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FINITE p-GROUPS WITH NORMAL NORMALISERS

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We consider the class N of groups in which the normaliser of every subgroup is normal, and the class C of groups in which the commutator subgroup normalises every subgroup. It is clear that $C \subseteq N$, and it is known that groups in the class N are nilpotent of class at most 3. We show that every finite p-group in N is also in C, provided that $p \geqslant 5$, and we give an example showing that this is not true for $p = 2$.

1. INTRODUCTION

We consider the class N of groups in which the normaliser of every subgroup is normal, and the class C of groups in which the commutator subgroup normalises every subgroup. Clearly $C \subseteq N$.

By a result of Heineken [3] and Mahdavianary [5], groups in the class N are nilpotent with nilpotency class at most 3.

In this paper we prove:

THEOREM. *If $p \geqslant 5$ and P is a finite p-group in the class N, then P is in the class C.*

For 2-generators groups this result was obtained by Hobby [2], and Mahdavianary [6] proved a corresponding result for finite 2-generator 3-groups. Moreover Parmeggiani proved in [7] that for $p \geqslant 3$ finite p-groups in N are also in C, if they have exponent at most p^2. In that paper she also gave an example of a p-group of odd order not in C which she erroneously claimed to be in N.

Bryce and Cossey in [1] gave an example of a 2-group in N but not in C when they found a minimal 2-group of Wielandt length 2 that is not in C.

We recall that the Wielandt subgroup of a finite p-group is the intersection of the normalisers of all the subgroups of the group. Hence a finite p-group of Wielandt length 2 is a finite p-group in which the quotient over the Wielandt subgroup is a group with all subgroups normal. Clearly p-groups in C have Wielandt length 2 and finite

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p-groups of Wielandt length 2 are in N. Moreover for p odd a finite p-group has Wielandt length 2 if and only if it is in C.

In this paper we give an example of a family of 2-groups in N that do not have Wielandt length 2.

2. Proof of the Theorem

We recall that if G is a nilpotent group of class at most 3 and $x, x_1, \ldots, x_n$, $y, y_1, \ldots, y_m$, $z, z_1, \ldots, z_s \in G$ then

$$\left[\prod_{i=1}^{n} x_i, \prod_{j=1}^{m} y_j, \prod_{k=1}^{s} z_k\right] = \prod_{i=1}^{n}\prod_{j=1}^{m}\prod_{k=1}^{s}[x_i, y_j, z_k]$$

and

$$[x,y,z][y,z,x][z,x,y] = 1 \quad \text{(Jacobi identity)}.$$

It is well known (see for example Huppert [**4**, Chapter III, 10.2 (a) and 10.6 (a)]) that p-groups with nilpotency class less than p are regular, and that if P is a finite regular p-group and a, $b \in P$ then

$$(*) \qquad \left(ab^{-1}\right)^{p^k} = 1 \iff a^{p^k} = b^{p^k}$$

For regular p-groups the following Lemma holds:

Lemma 2.1. *Let P be a regular p-group and $a, b \in P$. Then*

(i) $(ab)^{|a|} = b^{|a|} = (ba)^{|a|}$.

(ii) *If $\langle a\rangle \cap \langle b\rangle = \langle a^{p^t}\rangle = \langle b^{p^t}\rangle \neq 1$ and $\alpha \in \mathbb{N}$ not divisible by p is such that $a^{p^t} = \left(b^{p^t}\right)^{\alpha}$, then $|ab^{-\alpha}| = p^t$ and $\langle ab^{-\alpha}\rangle \cap \langle a\rangle = 1 = \langle ab^{-\alpha}\rangle \cap \langle b\rangle$.*

(iii) *Assume $K \leqslant P$ with $|K| = p$, $K \nleqslant \langle a\rangle$ and $K \nleqslant \langle b\rangle$. Then either $K \nleqslant \langle ab\rangle$ or $K \nleqslant \langle ab^2\rangle$.*

Proof: (i) and (ii) are direct consequences of (*).

