^q \mathfrak{I}^\cdot_\cdot = \mathfrak{I}^\cdot_\cdot$ *for* $q \ll 0$;

(ii) $\mathrm{Fil}^q \mathfrak{I}^r_\cdot = 0$ *for all* $r - q < s$;

(iii) *the induced homomorphism*

$$\mathrm{Gr}^q_{\mathrm{Fil}} \mathscr{F}^\cdot_\cdot \to \mathrm{Gr}^q_{\mathrm{Fil}} \mathfrak{I}^\cdot_\cdot$$

is an injective resolution for all q.

The injective resolution as above is independent of the choice in the derived category $D^+(\mathbb{Z}_{]\overline{Y}_\cdot[_{\mathcal{Y}_\cdot}})$. *Moreover, the injective resolution as above is functorial in* $\mathscr{F}^\cdot_\cdot$, $\mathfrak{X}$ *and* $\mathfrak{Y}_\cdot$ *in* $D^+(\mathbb{Z}_{]\overline{Y}_\cdot[_{\mathcal{Y}_\cdot}})$.

PROOF. Let $\mathrm{Gr}^q_{\mathrm{Fil}} \mathscr{F}^\cdot_\cdot \to \mathfrak{J}(q)^\cdot_\cdot$ be an injective resolution as sheaves of abelian groups on $]\overline{Y}_\cdot[_{\mathcal{Y}_\cdot}$ such that $\mathfrak{J}(q)^r_\cdot = 0$ for any $r - q < s$. If $\mathrm{Gr}^q_{\mathrm{Fil}} \mathscr{F}^\cdot_\cdot = 0$, we put $\mathfrak{J}(q)^\cdot_\cdot = 0$. We fix a homomorphism $\mathscr{F}^\cdot_\cdot \to \mathfrak{J}(q)^\cdot_\cdot$ of complexes for all q such that it induces the injective resolution $\mathrm{Gr}^q_{\mathrm{Fil}} \mathscr{F}^\cdot_\cdot \to \mathfrak{J}(q)^\cdot_\cdot$. Such a homomorphism exists by the condition (i) of $\mathscr{F}^\cdot_\cdot$ and the choice of $\mathfrak{J}(q)^\cdot_\cdot$. We define a complex $\mathfrak{I}^\cdot_\cdot$ of sheaves of abelian groups on $]\overline{Y}_\cdot[_{\mathcal{Y}_\cdot}$ by

$$\mathfrak{I}^\cdot_\cdot = \prod_q \mathfrak{J}(q)^\cdot_\cdot$$

and define a decreasing filtration of $\mathfrak{I}^\cdot_\cdot$ by

$$\mathrm{Fil}^q \mathfrak{I}^\cdot_\cdot = \prod_{q'} \geqq q\, \mathfrak{J}(q')^\cdot_\cdot$$

for any q. Then $\mathrm{Gr}^q_{\mathrm{Fil}} \mathfrak{I}^\cdot_\cdot = \mathfrak{J}(q)^\cdot_\cdot$. Since a direct product of injective sheaves is also injective, $\mathfrak{I}^r_\cdot$ is injective for all r. By the universal property of direct products the fixed system of homomorphisms induces a homomorphism

$$\mathscr{F}^\cdot_\cdot \to \mathfrak{I}^\cdot_\cdot$$

of filtered complexes. Since $\mathfrak{I}^\cdot_\cdot$ is a finite direct product at each degree, the homomorphism above satisfies the conditions (i)-(iii) by construction. The natural homomorphism $\underline{H}^r(\mathcal{F}^{\bullet}_{\bullet}) \to \underline{H}^r(\mathcal{F}^{\bullet}_{\bullet}/\mathrm{Fil}^q\mathcal{F}^{\bullet}_{\bullet})$ (resp. $\underline{H}^r(\mathfrak{I}^{\bullet}_{\bullet}) \to \to \underline{H}^r(\mathfrak{I}^{\bullet}_{\bullet}/\mathrm{Fil}^q\mathfrak{I}^{\bullet}_{\bullet})$) is an isomorphism for $r < q + s - 1$ by the conditions (i)-(iii) for any q. Hence the homomorphism $\mathcal{F}^{\bullet}_{\bullet} \to \mathfrak{I}^{\bullet}_{\bullet}$ gives an injective resolution. The independence follows from Proposition 4.2.1 and the functorialities follow from 4.2 and 4.3. ■

Since the filtration in the injective resolution given by lemma 4.5.1 is, at each degree, made by finite product we have the following convergence of spectral sequences. Here we use injective resolutions as complexes of sheaves of $\widetilde{w}_{\bullet}^{-1}(j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$-modules.

4.5.2. Proposition. *With the notation at the beginning of* 4.5, *there exists a spectral sequence*

$$\underline{E}_1^{qr} = \underline{H}^{q+r}(\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\bullet}; \mathrm{Gr}^q_{\mathrm{Fil}}\mathcal{F}^{\bullet}_{\bullet})) \Rightarrow \underline{H}^{q+r}(\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\bullet}; \mathcal{F}^{\bullet}_{\bullet}))$$

of sheaves of $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules. This spectral sequence does not depend on the choice of injective resolutions as above. Moreover, the spectral sequence is functorial in $\mathcal{F}^{\bullet}_{\bullet}$, $\mathfrak{X}$ and $\mathfrak{Y}_{\bullet}$.

5. A vanishing theorem.

In this section we prove a vanishing theorem for the higher cohomology of sheaves of coherent $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules.

5.1. First we prove the following proposition.

5.1.1. Proposition. *Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ be a $\mathcal{V}$-triple of finite type such that $\mathcal{X}$ is affine. Choose a complement ∂X of X in $\overline{X}$ and pick lifts $g_1, \dots, g_s \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ of the generators of the ideal of definition of ∂X in $\overline{X}$ as in . Suppose that W is an open affinoid subvariety in $U_{g_i}^{\geqq\delta}$ for some i and for some $\delta \in \sqrt{|K|} \cap]0, 1[$ (see the notation in* 2.6). *If E is a sheaf of coherent $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules, then we have*

$$H^q(W, E) = 0$$

for any $q > 0$.

Proof. Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_U$-modules for a strict neighbourhood U of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ with $E \cong j_U^{\dagger}\mathcal{E}$. Then there exists a real number $\delta \in \sqrt{|K|} \cap]0, 1[$ such that $W \cap U_{\mathfrak{X}, \partial X}^{\geqq\delta} \subset W \cap U$ by Lemma 2.6.6 (2). By

Proposition 2.6.8 we have

$$H^q(W, E) \cong \lim_{\nu \to 1^-} H^q(W, j_{U \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X}*} \mathcal{E}).$$

Suppose that $\nu \geqq \delta$. Since $W \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X}$ is affinoid (Lemma 2.6.1) and $\mathcal{E}$ is coherent on $W \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X}$, $\mathbb{R}j^W_{U \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X}*} \mathcal{E} = j^W_{U \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X}*} \mathcal{E}$ by Tate's acyclicity theorem [21, Theorem 8.7]. Hence we have

$$H^q(W, j_{U \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X}*} \mathcal{E}) = H^q(W \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X}, \mathcal{E}) = 0 \qquad (q > 0)$$

by Tate's acyclicity theorem. This completes the proof of the proposition. ■

The following result is contained in Berthelot's unpublished note [6]. By Proposition 2.7.2 (2) we have

5.1.2. Corollary. *Let $\mathfrak{X}$ be a $\mathcal{V}$-triple of finite type such that $\mathcal{X}$ is affine and some complement ∂X of X in $\overline{X}$ is a hypersurface, that is, $s = 1$ in Proposition 5.1.1, and let W be an open affinoid subvariety in $]\overline{X}[_{\mathcal{X}}$. If E is a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules, then we have*

$$H^q(W, E) = 0$$

for any $q > 0$.

5.2. We give some relative vanishing theorems.

5.2.1. Theorem. *Let $w : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of $\mathcal{V}$-triples locally of finite type such that*

(i) *all of $\overset{\circ}{w} : Y \to X$, $\overline{w} : \overline{Y} \to \overline{X}$ and $\widehat{w} : \mathcal{Y} \to \mathcal{X}$ are open immersions;*

(ii) *$\widehat{w} : \mathcal{Y} \to \mathcal{X}$ is affine.*

If E is a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}$-modules, then we have

$$\mathbb{R}^q \widetilde{w}_* E = 0$$

for any $q > 0$.

Before proving Theorem 5.2.1, we recall the notion of quasi-stein spaces in [16, Definition 2.3]. Let X be a rigid analytic space over $\operatorname{Spm} K$. X is quasi-stein if there is a countable increasing affinoid admissible covering $\{X_n\}_n$ of X such that the image of the restriction map $\Gamma(X_{n+1}, \mathcal{O}_X) \to \Gamma(X_n, \mathcal{O}_X)$ is dense for any n. One can easily see that

(1) an intersection of a finite number of quasi-stein subspaces in a separated rigid analytic space is quasi-stein;

(2) a rigid analytic space is affinoid if and only if it is quasi-stein and quasi-compact.

Let $\mathcal{P}$ be an affine formal scheme over Spf $\mathcal{V}$ and let Z be a k-closed subscheme of $\mathcal{P} \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}\, k$. We fix a system $f_1, \ldots, f_r \in \Gamma(\mathcal{P}, \mathcal{O}_{\mathcal{P}})$ of lifts of generators of the ideal of definition of Z in $\mathcal{P} \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}\, k$ and an increasing sequence $\eta = (\eta_n)_{n \geqq 0}$ ($\eta_n \in \sqrt{|K|} \cap]0, 1[$) with $\eta_n \to 1^-$ for $n \to \infty$. If we put

$$[Z]_{\mathcal{P}, \eta_n} = \{z \in]Z[_{\mathcal{P}} \,|\, |f_i(z)| \leqq \eta_n \text{ for any } i\},$$

then the tube $]Z[_{\mathcal{P}}$ is a quasi-stein space with an admissible covering $\{[Z]_{\mathcal{P}, \eta_n}\}_n$.

Proof of Theorem 5.2.1. Since the problem is local on $\mathcal{X}$, we may assume that $\mathcal{X}$ is affine. Then $\mathcal{Y}$ is affine by the assumption. Let us take a complement S (resp. T) of $\overline{Y}$ (resp. Y) in $\overline{X}$ and fix a system $f_1, \ldots, f_r \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ (resp. $g_1, \ldots, g_s \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$) of generators of the ideal of definition of S (resp. T) in $\overline{X}$. Note that the images of $g_1, \ldots, g_s \in \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ form a system of lifts of generators of the ideal of definition of the complement $\partial Y = T \cap \overline{Y}$ of Y in $\overline{Y}$.

Let W be an open affinoid subvariety of $]\overline{X}[_{\mathcal{X}}$. Since

$$\widetilde{w}^{-1}(W) = W \cap]\overline{Y}[_{\mathcal{Y}},$$

$\widetilde{w}^{-1}(W)$ is quasi-stein. Moreover, we have

$$\widetilde{w}^{-1}(W) = \{x \in W \,|\, |f_i(x)| = 1 \text{ for some } i\}.$$

In other words, $\widetilde{w}^{-1}(W)$ is a finite union of affinoid varieties. Hence it is quasi-compact. Therefore, $\widetilde{w}^{-1}(W)$ is an open affinoid subvariety of $]\overline{Y}[_{\mathcal{Y}}$.

Now we fix $\delta \in \sqrt{|K|} \cap]0, 1[$. Let $U^{\geqq \delta}_{\mathfrak{Y}, g_i}$ and $U^{\leqq \delta}_{\mathfrak{Y}, \partial Y}$ be admissible open subsets in $]\overline{Y}[_{\mathcal{Y}}$ for $\mathfrak{Y}$ with respect to g_i and ∂Y as in 2.6. We define admissible open subsets $V^{\geqq \delta}_{\mathfrak{X}, g_i}$ and $V^{\leqq \delta}_{\mathfrak{X}, T}$ by

$$V^{\geqq \delta}_{\mathfrak{X}, g_i} = \{x \in]\overline{X}[_{\mathcal{X}} \,|\, |g_i(x)| \geqq \delta, |g_i(x)| \geqq |g_{i'}(x)| \text{ for all } i' \text{ with } 1 \leqq i' \leqq s\}$$

$$V^{\leqq \delta}_{\mathfrak{X}, T} = \{x \in]\overline{X}[_{\mathcal{X}} \,|\, |g_i(x)| \leqq \delta \text{ for all } i \text{ with } 1 \leqq i \leqq s\}.$$

Then $\{V^{\geqq\delta}_{\mathfrak{X},g_i}\}_i \cup \{V^{\leqq\delta}_{\mathfrak{X},T}\}$ is an admissible covering of $]\overline{X}[_{\mathcal{X}}$, and we have

$$U^{\geqq\delta}_{\mathfrak{Y},g_i} = V^{\geqq\delta}_{\mathfrak{X},g_i} \cap]\overline{Y}[_{\mathcal{Y}}$$
$$U^{\leqq\delta}_{\mathfrak{Y},\partial Y} = V^{\leqq\delta}_{\mathfrak{X},T} \cap]\overline{Y}[_{\mathcal{Y}}.$$

Since $\mathbb{R}^q\widetilde{w}_*E$ is a sheaf associated to the presheaf

$$W \mapsto H^q(\widetilde{w}^{-1}(W), E)$$

for any admissible open subset W in $]\overline{X}[_{\mathcal{X}}$, it is sufficient to prove $H^q(\widetilde{w}^{-1}(W), E) = 0$ for any $q > 0$ and for any open affinoid subvariety W in $V^{\geqq\delta}_{\mathfrak{X},g_i}$ for some g_i. Indeed, if $W \subset V^{\leqq\delta}_{\mathfrak{X},T}$, then $E|_{W\cap]\overline{Y}[_{\mathcal{Y}}} = 0$ by Proposition 2.7.2 (2). If W is an open affinoid subvariety in $V^{\geqq\delta}_{\mathfrak{X},g_i}$, $\widetilde{w}^{-1}(W)$ is an open affinoid subvariety in $U^{\geqq\delta}_{\mathfrak{Y},g_i}$. The vanishing follows from 5.1.1. ■

5.2.2. Theorem. *Let $w : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of lft-triples such that w is strict as a morphism of triples and $\widehat{w}: \mathcal{Y} \to \mathcal{X}$ is affine. Then, for any sheaf of coherent $j^\dagger\mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}$-modules E, we have*

$$\mathbb{R}^q\widetilde{w}_*E = 0$$

for any $q > 0$.

Proof. Since the problem of vanishing is local on $\mathcal{X}$, we may assume that $\mathcal{X}$ is affine. Then $\mathcal{Y}$ is also affine by the assumption. Let us take a complement ∂X (resp. $\partial Y = \overline{w}^{-1}(\partial X)$) of X (resp. Y) in $\overline{X}$ (resp. $\overline{Y}$). We denote by $U^{\leqq\nu}_{\mathfrak{X},\partial X}$ and $U^{\leqq\nu}_{\mathfrak{Y},\partial Y}$ the admissible open subsets for $\mathfrak{X}$ and $\mathfrak{Y}$ with respect to ∂X and ∂Y as in 2.6, respectively. Then $\widetilde{w}^{-1}(U^{\geqq\nu}_{\mathfrak{X},\partial X}) = U^{\geqq\nu}_{\mathfrak{Y},\partial Y}$ and $\widetilde{w}^{-1}(U^{\leqq\nu}_{\mathfrak{X},\partial X}) = U^{\leqq\nu}_{\mathfrak{Y},\partial Y}$ by Lemma 2.6.3.

If W is an open affinoid subvariety of $]\overline{X}[_{\mathcal{X}}$, then $\widetilde{w}^{-1}(W) = W \times_{\mathcal{X}_K^{\mathrm{an}}} \mathcal{Y}_K^{\mathrm{an}}$ is also affinoid since w is strict as a morphism of triples ($\overline{X}$ and $\overline{Y}$ are defined by same equations in $\mathcal{X}$ and $\mathcal{Y}$, respectively). Now the assertion easily follows from Proposition 5.1.1. ■

6. Universally cohomological descent.

In this section we introduce a notion of universally cohomological descent for a morphism of triples.

6.1. We introduce a notion of «exact» for morphisms of diagrams of triples.

6.1.1. DEFINITION. (1) Let $w_.: \mathfrak{Y}_. \to \mathfrak{X}_.$ be a morphism of diagrams of lft-triples. We say that $w_.$ is exact (resp. faithful) if the functor $w_.^\dagger : \mathrm{Coh}(j^\dagger \mathcal{O}_{]\overline{X}_.[_{\mathcal{X}_.}}) \to \mathrm{Coh}(j^\dagger \mathcal{O}_{]\overline{Y}_.[_{\mathcal{Y}_.}})$ (see the definition in 3.3 and 3.5) is exact (resp. faithful).

(2) We say that a morphism $\mathfrak{Y} \to \mathfrak{X}$ of lft-triples is universally exact if, for any morphism $\mathfrak{Z} \to \mathfrak{X}$ of lft-triples, the base change morphism $\mathfrak{Y} \times \times_{\mathfrak{X}} \mathfrak{Z} \to \mathfrak{Z}$ is exact. Here w is regarded as a morphism of diagrams of triples indexed by the category $\underline{0}$ (Example 3.1.1 (1)). ∎

In general, $w_.^\dagger$ is a right exact functor by 2.10.1.

6.1.2. LEMMA. *Let $u : \mathfrak{Y} \to \mathfrak{X}$ and $w : \mathfrak{Z} \to \mathfrak{Y}$ be morphisms of lft-triples and put $v = uw$.*

(1) *If u and w are exact (resp. universally exact, resp. faithful), then v is also so.*

(2) *If w is exact and faithful and v is exact, then u is exact.*

PROOF. (1) is easy. (2) Let $\eta : E \to F$ be an injective homomorphism of sheaves of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules and let G be a kernel of $u^\dagger(\eta)$. Note that G is a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}$-modules. Since v is exact, $w^\dagger u^\dagger(\eta) = v^\dagger(\eta)$ is injective. Since w is exact, $w^\dagger G = 0$. The faithfulness of w implies $G = 0$. ∎

By definition we have

6.1.3. LEMMA. *Let $w : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of lft-triples. Suppose that w is universally exact. Then the Čech diagram $w_.: \mathfrak{Y}_. \to \mathfrak{X}$ is exact.*

6.1.4. PROPOSITION. *Let $\mathcal{V}$ be an object in* $\mathrm{CDVR}_{\mathbb{Z}_p}$ *with a uniformizer π. Suppose that $w : \mathfrak{Y} \to \mathfrak{X}$ is a morphism of $\mathcal{V}$-triples such that $\mathcal{Y} \times \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spec}\ \mathcal{V}/\pi^{e+1} \to \mathcal{X} \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spec}\ \mathcal{V}/\pi^{e+1}$ induced from $\widehat{w}$ is flat around Y for every e. Then w is universally exact. In particular, if $\widehat{w} : \mathcal{Y} \to \mathcal{X}$ is flat around Y, that is, there exists an open formal subscheme $\mathcal{U}$ of $\mathcal{Y}$ which contains Y such that $\widehat{w}\,|_{\mathcal{U}} : \mathcal{U} \to \mathcal{X}$ is flat, then w is universally exact.*

PROOF. Since the situation is unchanged after any base change of morphisms of lft-triples, it is sufficient to prove the exactness. Since the restriction functor j^* is exact and faithful by Lemma 2.9.1, we have only to prove the assertion in the case where $\overline{X} = X$ and $\overline{Y} = Y$ by Lemma

6.1.2. The assertion is local both on $\mathcal{X}$ and $\mathcal{Y}$, so that we may assume that both $\mathcal{X}$ and $\mathcal{Y}$ are affine, say $\mathcal{X} = \operatorname{Spf} \mathcal{A}$ and $\mathcal{Y} = \operatorname{Spf} \mathcal{B}$. Let $W = = \operatorname{Spm} \mathcal{A}\{t_1, \ldots, t_n\}/I \otimes_{\mathcal{V}} K$ be a rational subdomain of $]\overline{X}[_{\mathcal{X}}$. (See the notation in 2.2.) Then any sheaf of coherent $\mathcal{O}_W$-modules (resp. any homomorphism of sheaves of coherent $\mathcal{O}_W$-modules) arises from a finite $\mathcal{A}\{t_1, \ldots, t_n\}/I$-module (resp. a homomorphism of finite $\mathcal{A}\{t_1, \ldots, t_n\}/I$-modules) [8, 9.4.3 Theorem 3]. Since the localization $\widetilde{w}^{-1}(W) \to \to \mathcal{B}\{t_1, \ldots, t_n\}/I \otimes_{\mathcal{V}} K$ is flat [8, 7.3.2 Corollary 6], the flatness of $\mathcal{A}/\pi^{e+1} \to \mathcal{B}/\pi^{e+1}$ for every e implies the exactness of $w^{\dagger}$. ■

6.2. We introduce new notions: «cohomological descent» and «universally cohomological descent».

6.2.1. Definition. Let $\mathfrak{X}$ be a $\mathcal{V}$-triple locally of finite type and let $\mathfrak{Y}_{\cdot}$ be a simplicial $\mathfrak{X}$-triple locally of finite type with the structure morphism $w_{\cdot} : \mathfrak{Y}_{\cdot} \to \mathfrak{X}$.

(1) We say that $w_{\cdot}$ is cohomologically descendable if the following conditions (i) and (ii) are satisfied.

(i) $w_{\cdot}$ is exact.

(ii) For any sheaf E of coherent $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules, the canonical homomorphism

$$E \to \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\cdot}; w_{\cdot}^{\dagger} E)$$

is an isomorphism in $D^{+}(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$.

(2) We say that $w_{\cdot}$ is universally cohomologically descendable if, for any morphism $\mathfrak{Z} \to \mathfrak{X}$ of lft-triples, the base change morphism $\mathfrak{Y}_{\cdot} \times_{\mathfrak{X}} \mathfrak{Z} \to \to \mathfrak{Z}$ is cohomologically descendable. ■

6.2.2. Definition. Let $w : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of $\mathcal{V}$-triples locally of finite type and let $w_{\cdot} : \mathfrak{Y}_{\cdot} \to \mathfrak{X}$ be the Čech diagram for w as in Example 3.1.1 (4). We say that w is cohomologically descendable (resp. universally cohomologically descendable) if $w_{\cdot}$ is so. ■

6.2.3. Lemma. *Let $\mathfrak{X}$ be a $\mathcal{V}$-triple locally of finite type and let $\mathfrak{Y}_{\cdot}$ be a simplicial $\mathfrak{X}$-triple locally of finite type with the structure morphism $w_{\cdot} : \mathfrak{Y}_{\cdot} \to \mathfrak{X}$. Suppose that $w_{\cdot}$ satisfies the condition* (ii) *in Definition* 6.2.1 (1). *Then, $w_0 : \mathfrak{Y}_0 \to \mathfrak{X}$ is faithful.*

Proof. Let $\eta : E \to F$ be a homomorphism of coherent $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules such that $w_0^{\dagger}(\eta) = 0$. Then $w_{\cdot}^{\dagger}(\eta) = 0$. Hence, the induced homomorphism $\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_{\cdot}; w_{\cdot}^\dagger(\eta)) : \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_{\cdot}; w_{\cdot}^\dagger E) \to \mathbb{R}\mathcal{C}^\dagger(\mathcal{X}, \mathfrak{Y}_{\cdot}; w_{\cdot}^\dagger F)$ is a zero map in the derived category. Since $w_{\cdot}$ satisfies the condition (ii) in Definition 6.2.1 (1), we have $\eta = 0$. ■

6.2.4. EXAMPLE. The identity morphism $\mathrm{id}_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{X}$ of an lft-triple $\mathfrak{X}$ is universally cohomologically descendable. ■

6.2.5. PROPOSITION. *Let $w : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of $\mathcal{V}$-triples locally of finite type such that $\widehat{w}$ is a Zariski covering and w is strict as a morphism of triples.* (*See the definition in* 2.3.3 *and* 2.3.4.) *Then w is universally cohomologically descendable.*

PROOF. Since the situation is stable under any base change by a morphism of lft-triples, we have only to prove w is cohomologically descendable. Let us denote by $w_{\cdot} : \mathfrak{Y}_{\cdot} \to \mathfrak{X}$ the Čech diagram for $w : \mathfrak{Y} \to \mathfrak{X}$. Since $\widehat{w}$ is a Zariski covering, $w_{\cdot}^\dagger$ is exact by Proposition 6.1.4.

Now we prove the condition (ii) in Definition 6.2.1. We may assume that $\mathcal{X}$ is affine by Propositions 2.7.3 and 4.3.4. Let us take an affine Zariski covering $\mathcal{Z}$ of $\mathcal{Y}$ and we denote by $\mathfrak{Z}$ the triple over $\mathfrak{Y}$ induced from such a Zariski covering $\mathcal{Z}$ of $\mathcal{Y}$ as in 2.3.3. We denote by $\mathfrak{Z}_{\cdot}$ and $\mathfrak{Z}_{\cdot} \to \mathfrak{Y}_{\cdot}$ the Čech diagram for $\mathfrak{Z}$ over $\mathfrak{X}$ and the induced morphism, respectively. Let us denote by $\mathfrak{U}_{\cdot}$ the Čech diagram of $\mathfrak{Z}_{\cdot}$ over $\mathfrak{Y}_{\cdot}$ and by $v_{\cdot} : \mathfrak{U}_{\cdot} \to \mathfrak{Y}_{\cdot}$ and $u_{\cdot} : \mathfrak{U}_{\cdot} \to \mathfrak{X}_{\cdot}^{\cdot}(\Delta^2)^{\circ}$ the natural morphisms as in Example 3.1.2.

For each object (m, n) of $(\Delta^2)^{\circ}$, $u_{(m,n)}$ and $v_{(m,n)}$ satisfy the assumption of Theorem 5.2.1. The derived Čech diagram coincides with the Čech diagram for $u_{(m,\cdot)}$ and $v_{(\cdot,n)}$ by Corollary 4.4.3. Hence, we have only to prove that the canonical homomorphisms

$$\begin{array}{rcl} E & \to & \mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{U}_{(m,\cdot)}; u_{(m,\cdot)}^\dagger E) \\ w_n^\dagger E & \to & \mathcal{C}^\dagger(\mathfrak{Y}_n, \mathfrak{U}_{(\cdot,n)}; v_{(\cdot,n)}^\dagger w_n^\dagger E) \end{array}$$

are quasi-isomorphisms for any m and n by Proposition 4.4.4. This follows from Lemma 2.12.1. ■

6.2.6. PROPOSITION. *Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ be a $\mathcal{V}$-triple locally of finite type and let $\coprod_\alpha X_\alpha \to X$ be a finite Zariski covering. Put $\mathfrak{X}_\alpha = (X_\alpha, \overline{X}, \mathcal{X})$, and denote by $w : \coprod_\alpha \mathfrak{X}_\alpha \to \mathfrak{X}$ the structure morphism. Then w is universally cohomologically descendable.*

PROOF. Since the situation is stable under any base change by a morphism of lft-triples, we have only to prove w is cohomologically descendable. The assertion follows from Proposition 4.4.5. ■

6.3. Let $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ be $\mathcal{V}$-triples locally of finite type and let

$$\begin{array}{ccc} \mathfrak{Y} & \xleftarrow{w} & \mathfrak{Z} \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle v} & \\ & \mathcal{X} & \end{array}$$

be a commutative diagram of triples. We put $u_{\cdot}: \mathfrak{Y}_{\cdot} \to \mathfrak{X}$ and $v_{\cdot}: \mathfrak{Z}_{\cdot} \to \mathfrak{X}$ to be the Čech diagram for u and v, respectively, and denote by $w_{\cdot} = = \operatorname{cosk}_0^{\mathfrak{X}}(w): \mathfrak{Z}_{\cdot} \to \mathfrak{Y}_{\cdot}$ the induced morphism of diagrams of triples over $\mathfrak{X}$ from w.

We state the main theorem of this section. The proof is given at the end of this subsection. A generalization to the simplicial case will be given in 6.5.

6.3.1. THEOREM. *With the notation as above, suppose that w is universally cohomologically descendable. Then u is cohomologically descendable (resp. universally cohomologically descendable) if and only if v is so.*

6.3.2. PROPOSITION. *With the notation as above, suppose that there exists a section $s: \mathfrak{Y} \to \mathfrak{Z}$ over $\mathfrak{X}$, that is, $ws = \mathrm{id}_{\mathfrak{Y}}$ and $u = vs$. Then, for any sheaf E of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules, the canonical homomorphism*

$$\widetilde{w}_{\cdot}^{\sharp}: \mathbb{R}\mathcal{C}^\dagger(\mathcal{X}, \mathfrak{Y}_{\cdot}; u_{\cdot}^\dagger E) \to \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Z}_{\cdot}; v_{\cdot}^\dagger E)$$

induced by $w_{\cdot}$ is an isomorphism in $D^+(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$.

PROOF. Let E be a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules. Let

$$\tilde{s}^{\sharp}: \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Z}_{\cdot}; v_{\cdot}^\dagger E) \to \mathbb{R}\mathcal{C}^\dagger(\mathcal{X}, \mathfrak{Y}_{\cdot}; u_{\cdot}^\dagger E)$$

be the canonical homomorphism induced by s (see 4.3). Then $\tilde{s}^{\sharp}$ is a left inverse of $\widetilde{w}^{\sharp}$ by Proposition 4.3.1. Hence, it is sufficient to prove that $\widetilde{w}^{\sharp}\tilde{s}^{\sharp}$ coincides with the identity on $\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Z}_{\cdot}; v_{\cdot}^\dagger E)$.

Let $H(\mathfrak{Z}_{\cdot} \rightrightarrows \mathfrak{Z}_{\cdot})$ be the diagram of triples over $\mathfrak{X}$ indexed by $H(\Delta)$ which is induced by the homotopy from $s_{\cdot} w_{\cdot}$ to $\mathrm{id}_{Z_{\cdot}}$ (Example 3.10.1 and Proposition 3.10.3) and let $f_{\cdot}: H(\mathfrak{Z}_{\cdot} \rightrightarrows \mathfrak{Z}_{\cdot}) \to \mathfrak{X}$ be the structure morphism. We denote by $F_{\cdot}$ the induced sheaf $f_{\cdot}^\dagger E$ on the induced diagram of tubes for $H(\mathfrak{Z}_{\cdot} \rightrightarrows \mathfrak{Z}_{\cdot})$ (Lemmas 3.10.4 and 3.10.5). Let $F_{\cdot} \to \mathfrak{I}^{\cdot}_{\cdot}$ be an injective resolution (Proposition 4.2.1). Unfortunately, $v_{\cdot}^{\dagger} E = F_{(\cdot,0)} \to \mathfrak{I}^{\cdot}_{(\cdot,0)}$ might not be an injective resolution, although it is a quasi-isomorphism. However, $v_n^{\dagger} E = F_{(n,0)} \to \mathfrak{I}^{\cdot}_{(n,0)}$ is an injective resolution for each n by Proposition 3.8.7. Hence, there exists a canonical isomorphism $\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Z}_{\cdot}; v_{\cdot}^{\dagger} E) \cong \mathrm{tot}(\mathcal{C}(v_{\cdot *}^{\Delta^{\circ}} \mathfrak{I}^{\cdot}_{(\cdot,0)}))$ by Propositions 4.2.1, 4.2.2 and Lemma 4.4.1. We also know that $v_{\cdot}^{\dagger} E = F_{(\cdot,1)} \to \mathfrak{I}^{\cdot}_{(\cdot,1)}$ is an injective resolution by Proposition 3.8.7 and Lemma 3.10.7. The homotopy which is induced by the injective resolution $\mathfrak{I}^{\cdot}_{\cdot}$ (Lemma 3.10.5 (3)) gives an identification between $\widetilde{w}^{\sharp} \tilde{s}^{\sharp}$ and the identity on $\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Z}_{\cdot}; v_{\cdot}^{\dagger} E)$ by Proposition 3.10.8. This completes the proof. ■

Let $t_{\cdot} : (\mathfrak{U}_{\cdot}, (\Delta^2)^{\circ}) \to \mathfrak{X}$ be the 2-simplicial triple defined in Example 3.1.2 which is induced from the commutative diagram

$$\begin{array}{ccc} \mathfrak{Y}_{\cdot} & \xleftarrow{w_{\cdot}} & \mathfrak{Z}_{\cdot} \\ {\scriptstyle u_{\cdot}} \downarrow & & \downarrow {\scriptstyle v_{\cdot}} \\ \mathfrak{X} & = & \mathfrak{X}. \end{array}$$

6.3.3. Lemma. *With the notation as above, let E be a sheaf of coherent $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules. Then, the homomorphism*

$$\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{U}_{\cdot}; t_{\cdot}^{\dagger} E) \to \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Z}_{\cdot}; v_{\cdot}^{\dagger} E)$$

which is induced by the isomorphism $\mathrm{gr}^0_{\mathrm{fil}} \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{U}_{\cdot}; t_{\cdot}^{\dagger} E) \cong$ $\cong \mathbb{R}\mathcal{C}^{\dagger}(\mathcal{X}, \mathfrak{Z}_{\cdot}; v_{\cdot}^{\dagger} E)$ in Lemma 4.4.1 (*the case where* $(s, s') = (2, 1)$) *is an isomorphism in $D^{+}(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$. Assume furthermore that, for any object n of Δ, the Čech diagram for $w_n : \mathfrak{Z}_n \to \mathfrak{Y}_n$ satisfies the condition (ii) in Definition* 6.2.1 (1). *Then the homomorphism*

$$\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\cdot}; u_{\cdot}^{\dagger} E) \to \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Z}_{\cdot}; v_{\cdot}^{\dagger} E)$$

induced by $w_{\cdot}$ is an isomorphism in $D^{+}(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$.

Proof. Let $\eta^{1,l}_{n_1,n_2} : (n_1, n_2) \to (n_1+1, n_2)$ (resp. $\xi^{1,l}_{n_1,n_2} : (n_1, n_2) \to$ $\to (n_1 - 1, n_2)$) be a morphism in Δ^2 as in (resp. such that $\xi^{1,l}_{n_1,n_2} = \xi^{l}_{n_1} \times \mathrm{id}_{n_2}$, where $\xi^{l}_{n_1}$ is defined in 1.3.4). By applying Proposition 6.3.2 to the morphism $(\eta^{1,l}_{q,0})_{\mathfrak{U}_{\cdot}} : \mathfrak{U}_{(q+1,0)} \to \mathfrak{U}_{(q,0)}$ with sections $(\xi^{1,l-1}_{q+1,0})_{\mathfrak{U}_{\cdot}}$ and

$(\xi^{1,l}_{q+1,0})_{\mathfrak{U}_\cdot}$, the canonical homomorphism

$$(\eta^{1,l}_{q,\ \cdot})^\sharp_{\mathfrak{U}_\cdot}: \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{U}_{(q,\ \cdot)}; E) \to \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{U}_{(q+1,\cdot)}; E)$$

which is induced by $(\eta^{1,l}_{(q,\ \cdot)})_{\mathfrak{U}_\cdot}$ does not depend on l in $D^+(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$.

Let us now consider the spectral sequence

$$\underline{E}_1^{qr} = \underline{H}^r(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{U}_{(q,\ \cdot)}; t^\dagger_{(q,\ \cdot)}E)) \Rightarrow \underline{H}^{q+r}(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{U}_\cdot; t^\dagger_\cdot E))$$

of sheaves on $]\overline{X}[_{\mathcal{X}}$ which is induced from the filtration of $\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{U}_\cdot; t^\dagger_\cdot E)$. Then the edge homomorphism $d_1^{qr}: \underline{E}_1^{qr} \to \underline{E}_1^{q+1,r}$ is zero if q is even and is an isomorphism if q is odd. Since $\underline{E}_2^{0r} = \underline{E}_1^{0r}$ and $\underline{E}_2^{qr} = 0$ $(q \neq 0)$, the induced homomorphism

$$\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{U}_\cdot; t^\dagger_\cdot E) \to \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Z}_\cdot; v^\dagger_\cdot E)$$

is an isomorphism. The rest follows from Proposition 4.4.4. ■

6.3.4. Lemma. *With the notation as at the beginning of* 6.3, *suppose that, for any base change w' of w in the category of lft-triples, the Čech diagram $w'_\cdot$ for w' satisfies the condition (ii) in Definition* 6.2.1 (1). *Then, for any object n of Δ, the Čech diagram $(w_n)_\cdot: \mathfrak{U}_{(n,\ \cdot)} \to \mathfrak{Y}_n$ for $w_n: \mathfrak{Z}_n \to \mathfrak{Y}_n$ satisfies the same condition.*

Proof. We prove that $w_l \times \mathrm{id}^{n-l}_{\mathfrak{Y}}: \mathfrak{Z}_l \times_{\mathfrak{X}} \mathfrak{Y}_{n-l-1} \to \mathfrak{Y}_n$ satisfies the condition (ii) in Definition 6.2.1 (1) for $0 \leqq l \leqq n$ inductively on l ($\mathfrak{Z}_{-1} = \mathfrak{Y}_{-1} = \mathfrak{X}$). If $l = 0$, then $w \times \mathrm{id}^n_{\mathfrak{Y}}$ is a base change of $w: \mathfrak{Z} \to \mathfrak{Y}$ by the first projection $\mathfrak{Y}_n \to \mathfrak{Y}$. Hence, $w \times \mathrm{id}^n_{\mathfrak{Y}}$ satisfies the condition (ii).

Suppose that $w_{l-1} \times \mathrm{id}^{n-l+1}_{\mathfrak{Y}}$ satisfies the condition (ii). We want to apply Lemma 6.3.3 to the commutative diagram

$$\begin{tikzcd}[column sep=small]
\mathfrak{Z}_{l-1} \times_{\mathfrak{X}} \mathfrak{Y}_{n-l} \arrow[dr, "w_{l-1} \times \mathrm{id}^{n-l+1}_{\mathfrak{Y}}"'] & & \mathfrak{Z}_l \times_{\mathfrak{X}} \mathfrak{Y}_{n-l-1} \arrow[ll, "\mathrm{id}^l_{\mathfrak{Z}} \times w \times \mathrm{id}^{n-l}_{\mathfrak{Y}}"'] \arrow[dl, "w_l \times \mathrm{id}^{n-l}_{\mathfrak{Y}}"] \\
& \mathfrak{Y}_n. &
\end{tikzcd}$$

To do this, we have only to show that, for any m, the Čech diagram for the morphism

$$\mathrm{cosk}_0^{\mathfrak{Y}_n}(\mathrm{id}^l_{\mathfrak{Z}} \times w \times \mathrm{id}^{n-l}_{\mathfrak{Y}})_m: \mathrm{cosk}_0^{\mathfrak{Y}_n}(\mathfrak{Z}_l \times_{\mathfrak{X}} \mathfrak{Y}_{n-l-1})_m \to \mathrm{cosk}_0^{\mathfrak{Y}_n}(\mathfrak{Z}_{l-1} \times_{\mathfrak{X}} \mathfrak{Y}_{n-l})_m$$

satisfies the condition (ii). Let us consider the diagram

$$\begin{tikzcd}
\mathrm{cosk}_0^{\mathfrak{Y}_n}(\mathfrak{Z}_{l-1}\times_{\mathfrak{X}}\mathfrak{Y}_{n-l})_m \arrow[d] & \mathrm{cosk}_0^{\mathfrak{Y}_n}(\mathfrak{Z}_l\times_{\mathfrak{X}}\mathfrak{Y}_{n-l-1})_m \arrow[l, "\mathrm{cosk}_0^{\mathfrak{Y}_n}(\mathrm{id}_{\mathfrak{Z}}^l\times w\times \mathrm{id}_{\mathfrak{Y}}^{n-l})_m"'] \arrow[d] \\
\mathrm{cosk}_0^{\mathfrak{X}}(\mathrm{cosk}_0^{\mathfrak{Y}}(\mathfrak{Z})_m)_{l-1}\times_{\mathfrak{X}}\mathfrak{Y}_{n-l} & \mathrm{cosk}_0^{\mathfrak{X}}(\mathrm{cosk}_0^{\mathfrak{Y}}(\mathfrak{Z})_m)_l\times_{\mathfrak{X}}\mathfrak{Y}_{n-l-1} \arrow[l, "\mathrm{id}^l_{\mathrm{cosk}_0^{\mathfrak{Y}}(\mathfrak{Z})_m}\times w'_m\times\mathrm{id}_{\mathfrak{Y}}^{n-l}"]
\end{tikzcd}$$

of triples. Here $w'_m\colon \mathrm{cosk}_0^{\mathfrak{Y}}(\mathfrak{Z})_m \to \mathfrak{Y}$ is the structure morphism, the left vertical arrow is defined by

$$\prod_{i=0}^{m}(z_{0,i}, \ldots, z_{l-1,i}, y_{l,i}, \ldots, y_{n,i}) \mapsto \left(\prod_{j=0}^{l-1}(z_{j,0}, \ldots, z_{j,m})\right)\times(y_{l,0}, \ldots, y_{n,0})$$

(note that $(y_{l,i}, \ldots, y_{n,i})$ is independent of the choice of i), and the right vertical arrow is defined in the same way as in the left one. Then one can easily see that the diagram is commutative and both vertical arrows are isomorphisms. So it is sufficient to prove that the Čech diagram for the bottom arrow satisfies condition (ii). Now we consider the commutative diagram

$$\begin{tikzcd}
\mathrm{cosk}_0^{\mathfrak{X}}(\mathrm{cosk}_0^{\mathfrak{Y}}(\mathfrak{Z})_m)_{l-1}\times_{\mathfrak{X}}\mathfrak{Z}\times_{\mathfrak{X}}\mathfrak{Y}_{n-l-1} \arrow[dr, "\mathrm{id}_{\mathrm{cosk}_0^{\mathfrak{X}}(\mathrm{cosk}_0^{\mathfrak{Y}}(\mathfrak{Z})_m)_{l-1}}\times w\times\mathrm{id}_{\mathfrak{Y}}^{n-l}"'] & & \mathrm{cosk}_0^{\mathfrak{X}}(\mathrm{cosk}_0^{\mathfrak{Y}}(\mathfrak{Z})_m)_l\times_{\mathfrak{X}}\mathfrak{Y}_{n-l-1} \arrow[ll, "\mathrm{id}^l_{\mathrm{cosk}_0^{\mathfrak{Y}}(\mathfrak{Z})_m}\times\mathrm{pr}\times\mathrm{id}_{\mathfrak{Y}}^{n-l}"'] \arrow[dl, "\mathrm{id}^l_{\mathrm{cosk}_0^{\mathfrak{Y}}(\mathfrak{Z})_m}\times w'_m\times\mathrm{id}_{\mathfrak{Y}}^{n-l}"] \\
& \mathrm{cosk}_0^{\mathfrak{X}}(\mathrm{cosk}_0^{\mathfrak{Y}}(\mathfrak{Z})_m)_{l-1}\times_{\mathfrak{X}}\mathfrak{Y}_{n-l}, &
\end{tikzcd}$$

where $\mathrm{pr} : \mathrm{cosk}_0^{\mathfrak{Y}}(\mathfrak{Z})_m \to \mathfrak{Z}$ is the first projection. The diagonal morphism $\mathfrak{Z} \to \mathrm{cosk}_0^{\mathfrak{Y}}(\mathfrak{Z})_m$ induces a section of the horizontal morphism above. Since the left slanting arrow is a base change of $w : \mathfrak{Z} \to \mathfrak{Y}$, the Čech diagram for it satisfies condition (ii). Hence, the Čech diagram for the right slanting arrow satisfies condition (ii) by Proposition 6.3.2. This completes the proof of Lemma 6.3.4. ■

6.3.5. LEMMA. *With the notation as at the beginning of* 6.3,

(1) *if* w *is universally exact, then so is* w_n *for every object* n *of* Δ.

(2) *if* w *is universally cohomologically descendable, then* w_n *is cohomologically descendable for every object* n *of* Δ.

PROOF. One can easily prove (1) by induction on n.

