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Seiberg–Witten duality in Dijkgraaf–Vafa theory

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Abstract

We show that a suitable rescaling of the matrix model coupling constant makes manifest the duality group of the $N = 2$ SYM theory with gauge group $SU(2)$. This is done by first identifying the possible modifications of the SYM moduli preserving the monodromy group. Then we show that in matrix models there is a simple rescaling of the pair (S_D, S) which makes them dual variables with $\Gamma(2)$ monodromy. We then show that, thanks to a crucial scaling property of the free energy derived perturbatively by Dijkgraaf, Gukov, Kazakov and Vafa, this redefinition corresponds to a rescaling of the free energy which in turn fixes the rescaling of the coupling constant. Next, we show that in terms of the rescaled free energy one obtains a nonperturbative relation which is the matrix model counterpart of the relation between the u -modulus and the prepotential of $N = 2$ SYM. This suggests considering a dual formulation of the matrix model in which the expansion of the prepotential in the strong coupling region, whose QFT derivation is still unknown, should follow from perturbation theory. The investigation concerns the $SU(2)$ gauge group and can be generalized to higher rank groups.

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Recently Dijkgraaf and Vafa derived crucial relations between matrix models and SYM theories [1–3]. Subsequently, in [4] Dijkgraaf, Gukov, Kazakov and Vafa provided the explicit relationship between the $N = 2$ SYM theory [5] and matrix models. The original proposal was based on geometrical engineering analysis in string theory, while in [6,7] it has been argued that there exists a QFT proof of the Dijkgraaf–Vafa formulation. In these derivations a crucial role is played by holomorphy. This is a crucial issue as, for example, holomorphy and symmetries are at the basis of $N = 2$ SYM duality. Therefore, a basic question in considering the matrix model formulation, is to identify the duality

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structure which is the essence of Seiberg–Witten theory [5]. There are several reasons which suggest introducing the powerful tool of duality directly in the matrix model formulation. For example, an interesting question would be to understand the analogous of the nonperturbative relation between the u -modulus and the prepotential [8]. This should be useful for a proof of the relationship between matrix model and $N = 2$ SYM along the lines of [9]. We also note that this relation, which has been useful in investigating related issues [10], should help in deriving possible exact results in matrix models. Furthermore, this duality may help in understanding what is the QFT formulation of $N = 2$ SYM in the strong coupling region. In this context, one should expect that the expansion of the $N = 2$ SYM prepotential in the strong coupling region should be obtained by means of a perturbative calculation in a dual matrix model formulation.

The aim of this paper is to introduce such a duality in matrix models. We will start by showing that, on general grounds, in order to preserve the Seiberg–Witten duality, only a class of redefinitions of the moduli (a_D, a) is allowed. This is based on a mathematical general observation which involves the Picard–Fuchs equation.¹ In particular, it is shown that if

$$\tau(a) = \frac{\partial \mathcal{S}_D}{\partial \mathcal{S}} = \frac{\partial a_D}{\partial a},$$

with τ the $N = 2$ effective coupling constant, then $(\mathcal{S}_D, \mathcal{S})$ have the same monodromy of (a_D, a) on the u -plane if

$$\mathcal{S}_D = f a_D + 4(u^2 - \Lambda_{\text{SW}}^4) f' a'_D, \quad \mathcal{S} = f a + 4(u^2 - \Lambda_{\text{SW}}^4) f' a', \quad (1)$$

with f an arbitrary singlevalued function of u (note that a possible additional Z_2 monodromy leaves τ invariant).

Next, we identify the explicit relationship between the matrix model variables $(\mathcal{S}_D, \mathcal{S})$ and (a_D, a) . It turns out that $(\mathcal{S}_D, \mathcal{S})$ cannot have $\Gamma(2)$ -monodromy. Nevertheless, remarkably, the simple rescaling

$$\mathcal{S} = \left(\frac{\Lambda_{\text{SW}}}{2^{3/2} u^{1/2}} \right)^3 \mathcal{S},$$

restores duality, that is

$$\mathcal{S} = \frac{\Lambda_{\text{SW}}^3}{3 \cdot 2^6} [u^{-1/2} a - 2(u^2 - \Lambda_{\text{SW}}^4) u^{-3/2} a'],$$

satisfies (1) with $f = \frac{\Lambda_{\text{SW}}^3}{3 \cdot 2^6} u^{-1/2}$. On the other hand, this fixes \mathcal{S}^D to be

$$\mathcal{S}_D = \frac{\Lambda_{\text{SW}}^3}{3 \cdot 2^6} [u^{-1/2} a_D - 2(u^2 - \Lambda_{\text{SW}}^4) u^{-3/2} a'_D],$$

which, in turn, defines \mathcal{F}_0 by

$$\mathcal{S}_D = \frac{\partial \mathcal{F}_0}{\partial \mathcal{S}}.$$

¹ A suitable generalization of the method introduced here, suggests a possible application in investigating the Picard–Fuchs equations in the framework of the Mirror conjecture.