To prove (iii), assume that K is a subgroup of P of order p with $K \leqslant \langle ab\rangle \cap \langle ab^2\rangle$, $K \nleqslant \langle a\rangle$ and $K \nleqslant \langle b\rangle$. Then from (i) it follows that

$$p^n := |a| = |b| = |ab| = |ab^2|.$$

Since $K \leqslant Z(\langle ab, ab^2\rangle) = Z(\langle a, b\rangle)$, then $(ab)^{p^{n-1}}$, $(ab^2)^{p^{n-1}} \in Z(\langle a,b\rangle)$, and so

$$\left(ab^2\right)^{p^{n-1}} = (abb)^{p^{n-1}} = (ab)^{p^{n-1}} b^{p^{n-1}},$$

a contradiction to $b^{p^{n-1}} \notin K$. ∎

In particular the property (*) and Lemma 2.1 hold for finite p-groups in N if $p \geqslant 5$.

The proof of the Theorem is based on the following Lemma:

LEMMA 2.2. *Let P be a finite p-group in N and $A_0 \leqslant A \leqslant P$ such that $|A/\Phi(A)| = p^2$ and $\Phi(A) \leqslant A_0$. Then $[P, N_P(A), A_0] \leqslant A_0$. In particular, $[z,u,w] \in \langle w\rangle$ for every $z, u, w \in P$ such that $[w,u,u] = 1$ and $[w,u]^p = 1$.*

PROOF: Let $U := N_P(A)$ and $\overline{A} := A/\Phi(A)$. Choose $z \in P$ and $u \in U$. It suffices to show that $[z,u] \in N_P(A_0)$.

Note that $[z,u] \in U$ since $P \in N$. If $u \in C_U(\overline{A})$ or $[z,u] \in C_U(\overline{A})$, then $u \in N_P(A_0)$ or $[z,u] \in N_P(A_0)$, respectively. Thus we may assume that both u and $[z,u]$ are not in $C_U(\overline{A})$. As $\overline{A}$ has order p^2, we get that $|U/C_U(\overline{A})| = p$ and

$$U = \langle u\rangle C_U(\overline{A}) = \langle [z,u]\rangle C_U(\overline{A}).$$

Hence there exists $k \in \mathbb{N}$ such that $[z,u]^k u \in C_U(\overline{A})$ and thus

$$[z,u]^k u \in N_U(A_0).$$

Since $N_U(A_0)$ is normal in P we conclude that $\big[[z,u]^k u, z\big] \in N_U(A_0)$. On the other hand, since P has class at most 3, $\big[[z,u]^k, z\big] \in N_U(A_0)$. It follows that $[z,u] \in N_P(A_0)$.

In particular, if $w, u \in P$ are such that $[w,u,u] = 1$ and $[w,u]^p = 1$, we set $A = \langle w, [w,u]\rangle$ and $A_0 = \langle w\rangle$. Since $P \in N$, then $[w,u,w] \in \langle w^p\rangle$, hence $\Phi(A) = \langle w^p\rangle \leqslant A_0$ and $u \in N_P(A)$. Thus $[z,u,w] \in [P, N_P(A), A_0] \leqslant A_0 = \langle w\rangle$ for every $z \in P$. ∎

We can now prove the Theorem.

THEOREM. *Let P be a finite p-group in N with $p \geqslant 5$. Then $P \in C$.*

PROOF: Let P be a minimal counterexample. Then

$$\mathcal{S} := \Big\{ s \in P \mid P' \nleqslant N_P\big(\langle s\rangle\big)\Big\}$$

is not empty. Let $s \in \mathcal{S}$.