(2) The exactness of the Čech diagram $(w_n)_.$ for w_n follows from (1) and Lemmas 6.1.3. The condition (ii) in Definition 6.2.1 (1) follows from Lemma 6.3.4. Hence, w_n is cohomologically descendable. ■

Proof of Theorem 6.3.1. Since the situation is unchanged after any base change of morphisms of lft-triples, it is sufficient to prove the equivalence of the cohomological descendability. Since w_n is exact and faithful for any n by Lemmas 6.2.3 and 6.3.5, the exactness of $u_\bullet$ is equivalent to that of $v_\bullet$ by Lemma 6.1.2. The Čech diagram $(w_n)_\bullet$ for w_n satisfies condition (ii) in Definition 6.2.1 (1) for any n by Lemma 6.3.4. Hence, Lemma 6.3.3 implies that $u_\bullet$ satisfies condition (ii) if and only if $v_\bullet$ satisfies it. This completes the proof. ■

6.4. The notion of cohomological descent is independent of the choice of boundaries and embedding into formal schemes.

6.4.1. Proposition. *Let $w : \mathfrak{Y} \to \mathfrak{X}$ be a separated morphism of $\mathcal{V}$-triples locally of finite type such that*

(i) $\widehat{w} : \mathcal{Y} \to \mathcal{X}$ *is smooth around* X;

(ii) $\overline{w} : \overline{Y} \to \overline{X}$ *is proper*;

(iii) $\overset{\circ}{w} : Y \to X$ *is an isomorphism.*

Then w is universally cohomologically descendable.

6.4.2. Corollary. *Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$, $\mathfrak{Y} = (Y, \overline{Y}, \mathcal{Y})$ and $\mathfrak{Y}' = (Y, \overline{Y}', \mathcal{Y}')$ be $\mathcal{V}$-triples locally of finite type, and let $w : \mathfrak{Y} \to \mathfrak{X}$ and $w' : \mathfrak{Y}' \to \mathfrak{X}$ be separated morphisms of $\mathcal{V}$-triples such that*

(i) *both $\widehat{w}$ and $\widehat{w}'$ are smooth around* Y;

(ii) *both $\overline{w}$ and $\overline{w}'$ are proper*;

(iii) $\overset{\circ}{w} = \overset{\circ}{w}'$.

Then w is cohomologically descendable (resp. universally cohomologically descendable) if and only if w' is so.

First we prove the following lemma.

6.4.3. Lemma. *Under the assumption of Proposition* 6.4.1, *assume furthermore that $\overline{w} : \overline{Y} \to \overline{X}$ is an isomorphism. Then w is universally cohomologically descendable.*

Proof. The situation is unchanged under any base change by morphisms of lft-triples. Hence, we have only to prove that w is cohomologically descendable. The structure morphism $w_\bullet$ of Čech diagram is exact by Proposition 6.1.4. Hence, it is sufficient to prove that w satisfies the condition (ii) in Definition 6.2.1 (1). We may assume that $\mathcal{X}$ is affine by Proposition 4.3.4. Since $\overline{Y} = \overline{X}$, there exists an open formal subscheme $\mathcal{Y}'$ of $\mathcal{Y}$ of finite type over $\mathcal{X}$ such that $\overline{Y}$ is a closed subscheme in $\mathcal{Y}'$. Hence we may assume that $\mathcal{Y}$ is of finite type over $\mathcal{X}$.

First we prove the assertion in the case where $\widehat{w}: \mathcal{Y} \to \mathcal{X}$ is étale around Y. Let $w_{\textbf{.}}: \mathfrak{Y}_{\textbf{.}} \to \mathfrak{X}$ be a Čech diagram for $w: \mathfrak{Y} \to \mathfrak{X}$. Since $\overset{\circ}{w}_n$ and $\overline{w}_n$ are also isomorphisms and $\widehat{w}_n$ is also étale, one can use Propositions 2.7.2 (2) and 2.10.2 (2) for higher direct images of $\widetilde{w}_n$ for any n. By Corollary 4.4.3, it suffices to show that the canonical homomorphism $E \to \to \mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_{\textbf{.}}; w_{\textbf{.}}^\dagger E)$ is a quasi-isomorphism for any sheaf E of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules. The homomorphism $\widetilde{w}_{n*} w_n^\dagger E \to \widetilde{w}_{n+1*} w_{n+1}^\dagger E$ which is induced from any projection $\mathfrak{Y}_{n+1} \to \mathfrak{Y}_n$ coincides with the identity $E \to \to E$ by Proposition 2.10.2 (1). Hence, $E \to \mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_{\textbf{.}}; w_{\textbf{.}}^\dagger E)$ is a quasi-isomorphism and w is cohomologically descendable. Therefore, w is universally cohomologically descendable if $\widehat{w}$ is étale.

Now we reduce to the case where $\mathcal{Y}$ is affine. If one takes a suitable finite affine covering $\coprod_\alpha \overline{X}_\alpha$ of $\overline{X}$ one can find an affine open formal subscheme $\mathcal{X}_\alpha$ of $\mathcal{X}$ and an affine open formal subscheme $\mathcal{Y}_\alpha$ of $\mathcal{Y}$ with a commutative diagram

$$\begin{array}{ccc} & \nearrow & \mathcal{Y}_\alpha \\ \overline{X}_\alpha & & \downarrow \\ & \searrow & \mathcal{X}_\alpha \end{array}$$

such that both $\overline{X}_\alpha = \overline{X} \cap \mathcal{X}_\alpha \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spec}\, k$ and $\overline{X}_\alpha \to \mathcal{Y}_\alpha \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spec}\, k$ is a closed immersion for any α. Now we put $\mathcal{X}' = \bigcup_\alpha \mathcal{X}_\alpha$ and $\mathcal{Y}' = \bigcup_\alpha \mathcal{Y}_\alpha$, and define triples $\mathfrak{X}' = (X, \overline{X}, \mathcal{X}')$ and $\mathfrak{Y}' = (X, \overline{X}, \mathcal{Y}')$. We also put $X_\alpha = X \cap \cap \overline{X}_\alpha$ and define triples $\mathfrak{X}_\alpha = (X_\alpha, \overline{X}_\alpha, \mathcal{X}_\alpha)$ and $\mathfrak{Y}_\alpha = (X_\alpha, \overline{X}_\alpha, \mathcal{Y}_\alpha)$. Then we have only to prove that the natural structure morphism $\mathfrak{Y}' \to \mathfrak{X}'$ is cohomologically descendable by Lemma 4.3.2. Let us consider the natural commutative diagram:

$$\begin{array}{ccc} \mathfrak{Y}' \times_{\mathfrak{X}'} \mathfrak{X}_\alpha & \leftarrow & \mathfrak{Y}_\alpha \\ \downarrow & \swarrow & \\ \mathfrak{X}_\alpha. & & \end{array}$$

Since $\{\mathcal{X}_\alpha\}_\alpha$ is a Zariski covering of $\mathcal{X}'$ and $\mathcal{Y}_\alpha$ is an open formal subscheme of $\mathcal{Y}' \times_{\mathcal{X}'} \mathcal{X}_\alpha$, it is sufficient to prove that $\mathfrak{Y}_\alpha \to \mathfrak{X}_\alpha$ is cohomologically descendable for any α by Proposition 4.3.4 and Lemma 4.3.2. Hence, we may assume that $\mathcal{Y}$ is affine.

By the strong fibration theorem [7, 1.3.7 Théorème] there is a finite affine Zariski covering $\coprod_{\beta} X_\beta \to X$ such that $\widehat{w}$ is decomposed as follows:

$$\begin{array}{ccccc} & & & & \mathcal{Y} \\ & & & \nearrow & \downarrow{\scriptstyle \widehat{w}''_\beta} \\ X_\beta & \to & \overline{X} & \to & \widehat{\mathbb{A}}^{d_\beta}_{\mathcal{X}} \\ & & & \searrow & \downarrow{\scriptstyle \widehat{w}'_\beta} \\ & & & & \mathcal{X} \end{array} \qquad \widehat{w} = \widehat{w}'_\beta \widehat{w}''_\beta$$

for each β, where $\widehat{\mathbb{A}}^{d_\beta}_{\mathcal{X}}$ is the affine space over $\mathcal{X}$ of relative dimension d_β, $\widehat{w}'_\beta$ is the projection and $\widehat{w}''_\beta$ is étale around X_β. Since $\Gamma(\mathcal{X}, \mathcal{X}_{\mathcal{X}}) \to \to \Gamma(X, \mathcal{O}_X)$ is surjective, $\overline{X} \to \widehat{\mathbb{A}}^{d_\beta}_{\mathcal{X}} \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}\, k$ is a closed immersion. We denote by $\mathfrak{X}''$ (resp. $\mathfrak{Y}''$) the triple $\mathfrak{X}'' = \coprod_{\beta} (X_\beta, \overline{X}, \mathcal{X})$ (resp. $\mathfrak{Y}'' = = \coprod_{\beta} (X_\beta, \overline{X}, \mathcal{Y})$). Then we obtain a commutative diagram

$$\begin{array}{ccc} \mathfrak{Y} & \leftarrow & \mathfrak{Y}'' \\ \downarrow & & \downarrow \\ \mathfrak{X} & \leftarrow & \mathfrak{X}''. \end{array}$$

The horizontal maps are universally cohomologically descendable by Proposition 6.2.6. By Theorem 6.3.1 we are reduced to proving the cohomological descendability of the composition $\mathfrak{Y}'' \to \mathfrak{X}$. Let $\mathfrak{X}''_{\bullet} \to \mathfrak{X}$ (resp. $\mathfrak{Y}''_{\bullet} \to \mathfrak{X}$) be the Čech diagram for $\mathfrak{X}'' \to \mathfrak{X}$ (resp. $\mathfrak{Y}'' \to \mathfrak{X}$). Then each component of $\mathcal{Y}''_n \to \mathcal{X}''_n$ of the induced homomorphism $\mathfrak{Y}''_n \to \mathfrak{X}''_n$ is a composition of an étale morphism around Y''_n and a projection as above for all n. Hence, if one knows the cohomological descendability of morphisms $\mathfrak{Y}''_n \to \mathfrak{X}''_n$ for all n, then $\mathfrak{Y}'' \to \mathfrak{X}$ is cohomologically descendable by Proposition 6.1.4 and Lemma 6.3.3.

Now we suppose that $\widehat{w} : \mathcal{Y} \to \mathcal{X}$ is decomposed as $\widehat{w} = \widehat{w}' \widehat{w}''$ such that $\widehat{w}''$ is an étale morphism around X and $\widehat{w}'$ is a projection from the affine space $\widehat{\mathbb{A}}^d_{\mathcal{X}}$ as above. Since w'' is universally cohomologically descendable by what has just been proved above, we may assume that $w = w'$ and $\mathcal{Y} = \widehat{\mathbb{A}}^d_{\mathcal{X}}$ by Theorem 6.3.1. If we choose a suitable system of coordinates of $\widehat{\mathbb{A}}^d_{\mathcal{X}}$, then the 0-section $\widehat{s}$ of $\widehat{w}$ induces a section s of w as a morphism of triples. Now the assertion follows from Proposition 6.3.2. This completes the proof. ■

6.4.4. Proposition. *With the situation of Corollary* 6.4.2, *if we replace the condition* (ii) *with the condition* (ii)′ $\overline{Y} = \overline{Y}'$ *and* $\overline{w} = \overline{w}'$. *Then the assertion of Corollary* 6.4.2 *still holds.*

PROOF. Since the diagonal embedding $\overline{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}'$ is a closed immersion, we may assume that there exists a morphism $v : \mathfrak{Y}' \to \mathfrak{Y}$ over $\mathfrak{X}$ since $\overline{w} = \overline{w}'$ is proper. The assertion follows from Theorem 6.3.1 and Lemma 6.4.3. ■

PROOF OF PROPOSITION 6.4.1. We may assume that $\mathcal{X}$ is affine by Propositions 4.3.4. Since $\overline{Y}$ is of finite type over $\overline{X}$, there exists an open formal subscheme $\mathcal{Y}'$ of $\mathcal{Y}$ of finite type over $\mathcal{X}$ such that $\overline{Y}$ is a closed subscheme in $\mathcal{Y}'$. Hence we may assume that $\mathcal{Y}$ is of finite type over $\mathcal{X}$.

If $\widehat{w}$ is étale around Y, then some sufficiently small strict neighbourhood V of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ is isomorphic to the inverse image $\widetilde{w}^{-1}(V)$ of V by [7, 1.3.2 Théorème]. Hence we have the assertion.

In the general case we can use Berthelot's argument in [7, 2.3.5 Théorème]. There exists a projective scheme $\overline{Y}'$ with an open immersion $Y \to \overline{Y}'$ over $\overline{Y}$ such that $\overline{Y}'$ is also projective over $\overline{X}$ by the precise Chow's lemma [12, Corollaire 5.7.14]. We assume that $\overline{w}$ is projective by Theorem 6.3.1. Then there exists a triple $\coprod_\alpha (Y_\alpha, \overline{Y}_\alpha, \mathcal{Y}'_\alpha)$ of finite type over $\mathfrak{X}$ such that $\coprod_\alpha Y_\alpha$ is a Zariski covering of Y, $\overline{Y}_\alpha \to \overline{Y}$ is a closed immersion and $\mathcal{Y}'_\alpha \to \mathcal{X}$ is separated of finite type and étale around Y_α for each α by Lemma 6.4.5. Let us consider the commutative diagram below:

$$\begin{tikzcd}
(Y, \overline{Y}, \mathcal{Y}) \arrow[dr, "w"'] & \coprod_\alpha (Y_\alpha, \overline{Y}, \mathcal{Y}) \arrow[l, "6.2.6"'] & \coprod_\alpha (Y_\alpha, \overline{Y}_\alpha, \mathcal{Y}) \arrow[l, "\text{former part}"'] \arrow[dl, "6.4.4"] \\
& (X, \overline{X}, \mathcal{X}) & \\
\coprod_\alpha (Y_\alpha, \overline{X}, \mathcal{X}) \arrow[ur, "6.2.6"] & & \coprod_\alpha (Y_\alpha, \overline{Y}_\alpha, \mathcal{Y}'_\alpha), \arrow[ll, "\text{former part}"] \arrow[ul, "6.3.1"']
\end{tikzcd}$$

where $\overline{Y}_\alpha \to \overline{Y} \to \mathcal{Y}$ is the natural closed immersion. The universally cohomological descendability of arrows except w follows from each assertion indicated in the diagram. Observing the diagram from $\coprod_\alpha (Y_\alpha, \overline{X}, \mathcal{X}) \to (X, \overline{X}, \mathcal{X})$ to $w : (Y, \overline{Y}, \mathcal{Y}) \to (X, \overline{X}, \mathcal{X})$ counterclockwise, we obtain the universally cohomological descendability of w by Theorem 6.3.1. ■

PROOF OF COROLLARY 6.4.2. Let $\overline{Y}''$ be a Zariski closure of the diagonal embedding $Y \to \overline{Y} \times_{\overline{X}} \overline{Y}'$. Considering the triple $(Y, \overline{Y}'', \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}')$, we may assume that there exists a morphism $v : \mathfrak{Y}' \to \mathfrak{Y}$ over

$\mathfrak{X}$ by Theorem 6.3.1. Then the assertions follow from Theorem 6.3.1 and Proposition 6.4.1. ■

6.4.5. LEMMA. *Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ be a $\mathcal{V}$-triple of finite type and let $(Y, \overline{Y}) \to (X, \overline{X})$ be a morphism of pairs such that $\overline{Y} \to \overline{X}$ is projective and $Y \to X$ is étale. Then, for each point y of Y, there exists a triple $\mathfrak{U} = (U, \overline{U}, \mathcal{U})$ separated of finite type over $\mathfrak{X}$ with a morphism $(U, \overline{U}) \to (Y, \overline{Y})$ of pairs over $(X, \overline{X})$ such that U is an open subscheme of Y with $y \in U$, $\overline{U}$ is a closed subscheme of $\overline{Y}$ and $\mathcal{U} \to \mathcal{X}$ is étale around U.*

PROOF. Since the problem is local on $\mathcal{X}$, we may assume that $\mathcal{X}$ is affine. Let $\mathcal{P}$ be a formal projective space over $\mathcal{X}$ with an $\mathcal{X}$-closed immersion $\overline{Y} \to \overline{P} = \mathcal{P} \times_{\mathcal{X}} \overline{X}$, let $s \in \Gamma(\overline{P}, \mathcal{O}_{\overline{P}}(1))$ be a section with $s(y) \neq 0$ and let $t_1, \ldots, t_n$ be a regular sequence of $\overline{Y}$ at y. Then $s^m t_1, \ldots, s^m t_n$ is a global section of $\mathcal{O}_{\overline{P}}(m)$ for a sufficiently large integer m. Let us take lifts $u_1, \ldots, u_n$ of $s^m t_1, \ldots, s^m t_n$ in $\Gamma(\mathcal{P}, \mathcal{O}_{\mathcal{P}}(m))$. We define a formal subscheme $\mathcal{U}$ of $\mathcal{P}$ over $\mathcal{X}$ by $u_1 = \ldots = u_n = 0$ and define a closed subscheme $\overline{U}$ of $\overline{Y}$ by the pull back of $\overline{Y}$ under the closed immersion $\mathcal{U} \to \mathcal{P}$. By our choice of s and $t_1, \ldots, t_n$ there exists an open subscheme U of $\overline{U}$ with $y \in U$ such that $\mathcal{U}$ is étale over $\mathcal{X}$ around U. ■

6.5. We consider the cohomological descent for general simplicial triples.

6.5.1. PROPOSITION. *Let $\mathfrak{X}$ be a $\mathcal{V}$-triple locally of finite type and let*

$$\begin{array}{ccc} \mathfrak{Y}_{\bullet} & \xleftarrow{w_{\bullet}} & \mathfrak{Z}_{\bullet} \\ & {\scriptstyle u_{\bullet}}\searrow \quad \swarrow{\scriptstyle v_{\bullet}} & \\ & \mathfrak{X} & \end{array}$$

be a commutative diagram such that $\mathfrak{Y}_{\bullet}$ and $\mathfrak{Z}_{\bullet}$ are simplicial $\mathfrak{X}$-triples locally of finite type. Suppose that, for some nonnegative integer n,

(i) $\mathrm{cosk}_n^{\mathfrak{X}}(\mathfrak{Y}_{\bullet}^{(n)}) = \mathfrak{Y}_{\bullet}$ *and* $\mathrm{cosk}_n^{\mathfrak{X}}(\mathfrak{Z}_{\bullet}^{(n)}) = \mathfrak{Z}_{\bullet}$ (*i.e. they have dimension n*);

(ii) *w_l is an isomorphism for $l < n$, $\mathrm{cosk}_n^{\mathfrak{X}}(w_{\bullet}^{(n)}) = w_{\bullet}$ and w_l is cohomologically descendable* (*resp. universally cohomologically descendable*) *for all l.*

Then $u_{\bullet}$ is cohomologically descendable (resp. universally cohomologically descendable) if and only if $v_{\bullet}$ is so.

In Proposition 6.5.1 $w_n\colon \mathfrak{Z}_n \to \mathfrak{Y}_n$ is cohomologically descendable for all n by assumption. So we already have the property corresponding to Lemma 6.3.4. The proposition easily follows from Lemmas 6.1.2, 6.2.3 and the following two Lemmas.

6.5.2. LEMMA. *Let $\mathfrak{X}$ be a $\mathcal{V}$-triple locally of finite type and let $w_{\cdot}^{(i)}\colon \mathfrak{Z}_{\cdot} \to \mathfrak{Y}_{\cdot}$ $(i = 0, 1)$ be morphisms of simplicial $\mathfrak{X}$-triples locally of finite type. Suppose that, for some nonnegative integer n,*

(i) $\operatorname{cosk}_n^{\mathfrak{X}}(\mathfrak{Y}_{\cdot}^{(n)}) = \mathfrak{Y}_{\cdot}$ *and* $\operatorname{cosk}_n^{\mathfrak{X}}(\mathfrak{Z}_{\cdot}^{(n)}) = \mathfrak{Z}_{\cdot}$ *(i.e. they have dimension n);*

(ii) $w_l^{(0)} = w_l^{(1)}$ *for $l < n$ and* $\operatorname{cosk}_n^{\mathfrak{X}}((w_{\cdot}^{(i)})^{(n)}) = w_{\cdot}^{(i)}$ *for $i = 0, 1$.*

Then, for any sheaf of coherent $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules, two homomorphisms

$$(w_{\cdot}^{(0)})^{\sharp}, (w_{\cdot}^{(1)})^{\sharp}\colon \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\cdot}; u_{\cdot}^{\dagger}E) \to \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Z}_{\cdot}; v_{\cdot}^{\dagger}E)$$

which are induced by $w_{\cdot}^{(0)}$ and $w_{\cdot}^{(1)}$ coincide with each other in $D^{+}(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$.

PROOF. Let us consider a homotopy from $w_{\cdot}^{(0)}$ to $w_{\cdot}^{(1)}$ in Lemma 3.10.2 and the associated diagram $H(\mathfrak{Z}_{\cdot} \rightrightarrows \mathfrak{Y}_{\cdot})$ of triples to the homotopy. We denote by $f_{\cdot}\colon H(\mathfrak{Z}_{\cdot} \rightrightarrows \mathfrak{Y}_{\cdot}) \to \mathfrak{X}$ the structure morphism. Let $F_{\cdot}$ be the sheaf $f_{\cdot}^{\dagger}E$ on the tubes associated to $H(\mathfrak{Z}_{\cdot} \rightrightarrows \mathfrak{Y}_{\cdot})$ (Lemmas 3.10.4 and 3.10.5) and take an injective resolution $F_{\cdot} \to \mathfrak{I}_{\cdot}^{\cdot}$ (Proposition 4.2.1). Then we have a canonical commutative diagram

$$\begin{array}{ccc}
\operatorname{tot}(\mathcal{C}(u_{\cdot *}^{\Delta^{\circ}}\mathfrak{I}_{(\cdot,0)}^{\cdot})) & \xrightarrow{(w_{\cdot}^{(i)})^{\sharp}} & \operatorname{tot}(\mathcal{C}(v_{\cdot *}^{\Delta^{\circ}}\mathfrak{I}_{(\cdot,1)}^{\cdot})) \\
\downarrow & & \downarrow \\
\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\cdot}; u_{\cdot}^{\dagger}E) & \xrightarrow[(w_{\cdot}^{(i)})^{\sharp}]{} & \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Z}_{\cdot}; v_{\cdot}^{\dagger}E)
\end{array}$$

for each i by Propositions 4.2.1 and 4.2.2. By the same argument as in the proof of Lemma 6.3.2 both vertical arrows, which are independent of i, are isomorphisms. Since there exists a homotopy between two arrows with respect to the top horizontal arrows by Proposition 3.10.8, the two morphisms of the bottom horizontal arrows coincide with each other. ■

6.5.3. LEMMA. *Under the assumption of Proposition* 6.5.1, *let $\mathfrak{U}_{\cdot}$ be the* 2*-simplicial triple over $\mathfrak{X}$ defined in Example* 3.1.2 *which is induced* *from the commutative diagram*

$$\begin{array}{ccc} \mathfrak{Y}_{\bullet} & \leftarrow & \mathfrak{Z}_{\bullet} \\ \downarrow & & \downarrow \\ \mathfrak{X} & = & \mathfrak{X} \end{array}$$

and let $t_{\bullet}: \mathfrak{U}_{\bullet} \to \mathfrak{X}$ *be the structural morphism. Then, for any sheaf* E *of coherent* $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$*-modules, the homomorphism*

$$\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{U}_{\bullet}; t_{\bullet}^{\dagger}E) \to \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Z}_{\bullet}; v_{\bullet}^{\dagger}E)$$

which is induced by the isomorphism $\mathrm{gr}^0_{\mathrm{fil}}\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{U}_{\bullet}; t_{\bullet}^{\dagger}E) \cong$ $\cong \mathbb{R}\mathcal{C}^{\dagger}(\mathcal{X}, \mathfrak{Z}_{\bullet}; v_{\bullet}^{\dagger}E)$ *in Lemma* 4.4.1 (*the case where* $(s, s') = (2, 1)$) *is an isomorphism in* $D^{+}(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$.

PROOF. We use a spectral sequence arugument similar to that used in the proof of Lemma 6.3.3. To prove that the edge homomorphism d_1^{qr} of $\underline{E}_1$-terms is an isomorphism if q is odd and the 0-map if q is even, we apply Lemma 6.5.2. Since two of $\{\eta_{q,\bullet}^{1,l}\}_{0 \leqq l \leqq r+1}$ satisfy the assumption of Lemma 6.5.2, we have a result similar to Lemma 6.3.3. ■

6.5.4. COROLLARY. *Let* $\mathfrak{X}$ *be a* $\mathcal{V}$*-triple locally of finite type and let* $\mathfrak{Y}_{\bullet}$ *be a simplicial* $\mathfrak{X}$*-triple locally of finite type with the structure morphism* $w_{\bullet}$. *Suppose that, for any nonnegative integers* l *and* n, *the canonical morphism* $\mathrm{cosk}_n^{\mathfrak{X}}(\mathfrak{Y}_{\bullet}^{(n)})_l \to \mathrm{cosk}_{n-1}^{\mathfrak{X}}(\mathfrak{Y}_{\bullet}^{(n-1)})_l$ *is cohomologically descendable (resp. universally cohomologically descendable). If* $\mathrm{cosk}_m^{\mathfrak{X}}(w_{\bullet}^{(m)}): \mathrm{cosk}_m^{\mathfrak{X}}(\mathfrak{Y}_{\bullet}^{(m)}) \to \mathfrak{X}$ *is cohomologically descendable (resp. universally cohomologically descendable) for some nonnegative integer* m, *then* $w_{\bullet}: \mathfrak{Y}_{\bullet} \to \mathfrak{X}$ *is so.*

PROOF. Suppose that $\mathrm{cosk}_m^{\mathfrak{X}}(w_{\bullet}^{(m)}): \mathrm{cosk}_m^{\mathfrak{X}}(\mathfrak{Y}_{\bullet}^{(m)}) \to \mathfrak{X}$ is cohomologically descendable (resp. universally cohomologically descendable). Then $\mathrm{cosk}_n^{\mathfrak{X}}(w_{\bullet}^{(n)}): \mathrm{cosk}_n^{\mathfrak{X}}(\mathfrak{Y}_{\bullet}^{(n)}) \to \mathfrak{X}$ is cohomologically descendable (resp. universally cohomologically descendable) for any n by Proposition 6.5.1. There exists a commutative diagram

$$\begin{array}{ccc} \underline{E}_1^{qr} = \mathbb{R}^r\mathrm{cosk}_{n+1}^{\mathcal{X}}(w_{\bullet}^{(n+1)})_{q*}\mathrm{cosk}_{n+1}^{\mathfrak{X}}(w_{\bullet}^{(n+1)})_q^{\dagger}E & \Rightarrow & \underline{H}^{q+r}(\mathbb{R}\mathcal{C}^{\dagger}(\mathcal{X}, \mathrm{cosk}_{n+1}^{\mathfrak{X}}(\mathfrak{Y}_{\bullet}^{(n+1)}); \mathrm{cosk}_{n+1}^{\mathfrak{X}}(w_{\bullet}^{(n+1)})^{\dagger}E)) \\ \downarrow & & \downarrow \\ \underline{E}_1^{qr} = \mathbb{R}^r\widetilde{w}_{q*}w_q^{\dagger}E & \Rightarrow & \underline{H}^{q+r}(\mathbb{R}\mathcal{C}^{\dagger}(\mathcal{X}, \mathfrak{Y}_{\bullet}; w_{\bullet}^{\dagger}E)) \end{array}$$

of spectral sequences by Lemma 4.4.2 (the case where $s = s' = 1$). Since $\mathrm{cosk}_n^{\mathfrak{X}}(\mathfrak{Y}_{\bullet}^{(n)})^{(n)} = \mathfrak{Y}_{\bullet}^{(n)}$, the canonical homomorphism

$$\tau_{\leqq n}\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathrm{cosk}_{n+1}^{\mathfrak{X}}(\mathfrak{Y}_{\bullet}^{(n+1)}); \mathrm{cosk}_{n+1}^{\mathfrak{X}}(w_{\bullet}^{(n+1)})^{\dagger}E) \to \tau_{\leqq n}\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\bullet}; w_{\bullet}^{\dagger}E)$$

is an isomorphism. Here $\tau_{\leqq n}\mathcal{C}^{\bullet}$ is a complex defined by $(\tau_{\leqq n}\mathcal{C}^{\bullet})^r = \mathcal{C}^r$ $(r \leqq n)$, $(\tau_{\leqq n}\mathcal{C}^{\bullet})^{n+1} = \mathrm{im}(d^n)$ and $(\tau_{\leqq n}\mathcal{C}^{\bullet})^{r=0}$ $(r \geqq n+2)$. Hence, we have

$$\underline{H}^q(\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}_{\bullet}; w_{\bullet}^{\dagger}E)) =$$

$$= \underline{H}^q(\tau_{\leqq q}\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathrm{cosk}_{q+1}^{\mathfrak{X}}(\mathfrak{Y}_{\bullet}^{(q+1)}); \mathrm{cosk}_{q+1}^{\mathfrak{X}}(w_{\bullet}^{(q+1)})^{\dagger}E)) = \begin{cases} E & \text{if } q=0 \\ 0 & \text{if } q \neq 0. \end{cases}$$

The exactness (resp. universal exactness) follows from Lemmas 6.1.2 and 6.2.3. Therefore, $w_{\bullet}: \mathfrak{Y}_{\bullet} \to \mathfrak{X}$ is cohomologically descendable (resp. the universally cohomologically descendable). ■

7. Cohomological descent for étale morphisms.

In this section we prove two types of cohomological descent theorems for étale morphisms. For a module M over $\mathcal{V}$, we denote by $M^{\frown}$ the p-adic completion of M.

7.1. Let $\mathcal{A}$ and $\mathcal{B}$ be formal $\mathcal{V}$-algebras topologically of finite type and let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of formal $\mathcal{V}$-algebras (1.3.2) such that

(i) the induced morphism $\mathrm{Spec}\, \mathcal{B}/\pi^{e+1}\mathcal{B} \to \mathrm{Spec}\, \mathcal{A}/\pi^{e+1}\mathcal{A}$ is flat for any nonnegative integer e;

(ii) the induced morphism $\mathrm{Spec}\, \mathcal{B}/\pi\mathcal{B} \to \mathrm{Spec}\, \mathcal{A}/\pi\mathcal{A}$ is surjective.

Note that $\mathcal{A}$ and $\mathcal{B}$ are noetherian (see 2.2). We define $P = \mathrm{Spm}\, \mathcal{A} \otimes_{\mathcal{V}} K$ and $Q = \mathrm{Spm}\, \mathcal{B} \otimes_{\mathcal{V}} K$ to be the associated affinoid spaces and denote by $\widetilde{w}: Q \to P$ the induced morphism of rigid analytic spaces.

Let us consider the Čech diagram

$$P \leftarrow Q \;\substack{\leftarrow\\ \rightarrow\\ \leftarrow}\; Q \times_P Q \;\substack{\leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow}\; \cdots$$

of rigid analytic spaces induced by $\mathcal{A} \to \mathcal{B}$ and denote by $\widetilde{w}_{\bullet}: Q_{\bullet} \to P$ the structure morphism of the Čech diagram. As in the case of triples in 4.1 and 4.2, for a sheaf $\mathcal{E}$ of $\mathcal{O}_P$-modules, one can define the Čech complex $\mathcal{C}(P, Q_{\bullet}; \widetilde{w}_{\bullet}^{*}\mathcal{E})$ in the category of complexes of sheaves of $\mathcal{O}_P$-modules with respect to w (resp. the derived Čech complex

$\mathbb{R}\mathcal{C}(P, Q_{\bullet}; \widetilde{w}_{\bullet}^{*}\mathcal{E})$ in the derived category $D^{+}(\mathbb{Z}_P)$ of lower bounded complexes of sheaves of abelian groups on P).

7.1.1. Proposition. *With the notation as above, assume furthermore that $\mathcal{E}$ is a sheaf of coherent $\mathcal{O}_P$-modules.*

(1) *The inverse image functor $\widetilde{w}_{\bullet}^{*}$: $\mathrm{Coh}(\mathcal{O}_{\mathcal{P}}) \to \mathrm{Coh}(\mathcal{O}_{\mathcal{Q}_{\bullet}})$ is exact.*

(2) *The canonical homomorphism*

$$\mathcal{E} \to \mathcal{C}(P, Q_{\bullet}; \widetilde{w}_{\bullet}^{*}\mathcal{E})$$

is a quasi-isomorphism of complexes of sheaves of $\mathcal{O}_P$-modules.

(3) *The canonical homomorphism*

$$\mathcal{C}(P, Q_{\bullet}; \widetilde{w}_{\bullet}^{*}\mathcal{E}) \to \mathbb{R}\mathcal{C}(P, Q_{\bullet}; \widetilde{w}_{\bullet}^{*}\mathcal{E})$$

is an isomorphism in $D^{+}(\mathbb{Z}_P)$.

Proof. (1) Let $\mathcal{B}_n$ ($n \geqq 0$) be a tensor product of $n+1$-copies of $\mathcal{B}$ over $\mathcal{A}$ as a formal $\mathcal{V}$-algebra. Since $\mathcal{A}$ and $\mathcal{B}_n$ are separated in the p-adic topology, $\mathcal{B}_n$ is faithfully flat over $\mathcal{A}$ by the condition (i) and (ii). The assertion follows from this faithful flatness.

(2) Let V be a rational subdomain of P. Then there exist elements $f_1, \dots, f_r, g \in \mathcal{A}$ without common zeros such that

$$\begin{aligned} V &= \{x \in P \mid |f_i(x)| \leqq |g(x)| \ \text{ for any } i\} \\ &\cong \mathrm{Spm}\left(\frac{\mathcal{A}[t_1, \dots, t_r]^{\frown}}{(f_1 - gt_1, \dots, f_r - gt_r)} \otimes_{\mathcal{V}} K\right). \end{aligned}$$

If we put $W = \widetilde{w}^{-1}(V)$, then

$$\begin{aligned} W &= \{x \in Q \mid |f_i(x)| \leqq |g(x)| \ \text{ for any } i\} \\ &\cong \mathrm{Spm}\left(\frac{\mathcal{B}[t_1, \dots, t_r]^{\frown}}{(f_1 - gt_1, \dots, f_r - gt_r)} \otimes_{\mathcal{V}} K\right), \end{aligned}$$

where f_i (resp. g) is a image of the element f_i (resp. g) in $\mathcal{A}$ into $\mathcal{B}$. Since the natural homomorphism

$$\mathcal{A}[t_1, \dots, t_r]^{\frown}/(f_1 - gt_1, \dots, f_r - gt_r) \to \mathcal{B}[t_1, \dots, t_r]^{\frown}/(f_1 - gt_1, \dots, f_r - gt_r)$$

is obtained by the p-adic completion of a base change of $\mathcal{A} \to \mathcal{B}$, the morphism $\widetilde{w}\,|_W$: $W \to V$ of affinoid subvarieties also satisfies the conditions (i) and (ii). By the construction we have

$$\mathcal{C}(P, Q_{\bullet}; \widetilde{w}_{\bullet}^{*}\mathcal{E})\,|_V \cong \mathcal{C}(V, W_{\bullet}; (\widetilde{w}\,|_W)_{\bullet}^{*}(\mathcal{E}\,|_V)).$$

Here $W_{\bullet} \to V$ is the Čech diagram for $W \to V$. Since rational subdomains make a system of fundamental neighbourhoods of affinoid varieties in the weak Grothendieck topology [8, 7.3.5 Corollary 3], it is sufficient to prove that the canonical homomorphism

$$\Gamma(P, \mathcal{E}) \to \Gamma(P, \mathcal{C}(P, Q_{\cdot}; \widetilde{w}_{\cdot}^{*} \mathcal{E}))$$

is a quasi-isomorphism.

Since $\mathcal{E}$ is coherent, there exists a finitely generated $\mathcal{A}$-module M with $\Gamma(P, \mathcal{E}) \cong M \otimes_{\mathcal{V}} K$ by [8, 9.4.3 Theorem 3]. After tensoring K over $\mathcal{V}$, the Čech complex

$$M \otimes_{\mathcal{A}} \mathcal{B} \to M \otimes_{\mathcal{A}} (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B})^{\frown} \to M \otimes_{\mathcal{A}} (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B})^{\frown} \to \dots$$

is isomorphic to the complex $\Gamma(P, \mathcal{C}(P, Q_{\cdot}; \widetilde{w}_{\cdot}^{*} \mathcal{E}))$. Hence, we have only to prove that the canonical homomorphism $M \to M \otimes_{\mathcal{A}} \mathcal{B}$ $(m \mapsto m \otimes 1)$ induces a quasi-isomorphism between M and the Čech complex above.

Since M is finitely generated over $\mathcal{A}$, we have

$$M \otimes_{\mathcal{A}} (\mathcal{B} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{B})^{\frown} \cong \varprojlim_{e} M \otimes_{\mathcal{A}} \mathcal{B} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{B}/\pi^{e+1}.$$

Hence, it is sufficient to prove that the complex

$$0 \to M/\pi^{e+1} \to M \otimes_{\mathcal{A}} \mathcal{B}/\pi^{e+1} \to M \otimes_{\mathcal{A}} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}/\pi^{e+1} \to$$
$$\to M \otimes_{\mathcal{A}} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}/\pi^{e+1} \to \dots$$

is exact for any e since $M/\pi^{e+1} \to M/\pi^{e}$ is surjective. The exactness follows from the faithfully flat descent theorem for finitely generated modules over noetherian rings by the conditions (i) and (ii).

Since $\widetilde{w} : \mathcal{Q} \to \mathcal{P}$ is an affinoid morphism, the assertion (3) follows from Tate's acyclicity theorem [21, Theorem 8.7]. ■

7.1.2. Proposition. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be $\mathcal{V}$-triples locally of finite type and let $w : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of finite type which satisfies the following conditions:*

(i) *$w : \mathfrak{Y} \to \mathfrak{X}$ is strict as a morphism of triples*;

(ii) *$\mathcal{Y} \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spec}\ \mathcal{V}/\pi^{e+1} \to \mathcal{X} \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spec}\ \mathcal{V}/\pi^{e+1}$ which is induced from $\widehat{w}$ is flat for every e*;

(iii) *$\overline{w} : \overline{Y} \to \overline{X}$ is surjective.*

Then w is universally cohomologically descendable.

Proof. For any base change by a morphism $\mathfrak{X}' \to \mathfrak{X}$ of lft-triples, the conditions (i)-(iii) are preserved. It is sufficient to prove that w is cohomologically descendable. We may assume that $\mathcal{X}$ is affine by Proposition

4.3.4. Let us take a finite affine Zariski covering $\{\mathcal{Y}_\alpha\}_\alpha$ of $\mathcal{Y}$ and let us denote by $\mathfrak{Y}_\alpha$ the triple induced from the natural open immersion $\mathcal{Y}_\alpha \to \mathcal{Y}$. Since the horizontal arrow of the commutative diagram

$$\begin{array}{ccc} \mathfrak{Y} & \leftarrow & \coprod_\alpha \mathfrak{Y}_\alpha \\ w\downarrow & \swarrow w' & \\ \mathfrak{X} & & \end{array}$$

is universally cohomologically descendable by Proposition 6.2.5, w is cohomologically descendable if and only if w' is so by Theorem 6.3.1. Hence, we may assume that $\mathcal{Y}$ is affine.

Suppose that both $\mathcal{X}$ and $\mathcal{Y}$ are affine. The natural homomorphism $\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; w_.^\dagger E) \to \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; w_.^\dagger E)$ is an isomorphism for any sheaf E of coherent $j^\dagger\mathcal{O}_{]\overline{X}[_\mathcal{X}}$-modules by Theorem 5.2.2 and Corollary 4.4.3. Hence, we have only to prove that the structure morphism $w_.: \mathfrak{Y}_. \to \mathfrak{X}$ of the associated Čech diagram is exact and the canonical homomorphism $E \to \mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; w_.^\dagger E)$ is a quasi-isomorphism. The exactness follows from Proposition 6.1.4.

Let us put $\mathcal{A} = \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ and $\mathcal{B}_n = \Gamma(\mathcal{Y}_n, \mathcal{O}_{\mathcal{Y}_n})$ for any object n of Δ°. We fix a complement ∂X of X in $\overline{X}$. Then $\partial Y_n = \overline{w}_n^{-1}(\partial X)$ is a complement of Y_n in $\overline{Y}_n$ for any n. Let us fix a system of lifts $f_1, \ldots, f_r$ (resp. $g_1, \ldots, g_s$) in $\mathcal{A}$ of generators of the ideal of definition of $\overline{X}$ (resp. ∂X) in $\mathcal{X} \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}\,k$ (resp. $\overline{X}$). Then the image of $f_1, \ldots, f_r$ (resp. $g_1, \ldots, g_s$) in $\mathcal{B}_n$ is also a system of lifts of generators of the ideal of definition of $\overline{Y}_n$ (resp. ∂Y_n) in $\mathcal{Y}_n \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}\,k$ (resp. $\overline{Y}_n$).

We define open affinoid subvarieties

$$\begin{aligned}
[\overline{X}]_{\mathcal{X},\,|\pi|^{1/l+1}} &= \{x \in]\overline{X}[_\mathcal{X} \mid |f_i(x)| \leqq |\pi|^{1/l+1} \text{ for all } i\} \\
&\cong \mathrm{Spm}\left(\frac{\mathcal{A}[u_1, \ldots, u_r]^\frown}{(f_1^{l+1} - \pi u_1, \ldots, f_r^{l+1} - \pi u_r)} \otimes_\mathcal{V} K\right) \\
[\overline{Y}_n]_{\mathcal{Y}_n,\,|\pi|^{1/l+1}} &= \{x \in]\overline{Y}_n[_{\mathcal{Y}_n} \mid |f_i(x)| \leqq |\pi|^{1/l+1} \text{ for all } i\} \\
&\cong \mathrm{Spm}\left(\frac{\mathcal{B}_n[u_1, \ldots, u_r]^\frown}{(f_1^{l+1} - \pi u_1, \ldots, f_r^{l+1} - \pi u_r)} \otimes_\mathcal{V} K\right)
\end{aligned}$$

in $]\overline{X}[_\mathcal{X}$ and $]\overline{Y}[_{\mathcal{Y}_n}$ $(n \in \Delta^\circ)$ for any nonnegative integer l, respectively. Then $\{[\overline{X}]_{\mathcal{X},\,|\pi|^{1/l+1}}\}_l$ (resp. $\{[\overline{Y}_n]_{\mathcal{Y}_n,\,|\pi|^{1/l+1}}\}_l$) is an admissible covering of $]\overline{X}[_\mathcal{X}$ (resp. $]\overline{Y}_n[_{\mathcal{Y}_n}$) by the principle of maximum (Lemma 2.6.7) and we have

$$\tilde{w}_n^{-1}([\overline{X}]_{\mathcal{X},\,|\pi|^{1/l+1}}) = [\overline{Y}_n]_{\mathcal{Y}_n,\,|\pi|^{1/l+1}}.$$

We also define open affinoid subvarieties

$$V_{l,m}^{(j)}=\{x\in[\overline{X}]_{\mathcal{X},|\pi|^{1/l+1}}\,|\,|g_j(x)|\geqq|\pi|^{1/m+1}\text{ and }|g_j(x)|\geqq|g_{j'}(x)|\text{ for all }j'\neq j\}$$
$$\cong\mathrm{Spm}\left(\frac{\mathcal{A}[u_1,\dots,u_r,v_1,\dots,v_s]^{\frown}}{(f_i^{l+1}-\pi u_i(1\leqq i\leqq r),g_j^{m+1}v_j-\pi,g_jv_{j'}-g_{j'}(j'\neq j))}\otimes_{\mathcal{V}}K\right)$$
$$V_{l,m}^{(\infty)}=\{x\in[\overline{X}]_{\mathcal{X},|\pi|^{1/l+1}}\,|\,|g_{j'}(x)|\leqq|\pi|^{1/m+1}\text{ for all }j'\}$$
$$\cong\mathrm{Spm}\left(\frac{\mathcal{A}[u_1,\dots,u_r,v_1,\dots,v_s]^{\frown}}{(f_i^{l+1}-\pi u_i(1\leqq i\leqq r),g_j^{m+1}-\pi v_j(1\leqq j\leqq s))}\otimes_{\mathcal{V}}K\right)$$

in $[\overline{X}]_{\mathcal{X},|\pi|^{1/l+1}}$ for any nonnegative integers l, m and $1\leqq j\leqq s$ and

$$W_{n,l,m}^{(j)}=\{y\in[\overline{Y}_n]_{\mathcal{Y}_n,|\pi|^{1/l+1}}\,|\,|g_j(y)|\geqq|\pi|^{1/m+1}\text{and }|g_j(y)|\geqq|g_{j'}(y)|\text{ for all }j'\neq j\}$$
$$\cong\mathrm{Spm}\left(\frac{\mathcal{B}_n[u_1,\dots,u_r,v_1,\dots,v_s]^{\frown}}{(f_i^{l+1}-\pi u_i(1\leqq i\leqq r),g_j^{m+1}v_j-\pi,g_jv_{j'}-g_{j'}(j'\neq j))}\otimes_{\mathcal{V}}K\right)$$
$$W_{n,l,m}^{(\infty)}=\{y\in[\overline{Y}_n]_{\mathcal{Y}_n,|\pi|^{1/l+1}}\,|\,|g_{j'}(y)|\leqq|\pi|^{1/m+1}\text{for all }j'\}$$
$$\cong\mathrm{Spm}\left(\frac{\mathcal{B}_n[u_1,\dots,u_r,v_1,\dots,v_s]^{\frown}}{(f_i^{l+1}-\pi u_i(1\leqq i\leqq r),g_j^{m+1}-\pi v_j(1\leqq j\leqq s))}\otimes_{\mathcal{V}}K\right)$$

in $[\overline{Y}_n]_{\mathcal{Y}_n,|\pi|^{1/l+1}}$ for any nonnegative integers l, m, any object n in Δ°, and $1\leqq j\leqq s$. Then $\{V_{l,m}^{(j)},\ (1\leqq j\leqq s),\ V_{l,m}^{(\infty)}\}$ (resp. $\{W_{n,l,m}^{(j)},\ (1\leqq j\leqq s),\ W_{n,l,m}^{(\infty)}\}$) is an admissible affinoid covering of $[\overline{X}]_{\mathcal{X},|\pi|^{1/l+1}}$ (resp. $[\overline{Y}_n]_{\mathcal{Y}_n,|\pi|^{1/l+1}}$) and we have

$$\widetilde{w}_n^{-1}(V_{l,m}^{(j)})=W_{n,l,m}^{(j)}$$

for any $1\leqq j\leqq s$ or $j=\infty$. The simplicial rigid analytic space $W_{\bullet,l,m}^{(j)}$ over $V_{l,m}^{(j)}$ is the Čech diagram for $W_{0,l,m}^{(j)}\to V_{l,m}^{(j)}$.

