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# GARCH-type Models with Generalized Secant Hyperbolic Innovations

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Paola Palmitesta and Corrado Provasi

**Abstract**

GARCH-type models have been analyzed assuming various nongaussian distributions of errors. In general, the asymmetric generalized Student-t random variable seems to be the distribution which better captures the nonnormality features of financial data. However, a drawback of this distribution is represented by the technical dificulties due to the evaluation of moments, especially in the case of fractional degrees of freedom. In this paper we propose to model high frequency time series returns using GARCH-type models with a generalized secant hyperbolic (GSH) distribution. The main advantage of the GSH distribution over the Student-t distribution is that all the moments are finite for each value of the shape parameter. The distribution is symmetric with respect to the mean, but we show that it is still possible to obtain the density in a closed form introducing a skewness parameter according to the method proposed by Fernandez and Steel. We use a Monte Carlo experiment to validate this distribution in the context of GARCH models with maximum likelihood estimates of parameters. Finally, we show an application to log returns of a stock index.

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# 1 Introduction

The GARCH model introduced by Bollerslev (1986) as a generalization of the ARCH model proposed by Engle (1982) to interpret the evolution of the conditional variance using heteroskedastic autoregressive processes, has found various financial applications in order to capture relevant empirical aspects of high frequency time series of returns, in particular the symmetrical effects of volatility. Nevertheless, in risk analysis, for example, we have to model financial series taking into account the effects of extreme observations and skewness on returns (Lee and Tse, 1991). In this context, GARCH-type models have been analyzed assuming various nongaussian distributions of errors and empirical results show that if the estimation of parameters is done using skewed leptokurtic distributions (fat-tails distributions) we have better results with respect to estimates obtained assuming normality, because the estimated parameters are less influenced by adverse effects of outliers (for a review see, among others, Peters, 2001 and Verhoven and McAleer, 2003).

In general, the Student-$t$ random variable seems to be the distribution which better captures nonnormality aspects of innovations (see, among others, Bollerslev, 1987, Baille and Bollerslev, 1989, Kaiser, 1996, Beine, Laurent and Lecourt, 2000, and the extensions to the asymmetric case in the GARCH context by Lambert and Laurent, 2000, and Jondeau and Rockinger, 2003). Anyway, a drawback of the Student-$t$ distribution is given by the possible technical difficulties associated to the evaluation of moments, especially when the degrees of freedom are fractional.

In this paper we propose to model the innovations of high frequency time series of returns using GARCH-type models with a generalization of the hyperbolic secant distribution. This distribution has been studied by Vaughan (Vaughan, 2002), and it is called *generalized secant hyperbolic distribution*. The kurtosis goes from 1.8 to infinity and it includes the logistic distribution as a special case and the uniform distribution as limit case; further, it approximates the normal and Student-$t$ distributions with corresponding kurtosis. The main difference between the Student-$t$ and the generalized secant hyperbolic distribution is that all the moments of the latter are finite for any value of the shape parameter. The distribution is symmetric about the mean, but we are able to show that it is still possible to obtain a density in closed form introducing a skewness parameter, following the method proposed by Fernández and Steel (1998).

The paper is organized as follows. In the next section we present the GARCH model with a GSH distribution of innovations, while in the third section we validate this distribution in the context of GARCH models by means of a Monte Carlo experiment; in the fourth section we show an application to log returns of the MIBTEL, an italian value-weighted stock index. Conclusions are left to the last section.

# 2 GARCH Models

Without loss of generality, consider a $GARCH(p, q)$ model for the time series $\{\epsilon_t,\ t = 1, 2, \dots, T\}$,

$$\begin{aligned} \epsilon_t &= \sqrt{h_t} z_t, \\ h_t &= \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-1}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}, \end{aligned} \tag{2.1}$$

where $h_t$ is the conditional variance of $\epsilon_t$ given $\psi_{t-1} = \{\epsilon_{t-1}, \epsilon_{t-2}, \ldots\}$, and $z_t$ is a sequence of iid random variables with mean 0 and variance 1. Moreover, assume that the parameters $\{\omega, \alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots, \beta_q\}$ are such that the GARCH process is stationary (cf. Nelson and Cao, 1992). If the $z_t$ are normal, then $\epsilon_t$ is generated by the $GARCH(p, q)$ process of Bollerslev (1986).