It then follows that the new pair has $\Gamma(2)$ monodromy

$$\begin{pmatrix} \tilde{S}_D \\ \tilde{S} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} S_D \\ S \end{pmatrix}. \quad (2)$$

We then show that thanks to a remarkable scaling property of the genus zero free energy, passing to the new variables is equivalent to a simple rescaling, that is the change of variables ($\Lambda = 2^{-1/2} \Lambda_{\text{SW}}$)

$$S_k \longrightarrow \mathcal{S}_k = \left(\frac{\Lambda}{\Delta}\right)^3 S_k, \quad \Delta \longrightarrow \frac{\Lambda}{\Delta} \Delta = \Lambda, \quad \Lambda \longrightarrow \frac{\Lambda}{\Delta} \Lambda = \frac{\Lambda^2}{\Delta},$$

induces the scaling transformation

$$\mathcal{F}_0 \longrightarrow \mathcal{F}_0\left(\mathcal{S}_k, \Lambda, \frac{\Lambda}{\Delta}\right) = \left(\frac{\Lambda}{\Delta}\right)^6 \mathcal{F}_0,$$

which has no effect on the SYM coupling constant τ as it cancels with the Jacobian of $\partial^2/\partial S_1 \partial S_2$. As a result, even if the partition function remains invariant, we have the same rescaling for both the potential and the matrix coupling constant

$$g_S \longrightarrow g_{\mathcal{S}} = \left(\frac{\Lambda}{\Delta}\right)^3 g_S,$$

$$W \longrightarrow \mathcal{W}(\Phi) = \left(\frac{\Lambda}{\Delta}\right)^3 W(\Phi).$$

As a consequence

$$\tilde{\mathcal{F}}_g = \left(\frac{\Lambda}{\Delta}\right)^{3(2-2g)} \mathcal{F}_g = \mathcal{F}_g\left(\mathcal{S}_k, \Lambda, \frac{\Lambda^2}{\Delta}\right).$$

We then show that the new prepotential satisfies the nonperturbative relation

$$\left(\frac{\Lambda}{\Delta}\right)^4 = \frac{48\pi i}{\Lambda^6} \left(\mathcal{F}_0 - \frac{\mathcal{S}}{2} \frac{\partial \mathcal{F}_0}{\partial \mathcal{S}}\right).$$

Introducing duality then leads to consider a dual formulation of the matrix model that we propose should correspond to introduce the Legendre transform of the free energy

$$\mathcal{F}_{Dg} = \mathcal{F}_g - \sum_{i=1,2} \mathcal{S}_i \frac{\partial \mathcal{F}_g}{\partial \mathcal{S}_i},$$

where now $\mathcal{F}_g \equiv \mathcal{F}_g(\mathcal{S}_k, \Lambda, \frac{\Lambda^2}{\Delta})$.

Let us start by recalling that in matrix model the effective coupling constant of $N = 2$ SYM has the form [4]

$$\tau(a) = \frac{\partial^2 \mathcal{F}_0(S)}{\partial S^2}, \quad (3)$$

which should be compared with

$$\tau(a) = \frac{\partial^2 \mathcal{F}(a)}{\partial a^2}. \quad (4)$$

The problem is to find the relationship between $\mathcal{F}_0(S)$ and $\mathcal{F}(a)$. Let us introduce the dual

$$S_D = \frac{\partial \mathcal{F}_0(S)}{\partial S}, \quad (5)$$

so that

$$\tau(a) = \frac{\partial_u S_D}{\partial_u S} = \frac{\partial_u a_D}{\partial_u a}. \quad (6)$$

It is clear that the dual pairs (S_D, S) and (a_D, a) should have the same monodromy on the u -plane. As observed in [11] in considering a similar problem, we may use the differential equation [8,12]

$$\left(\partial_u^2 + \frac{1}{4(u^2 - \Lambda_{\text{SW}}^4)} \right) \begin{pmatrix} a_D \\ a \end{pmatrix} = 0, \quad (7)$$

to investigate the structure of the possible solutions of (6). Generalizing the analysis in [11] we set² ($' \equiv \partial_u$)

$$S_D = f_D a_D + g_D a'_D, \quad S = f a + g a', \quad (8)$$

where the two dual pairs (f_D, f) and (g_D, g) are functions of u . Note that if these functions are singlevalued with respect to u , then (S_D, S) would have the $\Gamma(2)$ monodromy of (a_D, a) . However, since a possible additional Z_2 monodromy of (S_D, S) with respect to (a_D, a) does not change the polymorphicity of $\mathcal{S}'_D/\mathcal{S}'$, the functions (f_D, f) and (g_D, g) should be singlevalued on the u -space except for a possible minus sign they may get winding around some point.