The inverse image of a system of fundamental strict neighbourhoods of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ by $\widetilde{w}_n$ is a system of fundamental strict neighbourhoods of $]Y_n[_{\mathcal{Y}_n}$ in $]\overline{Y}_n[_{\mathcal{Y}_n}$ for all n by condition (i). Hence, for any sheaf $\mathcal{E}$ of $\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules such that $\mathcal{E}$ is coherent on a strict neighbourhood of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$ and $E\cong j^{\dagger}\mathcal{E}$, we have isomorphisms

$$E|_{V_{l,m}^{(j)}}\cong\varinjlim_{m'\geqq m}j_{m,m'*}(\mathcal{E}|_{V_{l,m'}^{(j)}})$$
$$\mathcal{C}^{\dagger}(\mathfrak{X},\mathfrak{Y}_{\bullet};w_{\bullet}^{\dagger}E)|_{V_{l,m}^{(j)}}\cong\varinjlim_{m'\geqq m}j_{m,m'*}\mathcal{C}(V_{l,m'}^{(j)},W_{\bullet,l,m'}^{(j)};\widetilde{w}_{\bullet}^{*}\mathcal{E})$$

by Proposition 2.6.8 and Lemma 2.6.6 (2). Here $j_{m,m'}\colon V_{l,m'}^{(j)}\to V_{l,m}^{(j)}$ is the inclusion. Since E and $\mathcal{C}^{\dagger}(\mathfrak{X},\mathfrak{Y}_{\bullet};w_{\bullet}^{\dagger}E)$ vanish on $V_{l,m}^{(\infty)}$ for any m, we have only to prove that the morphism $W^{(j)}_{0,l,m} \to V^{(j)}_{l,m}$ of rigid analytic spaces arises from the morphism of formal $\mathcal{V}$-schemes which satisfies the assumption of Proposition 7.1.1 for any $1 \leqq j \leqq s$ and any m.

Since the homomorphism

$$(*) \qquad \frac{\mathcal{A}[u_1, \ldots, u_r, v_1, \ldots, v_s]^\frown}{(f_i^{l+1} - \pi u_i \ (1 \leqq i \leqq r),\ g_j^{m+1} v_j - \pi,\ g_j v_{j'} - g_{j'}\ (j' \neq j))} \to$$

$$\to \frac{\mathcal{B}[u_1, \ldots, u_r, v_1, \ldots, v_s]^\frown}{(f_i^{l+1} - \pi u_i \ (1 \leqq i \leqq r),\ g_j^{m+1} v_j - \pi,\ g_j v_{j'} - g_{j'}\ (j' \neq j))}$$

modulo π^{e+1} is a base change of the morphism $\mathcal{A}/\pi^{e+1} \to \mathcal{B}/\pi^{e+1}$ for any e, it is flat by (ii). The topological spaces of

$$\mathrm{Spec}\,(\mathcal{A}/\pi)[u, v]/(f_i^{l+1}, g_j^{m+1} v_j, g_j v_{j'} - g_{j'} \quad (j' \neq j))$$

and

$$\mathrm{Spec}\,(\mathcal{A}/\pi)[u, v]/(f_i, g_j v_j, g_j v_{j'} - g_{j'} \quad (j' \neq j))$$

are the same. The same holds for $\mathcal{Y}_0$. Note that $\overline{X} = \mathrm{Spec}\ \mathcal{A}/(\pi, f_i)$ and $\overline{Y} = \mathrm{Spec}\ \mathcal{B}_0/(\pi, f_i)$. The morphism $(*)$ modulo π is surjective by condition (iii). Hence, the morphism $\widetilde{w}_0\,|_{W^{(j)}_{0,l,m}}: W^{(j)}_{0,l,m} \to V^{(j)}_{l,m}$ satisfies both conditions (i) and (ii) of Proposition 7.1.1.

This completes the proof of Proposition 7.1.2. ■

7.2. We give definitions of two types of hypercovering of triples.

7.2.1. Definition. (1) Let (P) be a property of morphisms of (formal) schemes such that (i) it is stable under any base change and (ii) under the assumption f is (P), fg is (P) if and only if g is (P). A simplicial (formal) scheme $Y_.$ over a (formal) scheme X is (P) if Y_n is (P) over X for any n.

(2) Let $Y_.$ be a simplicial (formal) scheme separated of finite type over a (formal) scheme X. We say that $Y_. \to X$ is an étale (resp. proper) hypercovering if the canonical morphism $Y_n \to \mathrm{cosk}^X_{n-1}(Y^{(n-1)}_.)_n$ is étale surjective (resp. proper surjective) for any n.

(3) Let $(X, \overline{X})$ be a pair of schemes and let $(Y_., \overline{Y}_.)$ be a simplicial pair of schemes separated of finite type over $(X, \overline{X})$. We say that $(Y_., \overline{Y}_.) \to (X, \overline{X})$ is an étale-étale hypercovering if $Y_. \to X$ and $\overline{Y}_. \to \overline{X}$ are étale hypercoverings and $(Y_n, \overline{Y}_n) \to (X, \overline{X})$ is strict as a morphism of pairs for any n.

(4) Let $(X, \overline{X})$ be a pair of schemes and let $(Y_., \overline{Y}_.)$ be a simplicial pair of schemes separated of finite type over $(X, \overline{X})$. We say that $(Y_., \overline{Y}_.) \to$ $\to (X, \overline{X})$ is an étale-proper hypercovering if $Y_. \to X$ is an étale hypercovering and $\overline{Y}_. \to \overline{X}$ is proper.

We also define the n-truncated version of étale (proper) simplicial (formal) schemes, étale (resp. proper) hypercoverings, étale-étale hypercoverings, and étale-proper hypercoverings in the same way. ■

The definition (1) above for a simplicial (formal) scheme $Y_.$ is equivalent to the condition that for each n the canonical morphism $Y_n \to \to \mathrm{cosk}_{n-1}(Y_.^{(n-1)})_n$ is (P). As a corollary we get that for an étale (resp. proper) hypercovering $Y_.$, all the morphisms $Y_n \to Y_m$, $n \geq m$ which come from the simplicial structure are étale (resp. proper).

7.2.2. Definition. Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ be a $\mathcal{V}$-triple locally of finite type. A simplicial triple $\mathfrak{Y}_. = (Y_., \overline{Y}_., \mathcal{Y}_.)$ separated locally of finite type over $\mathfrak{X}$ is an étale-étale hypercovering (resp. an étale-proper hypercovering) if it satisfies the following conditions:

(i) $(Y_., \overline{Y}_.) \to (X, \overline{X})$ is an étale-étale hypercovering (resp. an étale-proper hypercovering);

(ii) $\mathrm{cosk}_n^{\mathcal{X}}(\mathcal{Y}_.^{(n)})_l \to \mathrm{cosk}_{n-1}^{\mathcal{X}}(\mathcal{Y}_.^{(n-1)})_l$ is smooth around $\mathrm{cosk}_n^X(Y_.^{(n)})_l$ for any n and l. ■

Condition (ii) above implies that $\mathcal{Y}_. \to \mathcal{X}$ (resp. $\mathrm{cosk}_n^{\mathcal{X}}(\mathcal{Y}_.^{(n)}) \to \mathcal{X}$) is smooth around $Y_.$ (resp. $\mathrm{cosk}_n^X(Y_.^{(n)})$).

By definition we have

7.2.3. Lemma. (1) *Let $(X, \overline{X})$ be a pair of schemes and let $(Y_., \overline{Y}_.)$ be an étale-étale (resp. étale-proper) hypercovering over $(X, \overline{X})$. Then* $\mathrm{cosk}_n^{(X,\overline{X})}((Y_.^{(n)}, \overline{Y}_.^{(n)}))$ *is also an étale-étale (resp. étale-proper) hypercovering over $(X, \overline{X})$ for all n.*

(2) *Let $\mathfrak{X}$ be a $\mathcal{V}$-triple locally of finite type and let $\mathfrak{Y}_.$ be an étale-étale (resp. étale-proper) hypercovering over $\mathfrak{X}$. Then* $\mathrm{cosk}_n^{\mathfrak{X}}(\mathfrak{Y}_.^{(n)})$ *is also an étale-étale (resp. étale-proper) hypercovering over $\mathfrak{X}$ for all n.*

7.3. We give a cohomological descent theorem for étale-étale coverings.

7.3.1. Theorem. *Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ and $\mathfrak{Y} = (Y, \overline{Y}, \mathcal{Y})$ be $\mathcal{V}$-triples locally of finite type and let $w : \mathfrak{Y} \to \mathfrak{X}$ be a separated morphism which satisfies the following conditions:*

(i) *$\widehat{w} : \mathcal{Y} \to \mathcal{X}$ is smooth around Y;*
(ii) *$\overline{w} : \overline{Y} \to \overline{X}$ is étale surjective;*
(iii) *$\overline{w}^{-1}(X) = Y$.*

Then w is universally cohomologically descendable.

Note that the Čech diagram for w is an étale-étale hypercovering.

PROOF. We may assume that $\mathcal{X}$ is affine by Theorem 6.3.1 and Proposition 6.2.5. We put $\mathcal{A} = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $A = \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$. It is sufficient to find a finite Zariski covering $\coprod_{\alpha} \overline{Y}_{\alpha}$ of $\overline{Y}$ and a formal scheme $\mathcal{Z}_{\alpha}$ flat of finite type over $\mathcal{X}$ with an isomorphism $\overline{Y}_{\alpha} \cong \mathcal{Z}_{\alpha} \times_{\mathcal{X}} \overline{X}$ for each α such that the structure morphism $\widehat{u}_{\alpha}: \mathcal{Z}_{\alpha} \to \mathcal{X}$ is smooth around $Y_{\alpha} = Y \cap \overline{Y}_{\alpha}$. Indeed, if $\mathcal{Y}_{\alpha}$ is an open formal subscheme of $\mathcal{Y}$ such that $\overline{Y}_{\alpha}$ is a closed subscheme of $\mathcal{Y}_{\alpha} \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spec}\, k$, then w is universally cohomologically descendable if and only if $\coprod_{\alpha} (Y_{\alpha}, \overline{Y}_{\alpha}, \mathcal{Y}_{\alpha}) \to \mathfrak{X}$ is so by Theorem 6.3.1, Propositions 6.2.5 and 4.3.2. Now $\coprod_{\alpha} (Y_{\alpha}, \overline{Y}_{\alpha}, \mathcal{Z}_{\alpha}) \to \mathfrak{X}$ is universally cohomologically descendable by Proposition 7.1.2 (note that we use the hypothesis (i) to have flatness, (ii) for surjectivity and strictness follows from our construction), hence the universal cohomological descendability of $\coprod_{\alpha} (Y_{\alpha}, \overline{Y}_{\alpha}, \mathcal{Y}_{\alpha}) \to \mathfrak{X}$ follows from Proposition 6.4.4. Here we use hypothesis (i).
Let $\coprod_{\alpha} \overline{Y}_{\alpha} \to \overline{Y}$ be a finite affine Zariski covering such that $\overline{Y}_{\alpha} \to \overline{X}$ is a standard étale morphism for any α. In other words, if we put $B_{\alpha} = \Gamma(\overline{Y}_{\alpha}, \mathcal{O}_{\overline{Y}_{\alpha}})$, then there exist a monic polynomial $p_{\alpha} \in A[t]$ and an element $h_{\alpha} \in A[t]/(p_{\alpha})$ such that the derivation p'_{α} of p_{α} is a unit of $(A[t]/(p_{\alpha}))[h_{\alpha}^{-1}]$ and there exists an isomorphism

$$B_{\alpha} \cong (A[t]/(p_{\alpha}))[h_{\alpha}^{-1}]$$

of A-algebras. Since $\overline{w}: \overline{Y} \to \overline{X}$ is étale of finite type, such a covering always exists [18, Theorem 3.14] and we fix it as above. We put $Y_{\alpha} = Y \cap \overline{Y}_{\alpha}$.

Let us fix a lift $\widehat{p}_{\alpha}$ (resp. $\widehat{h}_{\alpha}$) of p_{α} (resp. h_{α}) in $\mathcal{A}[t]$ (resp. $\mathcal{A}[t]/(\widehat{p}_{\alpha})$) such that $\widehat{p}_{\alpha}$ is monic for any α. We define a formal $\mathcal{V}$-algebra $\mathcal{B}_{\alpha}$ by

$$\mathcal{B}_{\alpha} = (\mathcal{A}[t]/\widehat{p}_{\alpha})[\widehat{h}_{\alpha}^{-1}]^{\frown}$$

and put $\mathcal{Z}_{\alpha} = \mathrm{Spf}\, \mathcal{B}_{\alpha}$. Since $\widehat{p}_{\alpha}$ is monic, $\mathcal{B}_{\alpha}$ is flat over $\mathcal{A}$. Since p'_{α} is a unit on Y_{α}, $\mathrm{Spf}\, \mathcal{B}_{\alpha}[(\widehat{p}'_{\alpha})^{-1}]^{\frown}$ is étale over $\mathrm{Spf}\, \mathcal{A}$ and it includes Y_{α}. Moreover, we have $\widehat{u}_{\alpha}^{-1}(\overline{X}) = \overline{Y}_{\alpha}$ by construction. Hence the hypothesis (i) of Proposition 7.1.2 holds.

This completes the proof of Theorem 7.3.1. ■

7.3.2. REMARK. We can take $\mathcal{B}_{\alpha} = (\mathcal{A}[t]/\widehat{p}_{\alpha})[(\widehat{h}_{\alpha}\widehat{p}'_{\alpha})^{-1}]^{\frown}$ in the proof above. Then $\widehat{w}_{\alpha}$ is étale. ■

By Corollary 6.5.4, Theorem 7.3.1 and Lemma 7.2.3 we have

7.3.3. Corollary. *Let* $\mathfrak{Y}_.$ *be an étale-étale hypercovering over a* $\mathcal{V}$-*triple* $\mathfrak{X}$ *locally of finite type. Then* $\mathfrak{Y}_. \to \mathfrak{X}$ *is universally cohomologically descendable.*

7.4. We give a cohomological descent theorem for étale-proper covering.

7.4.1. Theorem. *Let* $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ *and* $\mathfrak{Y} = (Y, \overline{Y}, \mathcal{Y})$ *be* $\mathcal{V}$-*triples locally of finite type and let* $w : \mathfrak{Y} \to \mathfrak{X}$ *be a separated morphism which satisfies the following conditions:*

(i) $\widehat{w} : \mathcal{Y} \to \mathcal{X}$ *is smooth around* Y;
(ii) $\overline{w} : \overline{Y} \to \overline{X}$ *is proper*;
(iii) $\overset{\circ}{w} : Y \to X$ *is étale surjective.*

Then w *is universally cohomologically descendable.*

Note that the Čech diagram for w is an étale-proper hypercovering. First we prove the following lemma.

7.4.2. Lemma. *Let* $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ *be a* $\mathcal{V}$-*triple of finite type such that* $\mathcal{X}$ *is affine with* $A = \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$ *and* $\mathcal{A} = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *and let* $\mathfrak{Y} = (Y, \overline{Y}, \mathcal{Y})$ *be an* $\mathfrak{X}$-*triple of finite type which satisfies the following conditions:*

(i) *there exists a monic polynomial* $p(x) \in A[x]$ *with* $\overline{Y} = \operatorname{Spec} B$ for $B = A[x]/(p(x))$;

(ii) $\mathcal{Y} = \operatorname{Spf} \mathcal{B}$ *with* $\mathcal{B} = \mathcal{A}[x]/(\widehat{p}(x))$ *for some lift* $\widehat{p}(x) \in \mathcal{A}[x]$ *of* $p(x)$ *as a monic polynomial*;

(iii) $Y \to X$ *is étale surjective.*

Then the structure morphism $w : \mathfrak{Y} \to \mathfrak{X}$ *is universally cohomologically descendable.*

Proof. We prove the assertion by the induction on the degree of $p(x)$. Note that the assumption implies that $\widehat{w} : \mathcal{Y} \to \mathcal{X}$ is étale around Y. If the degree of $p(x)$ is 1, then w is identity and there is nothing to prove. Suppose that the degree of $p(x)$ is greater than 1. We put $Z = \overline{w}^{-1}(X)$ and $\mathfrak{Z} = (Z, \overline{Y}, \mathcal{Y})$. Then the structure morphism $\mathfrak{Z} \to \mathfrak{X}$ is universally cohomologically descendable by Proposition 7.1.2. Hence the first projection $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z} \to \mathfrak{Y}$ is universally cohomologically descendable by definition.

Therefore, we have only to prove that the second projection $v : \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z} \to \to \mathfrak{Z}$ is universally cohomologically descendable by Theorem 6.3.1.

Let $p(y)$ (resp. $\widehat{p}(y)$) be a monic polynomial in $B[y]$ (resp. $\mathcal{B}[y]$) such that the variable x is replaced by y. Then we have

$$\overline{Y} \times_{\overline{X}} \overline{Y} \cong \operatorname{Spec} B[y]/(p(y))$$
$$\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \cong \operatorname{Spf} \mathcal{B}[y]/(\widehat{p}(y)) .$$

If we denote by $\overline{x}$ the image of x in B (resp. $\mathcal{B}$), then we have a decomposition $p(y) = (y - \overline{x})\, q(y)$ (resp. $\widehat{p}(y) = (y - \overline{x})\, \widehat{q}(y)$) with a monic polynomial $q(y) \in B[y]$ (resp. $\widehat{q}(y) \in \mathcal{B}[y]$ which is a lift of $q(y)$). Note that the ideal generated by the image of $y - \overline{x}$ and $q(y)$ in $\mathcal{O}_{\overline{Y} \times_{\overline{X}} \overline{Y}}$ is a unit ideal on $Y \times_X Z$. Indeed, $p'(y) = q'(y)(y - \overline{x}) + q(y)$ is a unit on $Y \times_X Z$ since the second projection $Y \times_X Z \to Z$ is étale.

We put $\mathfrak{W} = (W, \operatorname{Spec} B[y]/(q(y)), \operatorname{Spf} \mathcal{B}[y]/(\widehat{p}(y)))$, where W is the inverse image of $Y \times_X Z$ by the natural closed immersion $\operatorname{Spec} B[y]/(q(y)) \to \operatorname{Spec} B[y]/(p(y))$. Then we have a strict morphism

$$\mathfrak{Y} \coprod \mathfrak{W} \to \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z}$$

as a morphism of triples. (the first $\mathfrak{Y}$ is the diagonal component.) Since the ideal generated by the image of $y - \overline{x}$ and $q(y)$ in $\mathcal{O}_{\overline{Y} \times_{\overline{X}} \overline{Y}}$ is a unit ideal on $Y \times_X Z$, we have $Y \coprod W \cong Y \times_X Z$. Hence, it is sufficient to prove the composition morphism $v : \mathfrak{Y} \coprod \mathfrak{W} \to \mathfrak{Z}$ is universally cohomologically descendable by Corollary 6.4.2.

Now we decompose the morphism v as

$$\mathfrak{Y} \coprod \mathfrak{W} \to v(\mathfrak{Y}) \coprod v(\mathfrak{W}) \to \mathfrak{Z} ,$$

Then $\mathfrak{Y} \to v(\mathfrak{Y})$ is an isomorphism and $\mathfrak{W} \to v(\mathfrak{W}) = \mathfrak{Z}$ satisfies the assumption with the degree $\deg q(x) = \deg p(x) - 1$. Hence, the first morphism $\mathfrak{Y} \coprod \mathfrak{W} \to v(\mathfrak{Y}) \coprod v(\mathfrak{W})$ is universally cohomologically descendable by the hypothesis of induction. The second morphism $v(\mathfrak{Y}) \coprod v(\mathfrak{W}) \to \mathfrak{Z}$ is universally cohomologically descendable by Proposition 6.2.6. Hence v is so. This completes the proof. ■

PROOF OF THEOREM 7.4.1. We may assume that $\mathcal{X}$ is affine by Theorem 6.3.1 and Proposition 6.2.5. We put $\mathcal{A} = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $A = = \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$.

Let $\coprod_\alpha Y_\alpha \to Y$ be a finite affine Zariski covering such that $Y_\alpha \to \overline{X}$ is a standard étale morphism for any α. In other words there exist a monic polynomial $p_\alpha \in A[x]$ and an element $h_\alpha \in A[x]/(p_\alpha)$ such that the derivation p'_α of p_α is a unit of $(A[t]/(p_\alpha))[h_\alpha^{-1}]$ and there exists an isomorphism

$$\Gamma(Y_\alpha, \mathcal{O}_{Y_\alpha}) \cong (A[t]/(p_\alpha))[h_\alpha^{-1}]$$

of A-algebras. Take a lift $\widehat{p}_\alpha \in \mathcal{A}[x]$ such that $\widehat{p}_\alpha$ is monic. We put $\overline{Y}_\alpha = \mathrm{Spec}\, A[x]/(p_\alpha(x))$, $\mathcal{Y}_\alpha = \mathrm{Spf}\, \mathcal{A}[x]/(\widehat{p}_\alpha(x))$ and denote the triple $(Y_\alpha, \overline{Y}_\alpha, \mathcal{Y}_\alpha)$ over $\mathfrak{X}$ by $\mathfrak{Y}_\alpha$. By the hypotheses (i) and (ii) we can apply Corollary and we have only to prove that the morphism $\coprod_\alpha \mathfrak{Y}_\alpha \to \mathfrak{X}$ is universally cohomologically descendable by Theorem 6.3.1 and Proposition 6.2.6.

If we denote by X_α the image of Y_α in X, then X_α is open in X since $w : Y \to X$ is étale. The natural morphism $\mathfrak{Y}_\alpha \to \mathfrak{X}_\alpha = (X_\alpha, \overline{X}, \mathcal{X})$ is universally cohomologically descendable by Lemma 7.4.2 and the natural morphism $\coprod_\alpha \mathfrak{X}_\alpha \to \mathfrak{X}$ is universally cohomologically descendable by Proposition 6.2.6. Hence $\coprod_\alpha \mathfrak{Y}_\alpha \to \mathfrak{X}$ is also so by Theorem 6.3.1. This completes the proof of Theorem 7.4.1. ■

By Corollary 6.5.4, Theorem 7.4.1 and Lemma 7.2.3 we have

7.4.3. COROLLARY. *Let $\mathfrak{Y}_.$ be an étale-proper hypercovering over a $\mathcal{V}$-triple $\mathfrak{X}$ locally of finite type. Then $\mathfrak{Y}_. \to \mathfrak{X}$ is universally cohomologically descendable.*

8. De Rham descent.

In this section we introduce a notion of universally de Rham descent for a morphism of triples.

8.1. The following proposition guarantees the existence of smooth strict neighborhoods.

8.1.1. PROPOSITION [7, 2.2.1 Lemma]. *Let $w : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of $\mathcal{V}$-triples of finite type such that $\widehat{w} : \mathcal{Y} \to \mathcal{X}$ is smooth around Y. Then there exists a strict neighbourhood V of $]Y[_{\mathcal{Y}}$ in $]\overline{Y}[_{\mathcal{Y}}$ such that V is smooth over $]\overline{X}[_{\mathcal{X}}$.*

If $w : \mathfrak{Y} \to \mathfrak{X}$ is a morphism of $\mathcal{V}$-triples locally of finite type such that $\widehat{w} : \mathcal{Y} \to \mathcal{X}$ is smooth around Y, then one can find a smooth strict neighborhood V of $]Y[_{\mathcal{Y}}$ in $]\overline{Y}[_{\mathcal{Y}}$ over $]\overline{X}[_{\mathcal{X}}$ locally on $\mathcal{Y}$. Hence, the sheaf $j^\dagger \Omega^1_{]\overline{Y}[_{\mathcal{Y}}/]\overline{X}[_{\mathcal{X}}}$ of overconvergent 1-forms of $]\overline{Y}[_{\mathcal{Y}}$ over $]\overline{X}[_{\mathcal{X}}$ is locally free over $j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}$ by Proposition 2.7.3. Moreover, we have

8.1.2. Lemma. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be $\mathcal{V}$-triples of finite type and let $w : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of $\mathcal{V}$-triples which satisfies the conditions:*

(i) *both $\mathcal{X}$ and $\mathcal{Y}$ are affine;*

(ii) *there exists a finite number of sections $t_1, \dots t_d \in \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ which determines an étale morphism from $\mathcal{Y}$ to a formal affine space $\widehat{\mathbb{A}}^d_{\mathcal{X}}$ around Y such that $\widehat{w}$ is a composition of $\mathcal{Y} \to \widehat{\mathbb{A}}^d_{\mathcal{X}}$ and the projection $\widehat{\mathbb{A}}^d_{\mathcal{X}} \to \mathcal{X}$.*

Then $j^\dagger \Omega^1_{]\overline{Y}[_{\mathcal{Y}}/]\overline{X}[_{\mathcal{X}}}$ is free of rank d over $j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}$.

Proof. Let us denote by $\mathfrak{I}$ the sheaf of ideals of $\mathcal{O}_{]\overline{Y}[_{\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}}}$ which corresponds to the diagonal immersion $]\overline{Y}[_{\mathcal{Y}} \to]\overline{Y}[_{\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}}$. Since $\mathcal{Y}$ is étale over $\widehat{\mathbb{A}}^d_{\mathcal{X}}$ around Y, the images dt_i of $1 \otimes t_i - t_i \otimes 1$ $(1 \leqq i \leqq d)$ in $\mathfrak{I}/\mathfrak{I}^2$ form a basis on $]Y[_{\mathcal{Y}}$. Since $j^\dagger \Omega^1_{]\overline{Y}[_{\mathcal{Y}}/]\overline{X}[_{\mathcal{X}}} = j^\dagger(\mathfrak{I}/\mathfrak{I}^2)$ is a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}$-modules, $j^\dagger \Omega^1_{]\overline{Y}[_{\mathcal{Y}}/]\overline{X}[_{\mathcal{X}}}$ is free over $j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}$ with a basis dt_i $(1 \leqq i \leqq d)$ by Lemma 2.9.1. ■

Locally on $\mathcal{Y}$ and Y, the hypotheses of the previous lemma hold (see the proof of Theorem 8.5.1).

8.2. Let $(w_., t) : (\mathfrak{Y}_., J) \to (\mathfrak{X}_., I)$ be a morphism of diagrams of $\mathcal{V}$-triples locally of finite type such that $\widehat{w}_. : \mathcal{Y}_. \to \mathcal{X}_.$ is smooth around $Y_.$, that is, $\widehat{w}_n : \mathcal{Y}_n \to \mathcal{X}_{t(n)}$ is smooth around Y_n for each object n of J. For a sheaf $\mathcal{F}_.$ of $j^\dagger \mathcal{O}_{]\overline{Y}_.[_{\mathcal{Y}_.}}$-modules, we say that a $\tilde{w}^{-1}(j^\dagger \mathcal{O}_{]\overline{X}_.[_{\mathcal{X}_.}})$-linear homomorphism

$$\nabla_. : \mathcal{F}_. \to \mathcal{F}_. \otimes_{j^\dagger \mathcal{O}_{]\overline{Y}_.[_{\mathcal{Y}_.}}} j^\dagger \Omega^1_{]\overline{Y}_.[_{\mathcal{Y}_.}/]\overline{X}_.[_{\mathcal{X}_.}}$$

is a connection on $]\overline{Y}_.[_{\mathcal{Y}_.}$ over $]\overline{X}_.[_{\mathcal{X}_.}$ if (i) ∇_n is a connection on $]\overline{Y}_n[_{\mathcal{Y}_n}$ over $]\overline{X}_{t(n)}[_{\mathcal{X}_{t(n)}}$ for each object n of J and (ii) for each morphism $\eta : m \to n$ of J, the diagram

$$\begin{array}{ccc}
\tilde{\eta}_{\mathfrak{Y}_.}^{-1} \mathcal{F}_n & \xrightarrow{\tilde{\eta}_{\mathcal{Y}_.}^{-1}(\nabla_n)} & \tilde{\eta}_{\mathcal{Y}_.}^{-1}(\mathcal{F}_n \otimes_{j^\dagger \mathcal{O}_{]\overline{Y}_n[_{\mathcal{Y}_n}}} j^\dagger \Omega^1_{]\overline{Y}_n[_{\mathcal{Y}_n}/]\overline{X}_{t(n)}[_{\mathcal{X}_{t(n)}}}) \\
\mathcal{F}_.(\eta) \downarrow & & \downarrow \mathcal{F}_.(\eta) \otimes \tilde{\eta}^\sharp_{\mathfrak{Y}_.} \\
\mathcal{F}_m & \xrightarrow[\nabla_m]{} & \mathcal{F}_m \otimes_{j^\dagger \mathcal{O}_{]\overline{Y}_m[_{\mathcal{Y}_m}}} j^\dagger \Omega^1_{]\overline{Y}_m[_{\mathcal{Y}_m}/]\overline{X}_{t(m)}[_{\mathcal{X}_{t(m)}}}
\end{array}$$

is commutative. A homomorphism of sheaves of $j^\dagger \mathcal{O}_{]\overline{Y}_.[_{\mathcal{Y}_.}}$-modules with connections is a $j^\dagger \mathcal{O}_{]\overline{Y}.[_{\mathcal{Y}.}}$-homomorphism which commutes with connections.

Let $(\mathcal{F}., \nabla.)$ be a sheaf $\mathcal{F}.$ of $j^\dagger \mathcal{O}_{]\overline{Y}.[_{\mathcal{Y}.}}$-modules with a connection $\nabla.$ on $]\overline{Y}.[_{\mathcal{Y}.}$ over $]\overline{X}.[_{\mathcal{X}.}$. Then $\nabla.$ determines a sequence

$$\begin{array}{ccccc} \cdots \to \mathcal{F}. \otimes_{j^\dagger \mathcal{O}_{]\overline{Y}.[_{\mathcal{Y}.}}} j^\dagger \Omega^r_{]\overline{Y}.[_{\mathcal{Y}.}/]\overline{X}.[_{\mathcal{X}.}} & \to & \mathcal{F}. \otimes_{j^\dagger \mathcal{O}_{]\overline{Y}.[_{\mathcal{Y}.}}} j^\dagger \Omega^{r+1}_{]\overline{Y}.[_{\mathcal{Y}.}/]\overline{X}.[_{\mathcal{X}.}} & \to \cdots \\ f \otimes \omega & \mapsto & (-1)^r \nabla.(f) \otimes \omega + f \otimes d.^r \omega. & \end{array}$$

of sheaves of $\widetilde{w}.^{-1}(j^\dagger \mathcal{O}_{]\overline{X}.[_{\mathcal{X}.}})$-modules, where $d.^r: j^\dagger \Omega^r_{]\overline{Y}.[_{\mathcal{Y}.}/]\overline{X}.[_{\mathcal{X}.}} \to$ $\to j^\dagger \Omega^{r+1}_{]\overline{Y}.[_{\mathcal{Y}.}/]\overline{X}.[_{\mathcal{X}.}}$ is a $\widetilde{w}.^{-1}(j^\dagger \mathcal{O}_{]\overline{X}.[_{\mathcal{X}.}})$-homomorphism which is induced by the trivial connection $d.$ on $j^\dagger \mathcal{O}_{]\overline{X}.[_{\mathcal{X}.}}$. We say that a connection $\nabla.$ is integrable if the sequence above is a complex. In this case we denote the complex above by $\mathrm{DR}^\dagger(\mathfrak{Y}./\mathfrak{X}., (\mathcal{F}., \nabla.))$.

Let $\mathfrak{Z}. \to \mathfrak{Y}.$ be a morphism of diagrams of triples locally of finite type over $\mathfrak{X}.$ such that $\mathcal{Y}. \to \mathcal{X}.$ (resp. $\mathcal{Z}. \to \mathcal{Y}.$) is smooth around $Y.$ (resp. $Z.$). Let $(\mathcal{G}., \nabla.)$ be a sheaf $\mathcal{G}.$ of $j^\dagger \mathcal{O}_{]\overline{Z}.[_{\mathcal{Z}.}}$-modules with a connection on $]\overline{Z}.[_{\mathcal{Z}.}$ over $]\overline{X}.[_{\mathcal{X}.}$. Then $\nabla.$ induces a relative connection

$$\overline{\nabla}.: \mathcal{G}. \to \mathcal{G}. \otimes_{j^\dagger \mathcal{O}_{]\overline{Z}.[_{\mathcal{Z}.}}} j^\dagger \Omega^1_{]\overline{Z}.[_{\mathcal{Z}.}/]\overline{Y}.[_{\mathcal{Y}.}}$$

by $\mathcal{G}. \xrightarrow{\nabla} \mathcal{G}. \otimes_{j^\dagger \mathcal{O}_{]\overline{Z}.[_{\mathcal{Z}.}}} j^\dagger \Omega^1_{]\overline{Z}.[_{\mathcal{Z}.}/]\overline{X}.[_{\mathcal{X}.}} \to \mathcal{G}. \otimes_{j^\dagger \mathcal{O}_{]\overline{Z}.[_{\mathcal{Z}.}}} j^\dagger \Omega^1_{]\overline{Z}.[_{\mathcal{Z}.}/]\overline{Y}.[_{\mathcal{Y}.}}$, where the second morphism is the natural surjection. If ∇ is integrable, then $\overline{\nabla}$ is also integrable. We denote by $\mathrm{DR}^\dagger(\mathfrak{Z}./\mathfrak{Y}., (\mathcal{G}., \nabla.))$ the relative complex $\mathrm{DR}^\dagger(\mathfrak{Z}./\mathfrak{Y}., (\mathcal{G}., \overline{\nabla}.))$.

Let $u.: \mathfrak{Z}. \to \mathfrak{Y}.$ be a morphism of diagrams of triples locally of finite type over $\mathfrak{X}.$ such that $\mathcal{Y}. \to \mathcal{X}.$ (resp. $\mathcal{Z}. \to \mathcal{X}.$) is smooth around $Y.$ (resp. $Z.$). For a sheaf $\mathcal{F}.$ of $j^\dagger \mathcal{O}_{]\overline{Y}.[_{\mathcal{Y}.}}$-modules with a connection $\nabla.$ on $]\overline{Y}.[_{\mathcal{Y}.}$ over $]\overline{X}.[_{\mathcal{X}.}$, we define a connection on $]\overline{Z}.[_{\mathcal{Z}.}$ over $]\overline{X}.[_{\mathcal{X}.}$

$$\widetilde{u}.^* \nabla.: \widetilde{u}.^* \mathcal{F}. \to \widetilde{u}.^* \mathcal{F}. \otimes_{j^\dagger \mathcal{O}_{]\overline{Z}.[_{\mathcal{Z}.}}} j^\dagger \Omega^1_{]\overline{Z}.[_{\mathcal{Z}.}/]\overline{X}.[_{\mathcal{X}.}}$$

by $f \otimes a \mapsto \nabla(f) \otimes a + f \otimes da$ ($f \in \mathcal{F}.$, $a \in j^\dagger \mathcal{O}_{]\overline{Z}.[_{\mathcal{Z}.}}$). Then $\widetilde{u}.^* \nabla.$ is integrable if $\nabla.$ is so. In the case where $\mathcal{F}.$ is a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{Y}.[_{\mathcal{Y}.}}$-modules, we denote by $u^\dagger(\mathcal{F}., \nabla.)$ the inverse image instead of $\widetilde{u}^*(\mathcal{F}., \nabla.)$ (see 3.5). Moreover, if $v.: \mathfrak{W}. \to \mathfrak{Z}.$ is a morphism of diagrams of triples locally of finite type over $\mathfrak{X}.$ such that $\mathcal{W}. \to \mathcal{Z}.$ is smooth around $W.$, then we have $\widetilde{uv}^*(\mathcal{F}., \nabla.) = \widetilde{v}^* \widetilde{u}^*(\mathcal{F}., \nabla.)$.

Now let $(\mathcal{G}., \nabla.)$ be a sheaf of $j^\dagger \mathcal{O}_{]\overline{Z}.[_{\mathcal{Z}.}}$-modules with a connection on $]\overline{Z}.[_{\mathcal{Z}.}$ over $]\overline{X}.[_{\mathcal{X}.}$ and let $\varphi : \widetilde{u}.^*(\mathcal{F}., \nabla.) \to (\mathcal{G}., \nabla.)$ be a homomorphism of sheaves of $j^{\dagger}\mathcal{O}_{]\overline{Z}.[_{\mathcal{Z}.}}$-modules with connections. Then φ induces a homomorphism

$$\tilde{u}_{\cdot}^{-1}\mathrm{DR}^{\dagger}(\mathfrak{Y}./\mathfrak{X}.,(\mathcal{F}., \nabla.)) \to \mathrm{DR}^{\dagger}(\mathfrak{Z}./\mathfrak{X}.,(\mathcal{G}., \nabla.)).$$

of complexes of sheaves of $\tilde{u}_{\cdot}^{-1}\tilde{w}_{\cdot}^{-1}(j^{\dagger}\mathcal{O}_{]\overline{X}.[_{\mathcal{X}.}})$-modules.

8.3. Let $\mathfrak{X}$ be a $\mathcal{V}$-triple locally of finite type and let $\mathfrak{Y}.$ be a simplicial $\mathfrak{X}$-triples locally of finite type with a structure homomorphism $w.: \mathfrak{Y}. \to \mathfrak{X}$ such that $\mathcal{Y}.$ is smooth over $\mathcal{X}$ around $Y.$.

8.3.1. Definition. (1) We say that $w.$ is de Rham descendable if, for any sheaf E of coherent $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules, the canonical homomorphism

$$E \to \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X}, \mathfrak{Y}.; \mathrm{DR}^{\dagger}(\mathfrak{Y}./\mathfrak{X}, w_{\cdot}^{\dagger}(E, 0)))$$

is an isomorphism in $D^{+}(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$. Here 0 means the 0-connection $E \to 0$ on $]\overline{X}[_{\mathcal{X}}/]\overline{X}[_{\mathcal{X}}$.

(2) We say that $w.$ is universally de Rham descendable if, for every morphism $\mathfrak{Z} \to \mathfrak{X}$ of lft-triples, the base change morphism $\mathfrak{Y}. \times_{\mathfrak{X}} \mathfrak{Z} \to \mathfrak{Z}$ is de Rham descendable. ■

Note that $w_{\cdot}^{\dagger}$ is automatically exact by Proposition 6.1.4.

8.3.2. Definition. Let $w: \mathfrak{Y} \to \mathfrak{X}$ be a morphism of $\mathcal{V}$-triples locally of finite type and let $w.: \mathfrak{Y}. \to \mathfrak{X}$ be a Čech diagram for w. We say that w is de Rham descendable (resp. universally de Rham descendable) if $w.$ is so. ■

8.3.3. Example. (1) Let $w: \mathfrak{Y} \to \mathfrak{X}$ be a morphism of $\mathcal{V}$-triples locally of finite type such that $\widehat{w}: \mathcal{Y} \to \mathcal{X}$ is a Zariski covering and w is strict as a morphism of triples. Then w is universally de Rham descendable by Proposition 6.2.5 since $\Omega^{1}_{]\overline{Y}[_{\mathcal{Y}}/]\overline{X}[_{\mathcal{X}}} = 0$.

(2) Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ be a $\mathcal{V}$-triple locally of finite type and let $\coprod_{\alpha} X_{\alpha} \to X$ be a finite Zariski covering. Put $\mathfrak{X}_{\alpha} = (X_{\alpha}, \overline{X}, \mathcal{X})$, and denote by $w: \coprod_{\alpha} \mathfrak{X}_{\alpha} \to \mathfrak{X}$ the structure morphism. Then w is universally de Rham descendable by Proposition 6.2.6. ■

The definition of de Rham descendability for absolute de Rham cohomology implies the descendability for relative de Rham cohomology.

8.3.4. PROPOSITION. *Let $\mathfrak{X} \to \mathfrak{S}$ be a morphism of $\mathcal{V}$-triples locally of finite type such that $\mathcal{X}$ is smooth over $\mathcal{S}$ around X and let $\mathfrak{Y}_.$ be a simplicial triple locally of finite type over $\mathfrak{X}$ with a structure morphism $w_.$ such that $\widehat{w}_.: \mathcal{Y}_. \to \mathcal{X}$ is smooth around $Y_.$. Suppose that $w_.$ is universally de Rham descendable. Then, for a sheaf E of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules with an integrable connection ∇ on $]\overline{X}[_{\mathcal{X}}$ over $]\overline{S}[_{\mathcal{S}}$, the canonical homomorphism*

$$\mathrm{DR}^\dagger(\mathfrak{X}/\mathfrak{S}, (E, \nabla)) \to \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathrm{DR}^\dagger(\mathfrak{Y}_./\mathfrak{S}, w_.^\dagger(E, \nabla)))$$

is an isomorphism in $D^+(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$.

PROOF. We may assume that $\mathcal{S}$ and $\mathcal{X}$ are affine by Proposition 4.3.4. Let us take a finite Zariski covering $\{U_\alpha\}_\alpha$ of X such that $j_\alpha^\dagger \Omega^1_{]\overline{X}[_{\mathcal{X}}/]\overline{S}[_{\mathcal{S}}}$ is free over $j_\alpha^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$ (Lemma 8.1.2), where $j_\alpha^\dagger$ is the functor of taking overconvergent sections for the triple $\mathfrak{U}_\alpha = (U_\alpha, \overline{X}, \mathcal{X})$. We denote by $\mathfrak{U}_. \to \mathfrak{X}$ the Čech diagram which is induced from the natural morphism $\coprod_\alpha \mathfrak{U}_\alpha \to \mathfrak{X}$. Consider the commutative diagram

$$\begin{array}{ccc} \mathfrak{Y}_. & \leftarrow & \mathfrak{Y}_. \times_{\mathfrak{X}} \mathfrak{U}_. \\ \downarrow & & \downarrow \\ \mathfrak{X} & \leftarrow & \mathfrak{U}_. \end{array}$$

of diagrams of triples. Since $w_.: \mathfrak{Y}_. \to \mathfrak{X}$ is universally de Rham descendable, $\mathfrak{Y}_. \times_{\mathfrak{X}} \mathfrak{U}_m \to \mathfrak{U}_m$ is universally de Rham descendable for all m. Applying Propositions 22.12.2 and 4.4.5, we may assume that $j^\dagger \Omega^1_{]\overline{X}[_{\mathcal{X}}/]\overline{S}[_{\mathcal{S}}}$ is free over $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$.