Assume that the error $z_t$ in (2.1) follows a distribution belonging to the family introduced by Vaughan (2002), which represents a generalization of the secant hyperbolic (GSH) distribution (see also Fischer, 2002 and Fischer and Vaughan, 2002). Here, however, we use a different form of the Vaughan formulation[1].

## 2.1 The symmetric GSH distribution

The density function of a random variable $X$ following a GSH distribution is

$$g_{\text{GSH}}(x) = \frac{c_1}{2(a + \cosh(c_2 x))}, \qquad x \in \mathbb{R}, \tag{2.2}$$

with

$$\begin{array}{llll} a = \cos(\lambda), & c_2 = \sqrt{\dfrac{\pi^2 - \lambda^2}{3}}, & c_1 = \dfrac{\sin(\lambda)}{\lambda} c_2, & \text{for } -\pi < \lambda < 0, \\ a = 1, & c_2 = \sqrt{\dfrac{\pi^2}{3}}, & c_1 = c_2, & \text{for } \lambda = 0, \\ a = \cosh(\lambda), & c_2 = \sqrt{\dfrac{\pi^2 + \lambda^2}{3}}, & c_1 = \dfrac{\sinh(\lambda)}{\lambda} c_2, & \text{for } \lambda > 0. \end{array}$$

We can immediately note that $X$ is symmetric around zero, because $g_{\text{GSH}}(x) = g_{\text{GSH}}(-x)$ for any value of $x$, and, consequently, it has zero mean; moreover, when $\lambda = 0$ it has a logistic distribution and when $\lambda \to \infty$ the density tends to that of the uniform distribution on $(-\sqrt{3}, \sqrt{3})$. The cumulative distribution function is

$$G_{\text{GSH}}(x) = \begin{cases} \frac{1}{2} + \frac{1}{\lambda} \tan^{-1}\left[\tan\left(\frac{\lambda}{2}\right) \tanh\left(\frac{c_2}{2} x\right)\right], & \lambda \in (-\pi, 0), \\ \frac{1}{2}\left[1 + \tanh\left(\frac{c_2}{2} x\right)\right], & \lambda = 0, \\ \frac{1}{2} + \frac{1}{\lambda} \tanh^{-1}\left[\tanh\left(\frac{\lambda}{2}\right) \tanh\left(\frac{c_2}{2} x\right)\right], & \lambda > 0, \end{cases}$$

from which we can say that the quantile function for $0 < p < 1$ is given by

$$G^{-1}_{\text{GSH}}(p) = \begin{cases} \frac{2}{c_2} \tanh^{-1}\left[\cot\left(\frac{\lambda}{2}\right) \tan\left(\frac{\lambda}{2}(2p-1)\right)\right], & \lambda \in (-\pi, 0), \\ \frac{2}{c_2} \tanh^{-1}(2p-1), & \lambda = 0, \\ \frac{2}{c_2} \tanh^{-1}\left[\coth\left(\frac{\lambda}{2}\right) \tanh\left(\frac{\lambda}{2}(2p-1)\right)\right], & \lambda > 0. \end{cases}$$

The moment generating function of $X$ is

$$M_X(u) = \begin{cases} \frac{\pi}{\lambda}\sin\left(\frac{\lambda}{c_2}u\right)\csc\left(\frac{\pi}{c_2}u\right), & \lambda \in (-\pi, 0), \\ \sqrt{3}u\csc(\sqrt{3}u), & \lambda = 0, \\ \frac{\pi}{\lambda}\sinh\left(\frac{\lambda}{c_2}u\right)\csc\left(\frac{\pi}{c_2}u\right), & \lambda > 0. \end{cases}$$

From the expansion of $M_X$, Vaughan (2001) obtained the first four moments of (2.2), showing that $X$ has unit variance and kurtosis given by

$$Ku = \begin{cases} \frac{21\pi^2 - 9\lambda^2}{5(\pi^2 - \lambda^2)}, & \lambda \in (-\pi, 0], \\ \frac{21\pi^2 + 9\lambda^2}{5(\pi^2 + \lambda^2)}, & \lambda > 0. \end{cases}$$

Note that $Ku$ decreases as $\lambda \to \infty$ and $1.8 < Ku < \infty$. In particular, when $\lambda = \pi$, $Ku = 3$, which is the kurtosis of the normal distribution.

