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ON THE EXCEPTIONAL SET OF HARDY-LITTLEWOOD'S NUMBERS IN SHORT INTERVALS

By

A. LANGUASCO

1. Introduction

In 1923 Hardy and Littlewood [4] conjectured that every sufficiently large integer is either a k-power of an integer or a sum of a prime and a k-power of an integer, for $k = 2, 3$. Define a Hardy-Littlewood number (HL-number) to be an integer which is a sum of a prime and of a k-power of an integer, $k \in N$, $k \geq 2$. Let X be a sufficiently large parameter. Denote by E_k the set of integers which are neither an HL-number nor a power of an integer, let $E_k(X) = E_k \cap [1, X]$ and $E_k(X, H) = E_k \cap [X, X + H]$, where $H = o(X)$. Hardy-Littlewood's conjectures are equivalent to $E_k(X) \ll 1$.

The best known result on $E_2(X)$ was independently proved by Brünner-Perelli-Pintz [1] and A. I. Vinogradov [14]:

There exists a (small) positive constant δ such that

$$|E_2(X)| \ll X^{1-\delta}.$$

In 1992 Zaccagnini [15] proved that such a result holds in the general case $k \geq 2$ too. Concerning short intervals, Perelli-Pintz [10] and Mikawa [6] proved independently:

Let $A > 0$, $\varepsilon > 0$ be arbitrary constants and $H \geq X^{7/24+\varepsilon}$; then

$$|E_2(X, H)| \ll H \log^{-A} X.$$

In 1995 Perelli and Zaccagnini [11] proved that such a result holds in the general case $k \geq 2$ too.

The aim of this paper is to prove that we can save a power of H in the estimate of $|E_k(X, H)|$, $k \in N$, $k \geq 2$, for H in some suitable range.

THEOREM. *Let $k \geq 2$ be a fixed integer and $K = 2^{k-2}$. There exists a (small) positive absolute constant δ such that for $H \geq X^{7/12(1-1/k)+\delta}$*

$$|E_k(X, H)| \ll H^{1-\delta/(5K)}.$$

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To prove our result we follow the circle method setting used by Brünner-Perelli-Pintz [1], Zaccagnini [15], Perelli-Pintz [10] and Perelli-Zaccagnini [11] to treat the major arcs. So we estimate the contribution of the zeros of Dirichlet L-functions located in a suitable thin strip near $\sigma = 1$ as "secondary" main terms. In the body of the proof we will use the zero-density estimate

$$\sum_{q \le P} \sum_{\chi}{}^{*} N(\sigma, T, \chi) \ll (P^2 T)^{12/5(1-\sigma)} (\log PT)^{22}, \tag{1}$$

for $\sigma \in [1/2, 1]$, see Ramachandra [13], and the log-free zero-density estimate

$$\sum_{q \le P} \sum_{\chi}{}^{*} N(\sigma, T, \chi) \ll (P^2 T)^{(2+\varepsilon)(1-\sigma)}, \tag{2}$$

for $\sigma \in [4/5, 1]$, see Jutila [5], where * means that the summation is over primitive characters and $N(\sigma, T, \chi) = |\{\rho = \beta + i\gamma : L(\rho, \chi) = 0, \beta \ge \sigma \text{ and } |\gamma| \le T\}|$.

In the proof we insert a localization parameter $Y = o(X)$ for the primes and write an HL-number $n \in [X, X + H]$ as $p + m^k$ with $X - Y \le p \le X + Y$ and $Y/4 \le m^k \le Y$. The Theorem is obtained using $Y = X^{7/12+10\delta+\varepsilon}$ and $H = Y^{(1-1/k)+\delta}$. The meaning of the previously mentioned constants $7/12$ and $(1 - 1/k)$ can be explained as follows. In the error term of the explicit formula we have to choose $T \ge X^{1+7\delta} Y^{-1} \log^2 X$ and, to estimate the contribution of the secondary main terms using (2), we have to choose $T \le X^{1/2-\varepsilon-2\delta}$. Combining such relations we get $Y \ge X^{1/2+\varepsilon+9\delta}$ which is already satisfied since in the centre of the major arcs our treatment requires (1) and hence $Y \ge X^{7/12+\varepsilon+10\delta}$. Moreover, in the mean-square estimates of the minor arcs and of the periphery of major arcs, we will choose $H \ge Y^{(1-1/k)+\delta}$.

The paper is organized as follows: in section 2 we shall define the quantities involved; section 3 will be devoted to arithmetic and analytic lemmas while in section 4 we will prove suitable mean-square estimates for minor arcs; in section 5 we will treat the contribution of the major arcs and in section 6 we will study the singular series of this problem. Finally, in section 7, we will deduce the Theorem.

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2. Notations and Definitions

Let n be an integer,

$$R(n) = \sum_{\substack{h+m^k=n \\ X-Y\le h\le X+Y \\ Y/4\le m^k\le Y}} \Lambda(h) \quad \text{and} \quad M(n) = \sum_{\substack{h+m^k=n \\ X-Y\le h\le X+Y \\ Y/4\le m^k\le Y}} 1,$$

where $\Lambda(n)$ denotes the von Mangoldt function. Defining

$$S(\alpha) = \sum_{X-Y\le h\le X+Y} \Lambda(h)e(h\alpha), \quad F_k(\alpha) = \sum_{Y/4\le m^k\le Y} e(m^k\alpha), \quad e(\alpha) = e^{2\pi i\alpha},$$

it is an easy matter to see that $S(\alpha)F_k(\alpha) = \sum_{X-3Y/4\le n\le X+2Y} R(n)e(n\alpha)$ and $R(n) = \int_0^1 S(\alpha)F_k(\alpha)e(-n\alpha)\,d\alpha$. Let now $Q = 4Y^{1-1/k}$ and consider the Farey dissection of level Q of $[1/Q, 1+1/Q]$. Let a/q be a Farey fraction,

$$I'_{q,a} = \left\{\frac{a}{q} + \eta, \eta \in \xi'_q\right\}, \quad \text{where } \xi'_q = \left(-\frac{P^4}{qY}, \frac{P^4}{qY}\right),$$

$$\mathfrak{M} = \bigcup_{q\le P} \bigcup_{a=1}^{q}{}^{*} I'_{q,a}, \quad \text{and} \quad \mathfrak{m} = [1/Q, 1+1/Q]\backslash\mathfrak{M},$$

where * means $(a,q) = 1$ and $P < Q$ will be chosen later. Hence

$$R(n) = \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\right) S(\alpha)F_k(\alpha)e(-n\alpha)\,d\alpha = R_{\mathfrak{M}}(n) + R_{\mathfrak{m}}(n), \tag{3}$$

say.

Let now $P_1 = Y^{\delta}$, $0 < \delta < 1/2$. According to Lemma 13′ below, applied with $P' = P_1$ and $|t| \le T = XY^{-1}P_1^7 \log^2 X$, we denote by $\tilde{\beta}$ the Siegel zero, $\tilde{\chi}$ the Siegel character and by $\tilde{r}$ its modulus. Let now

$$P_2 = \begin{cases} P_1 & \text{if } \tilde{r} < P_1^{\nu} \\ P_1^{\nu} & \text{otherwise,} \end{cases}$$

where $\nu = \nu(k) < 1/2$ is a parameter which will be specified later. Now Lemma 13′ remains true for $P' = P_2$, with a suitable change in the constant c_1. Hence $\tilde{r} \le P_2^{\nu}$, if it exists.

Following [1] and [15], and according to Lemmas 13′–14′ below, we define the P_2*-excluded zeros* as the zeros of the functions $L(s,\chi)$, where χ is any primitive character (mod q), $q \le P_2$, lying in the region

$$\sigma \ge 1 - \frac{(12k+6)\log\log X}{\log X}, \quad |t| \le T, \quad \text{if the Siegel zero does not exist,} \tag{4}$$

and, if the Siegel zero exists,

$$\sigma \geq 1 - \frac{(12k+6)\log\log X}{\log X}\log\left(\frac{ec_1}{(1-\tilde{\beta})\log P_2}\right), \quad |t| \leq T, \tag{5}$$

excluding the Siegel zero relative to P_2. Then define the *P_2-excluded characters* as the primitive characters (mod q) for which $L(\rho,\chi)=0$, ρ being an excluded zero. The *P_2-excluded moduli* are the moduli of the excluded characters. Let now

$$\mathscr{E} = \{P_2\text{-excluded characters}\}, \quad \mathscr{E}' = \{P_2\text{-excluded zeros}\},$$

$$\mathscr{S} = \{\text{Siegel character}\} \quad \text{and} \quad \mathscr{S}' = \{\text{Siegel zero}\}.$$

Let further $P = P_2$.