We define a decreasing filtration of the complex

$$\mathrm{DR}^\dagger(\mathfrak{Y}_./\mathfrak{S}, w_.^\dagger(E, \nabla))$$

by the image

$$\mathrm{Fil}^q = \mathrm{im}(\mathrm{DR}^\dagger(\mathfrak{Y}_./\mathfrak{S}, w_.^\dagger(E, \nabla))[-q] \otimes_{\widetilde{w}^{-1}(j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}})} \widetilde{w}^{-1}(j^\dagger \Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{S}[_{\mathcal{S}}}) \to$$
$$\to \mathrm{DR}^\dagger(\mathfrak{Y}_./\mathfrak{S}, w_.^\dagger(E, \nabla))).$$

By definition $\mathrm{Fil}^0 = \mathrm{DR}^\dagger(\mathfrak{Y}_./\mathfrak{S}, w_.^\dagger(E, \nabla)))$ and $\mathrm{Fil}^q = 0$ for $q \gg 0$. One can easily check that Fil^q is a subcomplex of $\mathrm{DR}^\dagger(\mathfrak{Y}_./\mathfrak{S}, w_.^\dagger(E, \nabla))$. Since $\mathcal{Y}_n \to \mathfrak{S}$ (resp. $\mathfrak{X} \to \mathfrak{S}$) is smooth around Y_n for each n (resp. X), we have

$$\mathrm{Gr}^q_{\mathrm{Fil}} = \mathrm{DR}^\dagger(\mathfrak{Y}_./\mathfrak{X}, w_.^\dagger(E, 0))[-q] \otimes_{\widetilde{w}^{-1}(j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}})} \widetilde{w}^{-1}(j^\dagger \Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{S}[_{\mathcal{S}}}).$$

By Proposition 4.5.2 there exists a spectral sequence

$$\underline{E}_1^{qr} = \underline{H}^{q+r}(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathrm{Gr}^q_{\mathrm{Fil}})) \Rightarrow$$

$$\Rightarrow \underline{H}^{q+r}(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathrm{DR}^\dagger(\mathfrak{Y}_./\mathfrak{S}, w_.^\dagger(E, \nabla))).$$

Now we consider an injective resolution of $\mathrm{DR}^\dagger(\mathfrak{Y}_./\mathfrak{X}, w_.^\dagger(E, 0))$ as complexes of sheaves of $\widetilde{w}_.^{-1}(j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$-modules. Since $j^\dagger \Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{S}[_{\mathcal{S}}}$ is free over $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$, we have

$$\underline{E}_1^{qr} \cong \underline{H}^r(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathrm{DR}^\dagger(\mathfrak{Y}_./\mathfrak{X}, w_.^\dagger(E, 0)))) \otimes_{j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}} j^\dagger \Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{S}[_{\mathcal{S}}}.$$

Since $w_.$ is de Rham descendable, only $\underline{E}_1^{q0} = E \otimes_{j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}} j^\dagger \Omega^q_{]\overline{X}[_{\mathcal{X}}/]\overline{S}[_{\mathcal{S}}}$ $(q \geqq 0)$ appears in the $\underline{E}_1$-terms of the spectral sequence. Moreover, the canonical homomorphism commutes with the filtrations. Hence, we have the isomorphism. ■

We give another example of universally de Rham descendable morphisms. The following proposition was proved in Berthelot's unpublished note [6] (see also [4, 1.4 Théorème]).

8.3.5. Proposition. *Let $u : \mathfrak{Y} \to \mathfrak{X}$ be a separated morphism of $\mathcal{V}$-triples locally of finite type such that*

(i) $\widehat{u} : \mathcal{Y} \to \mathcal{X}$ *is smooth around* Y;
(ii) $\overline{u} : \overline{Y} \to \overline{X}$ *is proper*;
(iii) $\overset{\circ}{u} : Y \to X$ *is an isomorphism.*

Then the canonical homomorphism

$$E \to \mathbb{R}\tilde{u}_* \mathrm{DR}^\dagger(\mathfrak{Y}/\mathfrak{X}, u^\dagger(E, 0))$$

which is induced by the adjonction is an isomorphism.

8.3.6. Corollary. *With the notation as in Proposition* 8.3.5, *let* $u_. : \mathfrak{Y}_.^{\Delta^\circ} \to \mathfrak{X}$ *be a constant simplicial triple induced from* u (*Example* 3.1.1 (2)). *Then* $u_.$ *is universally de Rham descendable.*

Proof. The situation is unchanged after any base change by a morphism of lft-triples. Consider the spectral sequence for $u_.$ in Lemma 4.4.2 (the case where $s = s' = 1$). Since $\mathfrak{Y}_.^{\Delta^\circ}$ is constant, we have $\underline{E}_2^{0r} = \underline{E}_1^{0r}$ and $\underline{E}_2^{qr} = 0$ for $q \neq 0$. Hence the canonical homomorphism $E \to$ $\to \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.^{\Delta^\circ}; \mathrm{DR}^\dagger(\mathfrak{Y}_.^{\Delta^\circ}/\mathfrak{X}, w_.^\dagger(E, 0)))$ is an isomorphism. ■

PROOF OF PROPOSITION 8.3.5. Suppose that u is a morphism of triples which satisfies the hypotheses of Proposition 8.3.5. We say that u is acyclic if the canonical homomorphism $E \to \mathbb{R}\tilde{u}_* \mathrm{DR}^\dagger(\mathfrak{Y}/\mathfrak{X},\, u^\dagger(E,\,0))$ is an isomorphism for any sheaf E of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules.

(0) Let $u : \mathfrak{Y} \to \mathfrak{X}$ and $w : \mathfrak{Z} \to \mathfrak{Y}$ be morphisms of triples which satisfy the hypothesis of Proposition 8.3.5 and let us put $v = uw$. Suppose that w is acyclic. Then u is acyclic if and only if v is so. Indeed, let us consider the spectral sequence

$$\underline{E}_1^{qr} = \mathbb{R}^r \widetilde{w}_* \mathrm{DR}^\dagger(\mathfrak{Z}/\mathfrak{Y};\, v^\dagger(E,\,0)) \otimes_{j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}} j^\dagger \Omega^q_{]\overline{Y}[_{\mathcal{Y}}/]\overline{X}[_{\mathcal{X}}} \Rightarrow$$

$$\Rightarrow \mathbb{R}^{q+r} \widetilde{w}_* \mathrm{DR}^\dagger(\mathfrak{Z}/\mathfrak{X};\, v^\dagger(E,\,0))$$

which is induced by the decreasing filtration Fil^q on $\mathrm{DR}^\dagger(\mathfrak{Z}./\mathfrak{X};\, v^\dagger(E,\,0))$ as in the proof of Proposition 8.3.4. Since $j^\dagger \otimes^q_{]\overline{Y}[_{\mathcal{Y}}}$ is a locally free sheaf of $j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}$-modules for any q, we have

$$\underline{E}_1^{qr} = \begin{cases} u^\dagger E \otimes_{j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}} j^\dagger \Omega^q_{]\overline{Y}[_{\mathcal{Y}}/]\overline{X}[_{\mathcal{X}}} & \text{if } r = 0 \\ 0 & \text{if } r \neq 0 \end{cases}$$

by using the acyclicity of w. Hence,

$$\mathrm{DR}^\dagger(\mathfrak{Y}/\mathfrak{X};\, u^\dagger(E,\,0)) \xleftarrow{\cong} \mathbb{R}\widetilde{w}_* \mathrm{DR}^\dagger(\mathfrak{Z}/\mathfrak{X};\, v^\dagger(E,\,0)).$$

Therefore, we have an isomorphism

$$\mathbb{R}\tilde{u}_* \mathrm{DR}^\dagger(\mathfrak{Y}/\mathfrak{X};\, u^\dagger(E,\,0)) \xleftarrow{\cong} \mathbb{R}\tilde{v}_* \mathrm{DR}^\dagger(\mathfrak{Z}/\mathfrak{X};\, v^\dagger(E,\,0)).$$

Now we prove the assertion in several steps. We may assume that $\mathcal{X}$ is affine of finite type over Spf $\mathcal{V}$ and $\mathcal{Y}$ is of finite type over $\mathcal{X}$.

(1) If $\widehat{u}$ is étale around Y, then $\mathrm{DR}^\dagger(\mathfrak{Y}/\mathfrak{X};\, u^\dagger(E,\,0)) = u^\dagger E$. The assertion follows from Proposition 2.7.2 (2) and 2.10.2.

(2) Now $\widehat{u}$ is general. We reduce to the case where $\overline{u}$ is projective.

Indeed, there exists a projective scheme $\overline{Z}$ over $\overline{Y}$ with a $\overline{Y}$-open immersion $Y = Z \to \overline{Z}$ such that $\overline{Z}$ is also projective over $\overline{X}$ by the precise Chow's lemma [12, Corollaire 5.7.14]. Let us take a triple $\mathfrak{Z}$ over $\mathfrak{Y}$ such that $Z = Y$, $\overline{Z}$ is projective over $\overline{X}$ (hence it is so over $\overline{Y}$), and $\mathcal{Z}$ is smooth of finite type over $\mathcal{Y}$ around Z. We put $w : \mathfrak{Z} \to \mathfrak{Y}$ and $v = uw$ to be structure morphisms. By using our assumption that v and w are acyclic, u is acyclic by (0). Hence, we may assume that $\overline{u}$ is projective.

(3) We reduce to the case where $\overline{u}$ is an isomorphism.

Since $\overline{u}$ is projective, there exists a triple $\coprod_{\alpha} (Y_\alpha,\, \overline{Y}_\alpha,\, \mathcal{Y}'_\alpha)$ of finite type over $\mathfrak{X}$ such that $\coprod_\alpha Y_\alpha$ is a finite Zariski covering, $\overline{Y}_\alpha \to \overline{Y}$ is a closed immersion and $\mathcal{Y}'_\alpha \to \mathcal{X}$ is étale around Y_α for any α by Lemma 6.4.5. Let us consider the following diagram:

$$\begin{array}{ccccccc}
(Y, \overline{Y}, \mathcal{Y}) & \leftarrow & \coprod_\alpha (Y_\alpha, \overline{Y}, \mathcal{Y}) & \leftarrow & \coprod_\alpha (Y_\alpha, \overline{Y}_\alpha, \mathcal{Y}) & \overset{\coprod_\alpha \mathrm{pr}_{1\alpha}}{\longleftarrow} & \coprod_\alpha (Y_\alpha, \overline{Y}_\alpha, \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}'_\alpha) \\
u \downarrow & & \downarrow & & & & \downarrow \coprod_\alpha \mathrm{pr}_{2\alpha} \\
(X, \overline{X}, \mathcal{X}) & \leftarrow & \coprod_\alpha (Y_\alpha, \overline{X}, \mathcal{X}) & & \leftarrow & & \coprod_\alpha (Y_\alpha, \overline{Y}_\alpha, \mathcal{Y}'_\alpha),
\end{array}$$

where $\mathrm{pr}_{i\alpha}$ is a morphism of triple which is induced by the i-th projection for $i = 1, 2$. The composition $(Y_\alpha, \overline{Y}_\alpha, \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}'_\alpha) \to (Y_\alpha, \overline{Y}_\alpha, \mathcal{Y})$ is acyclic by (1) and the composition $(Y_\alpha, \overline{Y}_\alpha, \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}'_\alpha) \to (Y_\alpha, \overline{X}_\alpha, \mathcal{X})$ is acyclic by (0), (1) and our assumption. Hence, $(Y_\alpha, \overline{Y}, \mathcal{Y}) \to (Y_\alpha, \overline{X}, \mathcal{X})$ is acyclic by (0) for any α. Note that, for any open subscheme Y' of some Y_α, $(Y', \overline{Y}, \mathcal{Y}) \to$ $\to (Y', \overline{X}, \mathcal{X})$ is acyclic by the choice of $\overline{Y}_\alpha$ and $\mathcal{Y}'_\alpha$.

We put $\mathfrak{X}_\alpha = (Y_\alpha, \overline{X}, \mathcal{X})$ and $\mathfrak{Y}_\alpha = (Y_\alpha, \overline{Y}, \mathcal{Y})$ and define $\mathfrak{X}_{\alpha_0 \dots \alpha_r}$ (resp. $\mathfrak{Y}_{\alpha_0 \dots \alpha_r}$) to be the fiber product of $\mathfrak{X}_{\alpha_0}, \dots, \mathfrak{X}_{\alpha_r}$ over $\mathfrak{X}$ (resp. $\mathfrak{Y}_{\alpha_0}, \dots, \mathfrak{Y}_{\alpha_r}$ over $\mathfrak{Y}$) with the structure morphism $v_{\alpha_0 \dots \alpha_r}: \mathfrak{X}_{\alpha_0 \dots \alpha_r} \to \mathfrak{X}$ (resp. $w_{\alpha_0 \dots \alpha_r}: \mathfrak{Y}_{\alpha_0 \dots \alpha_r} \to \mathfrak{Y}$). Then there exists a quasi-isomorphism

$$\mathrm{DR}^\dagger(\mathfrak{Y}/\mathfrak{X}; u^\dagger(E, 0)) \to \mathrm{tot}([\prod_{\alpha_0} \mathrm{DR}^\dagger(\mathfrak{Y}_{\alpha_0}/\mathfrak{X}_{\alpha_0}; (uw_{\alpha_0})^\dagger(E, 0)) \to$$

$$\to \prod_{\alpha_0, \alpha_1} \mathrm{DR}^\dagger(\mathfrak{Y}_{\alpha_0\alpha_1}/\mathfrak{X}_{\alpha_0\alpha_1}; (uw_{\alpha_0\alpha_1})^\dagger(E, 0)) \to \dots])$$

on $]\overline{Y}[_{\mathcal{Y}}$ by Proposition 2.12.2, where tot means the total complex of the double complex. Now we apply $\mathbb{R}\tilde{u}_*$ to both sides. Since each $\mathfrak{Y}_{\alpha_0 \dots \alpha_r} \to$ $\to \mathfrak{X}_{\alpha_0 \dots \alpha_r}$ is acyclic, the right-hand side is isomorphic to the complex

$$\Big[\prod_{\alpha_0} v^\dagger_{\alpha_0} E \to \prod_{\alpha_0} v^\dagger_{\alpha_0\alpha_1} E \to \prod_{\alpha_0\alpha_1\alpha_2} v^\dagger_{\alpha_0\alpha_1\alpha_2} E \to \dots\Big].$$

on $]\overline{X}[_{\mathcal{X}}$. Hence, u is acyclic by Proposition 2.12.2. Therefore, we may assume that $\overline{u}$ is an isomorphism.

(4) We may assume that the complement ∂X of X in $\overline{X}$ is defined by an equation $g = 0$ for an element $g \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Indeed, let us take a finite Zariski covering $\{X_\alpha\}_\alpha$ of X such that a complement of X_α in $\overline{X}$ is defined by an equation for any α. Then we can use an argument similar to the last part of (3).

(5) We may assume that $\mathcal{Y}$ is étale around Y over a formal affine space $\widehat{\mathbb{A}}^d_{\mathcal{X}}$ over $\mathcal{X}$ by the strong fibration theorem [7, 1.3.7 Théorème] (a similar

argument to the proof of Lemma 6.4.3) and (0) using a similar argument to the last part of (3). Then we may assume that $\mathcal{Y}$ is a formal affine space $\widehat{\mathbb{A}}^d_{\mathcal{X}}$ over $\mathcal{X}$ by using (0) and (1). In particular, we may assume that $\mathcal{Y} = \widehat{\mathbb{A}}^1_{\mathcal{X}}$ by induction on d.

(6) Now our assumption is that $\mathcal{X}$ is affine, a complement ∂X of X in $\overline{X}$ is defined by $g = 0$ for some element $g \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, u and $\overline{u}$ are isomorphisms and $\widehat{u} : \mathcal{Y} = \widehat{\mathbb{A}}^1_{\mathcal{X}} \to \mathcal{X}$ is the projection. Let us denote $\partial Y = \overline{u}^{-1}(\partial X)$. We calculate cohomology sheaves concretely.

Let us denote by $t \in \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ a coordinate of $\widehat{\mathbb{A}}^1_{\mathcal{X}}$ over $\mathcal{X}$. We put

$$U^{\geqq \nu}_{\mathfrak{X}, \partial X} = \{x \in]\overline{X}[_{\mathcal{X}} \mid |g(x)| \geqq \nu\}$$
$$U^{\geqq \nu}_{\mathfrak{Y}, \partial Y} = \{y \in]\overline{Y}[_{\mathcal{Y}} \mid |g(y)| \geqq \nu\}$$

for a real number $\nu \in \sqrt{|K|} \cap]0, 1[$. Then $U^{\geqq \nu}_{\mathfrak{X}, \partial X}$ (resp. $U^{\geqq \nu}_{\mathfrak{Y}, \partial Y}$) is an admissible open subset of $]\overline{X}[_{\mathcal{X}}$ (resp. $]\overline{Y}[_{\mathcal{Y}}$). For any subset W of $]\overline{X}[_{\mathcal{X}}$,

$$\tilde{u}^{-1}(W \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X}) = \tilde{u}^{-1}(W) \cap U^{\geqq \nu}_{\mathfrak{Y}, \partial Y}.$$

For each nonnegative integer n, we define an admissible open subset V_n of $]\overline{Y}[_{\mathcal{Y}}$ by

$$V_n = \{y \in]\overline{Y}[_{\mathcal{Y}} \mid |t(y)| \leqq |\pi|^{1/n+1}\}.$$

Then $\{V_n\}_{n \geqq 0}$ is an admissible covering of $]\overline{Y}[_{\mathcal{Y}}$. If W is an affinoid subvariety of $]\overline{X}[_{\mathcal{X}}$, then $\tilde{u}^{-1}(W) \cap V_n$ is affinoid for all n.

Let W be an affinoid subvariety of $]\overline{X}[_{\mathcal{X}}$. Then $W \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X}$ is affinoid for all $\nu \in \sqrt{|K|} \cap]0, 1[$. We put

$$A_\nu = \Gamma(W \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X}, \mathcal{O}_{]\overline{X}[_{\mathcal{X}}})$$

and denote by $| \ |_{W, \nu}$ a Banach norm on A_ν which is induced by a fixed Banach norm on $\Gamma(W, \mathcal{O}_W)$.

Let E be a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules and let $\mathcal{E}$ be a sheaf of $\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules with $E \cong j^\dagger \mathcal{E}$ such that $\mathcal{E}$ is coherent on a strict neighbourhood of $]X[_{\mathcal{X}}$ in $]\overline{X}[_{\mathcal{X}}$. For an affinoid subvariety W of $]\overline{X}[_{\mathcal{X}}$, we may assume that $\mathcal{E}|_{W \cap U^{\geqq \lambda}_{\mathfrak{X}, \partial X}}$ is coherent for some $\lambda \in \sqrt{|K|} \cap]0, 1[$ by Lemma 2.6.6. We put

$$M_\nu = \Gamma(W \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X}, \mathcal{E})$$

and denote by $\| \ \|_\nu$ a Banach norm on M_ν which is which is induced by a fixed presentation of M_λ over A_λ and the Banach norm $| \ |_{W, \nu}$ on A_ν for

$\lambda \leqq \nu < 1$ ($\nu \in \sqrt{|K|} \cap]0, 1[$). Then we have

$$\begin{aligned} N_{n,\nu} &= \Gamma(\tilde{u}^{-1}(W) \cap U^{\geqq \nu}_{\mathfrak{Y}, \partial Y} \cap V_n, \tilde{u}^* \mathcal{E}) \\ &= \left\{ \sum_{l=0}^{\infty} a_l t^l \mid a_l \in M_\nu \| a_l \|_\nu |\pi|^{l/n+1} \to 0 \ (l \to \infty) \right\}. \end{aligned}$$

Since $\mathbb{R}^q \tilde{u}_* \mathrm{DR}^\dagger(\mathfrak{Y}/\mathfrak{X}, u^\dagger(E, 0))$ is the sheafification of the presheaf defined by

$$W \mapsto H^q(\tilde{u}^{-1}(W), \mathrm{DR}^\dagger(\mathfrak{Y}/\mathfrak{X}, u^\dagger(E, 0)))$$

for any admissible open subset W, we have only to prove

$$(*) \qquad H^q(\tilde{u}^{-1}(W), \mathrm{DR}^\dagger(\mathfrak{Y}/\mathfrak{X}, u^\dagger(E, 0))) = \begin{cases} \Gamma(W, E) & \text{if } q = 0 \\ 0 & \text{if } q > 0 \end{cases}$$

for any affinoid subvariety W of $]\overline{X}[_{\mathcal{X}}$.

Let W be an affinoid subvariety of $]\overline{X}[_{\mathcal{X}}$. Since

$$\begin{aligned} H^q(\tilde{u}^{-1}(W) \cap V_n, u^\dagger E) &= 0 \\ H^q(\tilde{u}^{-1}(W) \cap V_n, u^\dagger E \otimes_{j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}} j^\dagger \Omega^1_{]\overline{Y}[_{\mathcal{Y}}/]\overline{X}[_{\mathcal{X}}}) &= 0 \end{aligned}$$

for $q > 0$ by Corollary 5.1.2, we can calculate cohomology groups $H^q(\tilde{u}^{-1}(W), \mathrm{DR}^\dagger(\mathfrak{Y}/\mathfrak{X}, u^\dagger(E, 0)))$ by using the total complex of the double complex

$$\begin{array}{ccc} \prod_{n_0} \Gamma(\tilde{u}^{-1}(W) \cap V_{n_0}, u^\dagger E \otimes_{j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}} j^\dagger \Omega^1_{]\overline{Y}[_{\mathcal{Y}}/]\overline{X}[_{\mathcal{X}}}) & \to & \prod_{n_0 < n_1} \Gamma(\tilde{u}^{-1}(W) \cap V_{n_0}, u^\dagger E \otimes_{j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}} j^\dagger \Omega^1_{]\overline{Y}[_{\mathcal{Y}}/]\overline{X}[_{\mathcal{X}}}) \\ d_{n_0} \uparrow & & \uparrow d_{n_0} \\ \prod_{n_0} \Gamma(\tilde{u}^{-1}(W) \cap V_{n_0}, u^\dagger E) & \to & \prod_{n_0 < n_1} \Gamma(\tilde{u}^{-1}(W) \cap V_{n_0}, u^\dagger E) \end{array}$$

by Lemma 2.11.1, where $\prod_{n_0} \Gamma(\tilde{u}^{-1}(W) \cap V_{n_0}, u^\dagger E)$ is of degree $(0, 0)$, the vertical derivations d_n's are induced by the derivations of de Rham complex and the horizontal derivations are those of the alternating Čech complex. If we know that

$$\begin{aligned} \ker d_n &= \Gamma(W, E) \cong \lim_{\nu \to 1^-} \Gamma(W \cap U^{\geqq \nu}_{\mathfrak{X}, \partial X}, \mathcal{E}) = \lim_{\nu \to 1^-} M_\nu \\ \operatorname{coker} d_n &= 0 \end{aligned}$$

for any n (here we use Proposition 2.6.8), then we have the formula (*).

Now we have

$$\begin{aligned}
\Gamma(\tilde{u}^{-1}(W)\cap V_n, u^{\dagger}E) &= \lim_{\nu\to 1^-}\Gamma(\tilde{u}^{-1}(W)\cap U_{\mathfrak{Y},\partial Y}^{\geqq\nu}\cap V_n, \tilde{u}^*\mathcal{E}) = \lim_{\nu\to 1^-} N_{n,\nu}\\
\Gamma(\tilde{u}^{-1}(W)\cap V_n, u^{\dagger}E\otimes_{j^{\dagger}\mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}} j^{\dagger}\Omega^1_{]\overline{Y}[_{\mathcal{Y}}/]\overline{X}[_{\mathcal{X}}}) &= \lim_{\nu\to 1^-} N_{n,\nu}\otimes_{\Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})}\Gamma(\mathcal{Y},\Omega^1_{\mathcal{Y}/\mathcal{X}})
\end{aligned}$$

by Proposition 2.6.8. For any element

$$\sum_l a_l t^l\, dt \in N_{n,\nu}\otimes_{\Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})}\Gamma(\mathcal{Y},\Omega^1_{\mathcal{Y}/\mathcal{X}}), \qquad \sum_l a_l t^{l+1}/l+1$$

is an element of $N_n, \sqrt{\nu}$ since $|1/l+1|\varrho^l \to 0$ $(l\to\infty)$ for any $0<\varrho<1$. Hence, d_n is surjective. Since l is a unit for any positive integer, we have $\ker d_n = \Gamma(W, E)$.

This completes the proof of Proposition 8.3.5. ■

8.4. Let

$$\begin{array}{ccc}
\mathfrak{Y} & \overset{w}{\leftarrow} & \mathfrak{Z}\\
 & u\searrow \quad \swarrow v & \\
 & \mathfrak{X} &
\end{array}$$

be a commutative diagram of $\mathcal{V}$-triples locally of finite type such that $\widehat{u}$ (resp. $\widehat{v}$, resp. $\widehat{w}$) is smooth around Y (resp. Z, resp. Z). We denote the Čech diagram for u (resp. v) by $u_\cdot : \mathfrak{Y}_\cdot \to \mathfrak{X}$ (resp. $v_\cdot : \mathfrak{Z}_\cdot \to \mathfrak{X}$) and $w_\cdot = \mathrm{cosk}_0^{\mathfrak{X}}(w) : \mathfrak{Z}_\cdot \to \mathfrak{Y}_\cdot$.

8.4.1. Theorem. *With the notation as above, suppose that w is universally de Rham descendable. u is de Rham descendable (resp. universally de Rham descendable) if and only if v is so.*

Theorem 8.4.1 easily follows from Proposition 8.4.2, Lemmas 8.4.4 and 8.4.5 below. We will give a generalization to the simplicial case in 8.7.

8.4.2. Proposition. *With the notation as above, suppose that there exists a section $s : \mathfrak{Y}\to\mathfrak{Z}$ over $\mathfrak{X}$, that is, $ws = \mathrm{id}_{\mathfrak{Y}}$ and $u = vs$. Then, for* *a sheaf E of coherent $j^\dagger\mathcal{O}_{]\overline{X}[_\mathcal{X}}$-modules, the canonical homomorphism*

$$\widetilde{w}_{\cdot}^{\sharp} : \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_{\cdot}; \mathrm{DR}^\dagger(\mathfrak{Y}_{\cdot}/\mathfrak{X}, u_{\cdot}^\dagger(E, 0))) \to$$

$$\to \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Z}_{\cdot}; \mathrm{DR}^\dagger(\mathfrak{Z}_{\cdot}/\mathfrak{X}, v_{\cdot}^\dagger(E, 0)))$$

induced by w (see 4.3) is an isomorphism in $D^+(\mathbb{Z}_{]\overline{X}[_\mathcal{X}})$.

The proof of Proposition 8.4.2 is same as the proof of Proposition 6.3.4 except for the use of a homotopy between $\tilde{s}_{\cdot}^{\sharp}\,\widetilde{w}_{\cdot}^{\sharp}$ and the identity on $\mathrm{DR}^\dagger(\mathfrak{Z}_{\cdot}/\mathfrak{X}, w_{\cdot}^\dagger(E, 0))$ which is given by Lemma 8.4.3 below. By Lemma 3.10.4 we have

8.4.3. Lemma. *Let $\mathfrak{X}$ be a $\mathcal{V}$-triple locally of finite type and let $\mathfrak{Y}_{\cdot}$ and $\mathfrak{Z}_{\cdot}$ be simplicial $\mathcal{V}$-triples locally of finite type over $\mathcal{X}$ with structure morphisms $u_{\cdot}: \mathfrak{Y}_{\cdot} \to \mathfrak{X}$ and $v_{\cdot}: \mathfrak{Z}_{\cdot} \to \mathfrak{X}$ such that $\widehat{u}_{\cdot}$ and $\widehat{v}_{\cdot}$ are smooth*
around $Y_{\cdot}$ and $Z_{\cdot}$, respectively. Suppose that $\{h_n(\eta)\}_n$, η is a homotopy from $w_{\cdot}^{(0)}: \mathfrak{Z}_{\cdot} \to \mathfrak{Y}_{\cdot}$ to $w_{\cdot}^{(1)}: \mathfrak{Z}_{\cdot} \to \mathfrak{Y}_{\cdot}$ over $\mathfrak{X}$. For any sheaf $\mathcal{F}$ of $j^\dagger\mathcal{O}_{]\overline{X}[_\mathcal{X}}$-modules with a 0-connection, we define a collection of morphisms $\theta_n(\eta): \widetilde{h_n(\lambda)}^* \mathrm{DR}^\dagger(\mathfrak{Y}_{\cdot}/\mathfrak{X}, \tilde{u}_n^*(\mathcal{F}, 0)) \to \mathrm{DR}^\dagger(\mathfrak{Z}_{\cdot}/\mathfrak{X}, \tilde{v}_n^*(\mathcal{F}, 0))$ by the induced map by the identity $u_n h_n(\lambda) = v_n$, then the collection $\{\theta_n(\eta)\}_n$, η gives a homotopy from $\varphi_{\cdot}^{(1)^{\cdot}} = \theta_{\cdot}(\partial^{(1)})$ to $\varphi_{\cdot}^{(0)^{\cdot}} = \theta_{\cdot}(\partial^{(0)})$.

Let $t_{\cdot}: (\mathfrak{U}_{\cdot}, (\Delta^2)^\circ) \to \mathfrak{X}$ be the 2-simplicial triple for the commutative diagram

$$\begin{array}{ccc} \mathfrak{Y}_{\cdot} & \stackrel{w_{\cdot}}{\leftarrow} & \mathfrak{Z}_{\cdot} \\ {\scriptstyle u_{\cdot}}\downarrow & & \downarrow{\scriptstyle v_{\cdot}} \\ \mathfrak{X} & = & \mathfrak{X} \end{array}$$

of $\mathcal{V}$-triples locally of finite type as in Example 3.1.2. The same argument as in Lemmas 6.3.3 and 6.3.4 works for the following lemmas.

8.4.4. Lemma. *With the notation as above, let E be a sheaf of coherent $j^\dagger\mathcal{O}_{]\overline{X}[_\mathcal{X}}$-modules. Then, the canonical homomorphism*

$$\mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{U}_{\cdot}; \mathrm{DR}^\dagger(\mathfrak{U}_{\cdot}/\mathfrak{X}, t_{\cdot}^\dagger(E, 0))) \to \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Z}_{\cdot}; \mathrm{DR}^\dagger(\mathfrak{Z}_{\cdot}/\mathfrak{X}, v_{\cdot}^\dagger(E, 0)))$$

induced by the filtration in 4.4 is an isomorphism in $D^+(\mathbb{Z}_{]\overline{X}[_\mathcal{X}})$. Moreover, if w_n is de Rham descendable for any object n of Δ°, then the *canonical homomorphism*

$$\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X},\, \mathfrak{Y}.;\, \mathrm{DR}^{\dagger}(\mathfrak{Y}./\mathfrak{X},\, u.^{\dagger}(E,0))) \to \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X},\, \mathfrak{Z}.;\, \mathrm{DR}^{\dagger}(\mathfrak{Y}./\mathfrak{X},\, v.^{\dagger}(E,0)))$$

is an isomorphism in $D^{+}(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$.

8.4.5. LEMMA. *With the notation as above, if w is universally de Rham descendable, then w_n is de Rham descendable for any object n of Δ°.*

8.5. In the case of Čech diagrams the notion of de Rham descendability is equivalent to that of cohomological descendability.

8.5.1. THEOREM. *Let $w : \mathfrak{Y} \to \mathfrak{X}$ be a separated morphism of $\mathcal{V}$-triples locally of finite type such that*

(i) $\widehat{w} : \mathcal{Y} \to \mathcal{X}$ *is smooth around* Y;
(ii) $\overline{w} : \overline{Y} \to \overline{X}$ *is of finite type*

and let E be a sheaf of coherent $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules. Denote by $w. : \mathfrak{Y}. \to \mathfrak{X}$ the Čech diagram associated to $w : \mathfrak{Y} \to \mathfrak{X}$. Then the natural projection $\mathrm{DR}^{\dagger}(\mathfrak{Y}./\mathfrak{X},\, w.^{\dagger}(E,0)) \to w.^{\dagger}E$ induces an isomorphism

$$\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X},\, \mathfrak{Y}.;\, \mathrm{DR}^{\dagger}(\mathfrak{Y}./\mathfrak{X},\, w.^{\dagger}(E,0))) \to \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X},\, \mathfrak{Y}.;\, w.^{\dagger}E).$$

8.5.2. COROLLARY. *With the same notation as in Theorem* 8.5.1, *w is cohomologically descendable (resp. universally cohomologically descendable) if and only if w is de Rham descendable (resp. universally de Rham descendable).*

We first prove a special case of Theorem 8.5.1.

8.5.3. LEMMA. *Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ and $\mathfrak{Y} = \coprod_{\alpha} \mathfrak{Y}_{\alpha} = \coprod_{\alpha} (Y_{\alpha}, \overline{Y}_{\alpha}, \mathcal{Y}_{\alpha})$ be $\mathcal{V}$-triples separated of finite type and let $w : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of $\mathcal{V}$-triples which satisfy the conditions:*

(i) *both $\mathcal{X}$ and $\mathcal{Y}$ are affine*;
(ii) *there exists a finite number of sections $t_{1,\alpha}, \ldots t_{\alpha, d_{\alpha}} \in \Gamma(\mathcal{Y}_{\alpha}, \mathcal{O}_{\mathcal{Y}_{\alpha}})$ which determine an étale morphism from $\mathcal{Y}_{\alpha}$ to a formal affine space $\widehat{\mathbb{A}}^{d_{\alpha}}_{\mathcal{X}}$ around Y_{α} such that $\widehat{w}$ is a composition of $\mathcal{Y}_{\alpha} \to \widehat{\mathbb{A}}^{d_{\alpha}}_{\mathcal{X}}$ and the projection $\widehat{\mathbb{A}}^{d_{\alpha}}_{\mathcal{X}} \to \mathcal{X}$ for each α.*

Let us denote by $w. : \mathfrak{Y}. \to \mathfrak{X}$ the Čech diagram for $w : \mathfrak{Y} \to \mathfrak{X}$. For any sheaf E of coherent $j^{\dagger}\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules, we define a decreasing filtration $\{\mathrm{Fil}^q\}_q$ of $\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{X},\, \mathfrak{Y}.;\, \mathrm{DR}^{\dagger}(\mathfrak{Y}./\mathfrak{X},\, w.^{\dagger}(E,0)))$ which is induced by *the Hodge filtration* $\{\mathrm{DR}^\dagger(\mathcal{Y}_\cdot/\mathcal{X}, w_\cdot^\dagger(E,0))^{\cdot\geqq q}\}_q$ *of the de Rham complex* $\mathrm{DR}^\dagger(\mathcal{Y}_\cdot/\mathcal{X}, w_\cdot^\dagger(E,0))$ (*Lemma* 4.5.1). *Then we have*

$$\underline{H}^r(\mathrm{Gr}^q_{\mathrm{Fil}})=0$$

for any $q\neq 0$ *and* r. *In particular, the natural projection induces an isomorphism*

$$\mathbb{R}\mathcal{C}^\dagger(\mathcal{X},\mathcal{Y}_\cdot;\mathrm{DR}^\dagger(\mathcal{Y}_\cdot/\mathcal{X}, w_\cdot^\dagger(E,0)))\xrightarrow{\cong}\mathbb{R}\mathcal{C}^\dagger(\mathcal{X},\mathcal{Y}_\cdot;w_\cdot^\dagger E).$$

Proof. Let q be a positive integer. We have only to prove that there exists a homotopy $\{\theta_n(\chi)\}_{n,\chi}$ from the identity to the 0-map on $\mathrm{Gr}^q_{\mathrm{Fil}}=$ $=w_\cdot^\dagger E\otimes_{j^\dagger\mathcal{O}_{]\overline{Y}_\cdot[_{\mathcal{Y}_\cdot}}}j^\dagger\Omega^q_{]\overline{Y}_\cdot[_{\mathcal{Y}_\cdot}/]\overline{X}[_{\mathcal{X}}}$ as a sheaf on $]\overline{Y}_\cdot[_{\mathcal{Y}_\cdot}$ (the case where $w_\cdot^{(0)}=$ $=w_\cdot^{(1)}=\mathrm{id}_{\mathcal{Y}_\cdot}$ and $h_n(\chi)=\mathrm{id}_{\mathcal{Y}_n}$ for $\chi:n\to 1$ in 3.10). Indeed, if such a homotopy exists, then $\underline{H}^r(\mathrm{Gr}^q_{\mathrm{Fil}})=0$ for any $q\neq 0$ and r by Propositions 3.10.5 and 3.10.8 just as in the proof of Proposition 6.3.2. Since the Hodge filtration of $\mathrm{DR}^\dagger(\mathcal{Y}_\cdot/\mathcal{X}, w_\cdot^\dagger(E,0))$ satisfies the conditions (i) and (ii) in 4.5 (before Lemma 4.5.1), the natural projection

$$\mathbb{R}\mathcal{C}^\dagger(\mathcal{X},\mathcal{Y}_\cdot;\mathrm{DR}^\dagger(\mathcal{Y}_\cdot/\mathcal{X}, w_\cdot^\dagger(E,0)))\to\mathbb{R}\mathcal{C}^\dagger(\mathcal{X},\mathcal{Y}_\cdot;w_\cdot^\dagger E)$$

is an isomorphism by Proposition 4.5.2.

Now we construct a homotopy $\{\theta_n(\chi)\}_{n,\chi}$ for $q\geqq 1$. Since $\mathcal{Y}_\alpha\to\widehat{\mathbb{A}}^{d_\alpha}_{\mathcal{X}}$ is étale around Y_α, $j_\alpha^\dagger\Omega^1_{]\overline{Y}_\alpha[_{\mathcal{Y}_\alpha}/]\overline{X}[_{\mathcal{X}}}$ is free over $j_\alpha^\dagger\mathcal{O}_{]\overline{Y}_\alpha[_{\mathcal{Y}_\alpha}}$ with basis $dt_\alpha,1,\ldots,dt_\alpha,d_\alpha$ by Lemma 8.1.2. For an integer m $(0\leqq m\leqq n)$, we denote by $r_{n,m}:\mathcal{Y}_n\to\mathcal{Y}_m$ the projection onto the first $m+1$-components. Then the fixed basis $\{dt_{\alpha,1},\ldots,dt_{\alpha,d_\alpha}\}_\alpha$ determines a left inverse of the homomorphism

$$\tilde{r}^\sharp_{n,m}:r^\dagger_{n,m}(w_m^\dagger E\otimes_{j^\dagger\mathcal{O}_{]\overline{Y}_m[_{\mathcal{Y}_m}}}j^\dagger\Omega^q_{]\overline{Y}_m[_{\mathcal{Y}_m}/]\overline{X}[_{\mathcal{X}}})\to w_n^\dagger E\otimes_{j^\dagger\mathcal{O}_{]\overline{Y}_n[_{\mathcal{Y}_n}}}j^\dagger\Omega^q_{]\overline{Y}_n[_{\mathcal{Y}_n}/]\overline{X}[_{\mathcal{X}}}$$

of sheaves of $j^\dagger\mathcal{O}_{]\overline{Y}_n[_{\mathcal{Y}_n}}$-modules induced by $r_{m,n}$. We denote by $L_{n,m}$ the image of $\tilde{r}^\sharp_{n,m}$ and by $s_{n,m}:w_n^\dagger E\otimes_{j^\dagger\mathcal{O}_{]\overline{Y}_n[_{\mathcal{Y}_n}}}j^\dagger\Omega^q_{]\overline{Y}_n[_{\mathcal{Y}_n}/]\overline{X}[_{\mathcal{X}}}\to L_{n,m}$ the splitting.

Let $\chi:n\to 1$ be a morphism in Δ with $\chi(m)=0$ and $\chi(m+1)=1$ for $-1\leqq m\leqq n$. We define a homomorphism

$$\theta_n(\chi):w_n^\dagger E\otimes_{j^\dagger\mathcal{O}_{]\overline{Y}_n[_{\mathcal{Y}_n}}}j^\dagger\Omega^q_{]\overline{Y}_n[_{\mathcal{Y}_n}/]\overline{X}[_{\mathcal{X}}}\to w_n^\dagger E\otimes_{j^\dagger\mathcal{O}_{]\overline{Y}_n[_{\mathcal{Y}_n}}}j^\dagger\Omega^q_{]\overline{Y}_n[_{\mathcal{Y}_n}/]\overline{X}[_{\mathcal{X}}}$$

of sheaves of $j^\dagger\mathcal{O}_{]\overline{Y}_n[_{\mathcal{Y}_n}}$-modules by $\theta_n(\chi)=s_{n,m}$ for $m\geqq 0$ and by $\theta_n(\chi)=$ $=0$ for $m=-1$. Then $\theta_n(\partial^{(0)})=\mathrm{id}$ and $\theta_n(\partial^{(1)})=0$. One can easily check condition (ii) of the definition of homotopy in 3.10. Hence the collection $\{\theta_n(\chi)\}_{n,\chi}$ gives the desired homotopy.

This completes a proof of Proposition 8.5.3. ■

PROOF OF THEOREM 8.5.1. Since the problem is local on $\mathcal{X}$, we may assume that $\mathcal{X}$ is affine by Proposition 4.3.4. Since $\overline{w}: \overline{Y} \to \overline{X}$ is of finite type, we may assume that $\mathcal{Y}$ is of finite type over Spf $\mathcal{V}$.

Let $v: \mathfrak{Z} \to \mathfrak{Y}$ be a separated morphism of $\mathcal{V}$-triples of finite type such that $\hat{v}: \mathcal{Z} \to \mathcal{Y}$ is smooth around Z and let us denote by $u_.: \mathfrak{Z}_. \to \mathfrak{X}$ the Čech diagram for $u = wv$. If v is universally cohomologically descendable and universally de Rham descendable, then $v_n: \mathfrak{Z}_n \to \mathfrak{Y}_n$ is also universally cohomologically descendable and universally de Rham descendable by Theorems 6.3.1 and 8.4.1. The vertical arrows of the natural commutative diagram

$$\begin{array}{ccc} \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; \mathrm{DR}^\dagger(\mathfrak{Y}_./\mathfrak{X}, w_.^\dagger(E, 0))) & \to & \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_.; w_.^\dagger E) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Z}_.; \mathrm{DR}^\dagger(\mathfrak{Z}_./\mathfrak{X}, u_.^\dagger(E, 0))) & \to & \mathbb{R}\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Z}_.; u_.^\dagger E) \end{array}$$

are isomorphisms by Lemmas 6.3.3 and 8.4.4. Hence, Theorem 8.5.1 holds for w if and only if it holds for u, under the hypothesis that v is universally cohomologically descendable.