If K is a minimal normal subgroup of P, then the minimality of P yields $[P', \langle s\rangle] \leqslant K\langle s\rangle$, in particular $K \nleqslant \langle s\rangle$. Assume there exist two distinct minimal normal subgroups of P, K_1 and K_2. Then again the minimality of P yields

$$[P', \langle s\rangle] \leqslant K_1\langle s\rangle \cap K_2\langle s\rangle = \langle s\rangle\big(K_1 \cap K_2\langle s\rangle\big),$$

and $|K_1| = p$ gives $K_1 \leqslant K_2\langle s\rangle$ and so $K_1\langle s\rangle = K_2\langle s\rangle$. Since K_1K_2 is a subgroup of $Z(P)$ of order p^2, then

$$1 \neq (K_1K_2) \cap \langle s\rangle \leqslant Z(P) \cap \langle s\rangle,$$

a contradiction. Hence P has a unique minimal normal subgroup K. Since P has nilpotency class 3, $K = \Omega_1(\gamma_3(P))$. Moreover $K \nleq \langle d \rangle$ for every $d \in \mathcal{S}$.

Since 2-generators groups in the class N are in the class C, then $[P, \langle d \rangle, \langle d \rangle] \leqslant \gamma_3(P) \cap \langle d \rangle$ for every $d \in P$. Therefore

(1) $[P, \langle d \rangle, \langle d \rangle] = 1$ for every $d \in P$ such that $K \nleq \langle d \rangle$ (in particular for every $d \in \mathcal{S}$).

For every $s \in \mathcal{S}$ we set

$$\mathcal{L}(s) := \Big\{(h, g) \in P \times P \mid [h, g] \notin N_P(\langle s \rangle)\Big\}.$$

Note that the minimality of P gives

(2) $K = \langle [h, g, s] \rangle$ for every $s \in \mathcal{S}$ and every $(h, g) \in \mathcal{L}(s)$.

We now show:

(3) $[z, h, g][z, g, h] = 1$ for every $z, g, h \in P$ with $K \nleq \langle h \rangle$ and $K \nleq \langle g \rangle$.

From (1) and Lemma 2.2 (iii) it follows that there exists $\lambda \in \{1, 2\}$ such that

$$1 = [z, hg^\lambda, hg^\lambda] = [z, h, h][z, h, g]^\lambda [z, g, h]^\lambda [z, g, g]^{\lambda^2} = \big([z, h, g][z, g, h]\big)^\lambda,$$

thus (3) follows from $p \neq 2$.

From (3) and the Jacobi identity we get:

(4) $[h, g, z] = [h, z, g]^2$ for every $h, g, z \in P$ with $K \nleq \langle h \rangle$ and $K \nleq \langle g \rangle$.

Next we prove:

(5) $K \leq \langle \widetilde{h} \rangle$ or $K \leq \langle \widetilde{g} \rangle$ for every $(\widetilde{h}, \widetilde{g}) \in \mathcal{L}(\widetilde{s})$, where $\widetilde{s} \in \mathcal{S}$.

Assume $K \nleq \langle \widetilde{h} \rangle$ and $K \nleq \langle \widetilde{g} \rangle$. Since $K \nleq \langle \widetilde{s} \rangle$, an application of (4) with $h = \widetilde{h}$, $g = \widetilde{g}$ and $z = \widetilde{s}$ gives $[\widetilde{h}, \widetilde{g}, \widetilde{s}] = [\widetilde{h}, \widetilde{s}, \widetilde{g}]^2$, and another application of (4) with $h = \widetilde{h}$, $g = \widetilde{s}$ and $z = \widetilde{g}$ gives $[\widetilde{h}, \widetilde{s}, \widetilde{g}] = [\widetilde{h}, \widetilde{g}, \widetilde{s}]^2$. Hence $[\widetilde{h}, \widetilde{g}, \widetilde{s}] = [\widetilde{h}, \widetilde{g}, \widetilde{s}]^4$ and $p \neq 3$ yields $[\widetilde{h}, \widetilde{g}, \widetilde{s}] = 1$, a contradiction to $\widetilde{s} \in \mathcal{S}$ and $(\widetilde{h}, \widetilde{g}) \in \mathcal{L}(\widetilde{s})$. This proves (5).

For every $s \in \mathcal{S}$ we set

$$\mathcal{L}^*(s) := \big\{(h, g) \in \mathcal{L}(s) \mid K \leqslant \langle h \rangle \cap \langle g \rangle\big\}.$$

Since $(h, g) \in \mathcal{L}(s)$ implies $(h, hg), (hg, g) \in \mathcal{L}(s)$, then (5) gives that $\mathcal{L}^*(s)$ is not empty.