## 2.2 The skew GSH distribution

In literature, many methods can be found in order to transform a symmetric distribution in a skewed one. In the present context, we apply to the density (2.2) the procedure used by Fernandéz and Steel (1998) to design a skew-$t$ distribution (see also Lambert and Laurent, 2001).

The density function of a random variable $X$ following a skew generalized secant hyperbolic (SGSH) distribution with skewness parameter $\gamma > 0$ is

$$g_{\mathsf{SGSH}}(x) = \frac{c_1}{(\gamma + \frac{1}{\gamma})(a + \cosh(c_2\gamma^{-\mathsf{sign}(x)}x))}, \qquad x \in \mathbb{R}, \tag{2.3}$$

where symbols are explained above. This density is symmetric for $\gamma = 1$, right skewed for $\gamma > 1$ and left skewed for $0 < \gamma < 1$. In fig. 1 we show the density function of the SGSH distribution for $\lambda = -2$ and $\gamma = 0.7,\ 1,\ 1.3$.

The cumulative probability function and the quantile function of this distribution can be expressed in terms of $G_{\mathsf{GSH}}$ and $G^{-1}_{\mathsf{GSH}}$. We have

$$G_{\mathsf{SGSH}}(x) = \begin{cases} \frac{2}{1+\gamma^2}\, G_{\mathsf{GSH}}(\gamma x), & x < 0, \\ 1 - \frac{2}{1+\gamma^{-2}}\, G_{\mathsf{GSH}}(-\gamma^{-1}x), & x \geq 0, \end{cases}$$

for the cumulative distribution function and

$$G^{-1}_{\mathsf{SGSH}}(p) = \begin{cases} \frac{1}{\gamma}\, G^{-1}_{\mathsf{GSH}}(\frac{p}{2}(1+\gamma^2)), & p < \frac{1}{1+\gamma^2}, \\ -\gamma\, G^{-1}_{\mathsf{GSH}}(\frac{1-p}{2}(1+\gamma^{-2})), & p \geq \frac{1}{1+\gamma^2}, \end{cases}$$

for the quantile function.

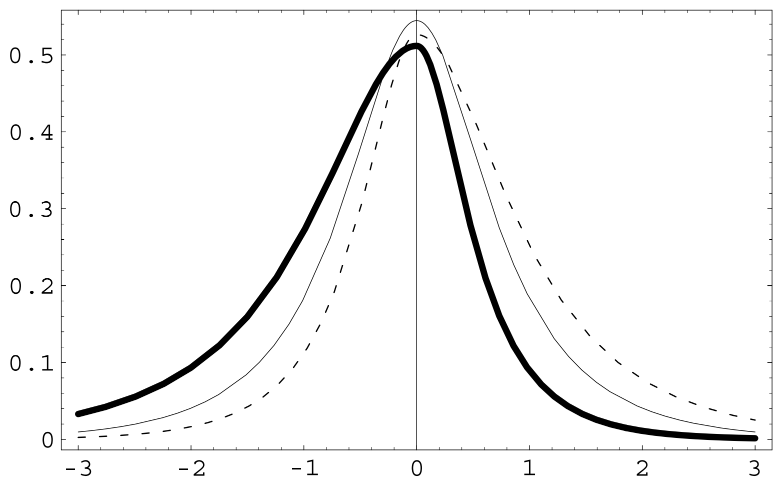


Figure 1: *Skew GSH densities with* $\lambda = -2$ *and* $\gamma = 0.7,\ 1,\ 1.3$ *(bold, plain, dashed)*

### 2.2.1 The moments

Following Fernandéz and Steel (1998), the moments of the SGSH distribution can be written as

$$E(X^r) = 2\,E^+(X^r)\frac{\gamma^{r+1} + \frac{(-1)^r}{\gamma^{r+1}}}{\gamma + \frac{1}{\gamma}}, \qquad r = 1, 2, \dots,$$

where

$$E^+(X^r) = \int_0^\infty x^r g_{\mathsf{GSH}}(x)dx$$

is the $r$-th moment of $g_{\mathsf{GSH}}$ truncated on real positive values. Evidently $E^+(X^r)$ assumes the value of the $r$-th moment of the GSH distribution divided by 2 when $r$ is even, being $g_{\mathsf{GSH}}$ symmetric. In order to write also the odd moments, we state the following proposition.