We now show that if the functions (f_D, f) and (g_D, g) solve a differential equation, then Eq. (6) is satisfied. By (7) and (8) we have

$$S'_D = \tilde{f}_D a_D + \tilde{g}_D a'_D, \quad S' = \tilde{f} a + \tilde{g} a',$$

where

$$\begin{aligned} \tilde{f}_D &= f'_D - \frac{1}{4(u^2 - \Lambda_{\text{SW}}^4)} g_D, & \tilde{g}_D &= f_D + g'_D, \\ \tilde{f} &= f' - \frac{1}{4(u^2 - \Lambda_{\text{SW}}^4)} g, & \tilde{g} &= f + g'. \end{aligned}$$

Imposing $\tilde{f}_D = 0 = \tilde{f}$

$$g_D = 4(u^2 - \Lambda_{\text{SW}}^4) f'_D, \quad g = 4(u^2 - \Lambda_{\text{SW}}^4) f', \quad (9)$$

and $\tilde{g}_D = \tilde{g}$, that is

$$\tilde{g}_D = f_D + 8u f'_D + 4(u^2 - \Lambda_{\text{SW}}^4) f''_D = f + 8u f' + 4(u^2 - \Lambda_{\text{SW}}^4) f'' = \tilde{g}, \quad (10)$$

² We are using the notation (S_D, S) rather than (S_D, S) since, as we will see, the pair (S_D, S) defined in matrix model has not $\Gamma(2)$ -monodromy.

we obtain

$$S'_D = ha'_D, \quad S' = ha',$$

where $h \equiv \tilde{g}_D = \tilde{g}$. Since f_D and f satisfy the same differential equation (10), it follows that once either f_D or f is given, say f , besides the choice $f_D = f$ (which would imply $g_D = g$), one can also choose f_D to be any other solution of (10). Summarizing, from (8) and (9) we have

$$S_D = f_D a_D + 4(u^2 - \Lambda_{\text{SW}}^4) f'_D a'_D, \quad S = f a + 4(u^2 - \Lambda_{\text{SW}}^4) f' a', \quad (11)$$

and $S'_D/S' = a'_D/a' = \tau$.

We now start considering the relationship between the $N = 2$ SYM and matrix model variables. We first set

$$\Lambda = 2^{-1/2} \Lambda_{\text{SW}}, \quad \Delta^2 = 4u,$$

in the loop expansion of S [4]

$$\frac{S}{2^3 u^{3/2}} = \frac{1}{2^6} \left(\frac{\Lambda_{\text{SW}}^2}{u} \right)^2 + \frac{3}{2^{11}} \left(\frac{\Lambda_{\text{SW}}^2}{u} \right)^4 + \frac{35}{2^{16}} \left(\frac{\Lambda_{\text{SW}}^2}{u} \right)^6 + \dots \quad (12)$$

We now show that rather than S itself, it is the right-hand side of (12) that matches with the expansion of \mathcal{S} in (11) with

$$f = \frac{1}{\sqrt{2} \cdot 48} u^{-1/2}. \quad (13)$$

Therefore, while $\frac{S}{2^3 u^{3/2}}$ is of the form that preserves duality, this is not the case for S itself. As we will see, this will lead to a natural rescaling of the coupling constant of the matrix model which will make the Seiberg–Witten duality manifest. In particular, we will see that one has to rescale S to

$$\left(\frac{\Lambda_{\text{SW}}}{2^{3/2} u^{1/2}} \right)^3 S = \frac{\Lambda_{\text{SW}}^3 u^{-3/2}}{3 \cdot 2^6} [ua - 2(u^2 - \Lambda_{\text{SW}}^4) a']. \quad (14)$$

In order to compare (12) and (14) we expand a for $u \rightarrow \infty$

$$\begin{aligned} a(u) &= \frac{\sqrt{2}}{\pi} \int_{-\Lambda_{\text{SW}}^2}^{\Lambda_{\text{SW}}^2} dx \frac{\sqrt{x-u}}{\sqrt{x^2 - \Lambda_{\text{SW}}^4}} \\ &= \sqrt{2} u \left(1 - \frac{1}{2^4} \left(\frac{\Lambda_{\text{SW}}^2}{u} \right)^2 - \frac{15}{2^{10}} \left(\frac{\Lambda_{\text{SW}}^2}{u} \right)^4 - \frac{105}{2^{14}} \left(\frac{\Lambda_{\text{SW}}^2}{u} \right)^6 + \dots \right), \end{aligned}$$