Among the elements in $\mathcal{S}$ we choose c of maximal order, and among the elements in $\mathcal{L}^*(c)$ we choose (x, y) such that $|x|\,|y|$ is maximal.

We now want to show:

(6) $|c| < |x| = |y|$.

If $|x| < |y|$, then $|xy|\,|y| > |x|\,|y|$ and Lemma 2.1 (i) gives $(xy, y) \in \mathcal{L}^*(c)$, a contradiction to the choice of (x, y). Interchanging x and y we obtain that $|x| = |y|$.

Assume $|c| \geqslant |x| = |y|$. Then from $K \not\leq \langle c\rangle$ and Lemma 2.1 (i) in the case $|x| = |y| < |c|$, and from $K \not\leq \langle c\rangle$, $K \leq \langle x\rangle \cap \langle y\rangle$ and (*) in the case $|x| = |y| = |c|$, it follows that $K \not\leq \langle cx\rangle$ and $K \not\leq \langle cy\rangle$. But (1) implies $(cx, cy) \in \mathcal{L}(c)$, a contradiction to (5). Thus (6) is proved.

Set

$$p^n := |x| = |y| \quad \text{and} \quad p^m := |c|.$$

If $[c,x]^{p^{m-1}} = 1 = [c,y]^{p^{m-1}}$, by (1) we get $[c^{p^{m-1}}, x] = 1 = [c^{p^{m-1}}, y]$. Hence $c^{p^{m-1}} \in Z(\langle c,x,y\rangle) = Z(P)$, a contradiction to $K \not\leq \langle c\rangle$. Thus there exists $a \in \{x,y\}$ such that

$$\big|[c,a]\big| = p^m.$$

Let b satisfy $\{a,b\} = \{x,y\}$.

Choose d so that $(a,d) \in \mathcal{L}(c)$, $K \not\leq \langle d\rangle$ and d has minimal order with respect to these properties. This is always possible since by Lemma 2.1 (ii) there exists an integer α such that $\langle a\rangle \cap \langle a^\alpha b\rangle = 1$, and $[a, a^\alpha b, c] = [a,b,c]$.

Set

$$p^k := |d|, \quad p^f := \big|[c,d]\big| \quad \text{and} \quad p^r := \big|\gamma_3(P)\big|.$$

Note that $f \leqslant k \leqslant m$ and that the minimality of P gives $P = \langle a,c,d\rangle$.

Next we show:

(7) $f \geqslant 2$.

Assume $[c,d]^p = 1$. Set $w = c$, $u = d$ and $z = a$. Then $[w,u,u] = 1$, $[w,u]^p = [c,d]^p = 1$ and $[z,u,w] = [d,a,c]^{-1} \neq 1$. Since by (2) $K = \langle [d,a,c]\rangle \not\leq \langle w\rangle = \langle c\rangle$, Lemma 2.2 leads to a contradiction. This proves (7).

Next we prove:

(8) $\{c^\alpha d^\beta, d^\beta c^\alpha \mid \alpha,\beta \in \mathbb{N}$ and p does not divide both α and $\beta\} \subseteq \mathcal{S}$.

Since $[a,c,c^\alpha d^\beta] = [a,c,d]^\beta = [a,c,d^\beta c^\alpha]$ and $[a,d,c^\alpha d^\beta] = [a,d,c]^\alpha = [a,d,d^\beta c^\alpha]$, from $(a,d) \in \mathcal{L}(c)$ and $(a,c) \in \mathcal{L}(d)$ it follows that to prove (8) it is sufficient to prove that $K \not\leq \langle c^\alpha d^\beta\rangle$ and $K \not\leq \langle d^\beta c^\alpha\rangle$ if p does not divide both α and β.

Assume $K \leq \langle c^\alpha d^\beta\rangle$ and set $h := c^\alpha d^\beta$ and $p^s := |h|$. From $K \not\leq \langle c\rangle$, $K \not\leq \langle d\rangle$ and Lemma 2.1 (i) we get that $|d^\beta| = |c^\alpha|$, and $|h| = p^s \leqslant p^k$.