**Proposition 1** *If the random variable* $X$ *has a GSH distribution with density function given by (2.2), then, for* $\lambda \neq 0$ *and* $r = 1, 2, \dots,$

$$E^+(X^r) = \frac{c_1\,\Gamma(r+1)}{2c_2^{r+1}\sqrt{a^2-1}}\left[L_{r+1}\left(-\frac{1}{\sqrt{a^2-1}+a}\right) - L_{r+1}\left(\frac{1}{\sqrt{a^2-1}-a}\right)\right], \tag{2.4}$$

*while for* $\lambda = 0$

$$E^+(X^r) = \begin{cases} \frac{\ln 2}{c_2}, & r = 1, \\ \frac{1}{c_2^r}(1 - \frac{1}{2^{r-1}})\Gamma(r+1)L_r(1), & r > 1, \end{cases} \tag{2.5}$$

*where $\Gamma(\cdot)$ indicates the gamma function and $L_r(\cdot)$ is the polylogarithmic function whose primary definition is*[2]

$$L_r(w) = \sum_{k=1}^{\infty} \frac{w^k}{k^r}, \qquad |w| < 1,\ w \in \mathbb{C}.$$

**Proof.** Let $\lambda \neq 0$ and $r \in \mathbb{N}_+$. Writing the density (2.2) in exponential terms, we have

$$E^+(X^r) = c_1 \int_0^\infty \frac{x^r \mathrm{e}^{c_2 x}}{\mathrm{e}^{2c_2 x} + 2a\mathrm{e}^{c_2 x} + 1}\, dx = \frac{c_1}{c_2^{r+1}} \int_0^\infty \frac{u^r \mathrm{e}^u}{e^{2u} + 2a\mathrm{e}^u + 1}\, du.$$

Multiplying the latter integrand by $\frac{2\sqrt{a^2-1}}{\Gamma(r+1)}$, we can write:

$$\begin{aligned}
E^+(X^r) =& \frac{c_1\Gamma(r+1)}{2c_2^{r+1}\sqrt{a^2-1}} \int_0^\infty \frac{2\sqrt{a^2-1}u^r\mathrm{e}^u}{\Gamma(r+1)(a\mathrm{e}^u - \sqrt{a^2-1}\mathrm{e}^u + 1)(a\mathrm{e}^u + \sqrt{a^2-1}\mathrm{e}^u + 1)}\, du \\
=& \frac{c_1\Gamma(r+1)}{2c_2^{r+1}\sqrt{a^2-1}} \left( -\frac{1}{\Gamma(r+1)(\sqrt{a^2-1}+a)} \int_0^\infty \frac{u^r}{\mathrm{e}^u + \frac{1}{\sqrt{a^2-1}+a}}\, du \right. \\
& \left. - \frac{1}{\Gamma(r+1)(\sqrt{a^2-1}-a)} \int_0^\infty \frac{u^r}{\mathrm{e}^u - \frac{1}{\sqrt{a^2-1}-a}}\, du \right).
\end{aligned}$$

The two expressions in brackets are the integral representations of the polylogarithmic function values[3]

$$L_{r+1}\left(-\frac{1}{\sqrt{a^2-1}+a}\right) = -\frac{1}{\Gamma(r+1)(\sqrt{a^2-1}+a)} \int_0^\infty \frac{u^r}{\mathrm{e}^u + \frac{1}{\sqrt{a^2-1}+a}}\, du$$

and

$$L_{r+1}\left(\frac{1}{\sqrt{a^2-1}-a}\right) = \frac{1}{\Gamma(r+1)(\sqrt{a^2-1}-a)} \int_0^\infty \frac{u^r}{\mathrm{e}^u - \frac{1}{\sqrt{a^2-1}-a}}\, du,$$

from which the result (2.4) follows. Moreover, (2.5) can be obtained as the limit of (2.4) as $\lambda \to 0$.