Now we write

$$S\left(\frac{a}{q}+\eta\right) = \frac{1}{\varphi(q)}\sum_{\chi \,(\mathrm{mod}\, q)} \chi(a)\tau(\bar{\chi})S(\chi,\eta) + O(\log^2 qX),$$

where

$$S(\chi,\eta) = \sum_{X-Y\leq l\leq X+Y} \Lambda(l)\chi(l)e(l\eta).$$

Let now

$$T(\eta) = \sum_{X-Y\leq l\leq X+Y} e(l\eta) \quad \text{and} \quad T_\rho(\eta) = \sum_{X-Y\leq l\leq X+Y} l^{\rho-1}e(l\eta).$$

Hence the corresponding approximation for $S(\alpha)$ becomes

$$S\left(\frac{a}{q}+\eta\right) = \frac{\mu(q)}{\varphi(q)}T(\eta) + D(a,q,\eta) + E(a,q,\eta) + O(\log^2 qX), \tag{6}$$

where

$$D(a,q,\eta) = \frac{1}{\varphi(q)}\sum_{\chi}\tau(\bar{\chi})\chi(a)W(\chi,\eta),$$

$$W(\chi,\eta) = \begin{cases} S(\chi_{0,q},\eta) - T(\eta) & \text{if } \chi = \chi_{0,q}, \\ S(\chi^*,\eta) + \sum_{\substack{\rho\in\mathscr{E}'\cup\mathscr{S}' \\ L(\rho,\chi^*)=0}} T_\rho(\eta) & \text{if } \chi = \chi_{0,q}\chi^*,\ \chi^*\in\mathscr{E}\cup\mathscr{S}, \\ S(\chi,\eta) & \text{otherwise,} \end{cases}$$

$$E(a,q,\eta) = -\sum_{\substack{\chi\in\mathscr{E}\cup\mathscr{S} \\ \mathrm{cond}\,\chi | q}}\ \sum_{\substack{\rho\in\mathscr{E}'\cup\mathscr{S}' \\ L(\rho,\chi)=0}} \frac{\chi_{0,q}\chi(a)\tau(\overline{\chi_{0,q}\chi})}{\varphi(q)}T_\rho(\eta),$$

$\mathrm{cond}\,\chi$ is the conductor of χ and $\chi_{0,q}$ is the principal character (mod q).

Moreover we define

$$L_p(Y,n) = \sum_{\substack{l+m^k=n \\ X-Y\le l\le X+Y \\ Y/4\le m^k\le Y}} l^{p-1} = \int_0^1 F_k(\eta)T_p(\eta)e(-n\eta)\,d\eta$$

and

$$L(Y,n) = \int_0^1 F_k(\eta)T(\eta)e(-n\eta)\,d\eta.$$

For $F_k(\alpha)$ we use the following approximation (see [15], Lemma 5.1 and p. 409):

$$F_k\left(\frac{a}{q}+\eta\right) = \frac{V_k(a,q)}{q}F_k(\eta) + \Delta_k(a,q,\eta),$$

where

$$\Delta_k(a,q,\eta) \ll q(1+|\eta|Y) \tag{7}$$

and (see eq. (10) of [11])

$$V_k(a,q) = \sum_{m \pmod q} e\left(m^k\frac{a}{q}\right) \ll q^{1-1/k}. \tag{8}$$

Let further

$$H_k(\chi,q,n) = \sum_{a=1}^{q}\chi(a)V_k(a,q)e\left(-\frac{na}{q}\right), \quad H_k(q,n) = H_k(\chi_{0,q},q,n),$$

$$T_k(\chi,r,n) = \frac{\tau(\bar\chi)H_k(\chi,r,n)}{r\varphi(r)}, \quad \mathfrak{S}_k(n,R,r) = \sum_{\substack{q\le R\\(q,r)=1}}\frac{\mu(q)}{q\varphi(q)}H_k(q,n),$$

$$\mathfrak{S}_k(n,R) = \mathfrak{S}_k(n,R,1) \quad \text{and} \quad \rho_k(d,n) = |\{h \pmod d), h^k \equiv n \pmod d)\}|.$$

3. Lemmas

In the following we denote by c a positive absolute constant, not necessarily the same at each occurrence.

Arithmetic Lemmas. We recall some lemmas of Zaccagnini [15] (in the case $k=2$ they are due to Brünner-Perelli-Pintz [1]).

LEMMA 1 (Lemma 4.1 of [15]). *Let* $(q_1, q_2) = 1$ *and* χ_{q_i} *be a character* (mod q_i), $i = 1, 2$. *Then*

$$H_k(\chi_{q_1}\chi_{q_2}, q_1q_2, n) = \chi_{q_1}(q_2)\chi_{q_2}(q_1)H_k(\chi_{q_1}, q_1, n)H_k(\chi_{q_2}, q_2, n).$$

LEMMA 2 (Lemma 4.2 of [15]). $H_k(p,n) = p(\rho_k(p,n) - 1)$. *If* $\mu(q) \neq 0$ *then* $|H_k(q,n)| \leq q(k-1)^{\omega(q)}$.

The next lemma is Lemma 5.2 of Montgomery-Vaughan [7].

LEMMA 3. *If* χ *is a character* (mod q) *induced by the primitive character* χ^* (mod r). *Then* $r|q$ *and* $\tau(\chi) = \mu(q/r)\chi^*(q/r)\tau(\chi^*)$ *and* $|\tau(\chi^*)| = r^{1/2}$.

LEMMA 4 (Lemma 4.4 of [15]). *Let* χ (mod r) *be a primitive character. Then*

$$|H_k(\chi, r, n)| \leq r^{3/2} \prod_{p|r} \left(1 - \frac{\rho_k(p,n)}{p}\right) \leq r^{3/2}.$$

In the next lemma we cite Lemma 4.5 of [15] and we state also a short interval version of it whose proof is totally analogous.

LEMMA 5. *Let* $A \in N$. *We have*

$$\sum_{j \leq T} A^{\omega(j)} \ll T(\log T)^{A-1}.$$

Let further $Y = o(X)$. *We have*

$$\sum_{X-(3/4)Y \leq n \leq X+2Y} A^{\omega(n)} \ll Y(\log X)^{A-1}.$$

LEMMA 6 (Lemma 4.6 of [15]). *Let* χ (mod r) *be a primitive character,* $Y = o(X)$ *and* $X - (3/4)Y \leq n \leq X + 2Y$. *Then*

$$\sum_{\substack{q \leq P \\ r|q}} \frac{|\tau(\chi_{0,q}\chi)|}{q\varphi(q)} |H_k(\chi_{0,q}\chi, q, n)| \ll (\log P)^k.$$

LEMMAS ON $F_k(\alpha)$. We first state Gallagher's famous lemma (Lemma 1 of [3]) which is a fundamental tool to estimate truncated L^2 norm of exponential sums.

LEMMA 7. *Let $u_1, \dots, u_N$ be arbitrary real numbers. Then for any $\theta > 0$*

$$\int_{-\theta}^{\theta} \left| \sum_{n \le N} u_n e(n\eta) \right|^2 d\eta \ll \int_{-\infty}^{\infty} \left| \theta \sum_{n=x}^{x+\theta^{-1}} u_n \right|^2 dx.$$

Using Lemma 7 the following lemmas can be proved.

LEMMA 8 (Lemma 5.2 of [15]). *For any integer $s \ge ck^2 \log k$, c being a suitable absolute constant, we have*

$$\int_0^1 |F_k(\eta)|^{2s} \, d\eta \ll Y^{(2s/k)-1}.$$

LEMMA 9 (Lemma 5.4 of [15]). *Let $0 < \theta < Y^{1/k-1}$. Then*

$$\int_{-\theta}^{\theta} |F_k(\eta)|^2 \, d\eta \ll Y^{2/k-1}.$$

The next lemma was proved by Perelli-Zaccagnini [11], eq. (39)–(40).