If α is not divisible by p, then $|c^\alpha| = |d^\beta|$ gives that also β is not divisible by p.

Choose t minimal so that

$$\langle h\rangle \cap \langle a\rangle = \langle h^{p^t}\rangle = \langle a^{p^{n-s+t}}\rangle.$$

Since $h^{p^t} \neq 1$, then $t < s$. Let $\delta \in \mathbb{N}$ such that $h^{p^t} = a^{p^{n-s+t}\delta}$ and set $d' := ha^{-p^{n-s}\delta}$. An application of Lemma 2.1 (ii) yields $|d'| = p^t$ and $\langle a^{p^{n-s}}\rangle \cap \langle d'\rangle = 1$. In particular $K \not\leq \langle d'\rangle$. Now β not divisible by p gives

$$[a,d',c] = [a,h,c] = [a,d,c]^\beta \neq 1,$$

and $t < s \leqslant k$ contradicts the choice of d. In a similar way one gets that $K \nleqslant \langle d^\beta c^\alpha \rangle$, and the proof of (8) is complete.

For the next step, we apply Lemma 2.2 to prove

(9) $\{[c,a,a],[d,a,a]\} \neq \{1\}$.

Assume $[c,a,a] = 1 = [d,a,a]$. Then by (2) $[d,a]^p, [c,a]^p \in Z(P)$. Hence $|[c,a]| = p^m \geqslant |[d,a]|$ gives that there exists $\mu \in \mathbb{N}$ such that $[d,a]^p = [c,a]^{p\mu}$. Set $w = c^{-\mu}d$, $u = a$ and $z = c$. Then $[w,u,u] = 1$, $[w,u]^p = [c,a]^{-p\mu}[d,a]^p = 1$ and $[z,u,w] = [c,a,d] \neq 1$. From (8) we have that $c^{-\mu}d \in \mathcal{S}$, hence $K \nleqslant \langle w \rangle = \langle c^{-\mu}d \rangle$. Now, as in (7), Lemma 2.2 gives a contradiction, and (9) is proved.

Set

$$\bar{c} := \begin{cases} c & \text{if} \quad |[d,a,a]| \leqslant |[c,a,a]| \\ dc & \text{if} \quad [c,a,a] = [d,a,a]^{p\lambda} \quad \text{and} \quad [dc,a]^{p^{m-1}} \neq 1 \\ dc^2 & \text{if} \quad [c,a,a] = [d,a,a]^{p\lambda} \quad \text{and} \quad [dc,a]^{p^{m-1}} = 1 \end{cases}$$

By the choice of $\bar{c}$ we have $P = \langle a, \bar{c}, d \rangle$ and:

$$\langle [d,a,a] \rangle \leqslant \langle [\bar{c},a,a] \rangle.$$

From (8) it follows that $\bar{c} \in \mathcal{S}$ and from (1) that $(a,d) \in \mathcal{L}(\bar{c})$.

Assume $[\bar{c},a,a] = 1$. Then $[c,a,a] = 1 = [d,a,a]$, a contradiction to (9). Thus

(10) $\gamma_3(P) = \langle [\bar{c},a,a] \rangle$.

Let $\tau \in \mathbb{N}$ such that $[d,a,a] = [\bar{c},a,a]^{-\tau}$, and set

$$\bar{d} := d\bar{c}^\tau.$$

From (8) we get that $\bar{d} \in \mathcal{S}$ and $[a,\bar{c},\bar{d}] = [a,\bar{c},d] = [a,c,d] \neq 1$. Hence $(a,\bar{c}) \in \mathcal{L}(\bar{d})$ and $P = \langle a,\bar{c},\bar{d} \rangle$. From the choice of $\bar{d}$ it also follows:

(11) $[\bar{d},a,a] = 1$.