■

At this point we can compute the first four moments $\mu_r = E(X^r)$, $r = 1, 2, 3, 4$, of the SGSH distribution with density function (2.3). Therefore, as $\lambda \neq 0$

$$\begin{aligned}
\mu_1 =& \frac{c_1}{c_2^2\sqrt{a^2-1}}\left[L_2\left(-\frac{1}{\sqrt{a^2-1}+a}\right) - L_2\left(\frac{1}{\sqrt{a^2-1}-a}\right)\right]\left(\gamma - \frac{1}{\gamma}\right),\\
\mu_2 =& \gamma^2 + \frac{1}{\gamma^2} - 1,\\
\mu_3 =& \frac{6\,c_1}{c_2^4\sqrt{a^2-1}}\left[L_4\left(-\frac{1}{\sqrt{a^2-1}+a}\right) - L_4\left(\frac{1}{\sqrt{a^2-1}-a}\right)\right]\\
&\cdot\left(\gamma^3 - \frac{1}{\gamma^3} - \gamma + \frac{1}{\gamma}\right),
\end{aligned}$$

$$\mu_4 = \begin{cases} \frac{21\pi^2-9\lambda^2}{5(\pi^2-\lambda^2)}\left(\gamma^4 + \frac{1}{\gamma^4} - \gamma^2 - \frac{1}{\gamma^2} + 1\right), & \lambda \in (-\pi, 0),\\ \frac{21\pi^2+9\lambda^2}{5(\pi^2+\lambda^2)}\left(\gamma^4 + \frac{1}{\gamma^4} - \gamma^2 - \frac{1}{\gamma^2} + 1\right), & \lambda > 0; \end{cases}$$

as $\lambda = 0$:

$$\begin{aligned}
\mu_1 =& \frac{\ln 4}{c_2}\left(\gamma - \frac{1}{\gamma}\right),\\
\mu_2 =& \gamma^2 + \frac{1}{\gamma^2} - 1,\\
\mu_3 =& \frac{9}{c_2^3}L_3(1)\left(\gamma^3 - \frac{1}{\gamma^3} - \gamma + \frac{1}{\gamma}\right),\\
\mu_4 =& \frac{21}{5}\left(\gamma^4 + \frac{1}{\gamma^4} - \gamma^2 - \frac{1}{\gamma^2} + 1\right).
\end{aligned}$$

Note that, since $E^+(X^r)$ is a real number for all $r \in \mathbb{N}_+$, the results obtained above in the set of complex numbers have the imaginary part equal to zero.

By means of standard relations it is immediate to derive the central moments using the moments about the origin and then obtain skewness and kurtosis of the SGSH distribution. Skewness and kurtosis implied by several combinations of $0.5 \leq \gamma \leq 1.5$ and $-3 \leq \lambda \leq 3$ are shown in fig. 2 and 3.

## 2.3 Maximum likelihood estimation

If $z_t$ in the equation (2.1) is GSH, indicating with $\theta$ the vector of parameters to be estimated, the loglikelihood function is $l(\theta) = \sum_t l_t(\theta)$, where

$$l_t(\theta) = \ln c_1 - \ln 2 - 0.5\ln h_t - \ln\left(a + \cosh(c_2\epsilon_t/\sqrt{h_t})\right).$$

In order to $z_t$ be SGSH, it is necessary to standardize the density (2.3). Then, on the basis of the previous section results, we can obtain the mean and variance of the SGSH

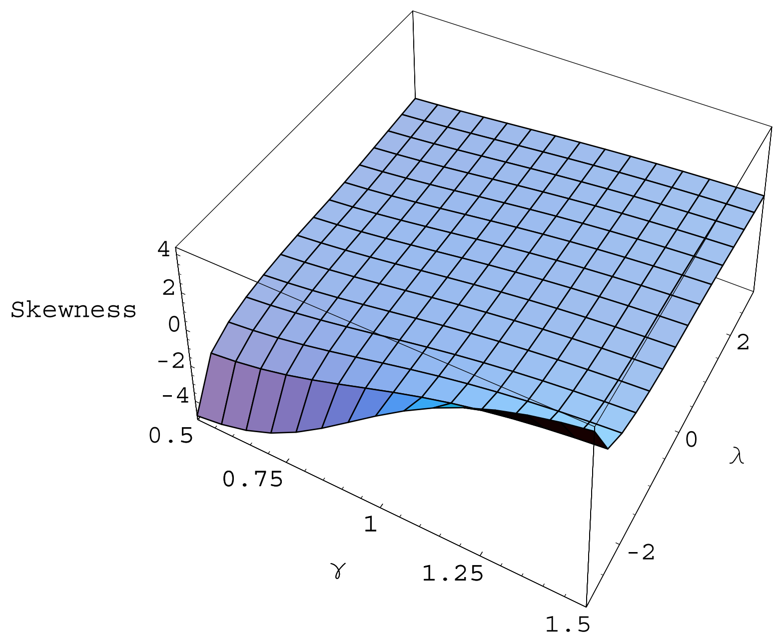


Figure 2: *Skewness implied by several combinations of* $0.5 \leq \gamma \leq 1.5$ *and* $-3 \leq \lambda \leq 3$