LEMMA 10. *Let $(a,q) = 1$ and $|\eta| < 1/qQ$. Then*

$$\left| F_k\left(\frac{a}{q} + \eta\right) \right| \ll \frac{Y^{1/k-1}}{|\eta|}.$$

The next Lemma is a modified version of the Lemma proved at page 199 of Perelli-Zaccagnini [11]. The difference is in the last term and follows from a different estimate of the divisor function involved in the proof in [11].

LEMMA 11. *Let $F(x,y) = x^g y + \sum_{j=0}^{g-1} b_j(y) x^j$ where $g \ge 2$ is a fixed integer and $b_j(y)$ are real-valued functions. Let $|\alpha - a/q| < 1/q^2$ and $(a,q) = 1$. Then for $T, R, q \le X$ and for every $\varepsilon > 0$ we have*

$$\sum_{1 \le d \le T} \left| \sum_{n \le R} e(\alpha F(n,d)) \right| \ll_g TR\left(\frac{1}{q} + \frac{1}{R} + \frac{q}{TR^g}\right)^{1/K} T^{\varepsilon/K} R^{(g-1)\varepsilon/K},$$

where $K = 2^{k-2}$.

ANALYTIC LEMMAS. In the following we state several results on the distribution of zeros of Dirichlet L-functions and on some summations involving such zeros. The following Lemma can be proved following the line of the proof of Lemma 12 of Brünner-Perelli-Pintz [1].

LEMMA 12. *Let* $|\gamma| < X/qQ$ *and* $1/qQ \leq |\eta| \leq 1/2$. *Then*

$$T_p(\eta) \ll \frac{X^{\beta-1}}{|\eta|}.$$

Now we recall some analytic results on zero-free regions for Dirichlet L-functions.

LEMMA 13. *Assume* $T \geq 0$. *There exists a constant* $c_1 > 0$ *such that* $L(\sigma + it, \chi) \neq 0$ *whenever*

$$\sigma \geq 1 - \frac{c_1}{\max\{\log P'; (\log(T+3) \log\log(T+3))^{3/4}\}}, \quad |t| \leq T$$

for all the primitive characters χ *modulo* $q \leq P'$, *with the possible exception of at most one primitive character* $\tilde{\chi}$ (mod $\tilde{r}$). *If it exists, the character* $\tilde{\chi}$ *is quadratic and the (unique) exceptional zero* $\tilde{\beta}$ *of* $L(s, \tilde{\chi})$ *is real, simple and satisfies*

$$\frac{c_2}{\tilde{r}^{1/2} \log^2 \tilde{r}} \leq 1 - \tilde{\beta} \leq \frac{c_1}{\max\{\log P'; (\log(T+3) \log\log(T+3))^{3/4}\}}.$$

The previous form of the zero-free region for Dirichlet L-functions can be found in Prachar [12], ch. 8, Satz 6.2. In our case we have $\log T \asymp \log P'$. Hence $(\log(T+3) \log\log(T+3))^{3/4} < \log P'$ for X sufficiently large. So in fact the zero free-region Lemma 13 becomes

LEMMA 13′. *Assume* $T \geq 0$ *and* $\log(T+3) \asymp \log P'$. *There exists a constant* $c_1 > 0$ *such that* $L(\sigma + it, \chi) \neq 0$ *whenever*

$$\sigma \geq 1 - \frac{c_1}{\log P'}, \quad |t| \leq T$$

for all the primitive characters χ *modulo* $q \leq P'$, *with the possible exception of at most one primitive character* $\tilde{\chi}$ (mod $\tilde{r}$). *If it exists, the character* $\tilde{\chi}$ *is quadratic and the (unique) exceptional zero* $\tilde{\beta}$ *of* $L(s, \tilde{\chi})$ *is real, simple and satisfies*

$$\frac{c_2}{\tilde{r}^{1/2} \log^2 \tilde{r}} \leq 1 - \tilde{\beta} \leq \frac{c_1}{\log P'}.$$

Concerning again the distribution of the zeros of Dirichlet L-functions, we state the following form of Deuring-Heilbronn phenomenon. It can be proved using the function $\max(\log P'; (\log(T+3)\log\log(T+3))^{3/4})$ instead of the function $\log P' + (\log(T+3)\log\log(T+3))^{3/4}$ in the proof of Lemma 2 of Peneva [8].

LEMMA 14. *Under the same hypotheses of Lemma* 13, *if* $\tilde{\beta}$ *exists, then for all the primitive characters* χ *modulo* $q \le P'$, $L(\sigma+it,\chi) \ne 0$ *whenever*

$$\sigma \ge 1 - \frac{c_1}{f(P',T)} \log\left(\frac{ec_1}{(1-\tilde{\beta})f(P',T)}\right), \quad |t| \le T,$$

where $f(P',T) = \max\{\log P'; (\log(T+3)\log\log(T+3))^{3/4}\}$ *and* $\tilde{\beta}$ *is still the only exception.*

Using the same remark after Lemma 13 we have.

LEMMA 14$'$. *Under the same hypotheses of Lemma* 13$'$, *if* $\tilde{\beta}$ *exists, then for all the primitive characters* χ *modulo* $q \le P'$, $L(\sigma+it,\chi) \ne 0$ *whenever*

$$\sigma \ge 1 - \frac{c_1}{\log P'} \log\left(\frac{ec_1}{(1-\tilde{\beta})\log P'}\right), \quad |t| \le T,$$

with $\tilde{\beta}$ *is still the only exception.*

The next lemma is a localized version of Lemma 4.3 of Montgomery-Vaughan [7].

LEMMA 15. *Let* $Y \ge X^{7/12+10\delta+\varepsilon}$, $T = X^{1+7\delta}Y^{-1}\log^2 X$ *and* P *be as defined in section* 2. *Then*

$$\sum_{r\le P}\sum_{\chi \,(\mathrm{mod}\, r)}^{*} \max_{X-2Y\le x\le X+2Y} \max_{Y/4<h\le Y} \left(h+\frac{Y}{P^4}\right)^{-1} \left|\sum_{l=x-h}^{x}{}^{\#} \Lambda(l)\chi(l)\right|$$
$$\ll G(\log Y)^{-2k-1} + P^{-1/3}, \tag{9}$$

where

$$\sum_{l=x-h}^{x}{}^{\#} \Lambda(l)\chi(l) = \begin{cases} \sum\limits_{l=x-h}^{x} (\Lambda(l)-1) & \textit{if } r=1 \\ \sum\limits_{l=x-h}^{x} \left(\Lambda(l)\chi(l) + \sum\limits_{\rho\in\mathscr{E}'\cup\mathscr{S}'} l^{\rho-1}\right) & \textit{if } r>1 \end{cases}$$

and
$$G=\begin{cases}(1-\tilde\beta)\log P & \text{if } \tilde\beta \text{ exists}\\ 1 & \text{if } \tilde\beta \text{ does not exist.}\end{cases}$$

PROOF. The proof follows the line of Theorem 7 of Gallagher [3] (in its effective form due to Montgomery-Vaughan [7], Lemma 4.3) in which x is localized near X. The differences here are that we sum only over the not-excluded zeros, the use of Lemmas 13′–14′ above and of the zero-density estimates (1) and (2). For seek of completeness, we sketch the proof. First we insert, in the left hand side of (9), the explicit formula, see, *e.g.*, Davenport [2], ch. 19,

$$\sum_{m\le x}\Lambda(m)\chi(m)=\delta_\chi x-\delta_{\chi,\tilde\chi}\frac{x^{\tilde\beta}}{\tilde\beta}-\sum_{|\rho|\le T}{}^{'}\frac{x^\rho}{\rho}+O\Big(\frac{x}{T}\log^2 qx+x^{1/4}\log x\Big), \qquad (10)$$

where $\delta_\chi=1$ if χ is the principal character, $\delta_\chi=0$ otherwise, $\delta_{\chi,\tilde\chi}=1$ if $\chi=\tilde\chi$ and $\delta_{\chi,\tilde\chi}=0$ otherwise. We get that the error term of the explicit formula furnishes a total contribution in (9) which is

$$\ll P^2\max_{X-2Y\le x\le X+2Y}\ \max_{Y/4<h\le Y}\Big(h+\frac{Y}{P^4}\Big)^{-1}\frac{x}{T}\log^2 x\ll\frac{P^6}{Y}\frac{X}{T}\log^2 X\ll P^{-1},$$

since for our choice of T the second error term in (10) is negligible. We remark that $(x^\rho-(x-h)^\rho)/\rho\ll hx^{\beta-1}$ and that $\max_{X-2Y\le x\le X+2Y}\max_{Y/4<h\le Y}(h+Y/P^4)^{-1}hx^{\beta-1}\ll(X-2Y)^{\beta-1}\ll X^{\beta-1}$, since $X-2Y\gg X$.