Note that (1) implies that there exists $\varepsilon \in \{1,2\}$ such that

$$[\bar{c},\bar{d}] = [c,d]^\varepsilon.$$

Assume $[\bar{c},a]^{p^{m-1}} = 1$. Then, by the choice of $\bar{c}$, we have $\bar{c} = dc^2$ and $[dc,a]^{p^{m-1}} = 1$. Since $[\bar{c},a]^{p^{m-1}} = [dc,a]^{p^{m-1}}[c,a]^{p^{m-1}} = 1$, also $[c,a]^{p^{m-1}} = 1$, a contradiction.

Together with $|\bar{c}| \leqslant |c| = p^m$, this gives

$$|[\bar{c},a]| = p^m, \qquad \text{and so} \qquad |\bar{c}| = |c| = p^m.$$

Note that (1), (2), (4) and $(a,c) \in \mathcal{L}(d)$ give:

(12) $[\overline{d},a]^p, [\overline{c},\overline{d}]^p \in Z(P)$.

The rest of the proof consists in four applications of Lemma 2.2. First we prove:

(13) There exists $\mu \in \mathbb{N}$ not divisible by p such that $[\overline{d},a]^{p\mu} = [\overline{c},\overline{d}]^p$.

By (12), to prove (13) it is sufficient to show that $\left|[\overline{d},a]\right| = \left|[\overline{c},\overline{d}]\right|$.

Assume $\left|[\overline{d},a]\right| \neq \left|[\overline{c},\overline{d}]\right|$. Then there exist $\lambda, \tau \in \mathbb{N}$ such that exactly one of them is divisible by p and $[\overline{d},a]^{p\lambda} = [\overline{c},\overline{d}]^{p\tau}$. Set $w = \overline{d}$, $u = a^{\lambda}\overline{c}^{\tau}$ and $z = \overline{c}a$. From (2), (3), (4) and (11) one gets

$$[w,u,u] = [\overline{d},a,\overline{c}]^{3\lambda\tau} = 1,$$
$$[w,u]^p = [\overline{c},\overline{d}]^{-p\tau}[\overline{d},a]^{p\lambda} = 1,$$
$$[z,u,w] = [\overline{c},a,\overline{d}]^{\lambda-\tau} \neq 1.$$

Hence Lemma 2.2 yields a contradiction, since $K \not\leqslant \langle\overline{d}\rangle = \langle w\rangle$, and the proof of (13) is complete.

Next we prove:

(14) Either $m-f < r-1$, or $r=1$ and $m=f$.

Assume $m-f \geqslant r-1$. Since $[\overline{c},a]^{p^r} \in Z(P)$ there exists $\lambda \in \mathbb{N}$ not divisible by p such that

$$[\overline{c},\overline{d}]^p = [\overline{c},a]^{p^{m-f+1}\lambda}.$$

Set $w = \overline{c}$, $u = a^{p^{m-f}\lambda}\overline{d}^{-1}$ and $z = a$. Then a direct calculation together with (3), (4), (7) and (12) give

$$[w,u,u] = [\overline{c},a,a]^{p^{2(m-f)}\lambda^2}[\overline{d},a,\overline{c}]^{3p^{m-f}\lambda},$$
$$[w,u]^p = [\overline{c},\overline{d}]^{-p}[\overline{c},a]^{p^{m-f+1}\lambda} = 1,$$
$$[z,u,w] = [\overline{d},a,\overline{c}] \neq 1.$$

If $[w,u,u]=1$ Lemma 2.2 yields a contradiction since $K \not\leqslant \langle\overline{c}\rangle = \langle w\rangle$. Thus $[w,u,u] \neq 1$.

Assume $m>f$. Then from $[\overline{c},a,a]^{p^{2(m-f)}\lambda^2} \neq 1$ it follows $2(r-1) \leqslant 2(m-f) < r$, a contradiction to $m>f$. We have shown that $m=f$, and $[\overline{c},a,a]^{\lambda}[\overline{d},a,\overline{c}]^3 = 1$ gives also $r=1$. This completes the proof of (14). We can now prove:

(15) $r=1$ and $m=f$.

Assume not. Then by (14) $m-f<r-1$, hence $f-m+r>1$.