distribution as $\mu = \mu_1$ and $\sigma^2 = (\gamma + \frac{1}{\gamma^2} - 1) - \mu^2$, respectively; therefore, we consider the transformation

$$z_t = \frac{X - \mu}{\sigma}.$$

Consequently, $z_t$ will have zero mean and unit variance with density function

$$f_{\mathsf{SGSH}}(z_t) = \frac{\sigma c_1}{(\gamma + \frac{1}{\gamma})[a + \cosh(c_1 \gamma^{-\mathsf{sign}(\mu + \sigma z_t)}(\mu + \sigma z_t))]},$$

from which it follows that the maximum likelihood estimation of the unknown parameters in $\theta$ of a $GARCH(p,q)$ model with innovations SGSH on the basis of $T$ observations, can be obtained maximizing the loglikelihood function $l(\theta)$ with components

$$l_t(\theta) = \ln \sigma + \ln c_1 - \ln(\gamma + 1/\gamma) - 0.5 \ln h_t - \ln(a + \cosh(c_1 \gamma^{-I_t}(\mu + \sigma \epsilon_t / \sqrt{h_t}))),$$

where $I_t = \mathsf{sign}(\mu + \sigma \epsilon_t / \sqrt{h_t})$.

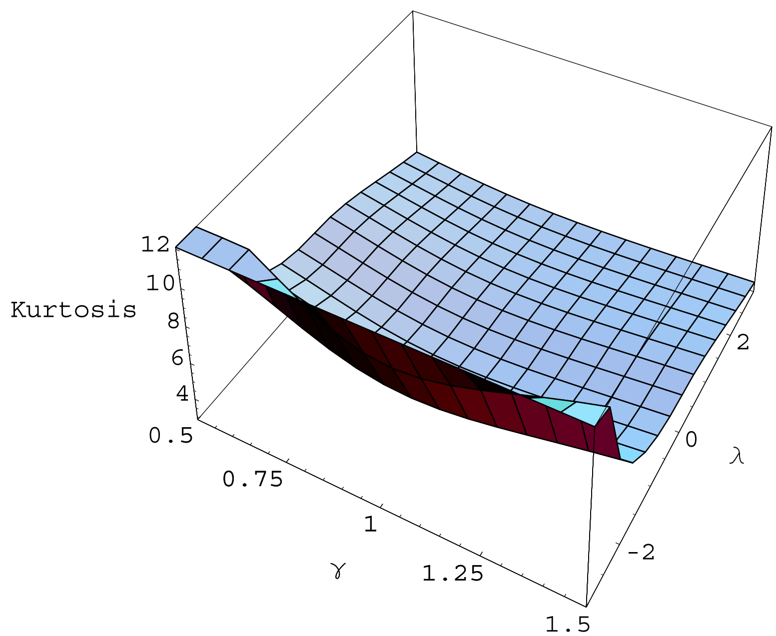


Figure 3: *Kurtosis implied by several combinations of* $0.5 \leq \gamma \leq 1.5$ *and* $-3 \leq \lambda \leq 3$

## 3 A Monte Carlo Experiment

Maximum likelihood estimation of the unknown parameters of a GARCH-type model with density function of the innovations completely specified produces optimal estimators under a set of regularity conditions. Maximum likelihood estimators are consistent and asymptotically efficient since they achieve the Cramér-Rao lower bound (cf. González-Rivera and Drost, 1999). However, little is known on the properties of these estimators when we observe a finite segment of (2.1), mainly when the stochastic component is not normal. Therefore, we have done a Monte Carlo experiment using a skew generalized secant hyperbolic distribution as data generating process with the aim to analyze the sampling properties of the maximum likelihood estimators of the parameters of a GARCH model.