Since strict Zariski coverings as triples are universally cohomologically descendable and universally de Rham descendable by Proposition 6.2.5 and Example 8.3.3 (1), we may assume that $\mathcal{Y}$ is affine by Theorems 6.3.1 and 8.4.1. Then there exists a finite Zariski covering $\{Y_\alpha\}_\alpha$ of Y such that such that there exists a finite number of sections $t_{\alpha,1}, \dots t_{\alpha,d_\alpha} \in$ $\in \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ which determines an étale morphism $\mathcal{Y} \to \widehat{\mathbb{A}}_{\mathcal{X}}^{d_\alpha}$ around Y_α. Such a covering $\{Y_\alpha\}_\alpha$ of Y always exists since $\mathcal{Y}$ is affine and $\mathcal{Y} \to \mathcal{X}$ is smooth around Y. We put $\mathfrak{Z} = \coprod_\alpha \mathfrak{Y}_\alpha = \coprod_\alpha (Y_\alpha, \overline{Y}, \mathcal{Y})$ and denote the structure morphism by $w_\alpha: \mathfrak{Y}_\alpha \to \mathfrak{X}$. Since $v: \mathfrak{Z} \to \mathfrak{Y}$ is universally cohomological descendable and universally de Rham descendable by Proposition 6.2.6 and Example 8.3.3 (2), we have only to prove the assertion for $u =$ $= \coprod_\alpha w_\alpha: \mathfrak{Z} \to \mathfrak{X}$. This has already been done in Lemma 8.5.3. This completes the proof of Theorem 8.5.1. ■

8.6. The notion of de Rham descent is independent of the choice of boundaries and embedding into formal schemes by Corollary 8.5.2 and Proposition 6.4.1. Thus, we have shown:

8.6.1. PROPOSITION. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be* $\mathcal{V}$*-triples locally of finite type and let* $w : \mathfrak{Y} \to \mathfrak{X}$ *be a separated morphism of triples such that*

(i) $\widehat{w} : \mathcal{Y} \to \mathcal{X}$ *is smooth around* Y;
(ii) $\overline{w} : \overline{Y} \to \overline{X}$ *is proper*;
(iii) $\overset{\circ}{w} : Y \to X$ *is an isomorphism.*

Then w *is universally de Rham descendable.*

8.6.2. COROLLARY. *Let* $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$, $\mathfrak{Y} = (Y, \overline{Y}, \mathcal{Y})$ *and* $\mathfrak{Y}' = (Y, \overline{Y}', \mathcal{Y}')$ *be* $\mathcal{V}$*-triples locally of finite type, and let* $w : \mathfrak{Y} \to \mathfrak{X}$ *and* $w' : \mathfrak{Y}' \to \mathfrak{X}$ *be separated morphisms of triples such that*

(i) *both* $\widehat{w}$ *and* $\widehat{w}'$ *are smooth around* Y;
(ii) *both* $\overline{w}$ *and* $\overline{w}'$ *are proper*;
(iii) *both* $\overset{\circ}{w} = \overset{\circ}{w}'$.

Then w *is de Rham descendable (resp. universally de Rham descendable) if and only if* w' *is so.*

8.6.3. PROPOSITION. *With the situation of Corollary* 8.6.2, *if we remove the condition* (ii) *and we assume that* (iii) $\overline{Y} = \overline{Y}'$ *and* $\overline{w} = \overline{w}'$. *Then the assertion of Corollary* 8.6.2 *still holds.*

8.7. We consider the de Rham descent theorem for general simplicial triples. The argument of the proof is same as in the proof of Proposition 6.5.1

8.7.1. PROPOSITION. *Let* $\mathfrak{X}$ *be a* $\mathcal{V}$*-triple locally of finite type and let*

$$\begin{array}{ccc} \mathfrak{Y}. & \overset{w.}{\longleftarrow} & \mathfrak{Z}. \\ & {\scriptstyle u.}\searrow \quad \swarrow{\scriptstyle v.} & \\ & \mathfrak{X} & \end{array}$$

be a morphism of simplicial triples locally of finite type over $\mathfrak{X}$ *such that* u_l *(resp.* v_l*, resp.* w_l*) is smooth around* Y_l *(resp.* Z_l*, resp.* Z_l*) for any* l*. Suppose that, for any nonnegative integer* n,

(i) $\mathrm{cosk}_n^{\mathfrak{X}}(\mathfrak{Y}.^{(n)}) = \mathfrak{Y}.$ *and* $\mathrm{cosk}_n^{\mathfrak{X}}(\mathfrak{Z}.^{(n)}) = \mathfrak{Z}.$ (*i.e. they have dimension* $\leqslant n$ [13]);

(ii) w_l *is an isomorphism for* $l < n$, $\mathrm{cosk}_n^{\mathfrak{X}}(w.^{(n)}) = w.$ *and* w_l *is de Rham descendable (resp. universally de Rham descendable) for any* l.

Then $u_.$ is de Rham descendable (resp. universally de Rham descendable) if and only if $v_.$ is so.

8.7.2. COROLLARY. *Let $\mathfrak{X}$ be a $\mathcal{V}$-triple locally of finite type and let $\mathfrak{Y}_.$ be a simplicial triple locally of finite type over $\mathfrak{X}$ with the structure morphism $w_.$. Suppose that, for any nonnegative integers l and n, the canonical morphism $\operatorname{cosk}_n^{\mathfrak{X}}(\mathfrak{Y}_.^{(n)})_l \to \operatorname{cosk}_{n-1}^{\mathfrak{X}}(\mathfrak{Y}_.^{(n-1)})_l$ is de Rham descendable (resp. universally de Rham descendable). If $\operatorname{cosk}_m^{\mathfrak{X}}(\mathfrak{Y}_.^{(m)}) \to \mathfrak{X}$ is de Rham descendable (resp. universally de Rham descendable) for some nonnegative integer m, then $w_.: \mathfrak{Y}_. \to \mathfrak{X}$ is so.*

9. De Rham descent for étale morphisms.

9.1. We give our results about de Rham descent for étale hypercoverings.

9.1.1. THEOREM. *Let $\mathfrak{Y}_.$ be an étale-étale hypercovering or an étale-proper hypercovering over a $\mathcal{V}$-triple $\mathfrak{X}$ locally of finite type. Then $\mathfrak{Y}_. \to$ $\to \mathfrak{X}$ is universally de Rham descendable.*

PROOF. Applying Corollary 8.7.2, we have only to prove the assertion in the case where $\mathfrak{Y}_. \to \mathfrak{X}$ is the Čech diagram associated to a morphism $\mathfrak{Y} \to \mathfrak{X}$ which satisfies the hypotheses of Theorem 7.3.1 or Theorem 7.4.1. Since w is universally cohomological descendable, w is universally de Rham descendable by Corollary 8.5.2. ∎

Let $u : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of $\mathcal{V}$-triples locally of finite type such that $\widehat{u}: \mathcal{Y} \to \mathcal{X}$ is smooth around Y and let $w_.: \mathfrak{Z}_. \to \mathfrak{Y}$ be a morphism such that $w_.$ is an étale-étale hypercovering or an étale-proper hypercovering. By Proposition 8.3.4 we have

9.1.2. COROLLARY. *With the notation as above, if E is a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{Y}[_{\mathcal{Y}}}$-modules with a connection ∇ on $]\overline{Y}[_{\mathcal{Y}}$ over $]\overline{X}[_{\mathcal{X}}$, then the canonical homomorphism*

$$\mathrm{DR}^\dagger(\mathfrak{Y}/\mathfrak{X}, (E, \nabla)) \to \mathbb{R}\mathcal{C}^\dagger(\mathfrak{Y}, \mathfrak{Z}_.; \mathrm{DR}^\dagger(\mathfrak{Z}_./\mathfrak{X}, w_.^\dagger(E, \nabla)))$$

is an isomorphism in $D^+(\mathbb{Z}_{]\overline{Y}[_{\mathcal{Y}}})$.

10. Rigid cohomology.

In this section we introduce a rigid cohomology which is defined by using universally de Rham descendable hypercoverings (see the definition in 10.1) and prove that our rigid cohomology does not depend on the choices of hypercoverings. We prove that our definition of rigid cohomology is equivalent to Berthelot's original definition.

Throughout this section, let $\mathfrak{S} = (S, \overline{S}, \mathcal{S})$ be the induced $\mathcal{V}$-triple from a formal $\mathcal{V}$-scheme $\mathcal{S}$ locally of finite type and let $\mathfrak{T} = (T, \overline{T}, \mathcal{T})$ be a triple separated locally of finite type over $\mathfrak{S}$ such that $\mathcal{T}$ is smooth over $\mathcal{S}$ around T.

10.1. In order to construct rigid cohomology for hypercoverings, we need to generalize the formalism of triples.

10.1.1. DEFINITION. (1) Let $(X, \overline{X})$ be a pair separated over $(T, \overline{T})$. A triple $\mathfrak{Y}$ (resp. An lft-triple) over $\mathfrak{T}$ with a $(T, \overline{T})$-morphism $(Y, \overline{Y}) \to$ $\to (X, \overline{X})$ is called an $(X, \overline{X})$-triple (resp. an $(X, \overline{X})$-lft-triple) over $\mathfrak{T}$. A morphism $w : \mathfrak{Z} \to \mathfrak{X}$ of $(X, \overline{X})$-triples over $\mathfrak{T}$ is a morphism of lft-triples such that the natural diagram

$$\begin{array}{ccc} (Z, \overline{Z}) & \rightarrow & (Y, \overline{Y}) \\ & \searrow \quad \swarrow & \\ & (X, \overline{X}) & \end{array}$$

is commutative.

(2) A covariant functor from a small category to the category of $(X, \overline{X})$-triples (resp. $(X, \overline{X})$-lft-triples) over $\mathfrak{T}$ is called a diagram of $(X, \overline{X})$-triples (resp. $(X, \overline{X})$-lft-triples) over $\mathfrak{T}$. ■

Let $(\mathfrak{Y}_., I)$ and $(\mathfrak{Z}_., J)$ be diagrams of $(X, \overline{X})$-triples over $\mathfrak{T}$. We define a fiber product of diagrams of $(X, \overline{X})$-triples over $\mathfrak{T}$ by

$$\mathfrak{Y}_. \times_{(X, \overline{X}, \mathcal{T})} \mathfrak{Z}_. = (Y_. \times_X Z_., \overline{Y}_. \times_{\overline{X}} \overline{Z}_., \mathcal{Y}_. \times_{\mathcal{T}} \mathcal{Z}_.).$$

Since $\overline{X} \to \overline{T}$ is separated, $\overline{Y}_m \times_{\overline{X}} \overline{Z}_n \to \mathcal{Y}_m \times_{\mathcal{T}} \mathcal{Z}_n \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spec}\, k$ is a closed immersion for any $m \in \mathrm{Ob}\,(I)$ and $n \in \mathrm{Ob}\,(J)$. The fiber product $\mathfrak{Y}_. \times$ $\times_{(X, \overline{X}, \mathcal{T})} \mathfrak{Z}_.$ is a diagram of $(X, \overline{X})$-triples over $\mathfrak{T}$ indexed by $I \times J$ and both projections are morphisms of diagrams of $(X, \overline{X})$-triples over $\mathfrak{T}$. If $\mathfrak{Y}_.$ is locally of finite type over $\mathfrak{T}$, then $\mathfrak{Y}_. \times_{(X, \overline{X}, \mathcal{T})} \mathfrak{Z}_.$ is a triple locally of finite type over $\mathfrak{Z}_.$.

10.1.2. Definition. Let $(X, \overline{X})$ be a pair separated and locally of finite type over $(T, \overline{T})$. An $(X, \overline{X})$-triple $\mathfrak{U} = (U, \overline{U}, \mathcal{U})$ separated and locally of finite type over $\mathfrak{T}$ is a Zariski covering of $(X, \overline{X})$ over $\mathfrak{T}$ if it satisfies the following conditions:

(i) $\mathcal{U} \to \mathcal{T}$ is smooth around U;
(ii) $\overline{u}: \overline{U} \to \overline{X}$ is a Zariski covering and $U = \overline{u}^{-1}(X)$. ∎

If $\mathfrak{U}$ and $\mathfrak{V}$ are Zariski coverings of $(X, \overline{X})$ over $\mathfrak{T}$, then the fiber product $\mathfrak{U} \times_{(X, \overline{X}, \mathcal{T})} \mathfrak{V}$ as $(X, \overline{X})$-triples over $\mathfrak{T}$ is also a Zariski covering of $(X, \overline{X})$ over $\mathfrak{T}$.

Now we introduce the notion of universally de Rham descendable hypercovering.

10.1.3. Definition. Let $(X, \overline{X})$ be a pair separated locally of finite type over $(T, \overline{T})$ and let $\mathfrak{Y}_.$ be a simplicial $(X, \overline{X})$-triple locally of finite type over $\mathfrak{T}$ such that $\mathcal{Y}_n \to \mathcal{T}$ is smooth around Y_n for any n. We say that $\mathfrak{Y}_.$ is a universally de Rham descendable hypercovering of $(X, \overline{X})$ over $\mathfrak{T}$ if, for any $(X, \overline{X})$-lft-triple $\mathfrak{Z}$ over $\mathfrak{T}$, the base change $\mathfrak{Y}_. \times_{(X, \overline{X}, \mathcal{T})} \mathfrak{Z} \to \mathfrak{Z}$ (i.e., as diagrams of $(X, \overline{X})$-triples over $\mathfrak{T}$) is de Rham descendable in the sense of Definition 8.3.1 (1). ∎

Note that the morphism $\mathfrak{Y}_. \times_{(X, \overline{X}, \mathcal{T})} \mathfrak{Z} \to \mathfrak{Z}$ in the definition above is automatically universally de Rham descendable.

10.1.4. Proposition. *Let $(X, \overline{X})$ be a pair separated and locally of finite type over $(T, \overline{T})$ and let $\mathfrak{U}$ be a Zariski covering of $(X, \overline{X})$ over $\mathfrak{T}$. Then the Čech diagram*

$$\mathfrak{U}_. = \mathrm{cosk}_0^{(X, \overline{X}, \mathcal{T})}(\mathfrak{U}) = (\mathrm{cosk}_0^{X}(U), \mathrm{cosk}_0^{\overline{X}}(\overline{U}), \mathrm{cosk}_0^{\mathcal{T}}(\mathcal{U}))$$

of $\mathfrak{U}$ as $(X, \overline{X})$-triples over $\mathfrak{T}$ is a universally de Rham descendable hypercovering of $(X, \overline{X})$ over $\mathfrak{T}$.

Proof. Let $\mathfrak{Z}$ be an $(X, \overline{X})$-lft-triple over $\mathfrak{T}$. Since $\mathrm{cosk}_0^{\mathcal{T}}(\mathcal{U}) \times_{\mathcal{T}} \mathcal{Z} = \mathrm{cosk}_0^{\mathcal{Z}}(\mathcal{U} \times_{\mathcal{T}} \mathcal{Z})$, the assertion follows from Example 8.3.3 (1) and Corollary 8.6.2. ∎

10.1.5. Corollary. *Let $(X, \overline{X})$ be a pair separated locally of finite type over $(T, \overline{T})$. Then there exists a universally de Rham descendable hypercovering of $(X, \overline{X})$ over $\mathfrak{T}$.*

10.1.6. EXAMPLE. Let $(X, \overline{X})$ be a pair separated locally of finite type over $(T, \overline{T})$.

(1) We say that a simplicial $(X, \overline{X})$-triple $\mathfrak{Y}_.$ over $\mathfrak{T}$ which is separated and locally of finite type over $\mathfrak{T}$ is an étale-étale (resp. étale-proper) hypercovering of $(X, \overline{X})$ over $\mathfrak{T}$ if it satisfies the following conditions:

(i) $(Y_., \overline{Y}_.) \to (X, \overline{X})$ is an étale-étale hypercovering (resp. an étale-proper hypercovering) of pairs;

(ii) $\mathrm{cosk}_n^{\mathcal{T}}(\mathcal{Y}_.^{(n)})_l \to \mathrm{cosk}_{n-1}^{\mathcal{T}}(\mathcal{Y}_.^{(n-1)})_l$ is smooth around $\mathrm{cosk}_n^X(Y_.^{(n)})_l$ for any n and l.

Since any base change $\mathfrak{Y}_. \times_{(X, \overline{X}, \mathcal{T})} \mathfrak{Z} \to \mathfrak{Z}$ by an $(X, \overline{X})$-lft-triple $\mathfrak{Z}$ over $\mathfrak{T}$ as diagrams of $(X, \overline{X})$-triples over $\mathfrak{T}$ is an étale-étale hypercovering (resp. an étale-proper hypercovering) in the sense of Definition 7.2.2, any étale-étale (resp. étale-proper) hypercovering $\mathfrak{Y}_.$ of $(X, \overline{X})$ over $\mathfrak{T}$ is a universally de Rham descendable hypercovering of $(X, \overline{X})$ over $\mathfrak{T}$ by Theorem 9.1.1.

(2) Let $\mathfrak{Y}$ be an $(X, \overline{X})$-triple separated locally of finite type over $\mathfrak{T}$ such that

(i) $\mathcal{Y} \to \mathcal{T}$ is smooth around Y;
(ii) $\overline{Y} \to \overline{X}$ is proper;
(iii) $Y \to X$ is an isomorphism.

Then the constant simplicial triple $\mathfrak{Y}_.^{\Delta^\circ}$ is a universally de Rham descendable hypercovering of $(X, \overline{X})$ over $\mathfrak{T}$ by Corollary 8.3.6. ■

10.2. We recall the definition of overconvergent isocrystals which was introduced by Berthelot [7, 2.2, 2.3].

First we recall the definition of overconvergent connections. Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ be a triple separated locally of finite type over $\mathfrak{S}$ such that $\mathcal{X} \to \mathcal{S}$ is smooth around X and let ∂X be a complement of X in $\overline{X}$. Let us put $\mathfrak{X}^2 = \mathfrak{X} \times_{(X, \overline{X}, \mathcal{S})} \mathfrak{X}$ and $\mathfrak{X}^3 = \mathfrak{X} \times_{(X, \overline{X}, \mathcal{S})} \mathfrak{X} \times_{(X, \overline{X}, \mathcal{S})} \mathfrak{X}$ to be the fiber products as $(X, \overline{X})$-triples over $\mathfrak{S}$. We denote by p_i: $\mathfrak{X}^2 \to \mathfrak{X}$ $(i=1, 2)$ (resp. q_i: $\mathfrak{X}^3 \to \mathfrak{X}$ $(i=1, 2, 3)$, resp. $r_ij : \mathfrak{X}^3 \to \mathfrak{X}^2$ $(1 \leqq i < j \leqq 3)$, $\delta : \mathfrak{X} \to \mathfrak{X}^2$) to be the i-th projection (resp. the i-th projection, resp. the (i, j)-th projection, resp. the diagonal morphism) of triples. We define the ideal $\mathcal{I}$ of $\mathcal{O}_{]\overline{X}[_{\mathcal{X}^2}}$ as the ideal of the diagonal immersion $\tilde{\delta}:]\overline{X}[_{\mathcal{X}} \to]\overline{X}[_{\mathcal{X}^2}$ and define an ideal $j^\dagger \mathcal{I}^{n+1}$ of $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}^2}}$ by the sheaf of overconvergent sections of $\mathcal{I}^{n+1}$ along ∂X for any nonnegative integer n.

Let E be a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules with an integrable connection $\nabla : E \to E \otimes_{j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}} j^\dagger \Omega^1_{]\overline{X}[_{\mathcal{X}}/]\overline{S}[_{\mathcal{S}}}$ on $]\overline{X}[_{\mathcal{X}}$ over $]\overline{S}[_{\mathcal{S}}$. ∇ is an overconvergent connection along ∂X if there exists an isomorphism

$$\varepsilon : p_2^\dagger E \xrightarrow{\cong} p_1^\dagger E$$

of sheaves of $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}^2}}$-modules which satisfies the conditions:

(i) the diagram

$$\begin{tikzcd} q_3^\dagger E \arrow[rr, "r_{13}^\dagger(\varepsilon)"] \arrow[dr, "r_{23}^\dagger(\varepsilon)"'] & & q_1^\dagger E \\ & q_2^\dagger E \arrow[ur, "r_{12}^\dagger(e)"'] & \end{tikzcd}$$

is commutative;

(ii) $\delta^\dagger(\varepsilon) = \mathrm{id}_E$;

(iii) the connection ∇ is induced by the isomorphism ε modulo $j^\dagger \mathcal{I}^{n+1}$ for all n in the sense of [7, 2.2.2], that is, ε modulo $j^\dagger \mathcal{I}^{n+1}$ gives the n-truncated Taylor's expansion

$$\varepsilon(1 \otimes e) \equiv \sum_{m_1 + \ldots + m_d \leqq n} 1/(m_1! \ldots m_d!)(\nabla(\partial/\partial t_1)^{m_1} \ldots$$

$$\ldots \nabla(\partial/\partial t_d)^{m_d})(e)\, \xi_1^{m_1} \ldots \xi_d^{m_d} (\mathrm{mod}\, j^\dagger \mathcal{I}^{n+1})$$

($\xi_i = 1 \otimes t_i - t_i \otimes 1$) in the situation of Lemma 8.1.2.

A morphism of overconvergent connections is a horizontal $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-homomorphism [7, 2.2.5 Définition].

Since $\bigcap_n j^\dagger \mathcal{I}^{n+1} = (0)$ by the smoothness of $\mathcal{X} \to \mathcal{S}$ around X and Lemma 2.9.1, the isomorphism ε is unique for the overconvergent connection ∇. On the other hand, an isomorphism ε which satisfies the conditions (i) and (ii) above determines an overconvergent connection on $]\overline{X}[_{\mathcal{X}}$ over $]\overline{S}[_{\mathcal{S}}$ along ∂X by the meaning of (iii). A homomorphism of overconvergent connections commutes with these isomorphisms ε.

From now on, let $(X, \overline{X})$ be a pair separated locally of finite type over $(S, \overline{S})$. (Note that we do not assume the existence of a global embedding of $\overline{X}$ into formal schemes over $\mathcal{S}$ smooth around X.)

Let $\mathfrak{U} = (U, \overline{U}, \mathcal{U})$ be a Zariski covering of $(X, \overline{X})$ over $\mathfrak{S}$. We put $\mathfrak{U}^2 = \mathfrak{U} \times_{(X, \overline{X}, \mathcal{S})} \mathfrak{U}$ and $\mathfrak{U}^3 = \mathfrak{U} \times_{(X, \overline{X}, \mathcal{S})} \mathfrak{U} \times_{(X, \overline{X}, \mathcal{S})} \mathfrak{U}$. We define by $p_i : \mathfrak{U}^2 \to \mathfrak{U}$ $(i = 1, 2)$ (resp. $q_i : \mathfrak{U}^3 \to \mathfrak{U}$ $(i = 1, 2, 3)$, resp. $r_ij : \mathfrak{U}^3 \to \mathfrak{U}^2$ $(1 \leqq i < j \leqq 3)$, $\delta : \mathfrak{U} \to \mathfrak{U}^2$) to be the i-th projection (resp. the i-th projection, resp. the (i, j)-th projection, resp. the diagonal morphism) of triples.

A realization $(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}})$ of an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ over $\mathfrak{U}$ is a sheaf $E_{\mathfrak{U}}$ of coherent $j^\dagger\mathcal{O}_{]\overline{U}[_{\mathfrak{U}}}$-modules with an overconvergent connection $\nabla_{\mathfrak{U}}: E_{\mathfrak{U}} \to E_{\mathfrak{U}} \otimes_{j^\dagger\mathcal{O}_{]\overline{U}[_{\mathfrak{U}}}} j^\dagger \Omega^1_{]\overline{U}[_{\mathfrak{U}}/]\overline{S}[_{\mathcal{S}}}$ on $]\overline{U}[_{\mathfrak{U}}$ over $]\overline{S}[_{\mathcal{S}}$ along ∂U such that there exists a horizontal isomorphism

$$\varrho : p_2^\dagger(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}}) \xrightarrow{\cong} p_1^\dagger(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}})$$

of sheaves of $j^\dagger\mathcal{O}_{]\overline{U}^2[_{\mathfrak{U}^2}}$-modules which satisfies the conditions:

(i) the diagram

$$\begin{tikzcd} q_3^\dagger(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}}) \arrow[rr, "r_{13}^\dagger(\varrho)"] \arrow[dr, "r_{23}^\dagger(\varrho)"'] & & q_1^\dagger(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}}) \\ & q_2^\dagger(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}}) \arrow[ur, "r_{12}^\dagger(\varrho)"'] & \end{tikzcd}$$

is commutative;

(ii) $\delta^\dagger(\varrho) = \mathrm{id}_{(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}})}$.

A homomorphism of realizations of overconvergent isocrystals on $(X, \overline{X})/\mathcal{S}_K$ over $\mathfrak{U}$ is a horizontal $j^\dagger\mathcal{O}_{]\overline{U}[_{\mathfrak{U}}}$-homomorphism which commutes with ϱ. We denote by $\mathrm{Isoc}^\dagger((X, \overline{X}), \mathfrak{U}/\mathcal{S}_K)$ the category of realizations of overconvergent isocrystals on $(X, \overline{X})/\mathcal{S}_K$ over $\mathfrak{U}$ [7, 2.3.2 Définition (i), (iii)].

Let $(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}})$ and $(E'_{\mathfrak{U}}, \nabla'_{\mathfrak{U}})$ be realizations of overconvergent isocrystals. The category of realizations of overconvergent isocrystals is closed under the tensor product

$$(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}}) \otimes (E'_{\mathfrak{U}}, \nabla'_{\mathfrak{U}}) = (E_{\mathfrak{U}} \otimes_{j^\dagger\mathcal{O}_{]\overline{U}[_{\mathfrak{U}}}} E'_{\mathfrak{U}}, 1 \otimes \nabla'_{\mathfrak{U}} + \nabla_{\mathfrak{U}} \otimes 1)$$

and the internal hom

$$\underline{\mathrm{Hom}}((E_{\mathfrak{U}}, \nabla_{\mathfrak{U}}), (E'_{\mathfrak{U}}, \nabla'_{\mathfrak{U}})) = (\underline{\mathrm{Hom}}_{j^\dagger\mathcal{O}_{]\overline{U}[_{\mathfrak{U}}}}(E_{\mathfrak{U}}, E'_{\mathfrak{U}}), \nabla),$$

where the connection ∇ is defined by

$$\nabla(\eta)(s) = \nabla'_{\mathfrak{U}}(\eta(s)) - (\eta \otimes \mathrm{id}_{j^\dagger\Omega^1_{]\overline{U}[_{\mathfrak{U}}/]\overline{S}[_{\mathcal{S}}}})(\nabla_{\mathfrak{U}}(s))$$

for $\eta : E_{\mathfrak{U}} \to E'_{\mathfrak{U}}$ and $s \in E_{\mathfrak{U}}$ [7, 2.2.10 Corollaire].

10.2.1. Proposition. *Let $\mathfrak{X}$ be a triple separated locally of finite type over $\mathfrak{S}$ such that $\mathcal{X} \to \mathcal{S}$ is smooth around X and let ∂X be a complement* *of X in $\overline{X}$. Then the natural forgetful functor*

$$\begin{array}{ccc} \mathrm{Isoc}^\dagger((X,\overline{X}),\mathfrak{X}/\mathcal{S}_K) & \to & \left(\begin{array}{l}\textit{sheaves } E \textit{ of coherent } j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}}}\textit{-modules with an} \\ \textit{overconvergent connection } \nabla \textit{ on }]\overline{X}[_{\mathcal{X}} \textit{ over }]\overline{S}[_{\mathcal{S}} \textit{ along } \partial X\end{array}\right) \\ (E_{\mathfrak{X}},\nabla_{\mathfrak{X}},\varepsilon,\varrho) & \mapsto & (E_{\mathfrak{X}},\nabla_{\mathfrak{X}},\varepsilon) \end{array}$$

is an equivalence of categories.

Under our assumption the two isomorphisms ε and ϱ have the same source $p_2^\dagger E_{\mathfrak{X}}$ and the same target $p_1^\dagger E_{\mathfrak{X}}$. The proposition above follows from the lemma below.

10.2.2. LEMMA. *With the notation as in Proposition* 10.2.1, *if* $(E_{\mathfrak{X}}, \nabla_{\mathfrak{X}}, \varepsilon, \varrho)$ *is a realization of an overconvergent isocrystal on* $(X,\overline{X})/\mathcal{S}_K$ *over* $\mathfrak{X}$, *then we have* $\varrho = \varepsilon$.

PROOF. One may assume the situation of Lemma 8.1.2. If $\eta : p_2^\dagger(E_{\mathfrak{X}}, \nabla_{\mathfrak{X}}) \to p_1^\dagger(E_{\mathfrak{X}}, \nabla_{\mathfrak{X}})$ is a horizontal $j^\dagger\mathcal{O}_{]\overline{X}[_{\mathcal{X}^2}}$-homomorphism such that $\delta^\dagger(\eta) = 0$, then $\eta = 0$. Indeed, we have $\bigcap_n j^\dagger \mathfrak{I}^{n+1} = 0$ by the smoothness of $\mathcal{X}$ over $\mathcal{S}$ around X and, for any $e \in j^\dagger\mathfrak{I}^{n+1} p_1^\dagger E_{\mathfrak{X}} \setminus j^\dagger \mathfrak{I}^{n+2} p_1^\dagger E_{\mathfrak{X}}$, there exists i such that $p_1^\dagger \nabla_{\mathfrak{X}}(\partial/\partial t_i \otimes 1)(e) \in j^\dagger \mathfrak{I}^n p_1^\dagger E_{\mathfrak{X}} \setminus j^\dagger\mathfrak{I}^{n+1} p_1^\dagger E_{\mathfrak{X}}$. Hence, ϱ is unique for $(E_{\mathfrak{X}}, \nabla_{\mathfrak{X}}, \varepsilon)$. Therefore, if $\varepsilon : p_2^\dagger E_{\mathfrak{X}} \to p_1^\dagger E_{\mathfrak{X}}$ commutes with the induced connections, then we have $\varrho = \varepsilon$. One can easily see the commutativity of ε and the induced connections by using Taylor expansions. ■

10.2.3. PROPOSITION ([7, 2.1.16, 2.1.17 Proposition, 2.3.2 Définition (iv)]). *Let $\mathcal{V}'$ be an object over $\mathcal{V}$ in* $\mathrm{CDRV}_{\mathbb{Z}_p}$, *let $\mathfrak{S}'$ be a $\mathcal{V}'$-triple which is induced from a formal $\mathcal{V}'$-scheme $\mathcal{S}'$ locally of finite type with a commutative diagram*

$$\begin{array}{ccc} \mathcal{S} & \leftarrow & \mathcal{S}' \\ \downarrow & & \downarrow \\ \mathrm{Spf}\ \mathcal{V} & \leftarrow & \mathrm{Spf}\ \mathcal{V}'. \end{array}$$

Let

$$\begin{array}{ccc} (X,\overline{X}) & \xleftarrow{v} & (X',\overline{X}') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ (S,\overline{S}) & \xleftarrow[u]{} & (S',\overline{S}'). \end{array}$$

be a commutative diagram of pairs such that $(X, \overline{X})$ *(resp.* $(X', \overline{X}')$*) is a pair separated and locally of finite type over* $(S, \overline{S})$ *(resp.* $(S', \overline{S}')$*), and let*

$$\begin{array}{ccc} \mathfrak{U} & \xrightarrow{w} & \mathfrak{U}' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathfrak{S} & \underset{u}{\leftarrow} & \mathfrak{S}' \end{array} \qquad \begin{array}{ccc} (U, \overline{U}) & \xleftarrow{w} & (U', \overline{U}') \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ (X, \overline{X}) & \underset{v}{\leftarrow} & (X', \overline{X}') \end{array}$$

be a commutative diagram of triples and pairs, respectively, such that $\mathfrak{U}$ *(resp.* $\mathfrak{U}'$*) is a Zariski covering of* $(X, \overline{X})$ *(resp.* $(X', \overline{X}')$*) over* $\mathfrak{S}$ *(resp.* $\mathfrak{S}'$*).*

(1) *If* $(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}})$ *is a realization of an overconvergent isocrystal on* $(X, \overline{X})/\mathcal{S}_K$ *over* $\mathfrak{U}$*. Then the inverse image* $w^\dagger(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}})$ *is a realization of an overconvergent isocrystal on* $(X', \overline{X}')/\mathcal{S}'_K$ *over* $\mathfrak{U}'$*.*

(2) *Suppose that* $w' : \mathfrak{U}' \to \mathfrak{U}$ *is another morphism of triples such that two diagrams above are also commutative after replacing* w *by* w'*. Then, for any realization* $(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}})$ *of an overconvergent isocrystal on* $(X, \overline{X})/\mathcal{S}_K$ *over* $\mathfrak{U}$*, the map* $(w', w) : \mathcal{U}' \to \mathcal{U}^2$ *induces an isomorphism*

$$(w', w)^\dagger(\varrho) : w^\dagger(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}}) \xrightarrow{\cong} (w')^\dagger(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}})$$

of realizations of overconvergent isocrystals on $(X', \overline{X}')/\mathcal{S}'_K$ *over* $\mathfrak{U}'$*. Here* ϱ *is the isomorphism coming from the definition of realization. Moreover, this isomorphism satisfies the cocycle conditions:*

(i) $(w'', w')^\dagger(\varrho)(w', w)^\dagger(\varrho) = (w'', w)^\dagger(\varrho)$;

(ii) $\delta^\dagger(w, w)^\dagger(\varrho) = \mathrm{id}_{(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}})}$.

Proof. Let us put $(w, w) : \mathfrak{U}' \times_{(U', \overline{U}', S')} \mathfrak{U}' \to \mathfrak{U} \times_{(U, \overline{U}, S)} \mathfrak{U}$ and let ε be the isomorphism as in the definition of overconvergent connections. Then one can see that the isomorphism $(w, w)^\dagger(\varepsilon)$ determines the connection $w^\dagger\nabla$. The rest is easy by definition. ■

On the other hand, we note that the inverse image functor commutes with tensor products and internal homs.

10.2.4. Proposition. *Let* $\mathfrak{U}$ *and* $\mathfrak{V}$ *be Zariski coverings of* $(X, \overline{X})$ *over* $\mathfrak{S}$ *and let* $w : \mathfrak{V} \to \mathfrak{U}$ *be a morphism of* $(X, \overline{X})$*-triples over* $\mathfrak{S}$ *such that* $\widehat{w} : \mathcal{V} \to \mathcal{U}$ *is smooth around* V *and* $\overline{w} : \overline{V} \to \overline{U}$ *is a Zariski covering. Then the inverse image functor* $w^\dagger : \mathrm{Isoc}^\dagger((X, \overline{X}), \mathfrak{U}/\mathcal{S}_K) \to$ $\to \mathrm{Isoc}^\dagger((X, \overline{X}), \mathfrak{V}/\mathcal{S}_K)$ *is an equivalence.*

We give some lemmas which will be used in the proof of Proposition 10.2.4.

10.2.5. LEMMA. *With the notation as in Proposition* 10.2.4, *assume furthermore that* $\widehat{w}: \mathcal{V} \to \mathcal{U}$ *is a Zariski covering and* $w: \mathfrak{V} \to \mathfrak{U}$ *is strict as a morphism of triples. Then the inverse image functor* $w^\dagger$ *is an equivalence.*

PROOF. $w^\dagger$ is faithful by Proposition 6.1.4. We shall prove the essential surjectivity of $w^\dagger$. Suppose that $(E_{\mathfrak{V}}, \nabla_{\mathfrak{V}})$ is a realization over $\mathfrak{V}$, ε and ϱ are isomorphisms as in the definition. The inverse image $(\mathrm{id}_{\mathfrak{V}}, \mathrm{id}_{\mathfrak{V}})^\dagger(\varrho)$ is a gluing data for a coherent sheaf and we get a sheaf $E_{\mathfrak{U}}$ of coherent $j^\dagger \mathcal{O}_{]\overline{U}[_{\mathfrak{U}}}$-modules, where $(\mathrm{id}_{\mathfrak{V}}, \mathrm{id}_{\mathfrak{V}})$ denotes the map $\mathfrak{V} \times_{\mathfrak{U}} \mathfrak{V} \to \mathfrak{V} \times_{(X, \overline{X}, \mathcal{S})} \mathfrak{V}$. Let $\mathrm{pr}_i: \mathfrak{V} \times_{(X, \overline{X}, \mathcal{S})} \mathfrak{V} \to \mathfrak{V}$ be the i-th projection and

$$(\mathrm{pr}_i, \mathrm{pr}_i): (\mathfrak{V} \times_{(X, \overline{X}, \mathcal{S})} \mathfrak{V}) \times_{(V \times_X V, \overline{V} \times_{\overline{X}} \overline{X}, \mathcal{S})} (\mathfrak{V} \times_{(X, \overline{X}, \mathcal{S})} \mathfrak{V}) \to \mathfrak{V} \times_{(V, \overline{V}, \mathcal{S})} \mathfrak{V}$$

be the induced morphism for $i = 1, 2$. Since ϱ is horizontal, we have an identity $(\mathrm{pr}_2, \mathrm{pr}_2)^\dagger(\varepsilon) = (\mathrm{pr}_1, \mathrm{pr}_1)^\dagger(\varepsilon)$. Consider the natural morphism

$$q: (\mathfrak{V} \times_{(V, \overline{V}, \mathcal{S})} \mathfrak{V}) \times_{(\mathfrak{U} \times_{(U, \overline{U}, \mathcal{S})} \mathfrak{U})} (\mathfrak{V} \times_{(V, \overline{V}, \mathcal{S})} \mathfrak{V}) \to$$

$$\to (\mathfrak{V} \times_{(X, \overline{X}, \mathcal{S})} \mathfrak{V}) \times_{(V \times_X V, \overline{V} \times_{\overline{X}} \overline{X}, \mathcal{S})} (\mathfrak{V} \times_{(X, \overline{X}, \mathcal{S})} \mathfrak{V})$$

of triples. Then the inverse image of the identity $(\mathrm{pr}_2, \mathrm{pr}_2)^\dagger(\varepsilon) = (\mathrm{pr}_1, \mathrm{pr}_1)^\dagger(\varepsilon)$ by $q^\dagger$ provides the gluing data for ε. Hence it determines an overconvergent connection $\nabla_{\mathfrak{U}}$ on $E_{\mathfrak{U}}$. $(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}})$ is a realization over $\mathfrak{U}$ with $w^\dagger(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}}) \cong (E_{\mathfrak{V}}, \nabla_{\mathfrak{V}})$ by the cocycle conditions of ϱ and ε. Hence, $w^\dagger$ is essentially surjective. The fullness of $w^\dagger$ also follows from the same argument. ■

By the strong fibration theorem [7, 1.3.7 Théorème] we have

10.2.6. LEMMA. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be triples separated of finite type over* $\mathfrak{S}$ *such that* $\mathcal{X}$ *is smooth* $\mathcal{S}$ *around* X *and let* $w: \mathfrak{Y} \to \mathfrak{X}$ *be a morphism of finite type over* $\mathfrak{S}$ *which satisfies the conditions:*

(i) $\widehat{w}: \mathcal{Y} \to \mathcal{X}$ *is étale around* Y;
(ii) $\overline{w}: \overline{Y} \to \overline{X}$ *is proper*;
(iii) $\overset{\circ}{w}: Y \to X$ *is an isomorphism.*

Then the inverse image functor

$$w^{\dagger}: \mathrm{Isoc}^{\dagger}((X, \overline{X}), \mathfrak{X}/\mathcal{S}_K) \to \mathrm{Isoc}^{\dagger}((Y, \overline{Y}), \mathfrak{Y}/\mathcal{S}_K)$$

is an equivalence of categories.

10.2.7. Lemma ([7, 2.2.11 Proposition]). *Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ be a triple separated locally of finite type over $\mathfrak{S}$ such that $\mathcal{X} \to \mathcal{S}$ is smooth around X, let $\{U_\alpha\}_\alpha$ be a finite Zariski covering of X, let us put $\mathfrak{U} = \coprod_\alpha (U_\alpha, \overline{X}, \mathcal{X})$ and denote by $w : \mathfrak{U} \to \mathfrak{X}$ the structure morphism. Let $\mathfrak{U}^n$ be the fiber product of n copies of $\mathfrak{U}$ over $\mathfrak{X}$, let us denote by $p_i: \mathfrak{U}^2 \to \mathfrak{U}$ $(i = 1, 2)$ (resp. $r_ij : \mathfrak{U}^3 \to \mathfrak{U}^2$ $(1 \leqq i < j \leqq 3)$, $\delta : \mathfrak{U} \to \mathfrak{U}^2$) the i-th projection (resp. the (i, j)-th projection, resp. the diagonal morphism) of triples.*

(1) *Suppose that (F, ∇_F) is a realization of an overconvergent isocrystal on $\coprod_\alpha (U_\alpha, \overline{X})/\mathcal{S}_K$ over $\mathfrak{U}$ with an isomorphism ϱ : $p_2^{\dagger}(F, \nabla_F) \xrightarrow{\cong} p_1^{\dagger}(F, \nabla_F)$ of realizations of overconvergent isocrystals which satisfies the conditions:*

(i) $r_{13}^{\dagger}(\varrho) = r_{12}^{\dagger}(\varrho)\, r_{23}^{\dagger}(\varrho)$;
(ii) $\delta^{\dagger}(\varrho) = \mathrm{id}_{(F, \nabla_F)}$.

Then there exists a unique realization (E, ∇_E) of an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ over $\mathfrak{X}$ such that $w^{\dagger}(E, \nabla_E) = (F, \nabla_F)$.

(2) *Suppose that $\psi : (F_1, \nabla_{F_1}) \to (F_2, \nabla_{F_2})$ is a homomorphism of realizations of overconvergent isocrystals on $\coprod_\alpha (U_\alpha, \overline{X})/\mathcal{S}_K$ over $\mathfrak{U}$ such that $p_2^{\dagger}(\psi) = p_1^{\dagger}(\psi)$. Then there exists a unique homomorphism $\varphi : (E_1, \nabla_{E_1}) \to (E_2, \nabla_{E_2})$ of realizations of overconvergent isocrystals on $(X_\alpha, \overline{X})/\mathcal{S}_K$ over $\mathfrak{X}$ such that $w^{\dagger}(\varphi) = \psi$.*

Proof. We may assume that $\mathcal{X}$ is affine by Lemma 10.2.5. Passing to a finite refinement of the covering $\{U_\alpha\}_\alpha$ of X, we may assume that a complement of U_α in $\overline{X}$ is defined by a single equation $g_\alpha = 0$ for a global section $g_\alpha \in \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$ for all α. Let us denote by $\mathfrak{I}$ the sheaf of ideals of $\mathcal{O}_{\overline{X}}$ which is generated by the g_α. If $\mathfrak{I}$ is the unit ideal, then there exists a Zariski covering $\{\overline{U}_\alpha\}_\alpha$ of $\overline{X}$ such that $U_\alpha = X \cap \overline{U}_\alpha$. Hence, the assertion follows from Lemma 10.2.5.

Suppose that $\mathfrak{I}$ is not the unit ideal. Let us take a lift $\widehat{g}_\alpha \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ of g_α and denote by $\widehat{\mathfrak{I}}$ the sheaf of ideals of $\mathcal{O}_{\mathcal{X}}$ which is generated by the

$\widehat{g}_\alpha$'s. We denote by $\overline{v}: \overline{X}' \to \overline{X}$ (resp. $\widehat{v}: \mathcal{X}' \to \mathcal{X}$) the blowing-up with respect to the sheaf of the ideal $\mathfrak{J}$ (resp. $\widehat{\mathfrak{J}}$). Let us put $\mathfrak{X}' = (X, \overline{X}', \mathcal{X}')$ and let

$$\begin{array}{ccc} \mathfrak{U} & \xleftarrow{v'} & \mathfrak{U}' \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle w'} \\ \mathfrak{X} & \xleftarrow[v]{} & \mathfrak{X}' \end{array}$$

be a natural cartesian square of triples. Then $v^\dagger$ and $(v')^\dagger$ are equivalence of categories of realizations of overconvergent isocrystals by Lemma 10.2.6. Hence, we have only to prove that $(w')^\dagger$ gives an equivalence of categories of realizations of overconvergent isocrystals. Since $\overline{v}: \overline{X}' \to \to \overline{X}$ is obtained by a blowing-up with respect to $\mathfrak{J}$, there exists a Zariski covering $\{\overline{U}'_\alpha\}_\alpha$ of $\overline{X}'$ such that $U_\alpha = X \cap \overline{U}'_\alpha$. Indeed, $\overline{X}'$ is a closed subscheme of the projective scheme defined by the set $\{g_\alpha t_\beta = g_\alpha t_\alpha\}_{\alpha,\beta}$ of homogeneous equations in a projective space. Then $\overline{U}'_\alpha$ is defined by $t_\alpha \neq \neq 0$. Hence, the assertion follows from Lemma 10.2.5. This completes the proof. ■

PROOF OF PROPOSITION 10.2.4. Taking Zariski coverings of $\mathfrak{U}$ and $\mathfrak{V}$ as triples and applying Lemma 10.2.5 to them, we may assume that $(V, \overline{V}) = (U, \overline{U})$. We may assume that $\widehat{w}: \mathcal{V} \to \mathcal{U}$ factors through a formal affine space $\widehat{\mathbb{A}}^d_{\mathcal{U}}$ over $\mathcal{U}$ such that $\mathcal{V} \to \widehat{\mathbb{A}}^d_{\mathcal{U}}$ is étale around V by Lemmas 10.2.6 10.2.7. Using an argument similar to the proof of Proposition 2.10.2 (1) ([7, 1.3.5 Théorème]) we may assume that $\mathcal{V} = \widehat{\mathbb{A}}^d_{\mathcal{U}}$. Then there exists a section $s: \mathfrak{U} \to \mathfrak{V}$ of w as a morphism of $(X, \overline{X})$-triples over $\mathfrak{S}$. Obviously, $s^\dagger w^\dagger = \mathrm{id}$. Since $sw: \mathfrak{V} \to \mathfrak{V}$ satisfies the hypotheses of Proposition 10.2.3 (2), the functor $w^\dagger s^\dagger: \mathrm{Isoc}^\dagger((X, \overline{X}), \mathfrak{V}/\mathcal{S}_K) \to \to \mathrm{Isoc}^\dagger((X, \overline{X}), \mathfrak{V}/\mathcal{S}_K)$ coincides with the identity functor. Hence, $w^\dagger$ is an equivalence. ■

Let $\mathfrak{U}$ and $\mathfrak{V}$ be Zariski coverings of $(X, \overline{X})$ over $\mathfrak{S}$. We put $\mathfrak{Z} = \mathfrak{U} \times \times_{(X, \overline{X}, \mathcal{S})} \mathfrak{V}$ and have the two projections $\mathrm{pr}_1: \mathfrak{Z} \to \mathfrak{U}$ and $\mathrm{pr}_2: \mathfrak{Z} \to \mathfrak{V}$. The composite

$$\theta_{\mathfrak{V}\mathfrak{U}} = (\mathrm{pr}_2^\dagger)^{-1} \mathrm{pr}_1^\dagger: \mathrm{Isoc}^\dagger((X, \overline{X}), \mathfrak{U}/\mathcal{S}_K) \to \mathrm{Isoc}^\dagger((X, \overline{X}), \mathfrak{V}/\mathcal{S}_K)$$

of functors gives an equivalence by Proposition 10.2.4. Moreover, Proposition 10.2.3 implies the following:

(1) if there exists a third Zariski covering $\mathfrak{W}$, then the induced equivalences satisfy the cocycle condition

$$\theta_{\mathfrak{W}\mathfrak{U}} = \theta_{\mathfrak{W}\mathfrak{V}}\,\theta_{\mathfrak{V}\mathfrak{U}};$$

(2) if $\mathfrak{U} = \mathfrak{V}$, then $\theta_{\mathfrak{U}\mathfrak{U}} = (\mathrm{pr}_2^\dagger)^{-1}\mathrm{pr}_1^\dagger$ is the identity functor. Therefore, the category $\mathrm{Isoc}^\dagger((X, \overline{X}), \mathfrak{U}/\mathcal{S}_K)$ is independent of the choice of Zariski coverings $\mathfrak{U}$ of $(X, \overline{X})$ over $\mathfrak{S}$ up to the canonical equivalence.