that substituted in (14) exactly reproduces (12). Substituting (14) in (6) and using

$$S' = \frac{1}{\sqrt{2} \cdot 4} (a - 2ua'), \quad (15)$$

we see that the relation between (S'_D, S') and (a'_D, a') is rather involved

$$S'_D = \frac{1}{\sqrt{2} \cdot 4} (a - 2ua') \frac{a'_D}{a'}.$$

This is not only a formal question since S'_D and S' cannot have simultaneously $\Gamma(2)$ monodromy. Even if this is implicit in the above construction, it is instructive to illustrate it explicitly. In particular, if S has $\Gamma(2)$ monodromy, this cannot be the case for S_D . Since the monodromy commutes with the derivative, we show this for S'_D and S' . Under the action of $\Gamma(2)$ we have

$$S' \longrightarrow \gamma(S') = \frac{1}{\sqrt{2} \cdot 4} C(a_D - 2ua'_D) + \frac{1}{\sqrt{2} \cdot 4} D(a - 2ua'),$$

so S' has $\Gamma(2)$ monodromy iff we consider as its dual

$$\widehat{S}'_D = \frac{1}{\sqrt{2} \cdot 4} (a_D - 2ua'_D) \neq S'_D,$$

so that

$$\gamma(S') = C\widehat{S}'_D + DS'.$$

Of course, as follows by the previous analysis, even if \widehat{S}'_D and S' have $\Gamma(2)$ monodromy, their ratio cannot correspond to τ .

A similar reasoning holds for S'_D . Actually, since under $\Gamma(2)$

$$\frac{A\tau + B}{C\tau + D} = \frac{AS'_D + BS'}{CS'_D + DS'} = \frac{\gamma(S'_D)}{\gamma(S')},$$

we see that

$$\gamma(S'_D) = \frac{(AS'_D + BS')(C\widehat{S}'_D + DS')}{CS'_D + DS'},$$

which cannot correspond to the $\Gamma(2)$ monodromy, that is

$$\gamma(S'_D) \neq AS'_D + BS'.$$

Note that

$$\begin{aligned} S_D &= \frac{1}{\sqrt{2} \cdot 2^2} \int_{u_0}^u d\tilde{u} (\tau a - 2\tilde{u} \partial_{\tilde{u}} a_D) + S_D(u_0), \\ S &= \frac{1}{\sqrt{2} \cdot 6} (ua - 2(u^2 - \Lambda_{\text{SW}}^4) a'), \end{aligned} \tag{16}$$

and by (14) and (15)

$$a = \frac{\sqrt{2} \cdot 2}{\Lambda_{\text{SW}}^4} [3uS - 2(u^2 - \Lambda_{\text{SW}}^4) S'].$$

The fact that the Seiberg–Witten duality is not manifest with the pair (S_D, S) can be also seen by noticing that S solves the differential equation

$$\left(\partial_u^2 - \frac{3}{4(u^2 - \Lambda_{\text{SW}}^4)}\right)S = 0, \quad (17)$$

which is not satisfied by S_D , indicating once again that they cannot have the same monodromy on the u -plane. Inverting Eq. (17) we obtain

$$4(\mathcal{G}^2 - \Lambda_{\text{SW}}^4) \frac{\partial^2 \mathcal{G}}{\partial S^2} + 3S \left(\frac{\partial \mathcal{G}}{\partial S}\right)^3 = 0, \quad (18)$$

where

$$u = \mathcal{G}(S).$$

To select a dual pair with $\Gamma(2)$ monodromy and whose ratio corresponds to τ is essential to recognize the underlying geometry of $N = 2$ SYM. In particular, winding around the u -moduli space, the pair (S_D, S) will not preserve the analogous relations satisfied by (a_D, a) . In order to restore manifest duality we rescale S and define

$$\bar{S} = \left(\frac{\Lambda_{\text{SW}}}{2^{3/2}u^{1/2}}\right)^3 S, \quad (19)$$

that is

$$\bar{S} = \frac{\Lambda_{\text{SW}}^3}{3 \cdot 2^6} [u^{-1/2}a - 2(u^2 - \Lambda_{\text{SW}}^4)u^{-3/2}a'], \quad (20)$$

where the term Λ_{SW}^3 has been introduced to make S and \bar{S} of the same dimension. We now choose $f_D = f$, so that

$$S_D = \frac{\Lambda_{\text{SW}}^3}{3 \cdot 2^6} [u^{-1/2}a_D - 2(u^2 - \Lambda_{\text{SW}}^4)u^{-3/2}a'_D], \quad (21)$$

which, in turn, defines \mathcal{F}_0 by

$$S_D = \frac{\partial \mathcal{F}_0}{\partial \bar{S}}.$$

By construction the pair (S_D, \bar{S}) has the same monodromy of (a_D, a) on the u -plane except for a minus sign they get winding around $u = 0$, as observed this does not change the polymorphicity properties of τ .