The model considered is $GARCH(1,1)$:

$$\begin{aligned} y_t =& \mu + \epsilon_t, \\ \epsilon_t =& \sqrt{h_t} z_t, \\ h_t =& \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 h_{t-1}, \end{aligned}$$

with $z_t$ normal, symmetric (TST) and skew (STST) Student-$t$ (cf. Lambert and Laurent, 2000), symmetric (GSH) and skew (SGSH) generalized secant hyperbolic and $t = 1, 2, \ldots, T$, $T = 500, 1000, 2000, 3000$. The parameters of the model are typical of the empirical literature concerning financial returns: $\mu = 0$, $\omega = 0.01$, $\alpha_1 = 0.1$, $\beta_1 = 0.7$. The Monte Carlo experiment has been done running 5000 times the process generating data with $z_t$ standardized SGSH and skewness and kurtosis equals to 0.5 and 5, respectively. In order to avoid the start-up problem, for each replication we have rejected the first 1000 realizations. The results of the experiment are shown in tables 1 and 2 where the parameters of the model are reported in the second column, while mean and, in brackets, the standard deviation of their maximum likelihood estimation with the various specifications of $z_t$ are in the following columns. The last two lines report the generated and estimated skewness and kurtosis[4].

From the tables, it is clear that when the error distribution, that is the skew generalized secant hyperbolic distribution, is correctly specified, the maximum likelihood method works quite well for segments longer than $T = 2000$, in relation to which the bias of the estimators of $\beta_1$ and of the parameter governing kurtosis are remarkably reduced. However, there is an important analogy between the behaviour of the maximum likelihood estimators of the Student-$t$ distribution and the one of the generalized secant hyperbolic distribution. This fact confirms what we have underlined in the first section. Finally, the two tables show also the well known result by Weiss (1986) and Bollerslev and Wooldridge (1992): if mean and variance are correctly specified, the quasi-maximum likelihood estimators under the assumption of normality are consistent, but less efficient than the maximum likelihood estimators obtained with the correct specification of the error term.

## 4 An Application

The results of the previous section have been validated with an application to the returns of the MIBTEL index of Milan Stock Exchange of a GARCH-type model with normal, Student-$t$ and generalized secant hyperbolic innovations. In particular, we have considered daily returns from July 1993 to April 2003, a total of $T = 2532$ observations. The series have been transformed in log and multiplied by 100. Data are plotted in fig. 4, while the nonparametric density with Epanechnikov kernel of the unconditional distribution of the returns is given in fig. 5[5]. From the latter figure, also if the unconditional distribution has fat tails, it is difficult to identify the shape of the distribution. Some summary statistics and independence tests are shown in tab. 3. The sample skewness and kurtosis for these data are, respectively, -0.1424 and 4.9989 and the Jarque-Bera (J-B) normality test statistics is 429.749, which indicates high non-normality. There is also autocorrelation in returns as the Q-statistics by Box-Pierce quoted in the same table shows. However, an $AR(1)$ model can take account for this serial autocorrelation.

Therefore, the model hypothesized to interpret dynamics in mean and volatility of the daily log returns of the MIBTEL index is a $AR(1) - APARCH(1, 1)$ (see Ding, Granger