Using the definition of $\sum^{\#}$, the explicit formula and the previous remarks we have to estimate

$$\sum_{q\le P}\ \sum_{\substack{\chi\,(\mathrm{mod}\ q)\\ \chi\notin\mathscr{E}\cup\mathscr{S}}}\ \sum_{\rho\notin\mathscr{E}'\cup\mathscr{S}'}X^{\beta-1}\ll\int_{1/2}^{1-\eta(P,T)}X^{\sigma-1}\sum_{q\le P}\ \sum_{\substack{\chi\,(\mathrm{mod}\ q)\\ \chi\notin\mathscr{E}\cup\mathscr{S}}}N(\sigma,T,\chi)\log X\,d\sigma$$
$$+X^{-1/2}\sum_{q\le P}\ \sum_{\chi\,(\mathrm{mod}\ q)}N\Big(\frac12,T,\chi\Big), \qquad (11)$$

where $\eta(P,T)$ is one of the functions defined in (4)–(5) (according to the existence of the Siegel zero).

By (1) the second term in (11) is

$$\ll X^{-1/2}\Big(\frac{P_1^9X\log^2X}{Y}\Big)^{6/5}(\log X)^{22}\ll X^{-1/2}P^{-6/5}(X^{5/12-\varepsilon/2})^{6/5}(\log X)^{22}$$
$$\ll P^{-1}. \qquad (12)$$

Now we split the integral according to the range of validity of the zero-density estimates (1) and (2). For $\sigma \in [1/2, 4/5]$ we get

$$\int_{1/2}^{4/5} X^{\sigma-1} \sum_{q \le P} \sum_{\chi \ (\mathrm{mod}\ q)} N(\sigma, T, \chi) \log X \, d\sigma \ll (\log X)^{23} X^{-1/5} \left(\frac{P_1^9 X \log^2 X}{Y} \right)^{12/25}$$

$$\ll (\log X)^{23} X^{-1/5} (X^{5/12-\varepsilon/2})^{12/25} P^{-12/25} \ll P^{-1/3}. \qquad (13)$$

Let now $\sigma \in [4/5, 1 - \eta(P, T)]$. We have

$$\int_{4/5}^{1-\eta(P,T)} X^{\sigma-1} \sum_{q \le P} \sum_{\chi \ (\mathrm{mod}\ q)} N(\sigma, T, \chi) \log X \, d\sigma$$

$$\ll \int_{4/5}^{1-\eta(P,T)} X^{\sigma-1} \log X \left(\frac{P_1^9 X \log^2 X}{Y} \right)^{(2+\varepsilon)(1-\sigma)} d\sigma$$

$$\ll \int_{4/5}^{1-\eta(P,T)} X^{(1/6+2\delta)(\sigma-1)} \log X \, d\sigma \ll X^{-(1/6)\eta(P,T)}. \qquad (14)$$

If the Siegel zero does not exist, then

$$X^{-(1/6)\eta(P,T)} \ll \exp(-(1/6)(12k+6) \log\log X) = (\log X)^{-2k-1}. \qquad (15)$$

If the Siegel zero exists, then

$$X^{-(1/6)\eta(P,T)} \ll \exp\left(-(2k+1) \log\log X \log\left(\frac{ec_1}{(1-\tilde{\beta}) \log P} \right) \right)$$

$$\ll (1-\tilde{\beta}) \log P \exp(-(2k+1) \log\log X)$$

$$\ll G (\log X)^{-2k-1}. \qquad (16)$$

Lemma 15 now follows from $\log X \asymp \log Y$ and (11)–(16). □

Using the same argument in Lemma 15 one can obtain the following result on a sum over the excluded zeros.

LEMMA 16. *Let $T \le X^{1/2-\varepsilon} P^{-2}$. There exists a positive constant c such that*

$$\sum_{q \le P} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \in \mathscr{E}}} \sum_{\rho \in \mathscr{E}'} X^{\beta} \ll GX \exp\left(-c \frac{\log X}{\log P} \right),$$

where G is defined as in Lemma 15.

PROOF. We argue as in Lemma 15 starting from (11). We have

$$X\sum_{q\le P}\sum_{\substack{\chi\ (\mathrm{mod}\ q)\\ \chi\in\mathscr{E}}}\sum_{\rho\in\mathscr{E}'}X^{\beta-1}\ll X\int_{1-\eta_1(P,T)}^{1-\eta_2(P,T)}X^{\sigma-1}\sum_{q\le P}\sum_{\substack{\chi\ (\mathrm{mod}\ q)\\ \chi\in\mathscr{E}}}N(\sigma,T,\chi)\log X\,d\sigma$$

$$+X^{1-\eta_1(P,T)}\sum_{q\le P}\sum_{\chi\ (\mathrm{mod}\ q)}N(1-\eta_1(P,T),T,\chi), \tag{17}$$

where $\eta_1(P,T)$ is one of the functions defined in (4)–(5) and $\eta_2(P,T)$ is one of the functions defined in Lemmas 13′ and 14′ (according to the existence of the Siegel zero). Using the density estimate (2) and integrating, we have that the right hand side of (17) is $\ll X^{1-\varepsilon\eta_2(P,T)}+X^{1-(1/6)\eta_1(P,T)}\ll X^{1-\varepsilon\eta_2(P,T)}$. Hence, if the Siegel zero does not exist, we get that the right hand side of (17) is

$$\ll X\exp\left(-\varepsilon\frac{\log X}{\log P}\right).$$

If the Siegel zero exists, we have that the right hand side of (17) is

$$\ll X\exp\left(-\varepsilon\frac{\log X}{\log P}\log\left(\frac{ec_1}{(1-\tilde\beta)\log P}\right)\right)$$

$$\ll X(1-\tilde\beta)\log P\exp\left(-\varepsilon\frac{\log X}{\log P}\right)\ll GX\exp\left(-c\frac{\log X}{\log P}\right)$$

and Lemma 16 follows. □

The last two lemmas of this section will be useful to evaluate the behaviour of the main term and of the "secondary" main terms.