The category of overconvergent isocrystals on $(X, \overline{X})/\mathcal{S}_K$ is defined by the category of realizations of overconvergent isocrystals on $(X, \overline{X})/\mathcal{S}_K$ over a Zariski covering of $(X, \overline{X})$ over $\mathfrak{S}$. It does not depend on the choice of Zariski coverings up to the canonical equivalence. We denote it by $\mathrm{Isoc}^\dagger((X, \overline{X})/\mathcal{S}_K)$. The category of overconvergent isocrystals has tensor products and internal homs.

Now we recall the definition given in [7, 2.3.2 Definition (iv)] of the inverse image functor for overconvergent isocrystals. We suppose the situation to be as in the assumption of Proposition 10.2.3. Note that $\mathfrak{U}'$ always exists by Lemma 10.2.8 below. Let E be an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ and let $(E_{\mathfrak{U}}, \nabla_{\mathfrak{U}})$ be a realization of E over $\mathfrak{U}$. We define an inverse image functor

$$v^*\colon \mathrm{Isoc}^\dagger((X, \overline{X})/\mathcal{S}_K) \to \mathrm{Isoc}^\dagger((X', \overline{X}')/\mathcal{S}'_K)$$

which is induced by the inverse image functor $w^\dagger\colon \mathrm{Isoc}^\dagger((X, \overline{X}), \mathfrak{U}/\mathcal{S}_K) \to \to \mathrm{Isoc}^\dagger((X', \overline{X}'), \mathfrak{U}'/\mathcal{S}'_K)$. The definition of inverse image functor does not depend on the choices of Zariski coverings $\mathfrak{U}$ and $\mathfrak{U}'$ up to canonical isomorphisms by Proposition 10.2.3 (2). Moreover, for a third $(X'', \overline{X}'')$ over $\mathfrak{S}''$ with a morphism $v'\colon (X'', \overline{X}'') \to (X', \overline{X}')$ and commutative diagrams as in Proposition 10.2.3, we have $(vv')^* = (v')^* v^*$ where $v'\colon (X'', \overline{X}'') \to \to (X', \overline{X}')$ denotes the structure morphism. The inverse image functors commute with tensor products and internal homs.

10.2.8. Lemma. *Let $\mathfrak{S}$, $\mathfrak{S}'$, $(X, \overline{X})$ and $(X', \overline{X}')$ be as in Proposition 10.2.3 and let $\mathfrak{U}$ be a Zariski covering of $(X, \overline{X})$ over $\mathcal{S}$. Then there exists a Zariski covering $\mathfrak{U}'$ of $(X', \overline{X}')$ over $\mathfrak{S}'$ with commutative diagrams*

$$\begin{array}{ccccccc} \mathfrak{U} & \leftarrow & \mathfrak{U}' & \quad & (U, \overline{U}) & \leftarrow & (U', \overline{U}') \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathfrak{S} & \leftarrow & \mathfrak{S}' & & (X, \overline{X}) & \leftarrow & (X', \overline{X}') \end{array}$$

of triples and pairs, respectively. Moreover, if $\mathcal{S}'$ is smooth of finite type over $\mathcal{S}$, then one can take a Zariski covering $\mathfrak{U}'$ such that $\widehat{w} : \mathcal{U}' \to \mathcal{U}$ is of finite type and smooth around U'.

Overconvergent isocrystals satisfy gluing properties. From Proposition 10.2.4 we have

10.2.9. PROPOSITION ([7, 2.3.2 Definition (iii)]). *Let $(X, \overline{X})$ be a pair separated locally of finite type over $(S, \overline{S})$ and let $v : (Y, \overline{Y}) \to (X, \overline{X})$ be a Zariski covering. Let us denote by $(Y^n, \overline{Y}^n)$ the fiber product of n copies of $(Y, \overline{Y})$ over $(X, \overline{X})$ and by $p_i : (Y^2, \overline{Y}^2) \to (Y, \overline{Y})$ (resp. $q_i : (Y^3, \overline{Y}^3) \to (Y, \overline{Y})$, resp. $r_{ij} : (Y^3, \overline{Y}^3) \to (Y^2, \overline{Y}^2)$, resp. $\delta : (Y, \overline{Y}) \to$ $\to (Y^2, \overline{Y}^2)$) the i-th projection (resp. the i-th projection, resp. the (i, j)-th projection, resp. the diagonal morphism).*

(1) *Suppose that F is an overconvergent isocrystal on $(Y, \overline{Y})/\mathcal{S}_K$ with an isomorphism $\vartheta : p_2^* F \to p_1^* F$ as overconvergent isocrystals on $(Y^2, \overline{Y}^2)/\mathcal{S}_K$ which satisfies the conditions:*

(i) *the diagram*

$$\begin{array}{ccc} q_3^* F & \xrightarrow{r_{13}^*(\vartheta)} & q_1^* F \\ & {\scriptstyle r_{23}^*(\vartheta)} \searrow \quad \nearrow {\scriptstyle r_{12}^*(\vartheta)} & \\ & q_2^* F & \end{array}$$

is commutative as a diagram of overconvergent isocrystals on $(Y^3, \overline{Y}^3)/\mathcal{S}_K$;

(ii) $\delta^*(\vartheta) = \mathrm{id}_F$.

Then there exists a unique overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$ such that $F = v^* E$ and ϑ is an induced morphism by the identity $vp_1 = vp_2$.

(2) *Let $\psi : F_1 \to F_2$ be a homomorphism of overconvergent isocrystals on $(Y, \overline{Y})/\mathcal{S}_K$ such that $p_2^*(\psi) = p_1^*(\psi)$ via the isomorphism ϑ. Then, there exists a unique homomorphism $\varphi : E_1 \to E_2$ of overconvergent isocrystals on $(X, \overline{X})/\mathcal{S}_K$ such that $v^*(\varphi) = \psi$.*

The proposition below follows from Lemma 10.2.7.

10.2.10. PROPOSITION ([7, 2.2.11 Proposition]). *Let $(X, \overline{X})$ be a pair separated locally of finite type over $(S, \overline{S})$, let $\{U_\alpha\}_\alpha$ be a finite Zariski covering of X, and let us denote by $v : (U, \overline{U}) = \coprod_\alpha (U_\alpha, \overline{X}) \to (X, \overline{X})$ the structure morphism. Let us denote by $(U^n, \overline{U}^n)$ the fiber product of n*

copies of $(U,\overline{U})$ *over* $(X,\overline{X})$ *and by* $p_i\colon(U^2,\overline{U}^2)\to(U,\overline{U})$ *(resp.* $q_i\colon(U^3,\overline{U}^3)\to(U,\overline{U})$, *resp.* $r_{ij}:(U^3,\overline{U}^3)\to(U^2,\overline{U}^2)$, *resp.* $\delta:(U,\overline{U})\to$ $\to(U^2,\overline{U}^2))$
the i-th projection (resp. the i-th projection, resp. the (i,j)*-th projection, resp. the diagonal morphism).*

(1) *Suppose that* F *is an overconvergent isocrystal on* $(U,\overline{U})/\mathcal{S}_K$ *with an isomorphism* $\vartheta: p_2^*F\to p_1^*F$ *as overconvergent isocrystals on* $(U^2,\overline{U}^2)/\mathcal{S}_K$ *which satisfies the conditions:*

(i) *the diagram*

$$\begin{tikzcd} q_3^*F \arrow[rr, "r_{13}^*(\vartheta)"] \arrow[dr, "r_{23}^*(\vartheta)"'] & & q_1^*F \\ & q_2^*F \arrow[ur, "r_{12}^*(\vartheta)"'] & \end{tikzcd}$$

is commutative as a diagram of overconvergent isocrystals on $(U^3,\overline{U}^3)/\mathcal{S}_K$;

(ii) $\delta^*(\vartheta)=\mathrm{id}_F$.

Then there exists a unique overconvergent isocrystal E *on* $(X,\overline{X})/\mathcal{S}_K$ *such that* $F=v^*E$ *and* ϑ *is an induced morphism by the identity* $vp_1=vp_2$.

(2) *Let* $\psi:F_1\to F_2$ *be a homomorphism of overconvergent isocrystals on* $(U,\overline{U})/\mathcal{S}_K$ *such that* $p_2^*(\psi)=p_1^*(\psi)$ *via the isomorphism* ϑ. *Then, there exists a unique homomorphism* $\varphi:E_1\to E_2$ *of overconvergent isocrystals on* $(X,\overline{X})/\mathcal{S}_K$ *such that* $v^*(\varphi)=\psi$.

The category of overconvergent isocrystals does not depends on the choice of completions.

10.2.11. Proposition ([7, 2.3.5 Théorème]). *Let* $v:(Y,\overline{Y})\to(X,\overline{X})$ *be a morphism of pairs separated locally of finite type over* $(S,\overline{S})$. *Suppose that* $\mathring{v}\colon Y\to X$ *is an isomorphism and* $\overline{v}\colon\overline{Y}\to\overline{X}$ *is proper. Then the inverse image functor*

$$v^*\colon \mathrm{Isoc}^\dagger((X,\overline{X})/\mathcal{S}_K)\to\mathrm{Isoc}^\dagger((Y,\overline{Y})/\mathcal{S}_K)$$

is an equivalence.

Proof. We use a similar argument to that of Proposition 6.4.1. First we may assume that $\overline{v}$ is projective by the precise Chow's lemma [12, Corollaire 5.7.14]. Then we may assume that $\overline{v}$ is an isomorphism by the strong fibration theorem [7, 1.3.7 Théorème], Lemma 6.4.5 and Lemma

10.2.7. The assertion then follows from Proposition 10.2.1 and Lemma 10.2.6. ■

10.3. We introduce the notion of overconvergent isocrystals on diagrams of pairs.

10.3.1. DEFINITION. Let $(X_., \overline{X}_.)$ be a diagram of pairs separated locally of finite type over $(S, \overline{S})$ indexed by a small category I. We say that $E_.$ is an overconvergent isocrystal on $(X_., \overline{X}_.)/\mathcal{S}_K$ if it consists of data $\{(E_n)_{n \in \mathrm{Ob}(I)}, (E_.(\eta))_{\eta \in \mathrm{Mor}(I)}\}$ which satisfies the following conditions:

(i) E_n is an overconvergent isocrystal on $(X_n, \overline{X}_n)/\mathcal{S}_K$ for each object n of I;

(ii) $E_.(\eta): \eta^*_{(X_., \overline{X}_.)} E_n \to E_m$ is an isomorphism of overconvergent isocrystals on $(X_m, \overline{X}_m)/\mathcal{S}_K$ for any morphism $\eta : m \to n$ such that $E_.(\xi)\, \xi^*_{(X_., \overline{X}_.)}(E_.(\eta)) = E_.(\eta\xi)$ for any $\xi : l \to m$ and $\eta : m \to n$.

We denote by $\mathrm{Isoc}^\dagger((X_., \overline{X}_.)/\mathcal{S}_K)$ the category of overconvergent isocrystals on $(X_., \overline{X}_.)/\mathcal{S}_K$. ■

10.3.2. LEMMA. *Let $(X_., \overline{X}_.)$ be a diagram of pairs separated locally of finite type over $(S, \overline{S})$ indexed by a small category I. Let $\mathfrak{U}_.$ be a diagram of triples over $\mathfrak{S}$ indexed by I such that $\mathfrak{U}_n$ is a Zariski covering of $(X_n, \overline{X}_n)$ over $\mathfrak{S}$ and the diagram*

$$\begin{array}{ccc} (U_m, \overline{U}_m) & \xrightarrow{\eta_{(U_., \overline{U}_.)}} & (U_n, \overline{U}_n) \\ \downarrow & & \downarrow \\ (X_m, \overline{X}_m) & \xrightarrow[\eta_{(X_., \overline{X}_.)}]{} & (X_n, \overline{X}_n) \end{array}$$

is commutative for any morphism $\eta : m \to n$ of I. Then associating to each overconvergent isocrystal its realization on $\mathfrak{U}_.$ will induce an equivalence

$$\mathrm{Isoc}^\dagger((X_., \overline{X}_.)/\mathcal{S}_K) \to \mathrm{Isoc}^\dagger((X_., \overline{X}_.), \mathfrak{U}_./\mathcal{S}_K)$$

of categories. Here $\mathrm{Isoc}^\dagger((X_., \overline{X}_.), \mathfrak{U}_./\mathcal{S}_K)$ *denotes the category of realizations of overconvergent isocrystals on $(X_., \overline{X}_.)/\mathcal{S}_K$ over $\mathfrak{U}_.$.*

We say that the diagram $\mathfrak{U}_.$ in the lemma above is a Zariski covering of $(X_., \overline{X}_.)$ over $(S, \overline{S})$. In general such a diagram does not exist for arbitrary $(X_., \overline{X}_.)$. (See 11.4 and 11.5 in the case of étale-étale and étale-proper hypercoverings.)

In this paper we will not deal with descent theory for overconvergent isocrystals. We plan to discuss it in the future.

Let $\mathfrak{S}$ and $\mathfrak{S}'$ be as in Proposition 10.2.3 and let

$$\begin{array}{ccc} ((X_{.}, \overline{X}_{.}), I) & \overset{(v_{.}, t)}{\longleftarrow} & ((X'_{.}, \overline{X}'_{.}), I') \\ \downarrow & & \downarrow \\ (S, \overline{S}) & \longleftarrow & (S', \overline{S}') \end{array}$$

be a commutative diagram of pairs such that $(X_{.}, \overline{X}_{.}) \to (S, \overline{S})$ (resp. $(X'_{.}, \overline{X}'_{.}) \to (S', \overline{S}')$) is separated and locally of finite type. For an overconvergent isocrystal $E_{.}$ on $(X_{.}, \overline{X}_{.})/\mathcal{S}_K$, we define the inverse image functor

$$v_{.}^{*}: \mathrm{Isoc}^{\dagger}((X_{.}, \overline{X}_{.})/\mathcal{S}_K) \to \mathrm{Isoc}^{\dagger}((X'_{.}, \overline{X}'_{.})/\mathcal{S}'_K)$$

by $(v_{.}^{*} E_{.})_m = v_m^{*} E_{t(m)}$ for any object m in I' and $(v_{.}^{*} E_{.})(\eta) = v_m^{*}(E_{.}(\eta))$ for any morphism $\eta : m \to n$ in I'. One can easily see that it is well-defined. The inverse image functor coincides with the inverse image functor of categories of realizations via the canonical equivalence. Moreover, if, again in the hypotheses of Proposition 10.2.3 we have $\mathfrak{S}''$ and $((X''_{.}, \overline{X}''_{.}), I'')$ and a morphism $v'_{.} : (X'', \overline{X}'') \to (X', \overline{X}')$, which form a commutative diagrams with v, then we have $(v_{.} v'_{.})^{*} = (v'_{.})^{*} v_{.}^{*}$.

10.4. We now give a generalization of Berthelot's definition of rigid cohomology using the notion of universally de Rham descendable hypercoverings. Let $f : (X, \overline{X}) \to (T, \overline{T})$ be a morphism of pairs separated locally of finite type over $(S, \overline{S})$ and let E be an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$.

10.4.1. LEMMA. *With the notation as above, let $\mathfrak{Y}_{.}$ and $\mathfrak{Z}_{.}$ be universally de Rham descendable hypercoverings of $(X, \overline{X})$ over $\mathfrak{T}$ and consider the natural commutative diagram*

$$\begin{array}{ccccc} & & \mathfrak{W}_{.} & & \\ & \mathrm{pr}_{1.} \swarrow & & \searrow \mathrm{pr}_{2.} & \\ \mathfrak{Y}_{.} & & \downarrow & & \mathfrak{Z}_{.} \\ & \searrow & & \swarrow & \\ & & \mathfrak{T}, & & \end{array}$$

where $\mathfrak{W}_{.}$ is the 2-simplicial triples $\mathfrak{Y}_{.} \times_{(X, \overline{X}, \mathfrak{T})} \mathfrak{Z}_{.}$ over $\mathfrak{T}$ and $\mathrm{pr}_{i.}$ is the i-th projection. If we denote by $(E_{\mathfrak{Y}_{.}}, \nabla_{\mathfrak{Y}_{.}})$ (resp. $(E_{\mathfrak{Z}_{.}}, \nabla_{\mathfrak{Z}_{.}})$, resp. $(E_{\mathfrak{W}_{.}}, \nabla_{\mathfrak{W}_{.}})$) the realization of the inverse image of E over $\mathfrak{Y}_{.}$ (resp. $\mathfrak{Z}_{.}$, re-

sp. $\mathfrak{V}.$), *then canonical homomorphism*

$$\widetilde{\mathrm{pr}}^{\sharp}_{1.}:\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{T},\mathfrak{Y}.;\mathrm{DR}^{\dagger}(\mathfrak{Y}./\mathfrak{T},(E_{\mathfrak{Y}.},\nabla_{\mathfrak{Y}.})))\to\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{T},\mathfrak{V}.;\mathrm{DR}^{\dagger}(\mathfrak{V}./\mathfrak{T},(E_{\mathfrak{V}.},\nabla_{\mathfrak{V}.})))$$

induced by $\widetilde{\mathrm{pr}}_{1.}$ *is an isomorphism in* $D^+(\mathbb{Z}_{]\overline{T}[_{\mathfrak{T}}})$. *Here both* $\mathrm{DR}^{\dagger}$*'s are relative complexes as defined in* 8.2. *The same holds for* $\widetilde{\mathrm{pr}}_{2.}$.

Proof. Since $\mathrm{pr}_{1,(m,\ \cdot)}:\mathfrak{V}_{(m,\ \cdot)}\to\mathfrak{Y}_m$ is universally de Rham descendable for each m, the canonical homomorphism

$$\mathrm{DR}^{\dagger}(\mathfrak{Y}_m/\mathfrak{T},(E_{\mathfrak{Y}_m},\nabla_{\mathfrak{Y}_m}))\to\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{Y}_m,\mathfrak{V}_{(m,\cdot)};\mathrm{DR}^{\dagger}(\mathfrak{V}_{(m,\cdot)}/\mathfrak{T},(E_{\mathfrak{V}_{(m,\cdot)}},\nabla_{\mathfrak{V}_{(m,\cdot)}})))$$

is an isomorphism by Proposition 8.3.4. Then the assertion follows from Proposition 4.4.4. ■

With the notation as in the lemma above, we define an isomorphism

$$\varrho_{\mathfrak{Z}\mathfrak{Y}}:\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{T},\mathfrak{Y}.;\mathrm{DR}^{\dagger}(\mathfrak{Y}./\mathfrak{T},(E_{\mathfrak{Y}.},\nabla_{\mathfrak{Y}.})))\to\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{T},\mathfrak{Z}.;\mathrm{DR}^{\dagger}(\mathfrak{Z}./\mathfrak{T},(E_{\mathfrak{Z}.},\nabla_{\mathfrak{Z}.})))$$

in $D^+(\mathbb{Z}_{]\overline{T}[_{\mathfrak{T}}})$ by $\varrho_{\mathfrak{Z}\mathfrak{Y}}=(\widetilde{\mathrm{pr}}^{\sharp}_{2.})^{-1}\widetilde{\mathrm{pr}}^{\sharp}_{1.}$.

10.4.2. Lemma. *With the notation as above, we have*

(1) $\varrho_{\mathfrak{W}\mathfrak{Y}}=\varrho_{\mathfrak{W}\mathfrak{Z}}\varrho_{\mathfrak{Z}\mathfrak{Y}}$ *for universally de Rham descendable hypercoverings* $\mathfrak{Y}.$, $\mathfrak{Z}.$ *and* $\mathfrak{W}.$ *of* $(X,\overline{X})$ *over* $\mathfrak{T}$;

(2) $\varrho_{\mathfrak{Y}\mathfrak{Y}}=\mathrm{id}$ *for any universally de Rham descendable hypercovering* $\mathfrak{Y}.$ *of* $(X,\overline{X})$ *over* $\mathfrak{T}$.

Proof. (1) Consider projections from $\mathfrak{Y}.\times_{(X,\overline{X},\mathfrak{T})}\mathfrak{Z}.\times_{(X,\overline{X},\mathfrak{T})}\mathfrak{W}.$ to $\mathfrak{Y}.\times_{(X,\overline{X},\mathfrak{T})}\mathfrak{Z}.$, $\mathfrak{Z}.\times_{(X,\overline{X},\mathfrak{T})}\mathfrak{W}.$ and $\mathfrak{Y}.\times_{(X,\overline{X},\mathfrak{T})}\mathfrak{W}.$, and from $\mathfrak{Y}.\times_{(X,\overline{X},\mathfrak{T})}\mathfrak{Z}.$ to $\mathfrak{Y}.$ and $\mathfrak{Z}.$, $\dots$. Then each projection induces an isomorphism of derived Čech diagrams as in Lemma 10.4.1. The assertion follows from the commutativity of the diagram which consists of the projections.

(2) Since $\varrho_{\mathfrak{Y}\mathfrak{Y}}$ is an isomorphism, the assertion follows from the formula in (1). ■

By Lemmas 10.4.1 and 10.4.2 we have

10.4.3. Proposition. *With the notation as above,* $\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{T},\mathfrak{Y}.;\mathrm{DR}^{\dagger}(\mathfrak{Y}./\mathfrak{T},(\mathrm{E}_{\mathfrak{Y}.},\nabla_{\mathfrak{Y}.})))$ *is independent of the choices of universally de Rham descendable hypercoverings up to canonical isomorphisms in* $D^+(\mathbb{Z}_{]\overline{T}[_{\mathfrak{T}}})$.

Let $f:(X,\overline{X})\to(T,\overline{T})$ be a morphism of pairs separated locally of finite type over $(S,\overline{S})$ and let $\mathfrak{Y}_.$ be a universally de Rham descendable hypercovering of $(X,\overline{X})$ over $\mathfrak{T}$. A universally de Rham descendable hypercovering always exists by Corollary 10.1.5. We define the rigid cohomology complex for an overconvergent isocrystal E on $(X,\overline{X})/\mathcal{S}_K$ evaluated on $]\overline{T}[_{\mathcal{T}}$ by

$$\mathbb{R}f_{\mathrm{rig}\,\mathfrak{T}*}E=\mathbb{R}\mathcal{C}^\dagger(\mathfrak{T},\ \mathfrak{Y}_.;\ \mathrm{DR}^\dagger(\mathfrak{Y}_./\mathfrak{T},(E_{\mathfrak{Y}_.},\ \nabla_{\mathfrak{Y}_.})))$$

and the q-th rigid cohomology sheaf for an overconvergent isocrystal E on $(X,\overline{X})/\mathcal{S}_K$ evaluated on $]\overline{T}[_{\mathcal{T}}$ by

$$\mathbb{R}^qf_{\mathrm{rig}\,\mathfrak{T}*}E=\underline{H}^q(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{T},\ \mathfrak{Y}_.;\ \mathrm{DR}^\dagger(\mathfrak{Y}_./\mathfrak{T},(E_{\mathfrak{Y}_.},\ \nabla_{\mathfrak{Y}_.})))),$$

where $(E_{\mathfrak{Y}_.},\ \nabla_{\mathfrak{Y}_.})$ is a realization of the inverse image of E over $\mathfrak{Y}_.$. The q-th rigid cohomology sheaf is independent of the choice of universally de Rham descendable hypercoverings $\mathfrak{Y}_.$ up to canonical isomorphisms by Proposition 10.4.3. It is a sheaf of $j^\dagger\mathcal{O}_{]\overline{T}[_{\mathcal{T}}}$-modules by Proposition 4.2.4.

In the case $\mathfrak{T}=\mathfrak{S}=(\mathrm{Spec}\,k,\ \mathrm{Spec}\,k,\ \mathrm{Spf}\,\mathcal{V})$ we put

$$H^q_{\mathrm{rig}}((X,\overline{X})/K,\ E)=\mathbb{R}^qf_{\mathrm{rig}\,\mathfrak{S}*}E$$

and call it the q-th rigid cohomology group. In particular, if $\overline{X}$ is proper over $\mathrm{Spec}\,k$, we will see that the rigid cohomology does not depend on the choice of completion $\overline{X}$ of X in Proposition 10.5.3 below. In this case we denote by $H^q_{\mathrm{rig}}(X/K,\ E)$ the q-th rigid cohomology group.

10.5. We give some properties of rigid cohomology.

10.5.1. Proposition. *Let $f:(X,\overline{X})\to(T,\overline{T})$ be a morphism of pairs separated and locally of finite type over $(S,\overline{S})$.*

(1) *Let $\varphi:E\to F$ be a homomorphism of overconvergent isocrystals on $(X,\overline{X})/\mathcal{S}_K$ and let $\varphi_{\mathcal{Y}_.}:E_{\mathfrak{Y}_.}\to F_{\mathfrak{Y}_.}$ be the induced homomorphism of inverse images of φ on a universally de Rham descendable hypercovering $\mathfrak{Y}_.$. Then $\varphi_{\mathcal{Y}_.}$ induces a $j^\dagger\mathcal{O}_{]\overline{T}[_{\mathcal{T}}}$-homomorphism*

$$\mathbb{R}^qf_{\mathrm{rig}\,\mathfrak{T}*}(\varphi):\mathbb{R}^qf_{\mathrm{rig}\,\mathfrak{T}*}E\to\mathbb{R}^qf_{\mathrm{rig}\,\mathfrak{T}*}F$$

on q-th rigid cohomology sheaves and it is independent of the choices of universally de Rham descendable hypercoverings up to canonical isomorphisms. Rigid cohomology is functorial in its isocrystal argument.

(2) *Let* $0 \to E \to F \to G \to 0$ *be an exact sequence of overconvergent isocrystals on* $(X, \overline{X})/\mathcal{S}_K$. *Then there exists an exact sequence*

$$\begin{aligned} 0 &\to \mathbb{R}^0 f_{\mathrm{rig}\,\mathfrak{T}*} E \to \mathbb{R}^0 f_{\mathrm{rig}\,\mathfrak{T}*} F \to \mathbb{R}^0 f_{\mathrm{rig}\,\mathfrak{T}*} G \\ &\to \mathbb{R}^1 f_{\mathrm{rig}\,\mathfrak{T}*} E \to \quad \dots \\ &\to \mathbb{R}^q f_{\mathrm{rig}\,\mathfrak{T}*} E \to \mathbb{R}^q f_{\mathrm{rig}\,\mathfrak{T}*} F \to \mathbb{R}^q f_{\mathrm{rig}\,\mathfrak{T}*} G \\ &\to \quad \dots \end{aligned}$$

of sheaves of $j^\dagger \mathcal{O}_{]\overline{T}[_{\mathfrak{T}}}$*-modules. This long exact sequence is functorial in short exact sequences.*

Proof. (1) The existence of the $j^\dagger \mathcal{O}_{]\overline{T}[_{\mathfrak{T}}}$-homomorphisms on cohomology sheaves for a fixed hypercovering follows from Proposition 4.2.4. One can prove the independence of the choices of hypercoverings by an argument similar to the proof of Proposition 10.4.3. Assertion (2) follows from Proposition 4.2.4. ■

10.5.2. Proposition. *Let*

$$\begin{array}{ccc} \mathfrak{T} & \xleftarrow{u} & \mathfrak{T}' \\ \downarrow & & \downarrow \\ \mathfrak{S} & \leftarrow & \mathfrak{S}' \end{array}$$

be a commutative diagram of triples such that $\mathfrak{S}'$ *and* $\mathfrak{T}'$ *are* $\mathcal{V}'$*-triples locally of finite type as same as* $\mathfrak{S}$ *and* $\mathfrak{T}$ *at the beginning of this section, respectively, and let*

$$\begin{array}{ccc} (X, \overline{X}) & \xleftarrow{v} & (X', \overline{X}') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ (T, \overline{T}) & \xrightarrow{u} & (T', \overline{T}') \end{array}$$

be a commutative diagram of pairs such that $(X, \overline{X})$ *(resp.* $(X', \overline{X}')$*) is a pair separated and locally of finite type over* $(T, \overline{T})$ *(resp.* $(T', \overline{T}')$*). Then, for any overconvergent isocrystal* E *on* $(X, \overline{X})/\mathcal{S}_K$*, there exists a canonical homomorphism*

$$\mathbb{L}\tilde{u}^* \mathbb{R} f_{\mathrm{rig}\,\mathfrak{T}*} E \to \mathbb{R} f'_{\mathrm{rig}\,\mathfrak{T}'*} v^* E$$

in the derived category of complexes of sheaves of $j^\dagger \mathcal{O}_{]\overline{T}'[_{\mathfrak{T}'}}$*-modules. The triangle arising from a short exact sequence of overconvergent isocrys-*

tals in Proposition 10.5.1 *commutes with the canonical homomorphism.*

Assume furthermore that $\mathcal{C}' \to \mathcal{C}$ is flat around T'. Then the canonical homomorphism above induces, for all q, a $j^\dagger \mathcal{O}_{]\overline{T}'[_{\mathcal{C}'}}$-homomorphism

$$\tilde{u}^* \mathbb{R}^q f_{\mathrm{rig}\,\mathfrak{T}*} E \to \mathbb{R}^q f'_{\mathrm{rig}\,\mathfrak{T}'*} v^* E$$

which is functorial in pairs and in E.

PROOF. Let

$$\begin{array}{ccc} \mathfrak{U}_\bullet & \stackrel{w_\bullet}{\leftarrow} & \mathfrak{U}'_\bullet \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \mathfrak{T} & \underset{u}{\leftarrow} & \mathfrak{T}' \end{array} \qquad \begin{array}{ccc} (U_\bullet, \overline{U}_\bullet) & \stackrel{w_\bullet}{\leftarrow} & (U'_\bullet, \overline{U}'_\bullet) \\ \downarrow & & \downarrow \\ (X, \overline{X}) & \underset{v_\bullet}{\leftarrow} & (X', \overline{X}') \end{array}$$

be commutative diagrams of triples and pairs such that $\mathfrak{U}_\bullet$ (resp. $\mathfrak{U}'_\bullet$) is a universally de Rham descendable hypercovering of $(X, \overline{X})$ (resp. $(X', \overline{X}')$) over $\mathfrak{T}$ (resp. $\mathfrak{T}'$). If $\mathfrak{U}$ is a Zariski covering of $(X, \overline{X})$ over $\mathfrak{T}$, then there exists a Zariski covering of $(X', \overline{X}')$ over $\mathfrak{U} \times_{\mathfrak{T}\mathfrak{T}'}$. Hence, such $\mathfrak{U}_\bullet$ and $\mathfrak{U}'_\bullet$ always exist by Proposition 10.1.4 and Lemma 10.2.8. The existence of homomorphisms in the derived category for the fixed hypercoverings follows from 4.3. One can prove the independence of the choices of hypercoverings by an argument similar to the proof of Proposition 10.4.3. ■

10.5.3. PROPOSITION. *Let*

$$\begin{array}{ccc} (X, \overline{X}) & \stackrel{h}{\leftarrow} & (Y, \overline{Y}) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & (T, \overline{T}) & \end{array}$$

be a commutative diagram of pairs separated and locally of finite type over $(S, \overline{S})$ and let E be an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$. Suppose that $\overset{\circ}{h} : Y \to X$ is an isomorphism and $\overline{h} : \overline{Y} \to \overline{X}$ is proper. Then the canonical homomorphism

$$\mathbb{R}^q f_{\mathrm{rig}\,\mathfrak{T}*} E \to \mathbb{R}^q g_{\mathrm{rig}\,\mathfrak{T}*} h^* E$$

is an isomorphism for any q.

PROOF. We may assume that $\overline{h}$ is projective by the precise Chow's lemma ([12, Corollaire 5.7.14]) with the same argument as used for Proposition 6.4.1. We fix a closed immersion $\overline{Y} \to \mathbb{P}^r_{\overline{X}}$ over $\overline{X}$. Let $\mathfrak{U}$ be a Zariski covering of $(X, \overline{X})$ over $\mathfrak{T}$. We define a closed immersion

$$\overline{U} \times_{\overline{X}} \overline{Y} \to \widehat{\mathbb{P}}^r_{\mathcal{U}}$$

by $\overline{U} \times_{\overline{X}} \overline{Y} \to \overline{U} \times_{\overline{X}} \mathbb{P}^r_{\overline{X}} = \mathbb{P}^r_{\overline{U}} \to \widehat{\mathbb{P}}^r_{\mathcal{U}}$, where $\widehat{\mathbb{P}}^r_{\mathcal{U}}$ is a formal projective space over $\mathcal{U}$ of dimension r. Then $\mathfrak{V} = (U, \overline{U} \times_{\overline{X}} \overline{Y}, \widehat{\mathbb{P}}^r_{\mathcal{U}})$ is a Zariski covering of $(Y, \overline{Y})$ over $\mathfrak{T}$ and the natural diagram

$$\begin{array}{ccc} (U, \overline{U}) & \leftarrow & (V, \overline{V}) \\ \downarrow & & \downarrow \\ (X, \overline{X}) & \leftarrow & (Y, \overline{Y}) \end{array}$$

of pairs is commutative. Let us denote by $u_{\cdot} : \mathfrak{U}_{\cdot} \to \mathfrak{T}$ (resp. $v_{\cdot} : \mathfrak{V}_{\cdot} \to \mathfrak{T}$) the Čech diagram of $(X, \overline{X})$-triples over $\mathfrak{T}$ (resp. $(Y, \overline{Y})$-triples over $\mathfrak{T}$) and by $(E_{\mathfrak{U}_{\cdot}}, \nabla_{\mathfrak{U}_{\cdot}})$ (resp. $(E_{\mathfrak{V}_{\cdot}}, \nabla_{\mathfrak{V}_{\cdot}})$) the realization of E (resp. h^*E) over $\mathfrak{U}_{\cdot}$. (resp. $\mathfrak{U}_{\cdot}$). Then the commutativity of the above diagram induces a commutative diagram

$$\begin{array}{ccc} \underline{E}_1^{qr} = \mathbb{R}^r \tilde{u}_{q*} \mathrm{DR}^\dagger(\mathfrak{U}_q/\mathfrak{T}; (\mathrm{E}_{\mathfrak{U}_q}, \nabla_{\mathfrak{U}_q})) & \Rightarrow & \underline{H}^{q+r}(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{T}, \mathfrak{U}_{\cdot}; \mathrm{DR}^\dagger(\mathfrak{U}_{\cdot}/\mathfrak{T}, (E_{\mathfrak{U}_{\cdot}}, \nabla_{\mathfrak{U}_{\cdot}})))) \\ \downarrow & & \downarrow \\ \underline{E}_1^{qr} = \mathbb{R}^r \tilde{v}_{q*} \mathrm{DR}^\dagger(\mathfrak{V}_q/\mathfrak{T}; (E_{\mathfrak{V}_q}, \nabla_{\mathfrak{V}_q})) & \Rightarrow & \underline{H}^{q+r}(\mathbb{R}\mathcal{C}^\dagger(\mathfrak{T}, \mathfrak{V}_{\cdot}; \mathrm{DR}^\dagger(\mathfrak{V}_{\cdot}/\mathfrak{T}, (E_{\mathfrak{V}_{\cdot}}, \nabla_{\mathfrak{V}_{\cdot}})))) \end{array}$$

of spectral sequences for hypercoverings as in Lemma 4.4.2 (the case where $s = s' = 1$). Since $U_r = V_r$ and $\overline{V}_r \to \overline{U}_r$ is proper, $\underline{E}_1$-terms of both spectral sequences coincide with each other by Proposition 8.3.5. Hence, the targets of the two spectral sequences are isomorphic. This completes the proof. ■

10.5.4. COROLLARY. *Let $f : (X, \overline{X}) \to (T, \overline{T})$ be a morphism of pairs separated locally of finite type over $(S, \overline{S})$. If $\overline{X}$ is proper over $\overline{T}$, then the rigid cohomology $\mathbb{R}^q f_{\mathrm{rig}\,\mathfrak{T}*} E$ is independent of the choices of $\overline{X}$ up to canonical isomorphism.*

10.5.5. PROPOSITION. *Let $f : \mathfrak{X} \to \mathfrak{T}$ be a separated morphism of $\mathcal{V}$-triples locally of finite type such that $\mathcal{X} \to \mathcal{T}$ is smooth around X. Then the spectral sequence in Lemma* 4.4.2 (*the case where* $s = s' = 1$) *in-*

duces a canonical isomorphism

$$\mathbb{R}^q f_{\mathrm{rig}\,\mathfrak{T}*} E \xrightarrow{\cong} \mathbb{R}^q \tilde{f}_* \mathrm{DR}^\dagger(\mathfrak{X}/\mathfrak{T}; (E_{\mathfrak{X}}, \nabla_{\mathfrak{X}}))\,.$$

PROOF. The assertion easily follows from the fact that the constant simplicial triple $\mathfrak{X}_\bullet^{\Delta^\circ}$ is a universally de Rham descendable hypercovering of $(X, \overline{X})$ over $\mathfrak{T}$ (Example 10.1.6 (2)). ■

10.6. Let $f : (X, \overline{X}) \to (T, \overline{T})$ be a morphism of pairs separated locally of finite type over $(S, \overline{S})$. Since a universally de Rham descendable hypercovering of $(X, \overline{X})$ over $\mathfrak{T}$ always exists (Proposition 10.1.4), we can define a rigid cohomology $\mathbb{R}^q f_{\mathrm{rig}\,\mathfrak{T}*} E$ for an overconvergent F-isocrystal on $(X, \overline{X})/\mathcal{S}_K$. If $\overline{X}$ can be embedded into a formal $\mathcal{T}$-scheme $\mathcal{X}$ as a closed subscheme such that $\mathcal{X} \to \mathcal{T}$ is smooth around X, then our rigid cohomology is canonically isomorphic to the rigid cohomology as introduced by Berthelot (the right-hand side in Proposition 10.5.5). Hence, our rigid cohomology coincides with Berthelot's original definition of rigid cohomology in [3, Section 2]. Note that Berthelot also extended the definition to the non-embeddable case using Zariski coverings in [4, 1.5, Remarque].

10.6.1. THEOREM. *The rigid cohomology, defined above, canonically coincides with Berthelot's original rigid cohomology.*

11. Spectral sequences.

In this section we will prove the existence of spectral sequence for the rigid cohomology of étale-étale (resp. étale-proper) hypercoverings of a given scheme. In general there is no embedding of simplicial k-pairs into smooth formal $\mathcal{V}$-schemes. Even in the local situation, it is impossible to embed simplicial k-pairs into smooth formal $\mathcal{V}$-schemes. But, if one considers truncated simplicial pairs, then, locally, one can embed simplicial k-pairs into smooth formal $\mathcal{V}$-schemes. In 11.4 (the case of étale-étale hypercoverings) and 11.5 (the case of étale-étale hypercoverings) we will prove the existence of such truncated embeddings. We will use it to show the existence of spectral sequence for the rigid cohomology of étale-étale (resp. étale-proper) hypercoverings.

Throughout this section except in 11.2 and 11.3, let $\mathfrak{S} = (S, \overline{S}, \mathcal{S})$ be a $\mathcal{V}$-triple which is induced from a formal $\mathcal{V}$-scheme $\mathcal{S}$ of finite type and let $\mathfrak{T} = (T, \overline{T}, \mathcal{T})$ be a triple separated and of finite type over $\mathfrak{S}$ such that $\mathcal{T}$ is smooth over $\mathcal{S}$ around T.

11.1. First we state our main theorem in this section. In the case of Zariski-Zariski and Zariski-proper coverings Berthelot proved the existence of spectral sequences (Čech spectral sequences) in his unpublished note [6].

11.1.1. THEOREM. *Let $f : (X, \overline{X}) \to (T, \overline{T})$ be a separated morphism of finite type, let $g_{\bullet} : (Y_{\bullet}, \overline{Y}_{\bullet}) \to (X, \overline{X})$ be an étale-étale (resp. étale-proper) hypercovering and let us put $h_{\bullet} = fg_{\bullet}$. For any overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$, there exists a spectral sequence*

$$\underline{E}_1^{qr} = \mathbb{R}^r h_{q,\,\mathrm{rig}\,\mathfrak{T}*}(g_q^* E) \Rightarrow \mathbb{R}^{q+r} f_{\mathrm{rig}\,\mathfrak{T}*} E.$$

This spectral sequence is functorial in E, in $(X, \overline{X})$, in $(Y_{\bullet}, \overline{Y}_{\bullet})$ and in $\mathfrak{S}$ and $\mathfrak{T}$.

We reduce the theorem above to the case of truncated simplicial hypercoverings. We show that the next proposition implies Theorem 11.1.1 and we prove it in 11.6.

11.1.2. PROPOSITION. *Theorem* 11.1.1 *is valid if we suppose that*

$$\mathrm{cosk}_n^{(X,\overline{X})}((Y_{\bullet}^{(n)}, \overline{Y}_{\bullet}^{(n)})) = (Y_{\bullet}, \overline{Y}_{\bullet})$$

for some nonnegative integer n. The spectral sequence is functorial in E, in $(X, \overline{X})$, in n-truncated simplicial pair $(Y_{\bullet}^{(n)}, \overline{Y}_{\bullet}^{(n)})$, and in $\mathfrak{S}$ and $\mathfrak{T}$.