Since $[\overline{c},a]^{p^r} \in Z(P)$ there exists $\lambda \in \mathbb{N}$ not divisible by p such that

$$[\overline{c},a]^{p^r} = [\overline{c},\overline{d}]^{p^{f-m+r}\lambda}.$$

Moreover from (10) and (13) there exist $\tau, \mu \in \mathbb{N}$ not divisible by p such that

$$[\overline{c}, a, \overline{d}] = [\overline{c}, a, a]^{p^{r-1}\tau} \text{ and } [\overline{d}, a]^{p\mu} = [\overline{c}, \overline{d}]^p.$$

Set $s = \mu\big(p^{f-1} - 3p^{f-m+r-1}\lambda\tau + 1\big)$ and note that s is not divisible by p.

Set $w = \overline{c}^{3p^{r-1}\tau}\overline{d}^s$, $u = a^s\overline{c}$ and $z = \overline{c}$. From (3) and (11)

$$\begin{aligned}[w, u, u] &= [\overline{c}, a, a]^{3p^{r-1}\tau s^2}[\overline{d}, a, \overline{c}]^{3s^2} = \big([c, a, d][d, a, c]\big)^{3s^2} = 1,\\ [w, u]^p &= \big([\overline{c}, \overline{d}]^{-p}[\overline{c}, a]^{3p^r\tau}[\overline{d}, a]^{ps}\big)^s\\ &= \big([\overline{c}, \overline{d}]^{-p}[\overline{c}, \overline{d}]^{3p^{f-m+r}\lambda\tau}[\overline{c}, \overline{d}]^{p(p^{f-1}-3p^{f-m+r-1}\lambda\tau+1)}\big)^s = 1,\end{aligned}$$

and $[z, u, w] = [\overline{d}, a, \overline{c}]^{-s^2} \neq 1$, since s is not divisible by p. Now Lemma 2.2 gives a contradiction, since $K \not\leq \langle \overline{c}^{3p^{r-1}\tau}\overline{d}^s\rangle$ by (8), and (15) is proved.

We now obtain the final contradiction.

By (15) $r = 1$, hence $[\overline{c}, a]^p \in Z(P)$ and from (10) we get that there exists $\tau \in \mathbb{N}$ not divisible by p such that $[\overline{c}, a, \overline{d}] = [\overline{c}, a, a]^\tau$. Since $[\overline{c}, \overline{d}]^p \in Z(P)$ by (12), and $f = m$ by (15), then there exists $\lambda \in \mathbb{N}$ not divisible by p such that

$$[\overline{c}, \overline{d}]^p = [\overline{c}, a]^{p\lambda}.$$

Moreover from (13) we get $\mu \in \mathbb{N}$ not divisible by p such that

$$[\overline{c}, \overline{d}]^p = [\overline{d}, a]^{p\mu}.$$

Set $w = \overline{c}^{3\tau\lambda}\overline{d}^{\mu(\lambda-3\tau)}$, $u = a^{3\tau}\overline{d}^{-1}$ and $z = a$. Using (3) and (11) we get

$$\begin{aligned}[w, u, u] &= \big([\overline{c}, a, a]^\tau[\overline{c}, a, \overline{d}]^{-1}\big)^{27\tau^2\lambda} = 1,\\ [w, u]^p &= \big([\overline{c}, \overline{d}]^{-p\lambda}[\overline{c}, a]^{p3\tau\lambda}[\overline{d}, a]^{p\mu(\lambda-3\tau)}\big)^{3\tau}\\ &= \big([\overline{c}, \overline{d}]^{-p\lambda}[\overline{c}, \overline{d}]^{p3\tau}[\overline{c}, \overline{d}]^{p(\lambda-3\tau)}\big)^{3\tau} = 1\end{aligned}$$

and $[z, u, w] = [\overline{d}, a, \overline{c}]^{3\tau\lambda} \neq 1$ since τ and λ are not divisible by p.

Now Lemma 2.2 gives a contradiction, since $K \not\leq \langle \overline{c}^{3\tau\lambda}\overline{d}^{\mu(\lambda-3\tau)}\rangle$ by (8).