We can now use the method introduced in [8] to derive the exact relation between the prepotential and the modular invariant. In this case, by means of (S_D, \bar{S}) we may construct the modular invariant

$$v = \frac{2^{13} \cdot 3\pi i}{\Lambda_{\text{SW}}^6} \left(\mathcal{F}_0 - \frac{\bar{S}}{2} \frac{\partial \mathcal{F}_0}{\partial \bar{S}} \right), \quad (22)$$

which implies that the pair (S_D, \bar{S}) satisfies the differential equation

$$\left(\partial_v^2 + \frac{1}{2}\{\sigma, v\}\right) \begin{pmatrix} S_D \\ \bar{S} \end{pmatrix} = 0, \quad (23)$$

where $\{g(x), x\}$ denotes the Schwarzian derivative $g'''/g' - \frac{3}{2}(g''/g')^2$ and σ is an arbitrary Möbius transformation of the ratio $\mathcal{S}_D/\mathcal{S}$. Later we will see that a simple redefinition of the matrix model coupling constant precisely leads to the above duality structure. Furthermore, we will see that $v = \Lambda_{\text{SW}}^4/u^2$ and will find the explicit expression of $\mathcal{S}(v)$ and $\mathcal{S}_D(v)$.

We now show that thanks to a scaling property of \mathcal{F}_0 , it is possible to identify the right variables to make Seiberg–Witten duality in Dijkgraaf–Vafa theory manifest. First, we note that, by an overall rescaling, the loop expansion of the genus zero free energy in matrix model [4] reduces by one the number of variables ($\Lambda = 2^{-1/2}\Lambda_{\text{SW}}$)

$$\begin{aligned} \Delta^{-6} \mathcal{F}_0(S_k, \Delta, \Lambda) &= \frac{1}{2} \sum_{i=1,2} \left(\frac{S_i}{\Delta^3} \right)^2 \log \left(\frac{S_i}{\Delta^3} \right) - \left(\frac{S_1}{\Delta^3} + \frac{S_2}{\Delta^3} \right)^2 \log \left(\frac{\Lambda}{\Delta} \right) \\ &\quad + \sum_{n \geq 3} \sum_{i=3}^n c_{n,i} \left(\frac{S_1}{\Delta^3} \right)^{n-i} \left(\frac{S_2}{\Delta^3} \right)^i, \end{aligned} \quad (24)$$

where

$$c_{n,i} = (-1)^n c_{n,n-i}, \quad c_{n,i} = (-1)^i |c_{n,i}|, \quad (25)$$

so that, except for the first term, \mathcal{F}_0 is symmetric in S_1 and $-S_2$. By Euler theorem we have

$$\sum_{i=1,2} S_i \frac{\partial \mathcal{F}_0}{\partial S_i} + \frac{\Delta}{3} \frac{\partial \mathcal{F}_0}{\partial \Delta} + \frac{\Lambda}{3} \frac{\partial \mathcal{F}_0}{\partial \Lambda} = 2\mathcal{F}_0. \quad (26)$$

A property of (24) is that apparently the natural variables are S_k/Δ^3 rather than S_k . However, note that this would change the dimensional properties, so we should select $\Lambda^3 S_k/\Delta^3$. Furthermore, we should also choose the scale Λ as independent variable. So we should express \mathcal{F}_0 as a function of

$$S_k = \left(\frac{\Lambda}{\Delta} \right)^3 S_k, \quad \mu = ?, \quad \Lambda.$$

It remains to find μ which, of course, should depend on Δ and possibly on Λ . A closer look to (24) fixes it. Actually, Eq. (24) suggests considering a natural rescaling of all dimensional quantities of the arguments of \mathcal{F}_0 , by the dimensionless factor Λ/Δ . In particular, if $[x] = [\Lambda]^n$, then $x \rightarrow (\Lambda/\Delta)^n x$, that is

$$S_k \longrightarrow \left(\frac{\Lambda}{\Delta} \right)^3 S_k, \quad \Delta \longrightarrow \frac{\Lambda}{\Delta} \Delta = \Lambda, \quad \Lambda \longrightarrow \frac{\Lambda}{\Delta} \Lambda = \frac{\Lambda^2}{\Delta},$$

and the map we define is

$$\mathcal{F}_0(S_k, \Delta, \Lambda) \longrightarrow \mathcal{F}_0 \left(\left(\frac{\Lambda}{\Delta} \right)^3 S_k, \Lambda, \frac{\Lambda^2}{\Delta} \right),$$

showing that S_k , Δ and Λ combine in such a way that the natural variables for \mathcal{F}_0 are

$$S_1 = \left(\frac{\Lambda}{\Delta} \right)^3 S_1, \quad S_2 = \left(\frac{\Lambda}{\Delta} \right)^3 S_2, \quad \mu = \frac{\Lambda}{\Delta}, \quad \Lambda.$$