LEMMA 17. *Let $n\in[X-(3/4)Y,X+2Y]$ and $\tilde\chi\in\mathscr{S}$. Then there exists a constant $c>0$ such that $L(Y,n)-L_{\tilde\beta}(Y,n)\ge cGY^{1/k}$.*

PROOF. By Lagrange's theorem we have

$$L(Y,n)-L_{\tilde\beta}(Y,n)=\sum_{\substack{l+m^k=n\\ X-Y\le l\le X+Y\\ Y/4\le m^k\le Y}}(1-l^{\tilde\beta-1})\ge cY^{1/k}(1-P^{\tilde\beta-1})$$

$$\ge cY^{1/k}(1-\tilde\beta)\log P\ge cGY^{1/k}.$$ □

LEMMA 18. *Let $n\in[X-(3/4)Y,X+2Y]$. Then there exists an absolute constant $c>0$ such that $|L_\rho(Y,n)|\le cY^{1/k}X^{\beta-1}$.*

PROOF. We have

$$|L_p(Y,n)| \le \sum_{\substack{n-X-Y\le m^k\le n-X+Y\\ Y/4\le m^k\le Y}} |n-m^k|^{\beta-1} \le Y^{1/k}(X-Y)^{\beta-1} \ll Y^{1/k}X^{\beta-1},$$

since $X - Y \gg X$. □

4. Minor Arcs

Following the argument of Perelli-Pintz [10], we subdivide $[1/Q, 1+1/Q]$ in H adjacent intervals I_j, $j=1\dots,H$, and we use the estimate $K(\eta)=\sum_{X\le m\le X+H} e(m\eta) \ll \min(H, 1/|\eta|)$. Then we obtain, using also the Prime Number Theorem, that

$$\begin{aligned}\sum_{n=X}^{X+H} |R_{\mathfrak{m}}(n)|^2 &= \int_{\mathfrak{m}} S(\xi)F_k(\xi)\int_{\mathfrak{m}} \overline{S(\alpha)F_k(\alpha)}K(\alpha-\xi)\,d\alpha d\xi\\ &\ll \sum_{j=1}^{H}\sum_{i=1}^{H}\int_{I_j\cap\mathfrak{m}}|S(\xi)F_k(\xi)|\int_{I_i\cap\mathfrak{m}}|S(\alpha)F_k(\alpha)|\frac{H}{1+|i-j|}\,d\alpha d\xi\\ &\ll H\log H\sum_{j=1}^{H}\left(\int_{I_j\cap\mathfrak{m}}|S(\alpha)|^2\,d\alpha\right)\left(\int_{I_j\cap\mathfrak{m}}|F_k(\alpha)|^2\,d\alpha\right)\\ &\ll HY(\log X)^2\left(\max_{j=1,\dots,H}\int_{I_j\cap\mathfrak{m}}|F_k(\alpha)|^2\,d\alpha\right). \qquad (18)\end{aligned}$$

Recalling $Q=4Y^{1-1/k}$ and letting $H=QP$, we remark that, for $1\le q\le P$, we have $1/qQ \ge 1/H$ and that, if $P<q\le Q$, we have $1/qQ\le 1/H$. We will prove

$$\begin{aligned}\left(\int_{-1/qQ}^{-P^4/qY}+\int_{P^4/qY}^{1/qQ}\right)\left|F_k\left(\frac{a}{q}+\eta\right)\right|^2 d\eta &\ll Y^{2/k-1}P^{-1/2} \quad \text{for } 1\le q\le P,\\ \int_{-2/H}^{2/H}\left|F_k\left(\frac{a}{q}+\eta\right)\right|^2 d\eta &\ll Y^{2/k-1}P^{-1/2K} \quad \text{for } P<q\le Q. \qquad (19)\end{aligned}$$

Let $1\le q\le P$. By Lemma 10 we have

$$\left(\int_{-1/qQ}^{-P^4/qY}+\int_{P^4/qY}^{1/qQ}\right)\left|F_k\left(\frac{a}{q}+\eta\right)\right|^2 d\eta \ll Y^{2/k-2}\int_{P^4/qY}^{1/qQ}\frac{1}{|\eta|^2}\,d\eta \ll \frac{Y^{2/k-1}}{P^3}$$

and the first part of (19) follows.

Let now $P < q \le Q$. Arguing as in eq. (41)–(43) of Perelli-Zaccagnini [11], we get

$$\int_{-2/H}^{2/H} \left| F_k\left(\frac{a}{q}+\eta\right)\right|^2 d\eta \ll H^{-2} \sum_{1\le d\le H/Y^{1-1/k}} \left|\sum\nolimits_d\right| + \frac{Y^{1/k}}{H}, \tag{20}$$

where

$$\sum\nolimits_d = \sum_{Y_1\le n\le Y_2} e\left(P(n,d)\frac{a}{q}\right) \max\left(0, \frac{H}{4} - \sum_{j=0}^{k-1}\binom{k}{j} n^j d^{k-j}\right),$$

$P(n,d) = \sum_{j=0}^{k-1}\binom{k}{j} n^j d^{k-j}$ and $Y_1 = (Y/8 + H/4)^{1/k}$, $Y_2 = (2Y - H/4)^{1/k}$. Hence by Abel's inequality we have

$$\sum\nolimits_d \ll H \max_{Y_1\le y\le Y_2} \left| \sum_{n\le y} e\left(P(n,d)\frac{a}{q}\right)\right|. \tag{21}$$

Applying Lemma 11 with $g = k-1$, $\alpha = ka/q$, $F(x,y) = P(x,y)$, $T = H/Y^{1-1/k}$, $R = y$ into (21), we have, since $ka/q = a'/q'$ for some $q' \gg q$ and $(a', q') = 1$, that

$$\sum_{1\le d\le H/Y^{1-1/k}} \left|\sum\nolimits_d\right| \ll H^2 Y^{2/k-1+\varepsilon} P^{-1/K}. \tag{22}$$

Now by (20), (22) and $H = QP$, the second part of (19) follows. Using (18) and (19) we finally get

$$\sum_{n=X}^{X+H} |R_{\mathfrak{m}}(n)|^2 \ll \frac{HY^{2/k}(\log X)^2}{P^{1/(2K)}} \ll \frac{HY^{2/k}}{P^{2/(5K)}}. \tag{23}$$

5. Major Arcs

Using what we have seen in section 2, we can write

$$R_{\mathfrak{M}}(n) = \sum_{q\le P} \frac{\mu(q)}{q\varphi(q)} H_k(q,n) \int_{\xi_q'} F_k(\eta) T(\eta) e(-n\eta)\, d\eta$$

$$+ \sum_{q\le P} \frac{1}{q} \sum_{a=1}^{q}{}^{*} V_k(a,q) e\left(-n\frac{a}{q}\right) \int_{\xi_q'} F_k(\eta) D(a,q,\eta) e(-n\eta)\, d\eta$$

$$+ \sum_{q\le P} \frac{1}{q} \sum_{a=1}^{q}{}^{*} V_k(a,q) e\left(-n\frac{a}{q}\right) \int_{\xi_q'} F_k(\eta) E(a,q,\eta) e(-n\eta)\, d\eta$$

$$+\sum_{q\le P}\sum_{a=1}^{q}{}^{*} e\Big(-n\frac{a}{q}\Big)\int_{\xi_q'}\Delta_k(a,q,\eta)S\Big(\frac{a}{q}+\eta\Big)e(-n\eta)\,d\eta$$

$$= S_1+S_2+S_3+S_4, \tag{24}$$

say.

Estimate of S_4. Using (7), the Cauchy-Schwarz estimate and the Prime Number Theorem, we have that

$$S_4 \ll \sum_{q\le P}\sum_{a=1}^{q}{}^{*} q\Big(\int_{\xi_q'}(1+|\eta|Y)^2\,d\eta\Big)^{1/2}\Big(\int_{\xi_q'}\Big|S\Big(\frac{a}{q}+\eta\Big)\Big|^2\,d\eta\Big)^{1/2}$$

$$\ll \frac{P^6}{Y^{1/2}}\sum_{q\le P}\frac{\varphi(q)}{q^{1/2}}\Big(\int_0^1|S(\eta)|^2\,d\eta\Big)^{1/2} \ll \frac{(Y\log X)^{1/2}P^{15/2}}{Y^{1/2}} \ll \Big(\frac{Y}{P}\Big)^{1/2}, \tag{25}$$

for $\delta>0$ sufficiently small.