The proof of the Theorem is now complete. □

3. A family of 2-groups

We now give an example of a family of 2-groups in N that do not have Wielandt length 2.

For $r \geqslant 2$ let $H(r)$ be the group on generators a, b, c with the following relations:

$$a^{2^r} = b^{2^r} = c^{2^r}, \quad a^{2^{r+1}} = 1,$$
$$[b,a,a] = [b,a,b] = [c,a,a] = [c,a,c] = [c,b,b] = [c,b,c] = a^{2^r} = [b,a]^{2^{r-1}},$$
$$[c,a]^{2^{r-1}} = [c,b]^{2^{r-1}} = [c,a,b] = [c,b,a] = 1.$$

To show that $H(r) \in N$ we have to prove that $[H(r), N_{H(r)}(K), K] \leqslant K$ for every $K \leqslant H(r)$, and since $\gamma_3(H(r))$ has order 2, it is sufficient to consider subgroups $K \leqslant H(r)$ with $\gamma_3(H(r)) \nleqslant K$.

Let $K \leqslant H(r)$ with $\gamma_3(H(r)) \nleqslant K$. We have to show that $[H(r), y, z] = 1$ for every $z \in K$ and $y \in N_{H(r)}(K)$.

Since $\gamma_3(H(r))$ has order 2, then $\left[H(r), H(r), \Phi(H(r))\right] = 1 = [H(r), \Phi(H(r)), H(r)]$, hence it is sufficient to consider z, y among the nontrivial coset representatives of $\Phi(H(r))$, that is $\{a, b, c, ab, ac, bc, abc\}$.

From $\gamma_3(H(r)) \leqslant \langle t \rangle$ for every $t \in \{a, b, c, ab\}$ and $\gamma_3(H(r)) \nleqslant K$ it follows that $z \notin \{a, b, c, ab\}$.

Moreover $[H(r), H(r), abc] = 1$ allows us to restrict to the case $z \neq abc$.

Hence we can assume $z \in \{ac, cb\}$.

Let $z = ac$ and $y \in N_{H(r)}(K) \cap \{a, b, c, ab, ac, bc, abc\}$. Since $[y, ac]^{2^{r-1}} = [y, z]^{2^{r-1}} \in K \cap \gamma_3(H(r)) = 1$, then $y \in \{a, c, ac\}$. From $[z, y, y] \in K$ and $K \cap \gamma_3(H(r)) = 1$, it follows that $y \notin \{a, c\}$, hence $z = y = ac$.

Similarly, if $z = cb$ and $y \in N_{H(r)}(K) \cap \{a, b, c, ab, ac, bc, abc\}$, then $[y, cb]^{2^{r-1}} = [y, z]^{2^{r-1}} \in K \cap \gamma_3(H(r)) = 1$ gives $y \in \{c, b, cb\}$. From $[z, y, y] \in K$ and $K \cap \gamma_3(H(r)) = 1$, it follows that $y \notin \{c, b\}$, hence $z = y = cb$.

To complete the proof that $H(r) \in N$ we now observe that

$$[H(r), ac, ac] = 1 = [H(r), cb, cb].$$

Let $\omega(H(r))$ be the Wielandt subgroup of $H(r)$. Since a group with all the subgroups normal has a derived subgroup of order at most 2, then to prove that $H(r)$ has not Wielandt length 2, it is sufficient to show that $H(r)'/\left(\omega(H(r)) \cap H(r)'\right) \cong \left(H(r)/\omega(H(r))\right)'$ has order bigger than 2. Hence it is sufficient to show that $[c,a], [c,b], [c,a][c,b] \notin \omega(H(r))$.

Since $[c, a, bc] = a^{2^r} \notin \langle bc \rangle$ then $[c,a] \notin \omega(H(r))$. Similarly, from

$$[c, b, ac] = [[c,a][c,b], ac] = a^{2^r} \notin \langle ac \rangle,$$

we get $[c,b], [c,a][c,b] \notin \omega(H(r))$.

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