This also follows by the scaling law which is crucial for us

$$\mathcal{F}_0(\mu^3 S_k, \mu \Delta, \mu \Lambda) = \mu^6 \mathcal{F}_0(S_k, \Delta, \Lambda), \quad (27)$$

that we rewrite as

$$\mathcal{F}_0(S_k, \Lambda, \mu \Lambda) = \mu^6 \mathcal{F}_0(S_k, \Delta, \Lambda). \quad (28)$$

Note that

$$\begin{aligned} \mathcal{F}_0(S_k, \Lambda, \mu \Lambda) = \Lambda^6 \Big[& \frac{1}{2} \sum_{i=1,2} \left(\frac{S_i}{\Lambda^3} \right)^2 \log \left(\frac{S_i}{\Lambda^3} \right) - \left(\frac{S_1}{\Lambda^3} + \frac{S_2}{\Lambda^3} \right)^2 \log \mu \\ & + \sum_{n \geq 3} \sum_{i=3}^n c_{n,i} \left(\frac{S_1}{\Lambda^3} \right)^{n-i} \left(\frac{S_2}{\Lambda^3} \right)^i \Big], \end{aligned} \quad (29)$$

that differs from $\mathcal{F}_0(S_k, \Lambda, \Delta)$, which, we stress, is the original function with Λ and Δ interchanged and S_k replaced by \mathcal{S}_k , by a minus sign in front to the term $(S_1/\Lambda^3 + S_2/\Lambda^3)^2 \log \mu$.

Since $\mathcal{F}_0(S_k, \Lambda, \mu \Lambda)$ is a function of S_k , μ , and Λ , it follows by (28) that this is the case also for $\mathcal{F}_0(S_k, \Delta, \Lambda)$. Therefore, we consider the map $(S_k, \Delta, \Lambda) \rightarrow (S_k, \mu, \Lambda)$, as change of variables for $\mathcal{F}_0(S_k, \Delta, \Lambda)$. The relationships between the derivatives in the old and new variables are

$$\frac{\partial \mathcal{F}_0}{\partial S_1} = \mu^3 \frac{\partial \mathcal{F}_0}{\partial S_1}, \quad \frac{\partial \mathcal{F}_0}{\partial S_2} = \mu^3 \frac{\partial \mathcal{F}_0}{\partial S_2}, \quad (30)$$

$$\frac{\partial \mathcal{F}_0}{\partial \Delta} = -3 \frac{\mu}{\Lambda} S_1 \frac{\partial \mathcal{F}_0}{\partial S_1} - 3 \frac{\mu}{\Lambda} S_2 \frac{\partial \mathcal{F}_0}{\partial S_2} - \frac{\mu^2}{\Lambda} \frac{\partial \mathcal{F}_0}{\partial \mu}, \quad (31)$$

$$\frac{\partial \mathcal{F}_0}{\partial \Lambda} = \frac{\partial \mathcal{F}_0}{\partial \Lambda} + 3 \frac{S_1}{\Lambda} \frac{\partial \mathcal{F}_0}{\partial S_1} + 3 \frac{S_2}{\Lambda} \frac{\partial \mathcal{F}_0}{\partial S_2} + \frac{\mu}{\Lambda} \frac{\partial \mathcal{F}_0}{\partial \mu}, \quad (32)$$

where in the left-hand side the derivatives have been taken considering \mathcal{F}_0 as function of the old variables while on the right-hand side it is seen as function of (S_k, μ, Λ) . In the following we make an abuse of notation and drop a factor Λ , that is

$$\mathcal{F}_0(S_k, \Lambda, \mu) \equiv \mathcal{F}_0(S_k, \Lambda, \mu \Lambda) = \mu^6 \mathcal{F}_0(S_k, \Delta, \Lambda). \quad (33)$$

Minimizing

$$W_{\text{eff}} = \sum_{i=1,2} \frac{\partial \mathcal{F}_0}{\partial S_i} = S_{D1} + S_{D2},$$

we obtain, by (30) and (33)

$$\sum_{i=1,2} \frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial S_j} = \sum_{i=1,2} \mu^6 \frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial S_j} = \sum_{i=1,2} \frac{\partial^2 \mathcal{F}_0(S_k, \Lambda, \mu)}{\partial S_i \partial S_j} = 0,$$

which gives $S = S_1 = -S_2$, where [4]

$$S = \Lambda^3 (\mu^4 + 6\mu^8 + 140\mu^{12} + 4620\mu^{16} + \dots).$$

The effective coupling constant of $N = 2$ SYM with gauge group $SU(2)$ is given by