PROPOSITION 11.1.2 $\Rightarrow$ THEOREM 11.1.1. Let $g_{\bullet}^n : (\mathrm{cosk}_n^X(Y_{\bullet}^{(n)}), \mathrm{cosk}_n^{\overline{X}}(\overline{Y}_{\bullet}^{(n)})) \to (X, \overline{X})$ be the induced morphism of pairs and put $h_{\bullet}^n = fg_{\bullet}^n$ for a nonnegative integer n. Let

$${}^n\underline{E}_1^{qr} = \mathbb{R}^r h^n_{q,\,\mathrm{rig}\,\mathfrak{T}*}(g^n_{q*} E) \Rightarrow \mathbb{R}^{q+r} f_{\mathrm{rig}\,\mathfrak{T}*} E$$

be the spectral sequence for an overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$ with respect to $g_{\bullet}^n$. By the functoriality we have a canonical commutative diagram

$$\begin{array}{ccc} {}^n\underline{E}_1^{qr} & \Rightarrow & \mathbb{R}^{q+r} f_{\mathrm{rig}\,\mathfrak{T}*} E \\ \downarrow & & \downarrow \\ {}^{n+1}\underline{E}_1^{qr} & \Rightarrow & \mathbb{R}^{q+r} f_{\mathrm{rig}\,\mathfrak{T}*} E \end{array}$$

of spectral sequences for any n. Note ${}^n\underline{E}_1^{qr} = \mathbb{R}^r h_{q,\,\mathrm{rig}\,\mathfrak{T}*}(g_q^* E)$ for $q \leqq n$ by construction. If $q + r < (n-1)/2$, then ${}^m\underline{E}_s^{qr} = {}^n\underline{E}_s^{qr}$ for $m \geqq n$ and $s \geqq 1$. Since

$$ {}^n\underline{E}_n^{qr} = {}^n\underline{E}_{n+1}^{qr} = \ldots = {}^n\underline{E}_\infty^{qr}, $$

there exists a spectral sequence

$$ \underline{E}_1^{qr} = \mathbb{R}^r h_{q,\,\mathrm{rig}\,\mathfrak{T}*}(g_q^* E) \Rightarrow \mathbb{R}^{q+r} f_{\mathrm{rig}\,\mathfrak{T}*} E $$

with respect to the hypercovering $g_{\bullet} : (Y_{\bullet}, \overline{Y}_{\bullet}) \to (X, \overline{X})$.

Since the n-truncated version of the spectral sequences is functorial in E, the same holds for general cases. Since $(\mathrm{cosk}_n^X(Y_{\bullet})_m, \mathrm{cosk}_n^{\overline{X}}(\overline{Y}_{\bullet})_m) = (Y_m, \overline{Y}_m)$ for $m \leqq n$, one can obtain the functoriality in $(Y_{\bullet}, \overline{Y}_{\bullet})$, in $(X, \overline{X})$ and in $\mathfrak{S}$ and $\mathfrak{T}$. ■

11.2. In this subsection «scheme» means «formal scheme» and we construct simplicial schemes which play an important role for constructing refinements of hypercovering and embedding simplicial schemes into smooth formal schemes in latter subsections. We deal with simplicial schemes, simplicial pairs and simplicial triples as contravariant functors from Δ to certain categories and denote by Δ^{epi} and $\Delta[n]^{\mathrm{epi}}$ the subcategory of Δ and $\Delta[n]$ whose set of morphisms consists of all epimorphisms.

Let X be a scheme and let n be a nonnegative integer. Let $Y_{\bullet}$ be a simplicial scheme over X and let $Z_{\bullet}^1, Z_{\bullet}^2, \ldots, Z_{\bullet}^r$ be simplicial schemes over $Y_{\bullet}$. We define a simplicial scheme

$$ U_{\bullet} = Z_{\bullet}^1 \times_{Y_{\bullet}} \ldots \times_{Y_{\bullet}} Z_{\bullet}^r $$

over X as follows:

- $U_m = Z_m^1 \times_{Y_m} \ldots \times_{Y_m} Z_m^r$ for any object m of Δ;
- $\eta_{\mathfrak{U}_{\bullet}} = \eta_{Z_{\bullet}^1} \times \ldots \times \eta_{Z_{\bullet}^r}$ for any morphism η of Δ.

One can check that $U_{\bullet}$ is a simplicial scheme over X and the natural morphism $U_{\bullet} \to Y_{\bullet}$ (resp. the projection $U_{\bullet} \to Z_{\bullet}^i$) is a morphism of simplicial schemes over X (resp. $\mathfrak{Y}_{\bullet}$). We call $U_{\bullet}$ a fiber product of $Z_{\bullet}^1, \ldots, Z_{\bullet}^m$ over $Y_{\bullet}$. One can also define a fiber product in the category of simplicial pairs, simplicial triples and $(X, \overline{X})$-simplicial triples over $\mathfrak{T}$.

11.2.1. LEMMA. *With the notation as above, we have*

$$\mathrm{cosk}_n^X((Z_\cdot^1 \times_{Y_\cdot} \dots \times_{Y_\cdot} Z_\cdot^r)^{(n)}) \cong$$

$$\cong \mathrm{cosk}_n^X((Z_\cdot^1)^{(n)}) \times_{\mathrm{cosk}_n^X(Y_\cdot^{(n)})} \dots \times_{\mathrm{cosk}_n^X(Y_\cdot^{(n)})} \mathrm{cosk}_n^X((Z_\cdot^r)^{(n)})$$

as simplicial schemes over X.

PROOF. This follows from the universality of the coskeleton functor which can be constructed as an inverse limit and from the fact that inverse limits commute with products. ■

We give a construction of simplicial schemes following [1, $\mathrm{V}^{\mathrm{bis}}$, 5.1]. We omit the proof of the next lemma

11.2.2. LEMMA. (1) *Let η be a morphism of Δ. Then, there exist a monomorphism η^{mon} and an epimorphism η^{epi} of Δ such that $\eta = \eta^{\mathrm{mon}}\eta^{\mathrm{epi}}$. Such a decomposition for η is unique.*

(2) *For morphisms ξ and η of Δ, we have $(\eta\xi)^{\mathrm{mon}} = \eta^{\mathrm{mon}}(\eta^{\mathrm{epi}}\xi)^{\mathrm{mon}}$ and $(\eta\xi)^{\mathrm{epi}} = (\eta\xi^{\mathrm{mon}})^{\mathrm{epi}}\xi^{\mathrm{epi}}$.*

Let S be a scheme. An n-truncated simplicial S-scheme $X_\cdot$ is split (σ-split in [1, $\mathrm{V}^{\mathrm{bis}}$, Définition 5.1.1]) if there exists a sequence $U_0, U_1, \dots, U_n$ of S-schemes such that, if we denote by $W_\beta = U_b$ for an epimorphism $\beta : m \to b$ of Δ, then

(i) for each object m of $\Delta[n]$

$$X_m = \coprod_{\beta : m \to b \in \mathrm{Mor}(\Delta[n]^{\mathrm{epi}})} W_\beta,$$

where β runs through all epimorphisms $\beta : m \to b$ of $\Delta[n]$ in the above disjoint sum;

(ii) for any epimorphism $\beta : m \to b$ of $\Delta[n]$, the diagram

$$\begin{tikzcd} & W_{\mathrm{id}_b} \arrow[r] & X_b \arrow[d, "\beta_{X_\cdot}"] \\ U_b \arrow[ur] \arrow[dr] & & \\ & W_\beta \arrow[r] & X_m \end{tikzcd}$$

is commutative.

A split simplicial scheme is defined similarly.

Let $X_.$ be a split n-truncated simplicial scheme over S and let $U_0, U_1, \dots, U_n$ and W_β ($\beta \in \mathrm{Mor}(\Delta[n]^{\mathrm{epi}})$) as above. Let U_{n+1} be an S-scheme with an S-morphism $U_{n+1} \to \mathrm{cosk}_n^S(X_.)_{n+1}$ and let us put $W_\beta = U_b$ for an epimorphism morphism $\beta : n+1 \to b$. We define an $(n+1)$-truncated simplicial scheme $Y_. = \Omega_{n+1}^S(X_., U_0, \dots, U_n, U_{n+1})$ over S as follows:

– for each object m of $\Delta[n+1]$,

$$Y_m = \begin{cases} X_m & \text{if } m \leqq n \\ \displaystyle\coprod_{\beta : n+1 \to b \in \mathrm{Mor}(\Delta[n+1]^{\mathrm{epi}})} W_\beta & \text{if } m = n+1, \end{cases}$$

where β runs through all epimorphisms $\beta : n+1 \to b$ of $\Delta[n+1]$ in the disjoint sum above;

– for each morphism $\eta : l \to m$ with $l, m \leqq n$,

$$\eta_{Y.} = \eta_{X.} : Y_m \to Y_l$$

– for each morphism $\eta : n+1 \to m$ with $m \leqq n$, $\eta_{Y.}$ is a composition

$$Y_m \overset{(\eta^{\mathrm{mon}})_{Y.}}{\longrightarrow} Y_l = \coprod_{\alpha : l \to a \in \mathrm{Mor}(\Delta[n+1]^{\mathrm{epi}})} W_\alpha = \coprod_{\alpha : l \to a \in \mathrm{Mor}(\Delta[n+1]^{\mathrm{epi}})} W_{\alpha\eta^{\mathrm{epi}}} \to Y_{n+1},$$

where $\eta^{\mathrm{mon}} : l \to m$ and $\eta^{\mathrm{epi}} : n+1 \to l$;

– for any morphism $\eta : m \to n+1$ of with $m \leqq n$, $\eta_{Y.}$ is defined by compositions

$$W_\beta = U_b = U_{\mathrm{id}_b} \to Y_b \overset{(\beta\eta)_{Y.}}{\longrightarrow} Y_m$$

$$W_{\mathrm{id}_{n+1}} = U_{n+1} \to \mathrm{cosk}_n^S(X_.)_{n+1} \overset{\eta_{\mathrm{cosk}_n^S(X.)}}{\longrightarrow} X_m = Y_m;$$

for any morphism $\eta : n+1 \to n+1$,

$$\eta_{Y.} = \begin{cases} (\eta^{\mathrm{epi}})_{Y.}(\eta^{\mathrm{mon}})_{Y.} & \text{if } \eta \neq \mathrm{id}_{n+1} \\ \mathrm{id}_{Y_{n+1}} & \text{if } \eta = \mathrm{id}_{n+1}. \end{cases}$$

Here η^{mon} and η^{epi} are defined in Lemma 11.2.2.

Then $Y_.$ is well-defined. Indeed, one can see that it is sufficient to prove $(\eta\xi)_{Y.} = \xi_{Y.}\eta_{Y.}$ for a monomorphism $\xi : l \to n+1$ and an epimorphism $\eta : n+1 \to m$ with $l, m \leqq n$ by Lemma 11.2.2. The well-definedness follows from the commutativity of the diagram

$$\begin{tikzcd}
 & W_{\mathrm{id}_b} \arrow[r] \arrow[d, equal] & Y_b \arrow[dd, "\beta_{Y.}"'] \arrow[dr, "(\beta\eta\xi)_{Z.}"] & \\
U_b \arrow[ur] \arrow[r] \arrow[dr] & W_{\beta\eta} \arrow[d, equal] & & Y_l \\
 & W_\beta \arrow[r] & Y_m \arrow[ur, "(\eta\xi)_{Z.}"'] &
\end{tikzcd}$$

for any $\beta : m \to b$. Hence we have the following lemma ([1, $\mathrm{V}^{\mathrm{bis}}$, Proposition 5.1.3]).

11.2.3. Lemma. *With the notation as above,*

(1) $Y. = \Omega^S_{n+1}(X., U_0, \ldots, U_n, U_{n+1})$ *is a split* $(n+1)$*-truncated simplicial scheme over* S.

(2) *Let (P) be a property of morphisms of schemes as in Definition* 7.2.1 (1). *If* $X.$ *is (P) over S and* $U_{n+1} \to \mathrm{cosk}^S_n(X.)_{n+1}$ *is (P), then* $Y.$ *is (P) over* S.

(3) *If* $X.$ *is an étale hypercovering (resp. a proper hypercovering) of* S *and* $U_{n+1} \to \mathrm{cosk}^S_n(X.)_{n+1}$ *is étale surjective (resp. proper surjective), then* $Y.$ *is an étale hypercovering (resp. a proper hypercovering) of* S.

Let $X.$ be a split n-truncated simplicial scheme over S, let $U_0, U_1, \ldots, U_n$ and W_β ($\beta \in \mathrm{Mor}(\Delta[n]^{\mathrm{epi}})$) as in the definition, and let us put $Y. = \Omega^S_{n+1}(X., U_0, \ldots, U_{n+1})$. Let $Z.$ be an $(n+1)$-truncated simplicial scheme over S. We suppose that there exists a morphism $f.: X. \to$ $\to Z.^{(n)}$ of n-truncated simplicial schemes over S and a morphism $h : U_{n+1} \to Z_{n+1}$ over S such that the diagram

$$\begin{array}{ccc}
U_{n+1} & \to & Z_{n+1} \\
\downarrow & & \downarrow \\
\mathrm{cosk}^S_n(X.)_{n+1} & \to & \mathrm{cosk}^S_n(Z.)_{n+1}
\end{array}$$

is commutative. We define a morphism

$$g.: Y. \to Z.$$

of $(n+1)$-truncated simplicial schemes as follows: $g_m = f_m$ for $m \leqq n$ and g_{n+1} is a disjoint sum of compositions of morphisms

$$W_\beta = U_b = W_{\mathrm{id}_b} \to X_b \xrightarrow{f_b} Z_b \xrightarrow{\beta_{Z.}} Z_{n+1},$$

for the W_β-component with the epimorphism $\beta : n+1 \to b$ ($\beta \neq \mathrm{id}_{n+1}$) and $h : W_{\mathrm{id}_{n+1}} = U_{n+1} \xrightarrow{h} Z_{n+1}$ for the $W_{\mathrm{id}_{n+1}}$-component. Then $g.$ is well-defined. Indeed, let us consider the following commutative diagram

$$
\begin{tikzcd}
 & W_{\mathrm{id}_b} \arrow[r] \arrow[d, equal] & Y_b \arrow[r, "g_b"'] \arrow[dd, "\beta_{Y.}"] & Z_b \arrow[dd, "\beta_{Z.}"] \arrow[dr, "(\beta\eta)_{Z.}"] & \\
U_b \arrow[ur] \arrow[r] \arrow[dr] & W_{\beta\eta} \arrow[r] \arrow[d, equal] & Y_{n+1} & & Z_{n+1} \\
 & W_\beta \arrow[r] & Y_m \arrow[r, "g_m"] & Z_m \arrow[ur, "\eta_{Z.}"'] &
\end{tikzcd}
$$

for epimorphisms $\eta : n+1 \to m$ ($m \neq n+1$) and $\beta : m \to b$. Since $Y.$ is a disjoint sum of W_γ's and the bottom arrows is the definition of g_{n+1}, we have $g_{n+1}\eta_{Y.} = \eta_{Z.}g_m$. One can also check the commutativity between $g.$ and a monomorphism $\eta : m \to n+1$ similarly. Hence, $g.$ is a morphism of $(n+1)$-truncated simplicial schemes over S.

Now we construct another simplicial scheme. Let n and q be objects of Δ. We denote by $\mathrm{Mor}_\Delta^{\leqq q}(n, m)$ the set of all morphisms $\gamma : n \to m$ such that the cardinal of the image $\gamma(n)$ is equal to or less than q. Then $\mathrm{Mor}_\Delta^{\leqq q}(n, m) = \mathrm{Mor}_\Delta(n, m)$ if $q \geqq n$.

Let W be a scheme over a scheme S. We define a simplicial scheme $\Gamma_n^S(W)^{\leqq q} = Z.$ over S as follows:

- for an object m,

$$Z_m = \prod_{\gamma : n \to m \in \mathrm{Mor}_\Delta^{\leqq q}(n, m)} W_\gamma,$$

where the product is a fiber product over S, γ runs through all morphisms $\gamma : n \to m$ in $\mathrm{Mor}_\Delta^{\leqq q}(n, m)$ and we put $W_\gamma = W$ for $\gamma : n \to m$;
- for a morphism $\eta : l \to m$ of Δ, we define

$$\eta_{Z.} : Z_m \to Z_l$$

by $(w_\gamma)_{\gamma : n \to m} \mapsto (v_\beta)_{\beta : n \to l}$ with $v_\beta = w_{\eta\beta}$.

Since the cardinal of the image $\eta\beta(n)$ is equal to or less than q for any morphism $\beta \in \mathrm{Mor}_\Delta^{\leqq q}(n, m)$, $Z. = \Gamma_n^S(W)^{\leqq q}$ is well-defined. $\Gamma_n^S(W)^{\leqq q}$ is functorial in W. If $q \geqq n$, then $\mathrm{Mor}_\Delta^{\leqq q}(n, m) = \mathrm{Mor}_\Delta(n, m)$ for any m. Hence, we have

$$\Gamma_n^S(W)^{\leqq q} = \Gamma_n^S(W)^{\leqq n}.$$

11.2.4. LEMMA. *Let $X_{\bullet}$ be an n-truncated scheme over S, let $f : X_n \to \to W$ be a morphism of S-schemes and let $Z_{\bullet} = \Gamma_n^S(W)^{\leqq n}$. If we define an S-morphism*

$$g_m : X_m \to Z_m$$

by $g_m = \prod_{\gamma : n \to m} f\gamma_{X_{\bullet}}$ for each m, then $g_{\bullet} : X_{\bullet} \to Z_{\bullet}^{(n)}$ is a morphism of n-truncated simplicial S-scheme. Moreover, if W is separated over S and f is a closed immersion, then g_n is a closed immersion. This construction is functorial.

PROOF. Since the diagram

$$\begin{array}{ccccc} X_m & \xrightarrow{g_m} & Z_m = \prod_{\gamma : n \to m} W_\gamma & \to & W_{\eta\alpha} \\ {\scriptstyle \eta_{X_\bullet}} \downarrow & & & & \downarrow {\scriptstyle \mathrm{id}_W} \\ X_l & \xrightarrow[g_l]{} & Z_l = \prod_{\beta : n \to l} W_\beta & \to & W_\alpha , \end{array}$$

where the right horizontal arrows are projections onto the $\eta\alpha$-component and onto the α-component, is commutative for any morphism $\eta : l \to m$ and $\alpha : n \to l$ in $\Delta[n]$, $g_{\bullet}$ is well-defined.

Let us regard Z_n as a scheme over W by the projection $Z_n \to W$ onto the W_{id_n}-component and let us observe the commutative diagram

$$\begin{array}{ccccc} X_n & \xrightarrow{\mathrm{id}_{X_n} \times g_n} & X_n \times_W Z_n & \xrightarrow{f \times \mathrm{id}_{Z_n}} & W \times_W Z_n = Z_n \\ {\scriptstyle g_n} \downarrow & & \downarrow {\scriptstyle g_n \times \mathrm{id}_{Z_n}} & & \\ Z_n & \xrightarrow[\mathrm{id}_{Z_n} \times \mathrm{id}_{Z_n}]{} & Z_n \times_W Z_n . & & \end{array}$$

Since W is separated over S and the square in the diagram above is cartesian, the morphism $\mathrm{id}_{X_n} \times g_n$ is a closed immersion. Since the composition of top horizontal morphisms is g_n and f is a closed immersion, g_n is a closed immersion. ■

11.2.5. PROPOSITION. *Let n, q be objects of Δ and let W be a scheme over S. Then*

$$\mathrm{cosk}_q^S((\Gamma_n^S(W)^{\leqq n})^{(q)}) = \Gamma_n^S(W)^{\leqq q}.$$

In particular, if $q \geqq n$, then we have

$$\mathrm{cosk}_q^S((\Gamma_n^S(W)^{\leqq n})^{(q)}) = \Gamma_n^S(W)^{\leqq n}.$$

PROOF. We have only to check the universality of the coskeleton functor. By our construction we have

$$(\Gamma_n^S(W)^{\leqq n})^{(q)} = (\Gamma_n^S(W)^{\leqq q})^{(q)}.$$

Let $X.$ be a simplicial scheme over S and let $f.: X. \to \Gamma_n^S(W)^{\leqq q}$ be a morphism of simplicial schemes over S. Let $m > q$ and let $\alpha : n \to m$ be an element in $\mathrm{Mor}_\Delta^{\leqq q}(n, m)$. Then there exists a unique pair (ξ, η) of morphisms in Δ with $\alpha = \eta\xi$ such that $\xi : n \to l$ ($l \leqq q$) is an epimorphism and $\eta : l \to m$ is a monomorphism by Lemma 11.2.2 (1). Then the diagram

$$\begin{array}{ccccccc}
X_m & \xrightarrow{f_m} & \Gamma_n^S(W)_m^{\leqq q} = & \prod\limits_{\gamma : n \to m \in \mathrm{Mor}_\Delta^{\leqq q}(n, m)} & W_\gamma & \to & W_\alpha \\
\eta_{X.}\downarrow & & & \eta_{Z.}\downarrow & & & \downarrow \mathrm{id}_W \\
X_l & \xrightarrow[f_l]{} & \Gamma_n^S(W)_l^{\leqq q} = & \prod\limits_{\beta : n \to l \in \mathrm{Mor}_\Delta^{\leqq q}(n, l)} & W_\beta & \to & W_\xi
\end{array}$$

is commutative, where the right horizontal arrows are projections to the $\alpha(=\eta\xi)$-component and to the ξ-component, respectively. Hence, the morphism $f.$ is determined by the q-th truncated morphism $f.^{(q)}$. The q-th skeleton functor induces a bijection $\mathrm{Mor}_S(X., \Gamma_n^S(W)^{\leqq q}) = \mathrm{Mor}_S(X.^{(q)}, (\Gamma_n^S(W)^{\leqq n})^{(q)})$ for any simplicial scheme $X.$ over S. Therefore, we have an isomorphism $\mathrm{cosk}_q^S((\Gamma_n^S(W)^{\leqq n})^{(q)}) = \Gamma_n^S(W)^{\leqq q}$. ∎

11.2.6. COROLLARY. *With the notation as above, suppose that W is smooth over S. Then the induced morphism*

$$\mathrm{cosk}_q^S((\Gamma_n^S(W)^{\leqq n})^{(q)})_m \to \mathrm{cosk}_{q-1}^S((\Gamma_n^S(W)^{\leqq n})^{(q-1)})_m$$

is smooth for any q and m.

PROOF. Since the morphism $\Gamma_n^S(W)_m^{\leqq q} \to \Gamma_n^S(W)_m^{\leqq q-1}$ is given by the projection

$$\prod_{\beta : n \to m \in \mathrm{Mor}_\Delta^{\leqq q}(n,m) - \mathrm{Mor}_\Delta^{\leqq q-1}(n,m)} W_\beta \times_S \prod_{\gamma : n \to m \in \mathrm{Mor}_\Delta^{\leqq q-1}(n,m)} W_\gamma \to \prod_{\gamma : n \to m \in \mathrm{Mor}_\Delta^{\leqq q-1}(n,m)} W_\gamma,$$

the induced morphism is smooth. ∎

Let $X.$ be a simplicial scheme over S such that $\mathrm{cosk}_n^S(X.^{(n)}) = X.$, let W_m be a scheme over S with a morphism $f_m : X_m \to W_m$ over S for $0 \leqq m \leqq n$ and let $Z. = \prod_{0 \leqq m \leqq n} \Gamma_m^S(W_m)^{\leqq m}$ be a product of simplicial schemes over S. We define a morphism

$$g_l \colon X_l \to Z_l$$

of schemes over S by

$$g_l = \prod_{0 \leqq m \leqq n} \prod_{\gamma : m \to l} f_m \gamma_{X.}.$$

By Lemma 11.2.4 we have

11.2.7. Proposition. *With the notation as above, the collection $g_.$ is a morphism of simplicial schemes over S. If W_m is separated over S and f_m is a closed immersion for all m, then g_l is a closed immersion for any object l of Δ.*

11.3. We give a definition of refinement of étale-étale hypercoverings (resp. étale-proper hypercoverings) of pairs and construct an affine (resp. projective) refinement for given hypercoverings.

11.3.1. Definition. (1) Let $Y_.$ be an étale hypercovering of a scheme X (Definition 7.2.1 (2)). We say that $Z_.$ is a refinement of $Y_.$ if $Z_.$ is an étale hypercovering of X with an X-morphism $Z_. \to Y_.$ such that the natural morphism

$$Z_n \to \mathrm{cosk}_{n-1}^X (Z_.^{(n-1)})_n \times_{\mathrm{cosk}_{n-1}^X (Y_.^{(n-1)})_n} Y_n$$

is étale surjective for each n. If Z_n is affine for any n, $Z_.$ is called an affine refinement of $Y_.$. (Note that Z_n is of finite type over X for each n by definition of étale hypercoverings.)

(2) Let $(Y_., \overline{Y}_.)$ be an étale-étale hypercovering of a pair $(X, \overline{X})$ of schemes. We say that $(Z_., \overline{Z}_.)$ is a refinement of $(Y_., \overline{Y}_.)$ if $(Z_., \overline{Z}_.)$ is an étale-étale hypercovering of $(X, \overline{X})$ with a morphism $(Z_., \overline{Z}_.) \to (Y_., \overline{Y}_.)$ over $(X, \overline{X})$ such that $\overline{Z}_.$ is a refinement of the étale hypercovering $\overline{Y}_.$. (Since $(Z_., \overline{Z}_.) \to (Y_., \overline{Y}_.)$ is strict as a morphism of pairs, $Z_.$ is a refinement of the étale hypercovering $\overline{Y}_.$ of X.)

(3) Let $(Y_., \overline{Y}_.)$ be an étale-proper hypercovering of a pair $(X, \overline{X})$ of schemes. We say that $(Z_., \overline{Z}_.)$ is a refinement of $(Y_., \overline{Y}_.)$ if $(Z_., \overline{Z}_.)$ is an étale-proper hypercovering of $(X, \overline{X})$ with a morphism $(Z_., \overline{Z}_.) \to \to (Y_., \overline{Y}_.)$ over $(X, \overline{X})$ such that $Z_.$ is a refinement of the étale hypercovering $Y_.$. If $\overline{Z}_n$ is projective over $\overline{X}$ for any n, $(Z_., \overline{Z}_.)$ is called a projective refinement of $(Y_., \overline{Y}_.)$.

One can also consider an n-truncated version of refinements in each case. ■

First we deal with refinements of étale-étale hypercoverings.

11.3.2. PROPOSITION. *Let X be a noetherian scheme.*

(1) *Let $Y_.$ be an étale hypercovering of X. Then there exists an affine refinement $Z_.$ of $Y_.$.*

(2) *Let $Y_.$ be an étale hypercovering over X and let $Z_.$ and $Z_.'$ be refinements of $Y_.$. Then the fiber product*

$$U_. = Z_. \times_{Y_.} Z_.'$$

of $Z_.$ and $Z_.'$ over $Y_.$ is a refinement of $Y_.$.

PROOF. (1) We will construct a sequence $\{Z_.^n\}_n$ inductively on n such that $Z_.^n$ is a split n-truncated simplicial scheme over X with $(Z_.^n)^{(n-1)} = Z_.^{n-1}$ and $Z_.^n$ is an affine refinement of the n-th truncated étale hypercovering $Y_.^{(n)}$ of X.

Suppose that $n = 0$. Then we have only to take a finite affine Zariski covering U_0 of Y_0 (this is possible since Y_0 is of finite type over X) and put $Z_.^0 = U_0$.

Suppose that we have proven the assertion for $(n-1)$-truncated simplicial schemes. Let $Z_.^m$ (resp. U_m) be as above (resp. a finite affine Zariski covering of $\operatorname{cosk}_{m-1}^X(Z_.^{m-1})_m \times_{\operatorname{cosk}_{m-1}^X(Y_.^{(m-1)})_m} Y_m$) for $0 \leqq m \leqq n-1$ such that $Z_.^m = \Omega_m^X(Z_.^{m-1}, U_0, \ldots, U_m)$. Now we take a finite affine Zariski covering

$$U_n \to \operatorname{cosk}_{n-1}^X(Z_.^{n-1})_n \times_{\operatorname{cosk}_{n-1}^X(Y_.^{(n-1)})_n} Y_n$$

and define a split $(n-1)$-truncated simplicial scheme $Z_.^n$ over X by

$$Z_.^n = \Omega_n^X(Z_.^{n-1}, U_0, \ldots, U_n).$$

Since every morphism which appears in the simplicial schemes $Y_.$, $Z_.^{n-1}$ and the morphism $Z_.^{n-1} \to Y_.^{(n-1)}$ is étale, $Z_.^n$ is a desired refinement of $Y_.^{(n)}$ by Lemma 11.2.3.

If we continue this construction, we get an affine refinement of $Y_.$.

(2) We have only to check étale surjectivities in the definition of étale

hypercoverings and refinements. Consider the commutative diagram

$$\begin{array}{ccc}
U_n & = & Z_n \times_{Y_n} Z'_n \\
\downarrow & & \downarrow \\
\operatorname{cosk}^X_{n-1}(U_.^{(n-1)})_n \times_{\operatorname{cosk}^X_{n-1}(Y_.^{(n-1)})_n} Y_n & \cong & (\operatorname{cosk}^X_{n-1}(Z_.^{(n-1)})_n \times_{\operatorname{cosk}^X_{n-1}(Y_.^{(n-1)})_n} Y_n) \times_{Y_n} (\operatorname{cosk}^X_{n-1}((Z_.')^{(n-1)})_n \times_{\operatorname{cosk}^X_{n-1}(Y_.^{(n-1)})_n} Y_n) \\
\downarrow & & \downarrow \\
\operatorname{cosk}^X_{n-1}(U_.^{(n-1)})_n & \cong & \operatorname{cosk}^X_{n-1}(Z_.^{(n-1)})_n \times_{\operatorname{cosk}^X_{n-1}(Y_.^{(n-1)})_n} (\operatorname{cosk}^X_{n-1}((Z_.')^{(n-1)})_n.
\end{array}$$

Lemma 11.2.1 gives two isomorphisms above. The first right vertical arrow is étale surjective since $Z_.$ and $Z_.'$ are refinements of $Y_.$ and the second right vertical arrow is étale surjective since $Y_.$ is an étale hypercovering of X. Hence, $U_.$ is a refinement of $Y_.$. ■

11.3.3. PROPOSITION. *Let X (resp. X') be a noetherian scheme and let*

$$\begin{array}{ccc}
Y_. & \leftarrow & Y_.' \\
\downarrow & & \downarrow \\
X & \leftarrow & X'
\end{array}$$

be a commutative diagram of diagrams of schemes such that $Y_.$ (resp. $Y_.'$) is an étale hypercovering over X (resp. X'). Then there exists a diagram

$$\begin{array}{ccc}
Z_. & \leftarrow & Z_.' \\
\downarrow & & \downarrow \\
Y_. & \leftarrow & Y_.'
\end{array}$$

of diagrams of schemes such that $Z_.$ (resp. $Z_.'$) is an affine refinement of $Y_.$ (resp. $Y_.'$).

PROOF. First we construct a finite affine refinement $Z_.$ of $Y_.$ by Proposition 11.3.2 (1). Then we consider a diagonal diagram $\operatorname{diag}(Z_. \times_X Y_.')$ (Example 3.1.1 (3)) and apply Proposition 11.3.2 (1) to $\operatorname{diag}(Z_. \times_X Y_.')$ over X'. Note that

$$\begin{aligned}
\operatorname{cosk}^{X'}_{n-1}(\operatorname{diag}(Z_. \times_X Y_.')^{(n-1)})_n &= \operatorname{cosk}^{X'}_{n-1}(Z_.^{(n-1)} \times_X X')_n \times_{X'} \operatorname{cosk}^{X'}_{n-1}((Y_.')^{(n-1)})_n \\
&= \operatorname{cosk}^{X}_{n-1}(Z_.^{(n-1)})_n \times_X \operatorname{cosk}^{X'}_{n-1}((Y_.')^{(n-1)})_n
\end{aligned}$$

by Lemma 11.2.1. The condition of étale surjectivity follows from the following commutative diagram

$$\begin{array}{ccccccc}
\mathrm{cosk}_{n-1}^{X'}((Z_\bullet')^{(n-1)})_n & \leftarrow & A & \leftarrow & B & \leftarrow & Z_n' \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{cosk}_{n-1}^{X'}(\mathrm{diag}\,(Z_\bullet \times_X Y_\bullet')^{(n-1)})_n & \leftarrow & \mathrm{cosk}_{n-1}^{X}(Z_\bullet^{(n-1)})_n \times_X Y_n' & \leftarrow & Z_n \times_X Y_n' & & \\
\downarrow & & \downarrow & & & & \\
\mathrm{cosk}_{n-1}^{X'}((Y_\bullet')^{(n-1)})_n & \leftarrow & Y_n', & & & &
\end{array}$$

where A and B make all squares cartesian: in fact all horizontal arrows are étale surjective by our construction. ∎

We deal now with refinements of étale-proper hypercoverings.

11.3.4. Proposition. *Let $(X, \overline{X})$ be a pair of noetherian schemes.*

(1) *Let $(Y_\bullet, \overline{Y}_\bullet)$ be an étale-proper hypercovering of $(X, \overline{X})$. Then there exists a projective refinement $(Z_\bullet, \overline{Z}_\bullet)$ of $(Y_\bullet, \overline{Y}_\bullet)$.*

(2) *Let $(Y_\bullet, \overline{Y}_\bullet)$ be an étale-proper hypercovering over $(X, \overline{X})$ and let $(Z_\bullet, \overline{Z}_\bullet)$ and $(Z_\bullet', \overline{Z}_\bullet')$ be refinements of $(Y_\bullet, \overline{Y}_\bullet)$. Then the fiber product*

$$(U_\bullet, \overline{U}_\bullet) = (Z_\bullet, \overline{Z}_\bullet) \times_{(Y_\bullet, \overline{Y}_\bullet)} (Z_\bullet', \overline{Z}_\bullet')$$

of $(Z_\bullet, \overline{Z}_\bullet)$ and $(Z_\bullet', \overline{Z}_\bullet')$ over $(Y_\bullet, \overline{Y}_\bullet)$ is a refinement of $(Y_\bullet, \overline{Y}_\bullet)$.

Proof. (1) The assertion can be proved by a method similar to that used in Proposition 11.3.2 (1). The only difference is that we need to take a pair $(U_n, \overline{U}_n)$ over $(X, \overline{X})$ instead of U_n such that (i) U_n is a finite affine Zariski covering of $\mathrm{cosk}_{n-1}^{X}(Z_\bullet^{n-1)_n} \times_{\mathrm{cosk}_{n-1}^{X}(Y_\bullet^{(n-1)})_n} Y_n$, and (ii) $\overline{U}_n$ is a completion of U_n over $\mathrm{cosk}_{n-1}^{\overline{X}}(\overline{Z}_\bullet^{n-1})_n \times_{\mathrm{cosk}_{n-1}^{\overline{X}}(\overline{Y}_\bullet^{(n-1)})_n} \overline{Y}_n$ and it is projective over $\overline{X}$.

Since U_n is affine of finite type over X, there exists a projective scheme $\overline{U}_n''$ over $\overline{X}$ such that $(U_n, \overline{U}_n'')$ is a pair over $(X, \overline{X})$. Now we define a scheme $\overline{U}_n'$ over $\overline{X}$ by the Zariski closure of U_n in $\mathrm{cosk}_{n-1}^{\overline{X}}(\overline{Z}_\bullet^{n-1})_n \times \times_{\mathrm{cosk}_{n-1}^{\overline{X}}(\overline{Y}_\bullet^{n-1})_n} \overline{Y}_n \times_{\overline{X}} \overline{U}_n''$. Then $(U_n, \overline{U}_n')$ is a pair over $(X, \overline{X})$ and the natural composition $\overline{U}_n' \to \overline{U}_n''$ is proper. Applying the precise version of Chow's lemma [12, Corollaire 5.7.14], there exists a pair $(U_n, \overline{U}_n)$ over $(U_n, \overline{U}_n')$ such that $\overline{U}_n$ is projective both over $\overline{U}_n'$ and over $\overline{X}$.

The assertion (2) is easy. ∎

As in Proposition 11.3.3 we have

11.3.5. Proposition. *Let* $(X, \overline{X})$ (*resp.* $(X', \overline{X}')$) *be a pair of noetherian schemes and let*

$$\begin{array}{ccc} (Y_., \overline{Y}_.) & \leftarrow & (Y'_., \overline{Y}'_.) \\ \downarrow & & \downarrow \\ (X, \overline{X}) & \leftarrow & (X', \overline{X}') \end{array}$$

be a commutative diagram of diagrams of schemes such that $(Y_., \overline{Y}_.)$ (*resp.* $(Y'_., \overline{Y}'_.)$) *is an étale-proper hypercovering over* $(X, \overline{X})$ (*resp.* $(X', \overline{X}')$). *Then there exists a diagram*

$$\begin{array}{ccc} (Z_., \overline{Z}_.) & \leftarrow & (Z'_., \overline{Z}'_.) \\ \downarrow & & \downarrow \\ (Y_., \overline{Y}_.) & \leftarrow & (Y'_., \overline{Y}'_.) \end{array}$$

of simplicial pairs such that $(Z_., \overline{Z}_.)$ (*resp.* $(Z'_., \overline{Z}'_.)$) *is a refinement of* $(Y_., \overline{Y}_.)$ (*resp.* $(Y'_., \overline{Y}'_.)$).

1.4. Let $(X, \overline{X})$ be a pair separated of finite type over $(T, \overline{T})$. We construct an étale-étale hypercovering of $(X, \overline{X})$ over $\mathfrak{T}$ which is a refinement of a given truncated étale-étale hypercovering of $(X, \overline{X})$. (See Example 10.1.6 (1) and Definition 11.3.1.)

11.4.1. Proposition. (1) *Let* $(Y_., \overline{Y}_.)$ *be an étale-étale hypercovering of* $(X, \overline{X})$ *such that*

$$\mathrm{cosk}_n^{(X, \overline{X})}((Y_.^{(n)}, \overline{Y}_.^{(n)})) = (Y_., \overline{Y}_.)$$

for some nonnegative integer n. *Then there exists an étale-étale hypercovering* $\mathfrak{Z}_.$ *of* $(X, \overline{X})$ *over* $\mathfrak{T}$ *such that* $(Z_., \overline{Z}_.)$ *is a refinement of* $(Y_., \overline{Y}_.)$.

(2) *Let* $(Y_., \overline{Y}_.)$ *be an étale-étale hypercovering of* $(X, \overline{X})$. *If* $\mathfrak{Z}_.$ *and* $\mathfrak{Z}'_.$ *are étale-étale hypercoverings of* $(X, \overline{X})$ *over* $\mathfrak{T}$ *such that both* $(Z_., \overline{Z}_.)$ *and* $(Z'_., \overline{Z}'_.)$ *are refinements of* $(Y_., \overline{Y}_.)$. *Then the fiber product*

$$\mathfrak{W}_. = \mathfrak{Z}_. \times_{(Y_., \overline{Y}_., \mathfrak{T})} \mathfrak{Z}'_. = (Z_. \times_{Y_.} Z'_., \overline{Z}_. \times_{\overline{Y}_.} \overline{Z}'_., \mathrm{diag}\,(\mathcal{Z}_. \times_{\mathfrak{T}} \mathcal{Z}'_.))$$

(*see* 11.2) *is also an étale-étale hypercovering of* $(X, \overline{X})$ *over* $\mathfrak{T}$ *such that* $(W_., \overline{W}_.)$ *is a refinement of* $(Y_., \overline{Y}_.)$.

Proof. (1) Let $(U_., \overline{U}_.)$ be an n-truncation of a refinement of $(Y_., \overline{Y}_.)$ such that $\overline{U}_m$ is affine for any m (Proposition 11.3.2 (1)). We take a separated smooth formal $\mathfrak{T}$-scheme $\mathcal{P}_m$ with a $\mathfrak{T}$-closed immersion

$\overline{U}_m \to \mathcal{W}_m \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}\,k$ for any m. Such $\mathcal{W}_m$'s always exist since $\overline{U}_m$ is of finite type over $\mathrm{Spec}\,k$. We define an $(X, \overline{X})$-triple $\mathfrak{Z}_{\bullet}$ over $\mathfrak{T}$ by

$$\mathfrak{Z}_{\bullet} = (Z_{\bullet}, \overline{Z}_{\bullet}, \mathcal{Z}_{\bullet}) = (\mathrm{cosk}_n^X(X \times_{\overline{X}} \overline{U}_{\bullet}), \mathrm{cosk}_n^{\overline{X}}(\overline{U}_{\bullet}), \prod_{0 \leqq m \leqq n} \Gamma_m^{\mathcal{C}}(\mathcal{W}_m)^{\leqq m}),$$

where $\mathrm{cosk}_n^{\overline{X}}(\overline{U}_{\bullet})_l \to \prod_{0 \leqq m \leqq n} \Gamma_m^{\mathcal{C}}(\mathcal{W}_m)_l^{\leqq m}$ is a closed immersion for all l by Proposition 11.2.7. In fact, $\mathfrak{Z}_{\bullet}$ is an $(X, \overline{X})$-triple over $\mathfrak{T}$ by Lemmas 11.2.1 and 11.2.4. $(Z_{\bullet}, \overline{Z}_{\bullet})$ is a refinement by construction. $\mathrm{cosk}_{m+1}^{\mathcal{C}}(\mathcal{Z}_{\bullet})_l \to \to \mathrm{cosk}_m^{\mathcal{C}}(\mathcal{Z}_{\bullet})_l$ is smooth for any l and m by Lemma 11.2.1 and Corollary 11.2.6 and by our construction. Hence, $\mathfrak{Z}_{\bullet}$ is the desired étale-étale hypercovering.

(2) Follows from Proposition 11.3.2 (2) and Lemma 11.2.1. ■

11.4.2. Proposition. *Let* $\mathfrak{S}$, $\mathfrak{T}$, $(X, \overline{X})$, $\mathfrak{S}'$, $\mathfrak{T}'$, *and* $(X', \overline{X}')$ *be as in Proposition* 10.5.2 *such that all schemes and formal schemes are of finite type over* Spf $\mathcal{V}$ *or* Spf $\mathcal{V}'$, *respectively, and let*

$$\begin{array}{ccc} (Y_{\bullet}, \overline{Y}_{\bullet}) & \leftarrow & (Y'_{\bullet}, \overline{Y}'_{\bullet}) \\ \downarrow & & \downarrow \\ (X, \overline{X}) & \leftarrow & (X', \overline{X}') \end{array}$$

be a commutative diagram of diagrams of pairs such that $(Y_{\bullet}, \overline{Y}_{\bullet})$ (*resp.* $(Y'_{\bullet}, \overline{Y}'_{\bullet})$) *is an étale-étale hypercovering of* $(X, \overline{X})$ (*resp.* $(X', \overline{X}')$) *and* $\mathrm{cosk}_n^{(X,\overline{X})}((Y_{\bullet}^{(n)}, \overline{Y}_{\bullet}^{(n)})) = (Y_{\bullet}, \overline{Y}_{\bullet})$ (*resp.* $\mathrm{cosk}_n^{(X',\overline{X}')}(((Y'_{\bullet})^{(n)}, (\overline{Y}'_{\bullet})^{(n)})) = (Y'_{\bullet}, \overline{Y}'_{\bullet})$) *for some nonnegative integer n. Then there exists an étale-étale hypercovering* $\mathfrak{Z}_{\bullet}$ (*resp.* $\mathfrak{Z}'_{\bullet}$) *of* $(X, \overline{X})$ (*resp.* $(X', \overline{X}')$) *over* $\mathfrak{T}$ *with commutative diagrams*

$$\begin{array}{ccc} \mathfrak{Z}_{\bullet} & \leftarrow & \mathfrak{Z}'_{\bullet} \\ \downarrow & & \downarrow \\ \mathfrak{T}_{\bullet} & \leftarrow & \mathfrak{T}'_{\bullet} \end{array} \qquad \begin{array}{ccc} (Z_{\bullet}, \overline{Z}_{\bullet}) & \leftarrow & (Z'_{\bullet}, \overline{Z}'_{\bullet}) \\ \downarrow & & \downarrow \\ (Y_{\bullet}, \overline{Y}_{\bullet}) & \leftarrow & (Y'_{\bullet}, \overline{Y}'_{\bullet}) \end{array}$$

such that $\mathcal{Z}' \to \mathcal{Z}$ *is smooth around* Z' *and* $(Z_{\bullet}, \overline{Z}_{\bullet})$ (*resp.* $(Z'_{\bullet}, \overline{Z}'_{\bullet})$) *is a refinement of* $(Y_{\bullet}, \overline{Y}_{\bullet})$ (*resp.* $(Y'_{\bullet}, \overline{Y}'_{\bullet})$).