Evaluation of S_1. First of all we remark that

$$S_1 = \sum_{q\le P}\frac{\mu(q)}{q\varphi(q)}H_k(q,n)\int_0^1 F_k(\eta)T(\eta)e(-n\eta)\,d\eta$$

$$+O\Big(\sum_{q\le P}\frac{\mu(q)^2}{q\varphi(q)}|H_k(q,n)|\Big(\int_{-1/2}^{-P^4/qY}+\int_{P^4/qY}^{1/2}\Big)|F_k(\eta)T(\eta)|\,d\eta\Big). \tag{26}$$

In the error term we estimate explicitely only the integral over $[P^4/qY,1/2]$. The other one can be estimated in a completely similar way. Applying Lemma 2, Hölder inequality with s as in Lemma 8, $T(\eta)\ll\min(Y,1/|\eta|)$ and Lemma 5, we get that the error term in (26) is

$$\ll \sum_{q\le P}\frac{(k-1)^{\omega(q)}}{\varphi(q)}\Big(\int_0^1|F_k(\eta)|^{2s}\,d\eta\Big)^{1/2s}\Big(\int_{P^4/qY}^{1/2}|\eta|^{-2s/(2s-1)}\,d\eta\Big)^{(2s-1)/2s}$$

$$\ll \frac{Y^{1/k-1/2s}Y^{1/2s}}{P^{3/2s}}\sum_{q\le P}\frac{(k-1)^{\omega(q)}}{\varphi(q)} \ll Y^{1/k}(\log P)^kP^{-3/2s} \ll Y^{1/k}P^{-1/s},$$

for $\delta>0$ sufficiently small. Hence (26) becomes

$$S_1 = L(Y,n)\sum_{q\le P}\frac{\mu(q)}{q\varphi(q)}H_k(q,n) + O(Y^{1/k}P^{-1/s})$$
$$= \mathfrak{S}_k(n,P)L(Y,n) + O(Y^{1/k}P^{-1/s}). \tag{27}$$

ESTIMATION OF S_2. Using (6), the Cauchy-Schwarz estimate and, for $\delta > 0$ sufficiently small, Lemma 9, we have that

$$S_2 \ll \sum_{q\le P}\frac{1}{q\varphi(q)}\sum_{\chi}|\tau(\bar\chi)H_k(\chi,q,n)|\left(\int_{\xi_q'}|F_k(\eta)|^2\,d\eta\right)^{1/2}\left(\int_{\xi_q'}|W(\chi,\eta)|^2\,d\eta\right)^{1/2}$$
$$\ll Y^{(2-k)/2k}\sum_{r\le P}\sum_{\substack{q\le P\\ r|q}}\frac{1}{q\varphi(q)}\sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ r)}|\tau(\overline{\chi_{0,q}\chi})H_k(\chi_{0,q}\chi,q,n)|\left(\int_{\xi_q'}|W(\chi_{0,q}\chi,\eta)|^2\,d\eta\right)^{1/2}. \tag{28}$$

We remark, for primitive characters χ, cond $\chi = r|q$, that $W(\chi\chi_{0,q},\eta) = W(\chi,\eta) + O((\log qX)^2)$. It is easy to see that such an error term is negligible. Using Lemma 6 and the explicit formula for $\psi(x,\chi)$, see (10), we get

$$S_2 \ll Y^{(2-k)/2k}\sum_{r\le P}\sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ r)}\left(\int_{\xi_r'}|W(\chi,\eta)|^2\,d\eta\right)^{1/2}\left(\sum_{\substack{q\le P\\ r|q}}\frac{|\tau(\overline{\chi_{0,q}\chi})H_k(\chi_{0,q}\chi,q,n)|}{q\varphi(q)}\right)$$
$$\ll Y^{(2-k)/2k}(\log Y)^k\sum_{r\le P}\sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ r)}\left(\int_{\xi_r'}|W(\chi,\eta)|^2\,d\eta\right)^{1/2}$$
$$\ll Y^{1/k}(\log Y)^k\left(\sum_{r\le P}\sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ r)}\max_{X-2Y\le x\le X+2Y}\ \max_{Y/4<h\le Y}\left(h+\frac{Y}{P^4}\right)^{-1}\left|\sideset{}{^\#}\sum_{l=x-h}^{x}\Lambda(l)\chi(l)\right|\right). \tag{29}$$

Choosing now $T = XY^{-1}P_1^7\log^2 X$ and using Lemma 15 with $Y = X^{7/12+\varepsilon}P_1^{10}$, we get, from (28)–(29), that

$$S_2 \ll G\frac{Y^{1/k}}{(\log Y)^{k+1}} + \frac{Y^{1/k}(\log Y)^k}{P^{1/3}}. \tag{30}$$

ESTIMATE OF S_3. First of all we remark that

$$S_3 = -\sum_{r \le P} \sum_{\substack{\chi \in \mathcal{E} \cup \mathcal{S} \\ \mathrm{cond}\, \chi = r|q}} \sum_{\substack{q \le P \\ r|q}} \frac{\tau(\overline{\chi_{0,q}\chi})}{q\varphi(q)} H_k(\chi_{0,q}\chi, q, n) \sum_{\rho \in \mathcal{E}' \cup \mathcal{S}'} \int_0^1 F_k(\eta) T_\rho(\eta) e(-n\eta)\, d\eta$$

$$+ O\left(\sum_{r \le P} \sum_{\substack{\chi \in \mathcal{E} \cup \mathcal{S} \\ \mathrm{cond}\, \chi = r|q}} \sum_{\substack{q \le P \\ r|q}} \frac{|\tau(\overline{\chi_{0,q}\chi})|}{q\varphi(q)} |H_k(\chi_{0,q}\chi, q, n)| \right.$$

$$\left. \times \sum_{\rho \in \mathcal{E}' \cup \mathcal{S}'} \left(\int_{-1/2}^{P^4/qY} + \int_{P^4/qY}^{1/2} \right) |F_k(\eta) T_\rho(\eta)|\, d\eta \right). \tag{31}$$

In the error term we estimate explicitely only the integral over $[P^4/qY, 1/2]$. The other one can be estimated in a completely similar way.

Now we split the interval of integration according to $P^4/qY \le \eta < 1/qQ$ and $1/qQ \le \eta \le 1/2$.

Using the Cauchy-Schwarz inequality, Lemma 10 and Parseval identity, the first integral is

$$\int_{P^4/qY}^{1/qQ} |F_k(\eta) T_\rho(\eta)|\, d\eta \le \left(\int_{P^4/qY}^{1/qQ} |F_k(\eta)|^2\, d\eta \right)^{1/2} \left(\int_0^1 |T_\rho(\eta)|^2\, d\eta \right)^{1/2}$$

$$\ll \left(\int_{P^4/qY}^{1/qQ} Y^{2/k-2} |\eta|^{-2}\, d\eta \right)^{1/2} (YX^{2\beta-2})^{1/2}$$

$$\ll P^{-3/2} Y^{1/k} X^{\beta-1}. \tag{32}$$

Using Hölder inequality with s as in Lemma 8, Lemma 8 and Lemma 12 (since $T = XY^{-1}P_1^7 \log^2 X \le X/qQ$ for $\delta > 0$ sufficiently small), the second integral is

$$\int_{1/qQ}^{1/2} |F_k(\eta) T_\rho(\eta)|\, d\eta \le \left(\int_0^1 |F_k(\eta)|^{2s}\, d\eta \right)^{1/2s} \left(\int_{1/qQ}^{1/2} |T_\rho(\eta)|^{2s/(2s-1)}\, d\eta \right)^{(2s-1)/2s}$$

$$\ll Y^{1/k-1/(2s)} X^{\beta-1} \left(\int_{1/qQ}^{1/2} |\eta|^{-2s/(2s-1)}\, d\eta \right)^{(2s-1)/2s}$$

$$\ll Y^{1/k-1/(2s)} X^{\beta-1} (qQ)^{1/(2s)} \ll P^{-1/(2s)} Y^{1/k} X^{\beta-1}. \tag{33}$$

Hence, using (32)–(33), Lemmas 6 and 16, the error term in (31) is

$$\ll P^{-1/(2s)} Y^{1/k} X^{-1} \sum_{r \le P} (\log P)^k \sum_{\chi \in \mathscr{E} \cup \mathscr{S}} \sum_{\rho \in \mathscr{E}' \cup \mathscr{S}'} X^{\beta}$$

$$\ll P^{-1/(3s)} Y^{1/k} X^{-1} \left(\sum_{r \le P} \sum_{\chi \in \mathscr{E}} \sum_{\rho \in \mathscr{E}'} X^{\beta} + X^{\tilde{\beta}} \right) \ll Y^{1/k} P^{-1/(3s)}, \tag{34}$$

since $\tilde{\beta} < 1$.