$$\tau = \frac{\partial^2 \mathcal{F}_0}{\partial S_1 \partial S_2} \Big|_{S_1 = -S_2 = S},$$

and by (30) and (33)

$$\tau = \frac{\partial^2 \mathcal{F}_0(S_k, \Lambda, \mu)}{\partial S_1 \partial S_2} \Big|_{S_1 = -S_2 = S},$$

where here \mathcal{F}_0 is rescaled by $1/\pi i$ with respect to the one in (24). So, we have seen that, thanks to the scaling property (33), one obtains the same effective coupling constant $\tau(a)$, if in the matrix model one considers as variables the old ones rescaled by $\mu^n = (\Lambda/\Delta)^n$, with n defined by $[x] = [\Lambda]^n$. As a consequence the duality structure of $N = 2$ SYM with gauge group $SU(2)$ is manifest. Before showing this explicitly we explain how the above rescaling of \mathcal{F}_0 simply amounts to a different choice of the matrix model coupling constant. Let us set

$$g_S = \mu^3 g_s, \quad \mathcal{W}(\Phi) = \mu^3 W(\Phi), \quad (34)$$

and note that

$$Z = \frac{1}{\text{vol}(G)} \int d\Phi \exp\left(-\frac{1}{g_S} \text{tr } W(\Phi)\right) = \frac{1}{\text{vol}(G)} \int d\Phi \exp\left(-\frac{1}{g_S} \text{tr } \mathcal{W}(\Phi)\right), \quad (35)$$

so that

$$Z = \exp\left(-\sum_{g \geq 0} g_S^{2g-2} \mathcal{F}_g\right) = \exp\left(-\sum_{g \geq 0} g_S^{2g-2} \tilde{\mathcal{F}}_g\right), \quad (36)$$

where

$$\tilde{\mathcal{F}}_g = \mu^{3(2-2g)} \mathcal{F}_g. \quad (37)$$

In particular, by (33) we see that $\tilde{\mathcal{F}}_0 = \mu^6 \mathcal{F}_0 = \mathcal{F}_0(S_k, \Lambda, \mu)$. This indicates that also the higher genus contributions should be considered as functions of the new variables, that is

$$\tilde{\mathcal{F}}_g = \mu^{3(2-2g)} \mathcal{F}_g = \mathcal{F}_g(S_k, \Lambda, \mu), \quad (38)$$

so we rewrite

$$Z = \exp\left(-\sum_{g \geq 0} g_S^{2g-2} \mathcal{F}_g\right), \quad (39)$$

where now $\mathcal{F}_g \equiv \mathcal{F}_g(S_k, \Lambda, \mu)$.

Let us now derive the explicit expression for \mathcal{S}_D and \mathcal{S} and show how the rescaling leads to make the $N = 2$ SYM duality manifest. The trick is to first consider the derivative of v with respect to u . In particular, by (20) and (21) we have

$$\mathcal{S}'_D = -\frac{\Lambda_{\text{SW}}^7}{64} u^{-5/2} a'_D, \quad \mathcal{S}' = -\frac{\Lambda_{\text{SW}}^7}{64} u^{-5/2} a', \quad (40)$$

and by (22)

$$v' = \frac{2^{10} \cdot 3\pi i}{\Lambda_{\text{SW}}^6} (S_D S' - S S'_D) = \pi i \Lambda_{\text{SW}}^4 (a'_D a - a_D a') u^{-3}. \quad (41)$$

On the other hand, since $aa'_D - a_D a' = 2i/\pi$, we have

$$v' = -2\Lambda_{\text{SW}}^4 u^{-3}, \quad (42)$$

that is

$$v = \left(\frac{\Lambda_{\text{SW}}^2}{u} \right)^2, \quad (43)$$

where the additive constant, that corresponds to fix the additive constant of \mathcal{F}_0 , has been set to zero. By construction we know that \mathcal{S} satisfies a second order differential equation with respect to v in which the first derivative term is absent. Actually, taking the second derivative of \mathcal{S} with respect to v , we have

$$\partial_v^2 \mathcal{S} = -(\partial_u v)^{-3} \partial_u^2 v \partial_u \mathcal{S} + (\partial_u v)^{-2} \partial_u^2 \mathcal{S} = \frac{3u^4}{16\Lambda_{\text{SW}}^4 (u^2 - \Lambda_{\text{SW}}^4)} \mathcal{S}, \quad (44)$$

that is (S_D, \mathcal{S}) satisfy the second order differential equation

$$\left(\partial_v^2 + \frac{3}{16v(v-1)} \right) \begin{pmatrix} S_D \\ \mathcal{S} \end{pmatrix} = 0, \quad (45)$$

whose solution is

$$\begin{aligned} S_D &= \frac{\Lambda_{\text{SW}}^3 \sqrt{v}}{48\pi} \int_{-1}^{1/\sqrt{v}} dx \frac{x - \sqrt{v}}{\sqrt{x^2 - 1} \sqrt{\sqrt{v}x - 1}}, \\ \mathcal{S} &= \frac{\Lambda_{\text{SW}}^3 \sqrt{v}}{48\pi} \int_{-1}^1 dx \frac{x - \sqrt{v}}{\sqrt{x^2 - 1} \sqrt{\sqrt{v}x - 1}}. \end{aligned} \quad (46)$$