Proof. An argument similar to the proof of Propositions 11.3.3 and 11.4.1 (1) can be used by means of Lemmas 11.2.1, 11.2.4 and Corollary 11.2.6. ■

11.4.3. Let $(Y_{\bullet}, \overline{Y}_{\bullet})$ be an étale-étale hypercovering over $(X, \overline{X})$ and suppose that $\mathfrak{Z}_{\bullet} = (Z_{\bullet}, \overline{Z}_{\bullet}, \mathcal{Z}_{\bullet})$ is an étale-étale hypercovering of $(X, \overline{X})$

over $\mathfrak{T}$ such that $(Z_{\bullet}, \overline{Z}_{\bullet})$ is a refinement of $(Y_{\bullet}, \overline{Y}_{\bullet})$. We define a 2-simplicial $(X, \overline{X})$-triple $\mathfrak{W}_{\bullet}$ over $\mathfrak{T}$ as follows:

– for any object (m, n) of Δ^2,

$$\mathfrak{W}_{(m,n)} = \mathrm{cosk}_0^{(Y_{\bullet}, \overline{Y}_{\bullet}, \mathscr{C}_{\bullet}^{\Delta^\circ})}(\mathfrak{Z}_{\bullet})_{(m,n)} = (\mathrm{cosk}_0^{Y_n}(Z_n)_m, \mathrm{cosk}_0^{\overline{Y}_n}(\overline{Z}_n)_m, \mathrm{cosk}_0^{\mathscr{C}}(\mathscr{Z}_n)_m)$$

– for any morphism $(\xi, \eta): (k, l) \to (m, n)$ of Δ^2, $(\xi, \eta)_{\mathfrak{W}_{\bullet}}: \mathfrak{W}_{(m,n)} \to \mathfrak{W}_{(k,l)}$ is defined by

$$(\xi, \eta)_{\mathfrak{W}_{\bullet}}(u_0, u_1, \ldots, u_m) = (\eta_{\mathfrak{Z}_{\bullet}}(u_{\xi(0)}), \eta_{\mathfrak{Z}_{\bullet}}(u_{\xi(1)}), \ldots, \eta_{\mathfrak{Z}_{\bullet}}(u_{\xi(k)})).$$

Then, by construction, the natural diagram

$$\begin{array}{ccc} (Y_{\bullet}, \overline{Y}_{\bullet}) & \leftarrow & (W_{\bullet}, \overline{W}_{\bullet}) \\ \downarrow & & \downarrow \\ (X, \overline{X}) & \leftarrow & (X_{\bullet}^{\Delta^\circ}, \overline{X}_{\bullet}^{\Delta^\circ}) \end{array}$$

is commutative.

11.4.4. Proposition. *With the notation as above, we have*
(1) $\mathfrak{W}_{(m,\bullet)}$ *is an étale-étale hypercovering of* $(X, \overline{X})$ *over* $\mathfrak{T}$ *for any* m;
(2) $\mathfrak{W}_{(\bullet,n)}$ *is an étale-étale hypercovering of* $(Y_n, \overline{Y}_n)$ *over* $\mathfrak{T}$ *for any* n.

Proof. (1) Since $(Z_{\bullet}, \overline{Z}_{\bullet})$ is an étale-étale hypercovering of $(X, \overline{X})$, $(\mathrm{cosk}_0^{Y_{\bullet}}(Z_{\bullet})_m, \mathrm{cosk}_0^{\overline{Y}_{\bullet}}(\overline{Z}_{\bullet})_m)$ is an étale-étale hypercovering of $(X, \overline{X})$ by Proposition 11.3.2 (2) and Lemma 11.2.1. The smoothness of the morphism $\mathrm{cosk}_q^{\mathscr{C}}(\mathrm{cosk}_0^{\mathscr{C}}(\mathscr{Z}_{\bullet})_m^{(q)})_l \to \mathrm{cosk}_{q-1}^{\mathscr{C}}(\mathrm{cosk}_0^{\mathscr{C}}(\mathscr{Z}_{\bullet})_m^{(q-1)})_l$ of formal schemes follows from Lemma 11.2.1.

The assertion (2) is trivial. ■

11.5. Let $(X, \overline{X})$ be a pair separated of finite type over $(T, \overline{T})$. We construct a refinement of a given truncated étale-proper hypercovering of $(X, \overline{X})$ over $\mathfrak{T}$. (See Example 10.1.6 (1) and Definition 11.3.1.)

11.5.1. Proposition. *With the notation as above, let* $(Y_{\bullet}, \overline{Y}_{\bullet})$ *be an étale-proper hypercovering of* $(X, \overline{X})$ *such that*

$$\mathrm{cosk}_n^{(X, \overline{X})}((Y_{\bullet}^{(n)}, \overline{Y}_{\bullet}^{(n)})) = (Y_{\bullet}, \overline{Y}_{\bullet}).$$

(1) *Suppose that* $\overline{X}$ *is affine. Then there exists an étale-proper hypercovering* $\mathfrak{Z}_{\bullet}$ *of* $(X, \overline{X})$ *over* $\mathfrak{T}$ *such that* $(Z_{\bullet}, \overline{Z}_{\bullet})$ *is a refinement of* $(Y_{\bullet}, \overline{Y}_{\bullet})$.

(2) *If* $\mathfrak{Z}_.$ *and* $\mathfrak{Z}'_.$ *are étale-proper hypercoverings of* $(X, \overline{X})$ *over* $\mathfrak{T}$ *such that both* $(Z_., \overline{Z}_.)$ *and* $(Z'_., \overline{Z}'_.)$ *are refinements of* $(Y_., \overline{Y}_.)$. *Then the fiber product*

$$\mathfrak{W}_. = \mathfrak{Z}_. \times_{(Y_., \overline{Y}_., \mathcal{T})} \mathfrak{Z}'_. = (Z_. \times_{Y_.} Z'_., \overline{Z}_. \times_{\overline{Y}_.} \overline{Z}'_., \mathrm{diag}(\mathcal{Z}_. \times_{\mathcal{T}} \mathcal{Z}'_.))$$

is also an étale-proper hypercovering of $(X, \overline{X})$ *over* $\mathfrak{T}$ *such that* $(W_., \overline{W}_.)$ *is a refinement of* $(Y_., \overline{Y}_.)$.

PROOF. (1) Let $(U_., \overline{U}_.)$ be an n-truncation of refinement of $(Y_., \overline{Y}_.)$ such that $\overline{U}_m$ is projective over $\overline{X}$ for each m (Proposition 11.3.4 (1)). We take a separated smooth formal $\mathcal{T}$-scheme $\mathcal{W}_m$ with a $\mathcal{T}$-closed immersion $\overline{U}_m \to \mathcal{W}_m \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}\, k$ for each m. Such $\mathcal{W}_m$'s always exist since $\overline{X}$ is affine and $\overline{U}_m$ is projective over $\overline{X}$. The rest is now as in the proof of Proposition 11.4.1 (1).

The assertion (2) follows from Proposition 11.3.4 (2) and Lemma 11.2.1. ∎

11.5.2. PROPOSITION. *Let* $\mathfrak{S}, \mathfrak{T}, (X, \overline{X}), \mathfrak{S}', \mathfrak{T}'$, *and* $(X', \overline{X}')$ *be as in Proposition* 10.5.2 *such that all schemes and formal schemes are of finite type over* Spf $\mathcal{V}$ *or* Spf $\mathcal{V}'$, *respectively, and let* $(Y_., \overline{Y}_.)$ (*resp.* $(Y'_., \overline{Y}'_.)$) *be an étale-proper hypercovering of* $(X, \overline{X})$ (*resp.* $(X', \overline{X}')$) *such that the diagram*

$$\begin{array}{ccc} (Y_., \overline{Y}_.) & \leftarrow & (Y'_., \overline{Y}'_.) \\ \downarrow & & \downarrow \\ (X, \overline{X}) & \leftarrow & (X', \overline{X}') \end{array}$$

of pairs is commutative and the coskeleton of n-th truncation of $(Y_., \overline{Y}_.)$ (*resp.* $(Y'_., \overline{Y}'_.)$) *over* $(X, \overline{X})$ (*resp.* $(X', \overline{X}')$) *is itself. Suppose that both* $\overline{X}$ *and* $\overline{X}'$ *are affine. Then there exists an étale-proper hypercovering* $\mathfrak{Z}_.$ (*resp.* $\mathfrak{Z}'_.$) *of* $(X, \overline{X})$ (*resp.* $(X', \overline{X}')$) *over* $\mathfrak{T}$ (*resp.* $\mathfrak{T}'$) *with a commutative diagram*

$$\begin{array}{ccc} \mathfrak{Z}_. & \leftarrow & \mathfrak{Z}'_. \\ \downarrow & & \downarrow \\ \mathfrak{T} & \leftarrow & \mathfrak{T}' \end{array} \qquad \begin{array}{ccc} (Z_., \overline{Z}_.) & \leftarrow & (Z'_., \overline{Z}'_.) \\ \downarrow & & \downarrow \\ (Y_., \overline{Y}_.) & \leftarrow & (Y'_., \overline{Y}'_.) \end{array}$$

such that $\mathcal{Z}'_. \to \mathcal{Z}_.$ *is smooth around* $Z'_.$ *and* $(Z_., \overline{Z}_.)$ (*resp.* $(Z'_., \overline{Z}'_.)$) *is a refinement of* $(Y_., \overline{Y}_.)$ (*resp.* $(Y'_., \overline{Y}'_.)$).

PROOF. A proof similar to that of Propositions 11.3.5 and 11.5.1 (1) works by using Lemmas 11.2.1, 11.2.4 and Corollary 11.2.6. ■

11.5.3. Let $(Y_., \overline{Y}_.)$ be an étale-proper hypercovering over $(X, \overline{X})$, let $\mathfrak{U} = (U, \overline{U}, \mathcal{U})$ be a Zariski covering of $(X, \overline{X})$ of finite type over $\mathfrak{T}$ such that $\overline{U}$ is affine, and let us put $(V_., \overline{V}_.) = (Y_., \overline{Y}_.) \times_{(X, \overline{X})} (U, \overline{U})$. We denote by $\mathfrak{U}_.$ the Čech diagram for $\mathfrak{U}$ as $(X, \overline{X})$-triples over $\mathfrak{T}$.

Suppose that $\mathfrak{Z}_.$ is an étale-proper hypercovering of $(U, \overline{U})$ over $\mathfrak{T}$ such that $(Z_., \overline{Z}_.)$ is a refinement of $(V_., \overline{V}_.)$. We define a 2-simplicial triple $\mathfrak{W}_.$ over $\mathfrak{T}$ as follows:

– for any object (m, n) of Δ^2,

$$\mathfrak{W}_{(m,n)} = (\mathrm{cosk}_0^{Y_n}(Z_n)_m, \mathrm{cosk}_0^{\overline{Y}_n}(\overline{Z}_n)_m, \mathrm{cosk}_0^{\mathcal{T}}(\mathcal{Z}_n \times_{\mathcal{T}} \mathcal{U}_n)_m)$$

– for any morphism $(\xi, \eta) : (k, l) \to (m, n)$ of Δ^2, $(\xi, \eta)_{\mathfrak{W}_.} : \mathfrak{W}_{(m,n)} \to \mathfrak{W}_{(k,l)}$ is defined by

$$(\xi, \eta)_{\mathfrak{W}_.}(u_0, u_1, \ldots, u_m) = (\eta_{\mathfrak{Z}_.}(u_{\xi(0)}), \eta_{\mathfrak{Z}_.}(u_{\xi(1)}), \ldots, \eta_{\mathfrak{Z}_.}(u_{\xi(k)})).$$

Since $\overline{V}_n \to \overline{U}$ is proper, $\mathrm{cosk}_0^{\overline{Y}_n}(\overline{Z}_n)_m \to \mathrm{cosk}_0^{\mathcal{T}}(\mathcal{Z}_n \times_{\mathcal{T}} \mathcal{U}_n)_m$ is a closed immersion. So $\mathfrak{W}_{(m,n)}$ is a triple for any m, n. By construction, there exist a natural commutative diagram

$$\begin{array}{ccc} (Y_., \overline{Y}_.) & \leftarrow & (W_., \overline{W}_.) \\ \downarrow & & \downarrow \\ (X, \overline{X}) & \leftarrow & (U_., \overline{U}_.) \end{array}$$

of diagrams of pairs and a natural morphism

$$\mathfrak{W}_. \to \mathfrak{U}_.$$

of $(X, \overline{X})$-triples over $\mathfrak{T}$.

11.5.4. PROPOSITION. *With the notation as above, we have*

(1) $\mathfrak{W}_{(m,\,.)}$ *is an étale-proper hypercovering of* $\mathfrak{U}_m$ *for any* m;

(2) $\mathfrak{W}_{(.,\,n)}$ *is a universally de Rham descendable hypercovering of* $(Y_n, \overline{Y}_n)$ *over* $\mathfrak{T}$ *for any* n.

PROOF. (1) Consider the commutative diagram

$$\begin{array}{ccc}
\operatorname{cosk}_0^{Y_n}(Z_n)_m & \rightarrow & \operatorname{cosk}_0^{\overline{Y}_n}(\overline{Z}_n)_m \\
\downarrow & & \downarrow \\
\operatorname{cosk}_{n-1}^{U_m}(\operatorname{cosk}_0^{Y.}(Z.)_m^{(n-1)})_n & \rightarrow & \operatorname{cosk}_{n-1}^{\overline{U}_m}(\operatorname{cosk}_0^{\overline{Y}.}(\overline{Z}.)_m^{(n-1)})_n \\
\cong\downarrow & & \downarrow\cong \\
\operatorname{cosk}_0^{\operatorname{cosk}_{n-1}^X(Y.^{(n-1)})_n}(\operatorname{cosk}_{n-1}^U(Z.^{(n-1)})_n)_m & \rightarrow & \operatorname{cosk}_0^{\operatorname{cosk}_{n-1}^{\overline{X}}(\overline{Y}.^{(n-1)})_n}(\operatorname{cosk}_{n-1}^{\overline{U}}(\overline{Z}.^{(n-1)})_n)_m \\
\downarrow & & \downarrow \\
U_m & \rightarrow & \overline{U}_m
\end{array}$$

with cartesian squares. Both middle vertical morphisms are isomorphisms since there is an isomorphism

$$\operatorname{cosk}_{n-1}^{U_m}(\operatorname{cosk}_0^{Y.}(Z.)_m^{(n-1)}) \rightarrow \operatorname{cosk}_0^{\operatorname{cosk}_{n-1}^X(Y.^{(n-1)})}(\operatorname{cosk}_{n-1}^U(Z.^{(n-1)}))_m$$

of simplicial schemes as in Lemma 11.2.1. By the hypothesis $Z_n \rightarrow \rightarrow \operatorname{cosk}_{n-1}^U(Z.^{(n-1)})_n$ (resp. $\overline{Z}_n \rightarrow \operatorname{cosk}_{n-1}^{\overline{U}}(\overline{Z}.^{(n-1)})_n$) is étale surjective (resp. proper), so that $W_{(m,\ .)} \rightarrow U_m$ (resp. $\overline{W}_{(m,\ .)} \rightarrow \overline{U}_m$) is an étale hypercovering (resp. proper).

The morphism $\operatorname{cosk}_n^{\mathcal{T}}(\mathfrak{W}_{(m,\ .)}^{(n)})_l \rightarrow \operatorname{cosk}_{n-1}^{\mathcal{T}}(\mathfrak{W}_{(m,\ .)}^{(n-1)})_l$ is smooth around $\operatorname{cosk}_n^{\mathcal{T}}(W_{(m,\ .)}^{(n)})_l$ for any n and l since $\operatorname{cosk}_0^{\mathcal{T}}(\mathcal{Z}_n \times_{\mathcal{T}} \mathcal{U}_n)_m \cong \operatorname{cosk}_0^{\mathcal{T}}(\mathcal{Z}_n)_m \times \times_{\mathcal{T}} \operatorname{cosk}_0^{\mathcal{T}}(\mathcal{U}_n)_m$. Therefore, $\mathfrak{W}_{(m,\ .)}$ is an étale-proper hypercovering of $\mathfrak{U}_m$ for any m.

(2) Let $\mathfrak{R} = (R, \overline{R}, \mathcal{R})$ be a $(Y_n, \overline{Y}_n)$-triple over $\mathfrak{T}$ and consider a sequence of morphisms of diagrams

$$\begin{array}{ccc}
(Z_n, \overline{Z}_n) & \qquad & (Z_n \times_{Y_n} R_n, \overline{Z}_n \times_{\overline{Y}_n} \overline{R}, \mathcal{Z}_n \times_{\mathcal{T}} \mathcal{U}_n \times_{\mathcal{T}} \mathcal{R}) \\
\downarrow & & \downarrow s \\
(V_n, \overline{V}_n) & & (V_n \times_{Y_n} R_n, \overline{V}_n \times_{\overline{Y}_n} \overline{R}, \mathcal{U}_n \times_{\mathcal{T}} \mathcal{R}) \\
\downarrow & & \downarrow t \\
(Y_n, \overline{Y}_n) & & (R, \overline{R}, \mathcal{R}).
\end{array}$$

s and t are universally cohomologically descendable by Theorem 9.1.1 since s (resp. t) is an étale-proper covering (resp. a Zariski covering), respectively. Hence, the composition ts is also universally cohomologically descendable by Theorem 8.4.1, so that the Čech diagram

$$\mathfrak{W}_{(\cdot,\ n)} = (\operatorname{cosk}_0^{Y_n}(Z_n), \operatorname{cosk}_0^{\overline{Y}_n}(\overline{Z}_n), \operatorname{cosk}_0^{\mathcal{T}}(\mathcal{Z}_n \times_{\mathcal{T}} \mathcal{U}_n) \times_{\mathcal{T}} \mathcal{R}) \rightarrow (R, \overline{R}, \mathcal{R})$$

with respect to ts is universally cohomologically descendable. Therefore,

$\mathfrak{W}_{(\cdot,\, n)}$ is a universally de Rham descendable hypercovering of $(Y_n, \overline{Y}_n)$ over $\mathfrak{T}$. ■

11.6. Now we prove Proposition 11.1.2. First we prove it in the case of étale-étale hypercoverings, and then we prove it in the case of étale-proper hypercoverings.

PROOF IN THE CASE OF ÉTALE-ÉTALE HYPERCOVERINGS. Let us take an étale-étale hypercovering $\mathfrak{Z}_{\cdot}$ of $(X, \overline{X})$ over $\mathfrak{T}$ such that $(Z_{\cdot}, \overline{Z}_{\cdot})$ is a refinement of $(Y_{\cdot}, \overline{Y}_{\cdot})$. Such $\mathfrak{Z}_{\cdot}$ always exists by Proposition 11.4.1 (1). Let $\mathfrak{W}_{\cdot}$ be a 2-simplicial $(X, \overline{X})$-triple over $\mathfrak{T}$ which is defined in 11.4.3 and let us denote by $w_{\cdot}: \mathfrak{W}_{\cdot} \to \mathfrak{T}$ the structure morphism. Let E be an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ with a realization $(E_{\mathfrak{W}_{\cdot}}, \nabla_{\mathfrak{W}_{\cdot}})$ of E over $\mathfrak{W}_{\cdot}$.

First we consider a spectral sequence

$$\underline{E}_1^{qr} = \underline{H}^r(\mathbb{R}\,\mathcal{C}^\dagger(\mathfrak{T}, \mathfrak{W}_{(q,\, \cdot)}; \mathrm{DR}^\dagger(\mathfrak{W}_{(q,\, \cdot)}/\mathfrak{T}, (E_{\mathfrak{W}_{(q,\, \cdot)}}, \nabla_{\mathfrak{W}_{(q,\, \cdot)}}))))$$

$$\Rightarrow \underline{H}^{q+r}(\mathbb{R}\,\mathcal{C}^\dagger(\mathfrak{T}, \mathfrak{W}_{\cdot}; \mathrm{DR}^\dagger(\mathfrak{W}_{\cdot}/\mathfrak{T}, (\mathrm{E}_{\mathfrak{W}_{\cdot}}, \nabla_{\mathfrak{W}_{\cdot}}))))$$

as in Lemma 4.4.2 (the case where $s = 2$ and $s' = 1$). $\underline{E}_1^{qr}$ gives the r-th rigid cohomology sheaf for E on $(X, \overline{X})/\mathcal{S}_K$ evaluated on $]\overline{T}[_{\mathfrak{T}}$ for any $r \geqq 0$ by Proposition 11.4.4 (1). Since our definition of rigid cohomology does not depend on the choice of universally cohomologically descendable hypercovering, the edge morphism d_1^{rq} is the 0-map if q is odd and the identity if q is even. Hence we have an isomorphism

$$\mathbb{R}f_{\mathrm{rig}\,\mathfrak{T}*}E \cong \mathbb{R}\,\mathcal{C}^\dagger(\mathfrak{T}, \mathfrak{W}_{\cdot}; \mathrm{DR}^\dagger(\mathfrak{W}_{\cdot}/\mathfrak{T}, (E_{\mathfrak{W}_{\cdot}}, \nabla_{\mathfrak{W}_{\cdot}}))).$$

On the other hand, we consider a spectral sequence

$$\underline{E}_{\mathrm{op}1}^{qr} = \underline{H}^r(\mathbb{R}\,\mathcal{C}^\dagger(\mathfrak{T}, \mathfrak{W}_{(\cdot,\, q)}; \mathrm{DR}^\dagger(\mathfrak{W}_{(\cdot,\, q)}/\mathfrak{T}, (E_{\mathfrak{W}_{(\cdot,\, q)}}, \nabla_{\mathfrak{W}_{(\cdot,\, q)}}))))$$

$$\Rightarrow \underline{H}^{q+r}(\mathbb{R}\,\mathcal{C}^\dagger(\mathfrak{T}, \mathfrak{W}_{\cdot}; \mathrm{DR}^\dagger(\mathfrak{W}_{\cdot}/\mathfrak{T}, (\mathrm{E}_{\mathfrak{W}_{\cdot}}, \nabla_{\mathfrak{W}_{\cdot}}))))$$

for the filtration on the opposite side. By Proposition 11.4.4 (2) we have

$$\underline{E}_{\mathrm{op}1}^{qr} \cong \mathbb{R}^r h_{q,\, \mathrm{rig}\,\mathfrak{T}*}(g_q^* E).$$

Hence, we obtain the spectral sequence for the étale-étale hypercovering.

Now we prove that the spectral sequence is independent of the choice of étale-étale hypercoverings $\mathfrak{Z}_{\cdot\cdot}$. Since the fiber product of two étale-étale hypercoverings is also so by Proposition 11.4.1 (2), we may assume that there exists a morphism between them. Since two spectral sequences in the proof above commute with the induced homomorphism in Lemma 10.4.2, the spectral sequence for étale-étale hypercoverings is independent of the choice of hypercoverings $\mathfrak{Z}_{\cdot\cdot}$.

The functoriality of the spectral sequence in overconvergent isocrystals E follows from Proposition 4.2.3, and the functorialities in X, in $\mathfrak{T}$ and in étale-étale hypercoverings $(Y_{\cdot}, \overline{Y}_{\cdot})$ follow from 4.3 and Proposition 11.4.2.

This completes the proof of Proposition 11.1.2 in the case of étale-étale hypercoverings. ■

PROOF IN THE CASE OF ÉTALE-PROPER HYPERCOVERINGS. Let us take a finite affine Zariski covering $\mathfrak{U}$ of $(X, \overline{X})$ over $\mathfrak{T}$ and an étale-proper hypercovering $\mathfrak{Z}_{\cdot}$ of $(U, \overline{U})$ such that $(Z_{\cdot}, \overline{Z}_{\cdot})$ is a refinement of $(Y_{\cdot} \times \times_X U, \overline{Y}_{\cdot} \times_{\overline{X}} \overline{U})$. Such $\mathfrak{Z}_{\cdot}$ always exists by Proposition 11.5.1 (1). Let $\mathfrak{W}_{\cdot}$ be a 2-simplicial triple as in 11.5.3, and let E be an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ with a realization $(E_{\mathfrak{U}_{\cdot}}, \nabla_{\mathfrak{U}_{\cdot}})$ (resp. $(E_{\mathfrak{W}_{\cdot}}, \nabla_{\mathfrak{W}_{\cdot}})$) of E over $\mathfrak{U}_{\cdot}$ (resp. $\mathfrak{W}_{\cdot}$). Since $\mathfrak{U}$ is a Zariski covering of $(X, \overline{X})$ over $\mathfrak{T}$, we have a canonical isomorphism

$$\mathbb{R}f_{\mathrm{rig}\mathfrak{T}*} E \cong \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{T}, \mathfrak{U}_{\cdot}; \mathrm{DR}^{\dagger}(\mathfrak{U}_{\cdot}/\mathfrak{T}, (E_{\mathfrak{U}_{\cdot}}, \nabla_{\mathfrak{U}_{\cdot}}))).$$

The canonical morphism induces an isomorphism

$$\mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{T}, \mathfrak{U}_{\cdot}; \mathrm{DR}^{\dagger}(\mathfrak{U}_{\cdot}/\mathfrak{T}, (E_{\mathfrak{U}_{\cdot}}, \nabla_{\mathfrak{U}_{\cdot}}))) \xrightarrow{\cong} \mathbb{R}\mathcal{C}^{\dagger}(\mathfrak{T}, \mathfrak{W}_{\cdot}; \mathrm{DR}^{\dagger}(\mathfrak{W}_{\cdot}/\mathfrak{T}, (E_{\mathfrak{W}_{\cdot}}, \nabla_{\mathfrak{W}_{\cdot}})))$$

by Propositions 4.4.4 and 11.5.4 (1). The rest is similar to the case of étale-étale hypercoverings. ■

11.7. We prove the existence of the spectral sequence of rigid cohomology for étale hypercoverings in the case where $\mathfrak{S} = \mathfrak{T} = (\mathrm{Spec}\, k, \mathrm{Spec}\, k, \mathrm{Spf}\, \mathcal{V})$ and $\overline{X}$ is a proper over $\mathrm{Spec}\, k$.

11.7.1. THEOREM. *Let X be a scheme separated of finite type over* $\mathrm{Spec}\, k$, *let $Y_{\cdot}$ be an étale hypercovering of X such that $Y_{\cdot}$ is split* (*see the definition in* 11.2), *and denote by $\overset{\circ}{g}_{\cdot} : Y_{\cdot} \to X$ the structure morphism.*

For any overconvergent isocrystal E on X/K, there exists a spectral sequence

$$E_1^{qr} = H^r_{\mathrm{rig}}(Y_q/K,\, g_q^* E) \Rightarrow H^{q+r}_{\mathrm{rig}}(X/K,\, E).$$

This spectral sequence is functorial in E, in X, in $Y_{\bullet}$ and in K.

11.7.2. Remark. In the first version of the paper the hypothesis of splitness on $Y_{\bullet}$ in Theorem 11.7.1 was omitted. However, in order to construct a «completion» for $Y_{\bullet}$ as simplicial schemes, we need the assumption of splitness on $Y_{\bullet}$ (see the proposition below).

We should mention that the spectral sequence for the étale hypercovering exists without the hypothesis of splitness. One can find a proof of that in [22, 7.5].

Theorem 11.7.1 follows from Theorem 11.1.1 and the next proposition. To prove it, we use M. Nagata's existence theorem for relative completion [19].

11.7.3. Proposition. *Let $(X, \overline{X})$ be a pair of schemes separated of finite type over a spectrum S of noetherian ring and let $Y_{\bullet}$ be an étale hypercovering of X such that $Y_{\bullet}$ is split. Then there exists a proper simplicial scheme $\overline{Y}_{\bullet}$ over $\overline{X}$ (Definition 7.2.1 (1)) with an $\overline{X}$-morphism $Y_{\bullet} \to \overline{Y}_{\bullet}$ such that $Y_m \to \overline{Y}_m$ is an open immersion for all m. In other words, $(Y_{\bullet}, \overline{Y}_{\bullet})$ is an étale-proper hypercovering of $(X, \overline{X})$.*

Proof. Suppose that we have $(n-1)$-truncated proper scheme $\overline{Z}_{\bullet}$ over $\overline{X}$ with a morphism $Y_{\bullet}^{(n-1)} \to \overline{Z}_{\bullet}$ such that $Y_m \to \overline{Z}_m$ is an open immersion for all $m < n$. We apply Nagata's theorem to the construction of a completion of the n-th data of splitting of $Y_{\bullet}$ over $\mathrm{cosk}^S_{n-1}(\overline{Z}_{\bullet})_n$. Then we obtain an n-truncated proper simplicial scheme over $\overline{X}$ which satisfies the desired property by Lemma 11.2.3. ■

11.8. We now prove a theorem «finite flat base change» type.

11.8.1. Theorem. *In the situation of Proposition* 10.5.2, *suppose that $u : \mathfrak{T}' \to \mathfrak{T}$ is strict as a morphism of triples, $\widehat{u} : \mathcal{T}' \to \mathcal{T}$ is finite and flat, and the commutative diagram*

$$\begin{array}{ccc} (X, \overline{X}) & \xleftarrow{v} & (X', \overline{X}') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ (T, \overline{T}) & \xleftarrow[u]{} & (T', \overline{T}') \end{array}$$

is cartesian. Then the canonical $j^\dagger \mathcal{O}_{]\overline{T}'[_{\mathcal{T}'}}$*-homomorphism*

$$\tilde{u}^* \mathbb{R}^q f_{\mathrm{rig}\,\mathfrak{T}*} E \to \mathbb{R}^q f'_{\mathrm{rig}\,\mathfrak{T}'*} v^* E$$

is an isomorphism for any overconvergent isocrystal E *on* $(X, \overline{X})/\mathcal{S}_K$ *and any* q.

Proof. We may assume that $\mathcal{T}$ is affine. Considering a Čech spectral sequence for an affine Zariski covering of $\overline{X}$, we may assume that $\overline{X}$ is affine. Let us take a finite Zariski covering $\{X_\alpha\}_\alpha$ of X such that a complement of X_α in $\overline{X}$ is a hypersurface in $\overline{X}$ and consider the spectral sequence for the étale-proper hypercovering induced from $\coprod_\alpha (X_\alpha, \overline{X}) \to \to (X, \overline{X})$. Then we may assume that some complement of X in $\overline{X}$ is a hypersurface in $\overline{X}$. Let $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ be a triple over $\mathfrak{T}$ such that $\mathcal{X}$ is a smooth affine formal scheme of finite type over $\mathcal{T}$. Such $\mathcal{X}$ exists since $\mathcal{T}$ and $\overline{X}$ are affine. We denote by $f : \mathfrak{X} \to \mathfrak{T}$ the structure morphism and define $f' : \mathfrak{X}' \to \mathfrak{T}'$ by the base change of f with respect to $v : \mathfrak{T}' \to \mathfrak{T}$.

Now let us fix lifts $g_1, \dots, g_s \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ of generators of the ideal of definition of $\overline{X}$ in $\mathcal{X} \times_{\mathrm{Spf}\,\mathcal{V}} \mathrm{Spec}\, k$ and let $U_{l_1 \dots l_s}$ be an affinoid admissible open subset of $]\overline{X}[_{\mathcal{X}}$ defined by

$$U_{l_1 \dots l_s} = \{x \in]\overline{X}[_{\mathcal{X}} \mid |\pi|^{1/l_i} \leqq |g_i(x)| \leqq |\pi|^{1/l_i + 1} \text{ for } 1 \leqq i \leqq s\}$$

for any nonnegative integers $l_1, \dots, l_s$, where π is a uniformizer of K and $|\pi|^{1/0} = 0$. We denote by $\iota_{l_1 \dots l_s} : U_{l_1 \dots l_s} \to]\overline{X}[_{\mathcal{X}}$ the open immersion. $\{U_{l_1 \dots l_s}\}_{l_1 \dots l_s}$ is an admissible covering of $]\overline{X}[_{\mathcal{X}}$ by the maximum principle (Lemma 2.6.7). Since a complement of X in $\overline{X}$ is a hypersurface in $\overline{X}$, we have $\mathbb{R}^q \iota_{l_1 \dots l_s *} \iota_{l_1 \dots l_s}^{-1} F = 0$ and $\mathbb{R}^q (\tilde{f} \iota_{l_1 \dots l_s})_* \iota_{l_1 \dots l_s}^{-1} F = 0$ for any sheaf F of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules and any $q > 0$ by Corollary 5.1.2. Hence the alternating Čech complex $\mathcal{C}^\bullet_{\mathrm{alt}}(\{U_{l_1 \dots l_s}\}, F^\bullet)$ of sheaves of $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules with respect to $\{U_{l_1 \dots l_s}\}$ gives an $\tilde{f}_*$-acyclic resolution of complexes $F^\bullet$ of sheaves of $\tilde{f}^{-1}(j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}}})$-modules such that each F^n is a sheaf of coherent $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules by Proposition 2.12.1. Since each $U_{l_1 \dots l_s}$ intersects with a finite number of other $U_{l_1 \dots l_s}$'s, the complex $\mathcal{C}^\bullet_{\mathrm{alt}}(\{U_{l_1 \dots l_s}\}, F^\bullet)$ is a bounded complex if $F^\bullet$ is bounded.

Let $L^\bullet \to j^\dagger \mathcal{O}_{]\overline{T}'[_{\mathcal{T}'}}$ be a resolution as sheaves of $\tilde{u}^{-1}(j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}}})$-modules such that each L^n $(n \leqq 0)$ is free of finite rank over $\tilde{u}^{-1}(j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}}})$. Such resolution exists since $\mathcal{T}$ is noetherian, $\mathcal{T}'$ is finite over $\mathcal{T}$, u is strict as a morphism of triples and $j^\dagger$ is an exact functor (Proposition 2.7.2 (6)). The exactness of $j^\dagger$ also implies that $j^\dagger \mathcal{O}_{]\overline{T}'[_{\mathcal{T}'}}$ is flat over $\tilde{u}^{-1}(j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}}})$. Since

$\mathcal{T}'$ is finite over $\mathcal{T}$ and f' is obtained by a base change of f with respect to u, the canonical homomorphism gives an isomorphism

$$\tilde{v}^* \mathcal{F} \cong (\tilde{f}')^{-1}(j^\dagger \mathcal{O}_{]\overline{T}'[_{\mathcal{T}'}}) \otimes_{(\tilde{u}\tilde{f}')^{-1}(j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}}})} \tilde{v}^{-1}\mathcal{F}$$

for any sheaf $\mathcal{F}$ of $j^\dagger \mathcal{O}_{]\overline{X}[_{\mathcal{X}}}$-modules. Moreover, we have

$$\mathbb{L}\tilde{v}^* \mathcal{F} = (\tilde{f}')^{-1}(L^\bullet) \otimes_{(\tilde{u}\tilde{f}')^{-1}(j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}}})} \tilde{v}^{-1}\mathcal{F} \cong \tilde{v}^* \mathcal{F}$$

in the derived category by the flatness.

Now let E be an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ and let (E, ∇) be a realization of E over $\mathfrak{X}$. Then we have quasi-isomorphisms

$$\begin{aligned}
&\mathrm{DR}^\dagger(\mathfrak{X}'/\mathfrak{T}', v^\dagger(E, \nabla)) \xrightarrow{\cong} \\
&\quad \xrightarrow{\cong} (\tilde{f}')^{-1}(j^\dagger \mathcal{O}_{]\overline{T}'[_{\mathcal{T}'}}) \otimes_{(\tilde{u}\tilde{f}')^{-1}(j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}}})} \tilde{v}^{-1}\mathrm{DR}^\dagger(\mathfrak{X}/\mathfrak{T}, (E, \nabla)) \\
&\quad \xleftarrow{\cong} \mathrm{tot}((\tilde{f}')^{-1}(L^\bullet) \otimes_{(\tilde{u}\tilde{f}')^{-1}(j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}}})} \tilde{v}^{-1}\mathrm{DR}^\dagger(\mathfrak{X}/\mathfrak{T}, (E, \nabla))) \\
&\quad \xrightarrow{\cong} \mathrm{tot}((\tilde{f}')^{-1}(L^\bullet) \otimes_{(\tilde{u}\tilde{f}')^{-1}(j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}}})} \tilde{v}^{-1}\mathcal{C}^\bullet_{\mathrm{alt}}(\{U_{l_1 \dots l_s}\}, \mathrm{DR}^\dagger(\mathfrak{X}/\mathfrak{T}, (E, \nabla))))
\end{aligned}$$

of complexes of sheaves of $(\tilde{f}')^{-1}(j^\dagger \mathcal{O}_{]\overline{T}_{\mathcal{T}}})$-modules bounded above, where the tensor products above are tensor products as complexes. Let us put $U'_{l_1 \dots l_s} = \tilde{v}^{-1}(U_{l_1 \dots l_s})$. Then $\{U'_{l_1 \dots l_s}\}$ is an affinoid covering of $]\overline{X}'[_{\mathcal{X}'}$. Since the last complex of the above formula is isomorphic to

$$\mathrm{tot}(\mathcal{C}^\bullet_{\mathrm{alt}}(\{U'_{l_1 \dots l_s}\}, (\tilde{f}')^{-1}(L^\bullet) \otimes_{(\tilde{u}\tilde{f}')^{-1}(j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}}})} \tilde{v}^{-1}\mathrm{DR}^\dagger(\mathfrak{X}/\mathfrak{T}, (E, \nabla))))$$

and a complement of X' in $\overline{X}'$ is a hypersurface in $\overline{X}'$, it consists of $\tilde{f}'_*$-acyclic sheaves. Since $L^\bullet \to j^\dagger \mathcal{O}_{]\overline{T}'[_{\mathcal{T}'}}$ is a free resolution, the canonical homomorphism gives an isomorphism

$$\tilde{u}^* \mathbb{R}^q f_{\mathrm{rig}\,\mathfrak{T}*} E \xrightarrow{\cong} \mathbb{R}^q f'_{\mathrm{rig}\,\mathfrak{T}'*} \tilde{v}^* E$$

for any q by Proposition 10.5.5. ■

11.8.2. Corollary [4, 1.8 Proposition]. *Let $\mathcal{V} \to \mathcal{V}'$ be a finite homomorphism in* $\mathrm{CDVR}_{\mathbb{Z}_p}$, *let $(X, \overline{X})$ be a k-pair separated and locally of finite type and let $(X', \overline{X}')$ be a k'-pair obtained by a base change of $(X, \overline{X})$ for the extension $k \to k'$. Suppose that E is an overconvergent isocrystal on $(X, \overline{X})/K$ and denote by E' the inverse image of E on*

$(X', \overline{X}')/K'$. *Then the canonical* K'*-homomorphism*

$$K' \otimes_K H^q_{\mathrm{rig}}((X, \overline{X})/K, E) \to H^q_{\mathrm{rig}}((X', \overline{X}')/K', E')$$

is an isomorphism.

12. Frobenius.

In this section we will study the action of the Frobenius on rigid cohomology.

12.1. Let $\mathcal{V}$ be an object of $\mathrm{CDVR}_{\mathbb{Z}_p}$ with a lift $\sigma : V \to V$ of Frobenius of $k = k(\mathcal{V})$. Let $\mathcal{S}$ be a formal $\mathcal{V}$-scheme locally of finite type with a lift $\sigma_{\mathcal{S}}$ of Frobenius endomorphism on $\mathcal{S} \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spec}\, k$ such that the diagram

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\sigma_{\mathcal{S}}} & \mathcal{S} \\ \downarrow & & \downarrow \\ \mathrm{Spf}\ \mathcal{V} & \underset{\sigma}{\to} & \mathrm{Spf}\ \mathcal{V} \end{array}$$

is commutative and let us denote by $\mathfrak{S}$ the induced $\mathcal{V}$-triple from $\mathcal{S}$ (see 2.3.3). We denote by Fr the absolute Frobenius on schemes (resp. pairs) of characteristic p.

We recall the definition of Frobenius on overconvergent isocrystals [7, 2.3.7]. Let $(X, \overline{X})$ be a pairs separated locally of finite type over $(S, \overline{S})$. A Frobenius homomorphism of overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$ is a homomorphism

$$\Phi : \mathrm{Fr}^* \mathrm{E} \to \mathrm{E}$$

as overconvergent isocrystals on $(X, \overline{X})/\mathcal{S}_K$. We define an overconvergent F-isocrystal on $(X, \overline{X})/\mathcal{S}_K$ as an overconvergent isocrystal on $(X, \overline{X})/\mathcal{S}_K$ endowed with a Frobenius homomorphism which is an isomorphism.

12.2. Let $\mathfrak{T}$ be a $\mathcal{V}$-triple locally of finite type over $\mathfrak{S}$ such that $\mathcal{T}$ is smooth over $\mathcal{S}$ around T. Suppose that there exists a $\sigma_{\mathcal{S}}$-endomorphism $\sigma_{\mathcal{T}}$ on $\mathcal{T}$ which is a lift of Frobenius on $\mathcal{T} \times_{\mathrm{Spf}\, \mathcal{V}} \mathrm{Spec}\, k$. Such $\sigma_{\mathcal{T}}$ exists locally if $\mathcal{T}$ is smooth over $\mathcal{S}$ [10, Théorème 3.1]. $\sigma_{\mathcal{T}}$ induces a σ-linear homomorphism $\sigma_{\mathcal{T}_K} : j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}}} \to j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{T}}}$.

Let $f : (X, \overline{X}) \to (T, \overline{T})$ be a morphism of pairs separated locally of finite type over $(S, \overline{S})$ and let us fix notation as follows:

$$\begin{array}{ccccc} (X, \overline{X}) & \xrightarrow{F_{X/T}} & (X^{(p)}, \overline{X}^{(p)}) & \xrightarrow{g_{X/T}} & (X, \overline{X}) \\ & f \searrow & f^{(p)} \downarrow & & \downarrow f \\ & & (T, \overline{T}) & \xrightarrow[\mathrm{Fr}]{} & (T, \overline{T}), \end{array}$$

where $g_{X/T} F_{X/T} = \mathrm{Fr}$ and the square is cartesian.

For any overconvergent isocrystal E on $(X, \overline{X})/\mathcal{S}_K$ with a Frobenius $\Phi : \mathrm{Fr}^* E \to E$, we have a homomorphism

$$\varphi : \mathbb{L}\sigma_{\mathcal{C}_K}^* \mathbb{R}f_{\mathrm{rig}\,\mathfrak{T}*} E \to \mathbb{R}f^{(p)}_{\mathrm{rig}\,\mathfrak{T}*}(g_{X/T}^* E) \to \mathbb{R}f_{\mathrm{rig}\,\mathfrak{T}*}(\mathrm{Fr}^* E) \to \mathbb{R}f_{\mathrm{rig}\,\mathfrak{T}*} E$$

in the derived category of complexes of sheaves of $j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{C}}}$-modules, where the first arrow is induced from $g_{X/T}$, the middle arrow is induced from $F_{X/T}$ (Proposition 10.5.2) and the last arrow is $\mathbb{R}^q f_{\mathrm{rig}\,\mathfrak{T}*}(\Phi)$ (Proposition 10.5.1 (1)). The homomorphism φ depends only on $\sigma_{\mathcal{C}}$. We say that φ is a Frobenius on the complex $\mathbb{R}f_{\mathrm{rig}\,\mathfrak{T}*} E$ of rigid cohomology.

If the Frobenius endomorphism $\sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is flat, then we have a $j^\dagger \mathcal{O}_{]\overline{T}[_{\mathcal{C}}}$-homomorphism on rigid cohomology sheaves:

$$\varphi^q : \sigma_{\mathcal{C}_K}^* \mathbb{R}^q f_{\mathrm{rig}\,\mathfrak{T}*} E \to \mathbb{R}^q f_{\mathrm{rig}\,\mathfrak{T}*} E\,.$$

Note that φ^q is not an isomorphism in general.

The Frobenius commutes with the triangle arising from short exact sequences of overconvergent F-isocrystals in Proposition 10.5.1 (2) by Proposition 10.5.2 and commutes with spectral sequences in Theorem 11.1.1 by Propositions 11.4.2 and 11.5.2. If one takes Frobenius endomorphisms on $\mathcal{S}'$ and $\mathcal{C}'$ compatible with those on $\mathcal{S}$ and $\mathcal{C}$ as in Proposition 10.5.2, then $v^* E$ is also an overconvergent F-isocrystal on $(X', \overline{X}')/\mathcal{S}'_K$ and the canonical homomorphism

$$\mathbb{L}\tilde{u}^* \mathbb{R}f_{\mathrm{rig}\,\mathfrak{T}*} E \to \mathbb{R}f'_{\mathrm{rig}\,\mathfrak{T}'*} v^* E$$

commutes with Frobenius.

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Manoscritto pervenuto in redazione il 20 agosto 2002.