Using Lemmas 1 and 3, the first term in (31) becomes

$$-\sum_{r \le P} \sum_{\substack{\chi \in \mathscr{E} \cup \mathscr{S} \\ \chi \ (\mathrm{mod}\ r)}} \frac{\tau(\bar{\chi}) H_k(\chi, r, n)}{r\varphi(r)} \sum_{j \le P/r} \frac{\mu(j)\bar{\chi}(j)}{j\varphi(j)} \chi(j)\chi_{0,j}(r) H_k(\chi_{0,j}, j, n) \sum_{\rho \in \mathscr{E}' \cup \mathscr{S}'} L_\rho(Y, n)$$

$$= -\sum_{r \le P} \sum_{\substack{\chi \in \mathscr{E} \cup \mathscr{S} \\ \chi \ (\mathrm{mod}\ r)}} \frac{\tau(\bar{\chi}) H_k(\chi, r, n)}{r\varphi(r)} \sum_{\substack{j \le P/r \\ (j,r)=1}} \frac{\mu(j)}{j\varphi(j)} H_k(j, n) \sum_{\rho \in \mathscr{E}' \cup \mathscr{S}'} L_\rho(Y, n)$$

$$= -\sum_{r \le P} \sum_{\substack{\chi \in \mathscr{E} \cup \mathscr{S} \\ \chi \ (\mathrm{mod}\ r)}} T_k(\chi, r, n) \mathfrak{S}_k\left(n, \frac{P}{r}, r\right) \sum_{\rho \in \mathscr{E}' \cup \mathscr{S}'} L_\rho(Y, n). \tag{35}$$

Hence, by (31)–(35), we finally get

$$S_3 = -\sum_{r \le P} \sum_{\substack{\chi \in \mathscr{E} \cup \mathscr{S} \\ \chi \ (\mathrm{mod}\ r)}} T_k(\chi, r, n) \mathfrak{S}_k\left(n, \frac{P}{r}, r\right) \sum_{\rho \in \mathscr{E}' \cup \mathscr{S}'} L_\rho(Y, n) + O(Y^{1/k} P^{-1/(3s)}). \tag{36}$$

Hence, from (24), (25), (27), (30) and (36) we get

$$R_{\mathfrak{M}}(n) = \mathfrak{S}_k(n, P) L(Y, n) - \sum_{r \le P} \sum_{\substack{\chi \in \mathscr{E} \cup \mathscr{S} \\ \chi \ (\mathrm{mod}\ r)}} T_k(\chi, r, n) \mathfrak{S}_k\left(n, \frac{P}{r}, r\right) \sum_{\rho \in \mathscr{E}' \cup \mathscr{S}'} L_\rho(Y, n)$$

$$+ O\left(Y^{1/k} P^{-1/(3s)} + G \frac{Y^{1/k}}{\log^{k+1} Y} \right). \tag{37}$$

6. The Singular Series

Here we follow closely the approach of Zaccagnini [15] and Perelli-Zaccagnini [11].

As in section 12 of Zaccagnini [15], let $P^* = P^{4/5}$ and write

$$\mathscr{F} = \{1\} \cup \{r \le P^*, r \text{ is an excluded or Siegel modulus}\}.$$

In the following we will call the $r \le P$ as *small moduli* and the $P^* < r \le P$ *large moduli*.

Large moduli. The contribution of the large moduli, *i.e.* $P^* < r \le P$, can be performed as in section 13 of [15]. We just sketch the main differences.

By eq. (13.1) of [15] we have

$$T_k(\chi, r, n) = \frac{\tau(\bar{\chi})\tau(\chi)}{r\varphi(r)}\sigma(r, \bar{\chi}, n), \quad \text{where } \sigma(r, \chi, n) = \sum_{h \pmod r} \chi(f(h)) \tag{38}$$

and $f(h) = h^k - n$. Now we need the following lemma.

Lemma 19. *Let* $\chi \pmod r$ *be a primitive character. Then for all but* $Hr^{-3/8}$ *integers* $n \in [X, X+H]$, *we have*

$$\sigma(r, \chi, n) \ll r^{1-1/(7(k-1))}$$

uniformly for $r \le X/100$.

In the proof we have to study

$$A(X, H, r) = |\{n \in [X, X+H] : (r, n) \ge r^{1/2}\}| \quad \text{and}$$
$$B(r) = |\{n \pmod r) : (r, n) \ge r^{1/2}\}|.$$

Since it is clear that $A(X, H, r) \ll (H/r + 1)B(r)$ and $B(r) \ll d(r)r^{1/2}$, we get

$$A(X, H, r) \ll \frac{H}{r^{3/8}}$$

which is the analogue of eq. (13.8) of [15]. The rest of the proof is the same of Lemma 13.1 of [15].

Using Lemma 19 we get that for all but $H(P^*)^{-1/4}$ integers $n \in [X, X+H]$, we have that

$$\sigma(r, \bar{\chi}, n) \ll r^{1-1/(7(k-1))} \tag{39}$$

holds for all excluded or Siegel moduli $r \in (P^*, P]$.

From Lemmas 2 and 5, we have, letting $R = P/r$, that

$$\mathfrak{S}_k(n, R, r) \ll \sum_{\substack{q \le R \\ (q, r) = 1}} \frac{\mu^2(q)}{\varphi(q)} \prod_{p|q} |\rho_k(p, n) - 1| \ll (\log R)^k. \tag{40}$$

Hence we get, by (38)–(40) and Lemmas 18 and 16, that

$$\sum_{P^*<r\le P}\sum_{\substack{\chi\in\mathscr{E}\cup\mathscr{S}\\ \chi\ (\mathrm{mod}\ r)}} T_k(\chi,r,n)\mathfrak{S}_k\left(n,\frac{P}{r},r\right)\sum_{\rho\in\mathscr{E}'\cup\mathscr{S}'} L_\rho(Y,n)$$

$$\ll Y^{1/k}X^{-1}(\log P)^{k+1}(P^*)^{-1/(7(k-1))}\left(\sum_{P^*<r\le P}\sum_{\substack{\chi\in\mathscr{E}\\ \chi\ (\mathrm{mod}\ r)}}\sum_{\rho\in\mathscr{E}'} X^\beta + X^{\tilde\beta}\right)$$

$$\ll Y^{1/k}P^{-1/(10(k-1))} \tag{41}$$

holds for all but $\ll HP^{-1/5}$ integers $n\in[X,X+H]$.

SMALL MODULI. The contribution of the small moduli follows the line of the Corollary in section 3 of Perelli-Zaccagnini [11]. We just sketch the main differences. Let $R=P/r$ and

$$A(n,q,r)=\frac{\mu(q)}{\varphi(q)}\mu((q,r)^2)\prod_{p|q}(\rho_k(p,n)-1).$$

Then

$$A(n,p,r)=-\frac{1}{p-1}\mu((p,r)^2)\sum_{\substack{\chi\ (\mathrm{mod}\ p)\\ \chi\ne\chi_0\\ \chi^k=\chi_0}}\chi(n)=\sum_{\chi\in\mathscr{A}(p)}c(\chi)\chi(n),$$

say, where $|\mathscr{A}(p)|\le k-1$ and $|c(\chi)|\le\varphi(p)^{-1}$. We approximate

$$\mathfrak{S}_k(n,R,r)=\sum_{\substack{q\le R\\ (q,r)=1}}\frac{\mu(q)}{\varphi(q)}\prod_{p|q}(\rho_k(p,n)-1)$$

by

$$\Pi(n,R',r)=\prod_{p\le R'}(1+A(n,p,r)),$$

say, where $R'=R^{1/2}$. Let $\mathscr{D}(R')=\{q\in N\backslash\{0\}:\mu(q)\ne 0, p|q\Rightarrow p\le R'\}$. Hence

$$\mathfrak{S}_k(n,R,r)-\Pi(n,R',r)=\sum_{\substack{R'<q\le R\\ q\notin\mathscr{D}(R')}}A(n,q,r)+\sum_{\substack{q>R\\ q\in\mathscr{D}(R')}}A(n,q,r)=\sum\nolimits_1+\sum\nolimits_2, \tag{42}$$

say.