Inverting Eq. (45) we obtain the differential equation for $v = \mathcal{H}(\mathcal{S})$

$$16\mathcal{H}(1 - \mathcal{H}) \frac{\partial^2 \mathcal{H}}{\partial \mathcal{S}^2} + 3\mathcal{S} \left(\frac{\partial \mathcal{H}}{\partial \mathcal{S}} \right)^3 = 0. \quad (47)$$

On the other hand, since

$$\mu = \left(\frac{\Lambda_{\text{SW}}^2}{2^3 u} \right)^{1/2} \longrightarrow v = 2^6 \mu^4,$$

we have

$$S_D = \frac{\Lambda_{\text{SW}}^3 \mu^2}{\sqrt{2} \cdot 12\pi} \int_{-1}^{1/8\mu^2} dx \frac{x - 8\mu^2}{\sqrt{x^2 - 1} \sqrt{8\mu^2 x - 1}},$$

$$S = \frac{\Lambda_{\text{SW}}^3 \mu^2}{\sqrt{2} \cdot 12\pi} \int_{-1}^1 dx \frac{x - 8\mu^2}{\sqrt{x^2 - 1} \sqrt{8\mu^2 x - 1}}. \quad (48)$$

In terms of μ the nonperturbative relation (22) reads

$$\mu^4 = \frac{3 \cdot 2^7 \pi i}{\Lambda_{\text{SW}}^6} \left(\mathcal{F}_0 - \frac{\mathcal{S}}{2} \frac{\partial \mathcal{F}_0}{\partial \mathcal{S}} \right), \quad (49)$$

which is the matrix model analog of the relation between the u -modulus and the Seiberg–Witten prepotential [8].

Introducing manifest duality has several interesting consequences. For example, one may investigate to what corresponds in matrix model the strong coupling region of $N = 2$ SYM. In particular, the QFT meaning of the strong coupling expansion of the prepotential at the points $u = \pm \Lambda_{\text{SW}}^2$ is a crucial open question. While in the weak coupling region the expansion of the SW prepotential corresponds to a one-loop term and to infinitely many instanton contributions, no QFT meaning is known for its expansion at string coupling. In $N = 2$ SYM, this region is investigated by performing a S -duality transformation on the fields. This corresponds to a Legendre transform of the prepotential. On the matrix model side one should consider a dual formulation corresponding to this region. It would be interesting whether perturbation theory would reproduce also in this region the $N = 2$ SYM theory. One should consider the Legendre transform

$$\mathcal{F}_{Dg} = \mathcal{F}_g - \sum_{i=1,2} \mathcal{S}_i \frac{\partial \mathcal{F}_g}{\partial \mathcal{S}_i}, \quad (50)$$

where $\mathcal{F}_g \equiv \mathcal{F}_g(\mathcal{S}_k, \Lambda, \mu)$, and

$$Z_D = \exp \left(- \sum_{g \geq 0} g_{\mathcal{S}_D}^{2g-2} \mathcal{F}_{Dg} \right), \quad (51)$$

which should induce the definition of \mathcal{W}_D

$$Z_D = \frac{1}{\text{vol}(G)} \int d\Phi_D \exp \left(- \frac{1}{g_{\mathcal{S}_D}} \text{tr} \mathcal{W}_D(\Phi_D) \right). \quad (52)$$

Before concluding, let us note that this approach should be related with the derivation of the structure of the instanton moduli space of $N = 2$ SYM obtained from the recursion relations for the instanton contributions to the prepotential [13]. In particular, it was shown how the analogs of the recursive structure of the Deligne–Knudsen–Mumford compactification of moduli space of Riemann surfaces and the Wolpert restriction phenomenon, essentially determine the structure of the instanton moduli spaces. These techniques are strictly related to the geometry of matrix models considered in the framework of Liouville quantum gravity [14]. So, it would be interesting to investigate whether there is a possible link between the matrix model approach to the $N = 2$ SYM and the geometrical approach considered in [13].

Finally, we note that making duality manifest, which generalizes to higher rank groups [15], may have possible relations with recent work on matrix models [16].

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