To estimate $\sum_2$ we argue as in equations (12.10)–(12.12) of [15]; we just choose differently the parameter λ there. We choose $\lambda=(\log P)^{-1/2}$ to obtain

$$\sum\nolimits_2\ll\exp(-c(\log P)^{1/2}). \tag{43}$$

To estimate the mean square of $\sum_1$ we argue as in eq. (53) of [11]. We finally get

$$\sum_{X\le n\le X+H}\left|\sum\nolimits_1\right|^2 \ll \left(\frac{H}{R'}+R\right)(\log X)^c \ll \frac{H}{P^{1/20}}$$

and hence, for all but $\ll H^{1-\delta/40}$ integers in $[X, X+H]$, we have

$$\sum\nolimits_1 \ll P^{-1/20}. \tag{44}$$

Using (42)–(44) we have, for all but $H^{1-\delta/40}$ integers in $[X, X+H]$ and all $r \in \mathscr{F}$, that

$$\begin{aligned}\mathfrak{S}_k(n,R,r) &= \prod_{p\le R'}(1+A(n,p,r)) + O(\exp(-c(\log P)^{1/2}))\\ &= \prod_{p\le R'}\left(\frac{p-\rho_k(p,n)}{p-1}\right)\prod_{\substack{p\le R'\\ p\mid r}}\left(\frac{p-1}{p-\rho_k(p,n)}\right) + \exp(-c(\log P)^{1/2}).\end{aligned} \tag{45}$$

Before ending this section we state other two lemmas on the singular series that we will use to finish the proof of the Theorem.

Lemma 20. *For all but $\ll H^{1-\delta/40}$ integers $n \in [X, X+H]$ and all $r \in \mathscr{F}\backslash\{1\}$, we have*

$$\left|T_k(\chi,r,n)\mathfrak{S}_k\left(n,\frac{P}{r},r\right)\right| \le c\prod_{p\le P}\left(\frac{p-\rho_k(p,n)}{p-1}\right) + O(\exp(-c(\log P)^{1/2})).$$

The proof of Lemma 20 is essentially the same of Lemma 14.1 of [15].

Lemma 21 (Lemma 14.2 of [15]).

$$\prod_{p\le P}\left(\frac{p-\rho_k(p,n)}{p-1}\right) \gg (\log P)^{-k}.$$

7. Proof the Theorem

Now we are ready to finish the proof.

From (23) we get

$$|R_{\mathfrak{m}}(n)| \ll Y^{1/k}P^{-1/(10K)} \tag{46}$$

for all but $\ll HP^{-1/(5K)}$ integers $n \in [X, X+H]$. Let $C(X,H)$ the union of all the exceptional set encountered in (41), (45), Lemma 20 and (46). It is clear that

$$|C(X,H)| \ll HP^{-1/(5K)}.$$

Now, from (37) and (41), we have, for every $n \in [X, X+H] \backslash C(X,H)$, that

$$R_{\mathfrak{M}}(n) = \mathfrak{S}_k(n,P)L(Y,n) - T_k(\tilde{\chi},\tilde{r},n)\mathfrak{S}_k(n,P/\tilde{r},\tilde{r})L_{\tilde{\beta}}(Y,n)$$
$$- \sum_{r \le P^*} \sum_{\substack{\chi \in \mathscr{E} \\ \chi \pmod r}} T_k(\chi,r,n)\mathfrak{S}_k(n,P/r,r) \sum_{\rho \in \mathscr{E}'} L_\rho(Y,n)$$
$$+ O\left(Y^{1/k}P^{-1/(3s)} + G\frac{Y^{1/k}}{\log^{k+1} Y}\right). \tag{47}$$

Now by (45), Lemmas 20 and 21 we obtain that

$$\mathfrak{S}_k(n,P) \ge \frac{1}{2} \prod_{p \le P} \frac{p - \rho_k(p,n)}{p-1} \tag{48}$$

and

$$\left| T_k(\chi,r,n)\mathfrak{S}_k\left(n,\frac{P}{r},r\right)\right| \le 2c \prod_{p \le P}\left(\frac{p-\rho_k(p,n)}{p-1}\right), \tag{49}$$

for $r \in \mathscr{F}\backslash\{1\}$, if P is sufficiently large.

Now, by Lemma 17 and (47)–(49), we have

$$R_{\mathfrak{M}}(n) \ge \frac{1}{2}\prod_{p \le P}\frac{p-\rho_k(p,n)}{p-1}\left(c'GY^{1/k} - 4c\sum_{r \le P^*}\sum_{\substack{\chi \in \mathscr{E} \\ \chi \pmod r}}\sum_{\rho \in \mathscr{E}'}|L_\rho(Y,n)|\right)$$
$$+ O\left(G\frac{Y^{1/k}}{\log^{k+1} Y} + Y^{1/k}P^{-1/(3s)}\right). \tag{50}$$

By Lemmas 18 and 16 we get

$$\sum_{r \le P^*}\sum_{\substack{\chi \in \mathscr{E} \\ \chi \pmod r}}\sum_{\rho \in \mathscr{E}'}|L_\rho(Y,n)| \le c''Y^{1/k}\sum_{r \le P^*}\sum_{\substack{\chi \in \mathscr{E} \\ \chi \pmod r}}\sum_{\rho \in \mathscr{E}'} X^{\beta-1} \le c(\delta)GY^{1/k}, \tag{51}$$

where $c(\delta)$ can be chosen arbitrarily small.

Recalling Lemma 13′ and the definition of G we obtain

$$\frac{G}{\log P} = 1 - \tilde{\beta} \ge \frac{c'''}{P^{\nu/2}\log X}.$$

Letting $\nu = \nu(k) = 1/(3s)$, from (47)-(51) and Lemma 21, we finally get

$$R_{\mathfrak{M}}(n) \gg \frac{GY^{1/k}}{(\log Y)^k} \tag{52}$$

for every $n \in [X, X+H] \setminus C(X, H)$.

Now, from (3), (46) and (52), we have that

$$R(n) \gg GY^{1/k}(\log Y)^{-k},$$

for every $n \in [X, X+H]$ with at most $O(H^{1-\delta/(5K)})$ exceptions. The Theorem follows.

References

[1] R. Brünner, A. Perelli, J. Pintz, The exceptional set for the sum of a prime and a square, Acta Math. Hungar. **53** (1989), 347–365.

[2] H. Davenport, Multiplicative Number Theory, 3-rd ed., Springer GTM, 2001.

[3] P. X. Gallagher, A large sieve density estimate near $\sigma = 1$, Invent. Math. **11** (1970), 329–339.

[4] G. H. Hardy, J. E. Littlewood, Some problems of "Partitio Numerorum"; III. On the expression of a number as a sum of primes, Acta Math. **44** (1923), 1–70.

[5] M. Jutila, On Linnik's constant, Math. Scand. **41** (1977), 45–62.

[6] H. Mikawa, On the sum of a prime and a square, Tsukuba J. Math. **17** (1993), 299–310.

[7] H. L. Montgomery, R. C. Vaughan, The exceptional set in Goldbach's problem, Acta Arith. **27** (1975), 353–370.

[8] T. P. Peneva, On the exceptional set for Goldbach's problem in short intervals, Monatsh. Math. **132** (2001), 49–65. Corrigendum to: On the exceptional set for Goldbach's problem in short intervals, Monatsh. Math. **141** (2004), 209–217.

[9] A. Perelli, J. Pintz, On the exceptional set for Goldbach's problem in short intervals, J. London Math. Soc. (2) **47** (1993), 41–49.

[10] A. Perelli, J. Pintz, Hardy-Littlewood numbers in short intervals, J. Number Theory **54** (1995), 297–308.

[11] A. Perelli, A. Zaccagnini, On the sum of a prime and a k-power, Izv. Vu. **59** (1995), 181–204.

[12] K. Prachar, Primzahlverteilung, Springer 1978.

[13] K. Ramachandra, On the number of Goldbach numbers in small intervals, J. Indian Math. Soc. **37** (1973), 157–170.

[14] A. I. Vinogradov, On a binary problem of Hardy and Littlewood, Acta Arith. **46** (1985), 33–56, (Russian).

[15] A. Zaccagnini, On the exceptional set for the sum of a prime and a k-power, Mathematika **39** (1992), 400–421.

Università di Padova
Dipartimento di Matematica Pura e Applicata
Via Belzoni 7
35131 Padova, Italy
e-mail: languasco@math.unipd.it