| | $T = 500$ | | | | | |
|---|---|---|---|---|---|---|
| | DGP | normal | TST | STST | GSH | SGSH |
| $\mu$ | 0.0 | -0.0004 | -0.0090 | 0.0003 | -0.0098 | $0.9634\ 10^{-5}$ |
| | | (0.0097) | (0.0090) | (0.0097) | (0.0091) | (0.0096) |
| $\omega$ | 0.01 | 0.0164 | 0.0158 | 0.0173 | 0.0148 | 0.0162 |
| | | (0.0126) | 0.0120 | (0.0135) | (0.0111) | (0.0125) |
| $\alpha_1$ | 0.1 | 0.1090 | 0.1081 | 0.1080 | 0.1057 | 0.1071 |
| | | (0.0641) | (0.0589) | (0.0595) | (0.0565) | (0.0566) |
| $\beta_1$ | 0.7 | 0.5530 | 0.5757 | 0.5455 | 0.5895 | 0.5605 |
| | | (0.2838) | (0.2656) | (0.2823) | (0.2525) | (0.2727) |
| Skew Parameter | 1.1783 | - | - | 1.1802 | - | 1.1802 |
| | | | | (0.0738) | | (0.0733) |
| Kurt Parameter | -1.4002 | - | 6.2342 | 6.4849 | -1.0363 | -1.0227 |
| | | | (3.4438) | (3.8754) | (0.8752) | (0.84991) |
| Skewness | 0.5 | 0 | 0 | 0.5890 | 0 | 0.4840 |
| Kurtosis | 5.0 | 3 | 9.4029 | 10.572 | 4.7285 | 4.9250 |
| | $T = 1000$ | | | | | |
| | DGP | normal | TST | STST | GSH | SGSH |
| $\mu$ | 0.0 | -0.0003 | -0.0094 | 0.0001 | -0.0103 | -0.0001 |
| | | (0.0068) | (0.0064) | (0.0068) | (0.0065) | (0.0067) |
| $\omega$ | 0.01 | 0.0132 | 0.0128 | 0.0132 | 0.0123 | 0.0127 |
| | | (0.0089) | (0.0082) | (0.0088) | (0.0074) | (0.0081) |
| $\alpha_1$ | 0.1 | 0.1055 | 0.1053 | 0.1060 | 0.1025 | 0.1041 |
| | | (0.0445) | (0.0400) | (0.0400) | (0.0388) | (0.0384) |
| $\beta_1$ | 0.7 | 0.6257 | 0.6417 | 0.6345 | 0.6482 | 0.6398 |
| | | (0.2034) | (0.1836) | (0.1919) | (0.1726) | (0.1837) |
| Skew Parameter | 1.1783 | - | - | 1.1770 | - | 1.1770 |
| | | | | (0.5076) | | (0.5044) |
| Kurt Parameter | -1.4002 | - | 5.6880 | 5.8805 | -1.2639 | -1.2372 |
| | | | (1.0487) | (1.1025) | (0.5387) | (0.5288) |
| Skewness | 0.5 | 0 | 0 | 0.5988 | 0 | 0.4908 |
| Kurtosis | 5.0 | 3 | 7.8827 | 8.6542 | 4.7928 | 4.9741 |

Table 1: *Results of the Monte Carlo experiment with* $T = 500, 1000$. *Data Generating Process:* $y_t = \epsilon_t = \sqrt{h_t} z_t$, $h_t = 0.01 + 0.1\epsilon_{t-1}^2 + 0.7h_{t-1}$ *and* $z_t$ *SGSH with skewness and kurtosis, respectively, 0.5 and 5.*

| | $T = 2000$ | | | | | |
|---|---|---|---|---|---|---|
| | DGP | normal | TST | STST | GSH | SGSH |
| $\mu$ | 0.0 | -0.3904 $10^{-4}$ | -0.0094 | 0.0003 | -0.0104 | 0.32479 $10^{-4}$ |
| | | (0.0049) | (0.0045) | (0.0049) | (0.0046) | (0.0048) |
| $\omega$ | 0.01 | 0.0112 | 0.0111 | 0.0111 | 0.0108 | 0.0109 |
| | | (0.0050) | (0.0045) | (0.0044) | (0.0041) | (0.0043) |
| $\alpha_1$ | 0.1 | 0.1026 | 0.1030 | 0.1039 | 0.1005 | 0.1019 |
| | | (0.0305) | (0.0276) | (0.0271) | (0.0268) | (0.0264) |
| $\beta_1$ | 0.7 | 0.6730 | 0.6801 | 0.6802 | 0.6811 | 0.6801 |
| | | (0.1163) | (0.1021) | ( 0.1003) | (0.0977) | (0.0993) |
| Skew Parameter | 1.1783 | - | - | 1.1778 | - | 1.1778 |
| | | | | (0.0353) | | (0.0350) |
| Kurt Parameter | -1.4002 | - | 5.5027 | 5.6863 | -1.3704 | -1.3416 |
| | | | (0.6672) | (0.6980) | (0.2972) | (0.2963) |
| Skewness | 0.5 | 0 | 0 | 0.6084 | 0 | 0.4993 |
| Kurtosis | 5.0 | 3 | 7.4583 | 8.1413 | 4.8198 | 4.9993 |
| | $T = 3000$ | | | | | |
| | DGP | normal | TST | STST | GSH | SGSH |
| $\mu$ | 0.0 | -0.2168 $10^{-4}$ | -0.0094 | 0.0003 | -0.0104 | 0.3370 $10^{-4}$ |
| | | (0.0040) | (0.0037) | (0.0039) | (0.0038) | (0.0039) |
| $\omega$ | 0.01 | 0.0106 | 0.0107 | 0.0107 | 0.0105 | 0.0105 |
| | | (0.0036) | (0.0032) | (0.0030) | (0.0031) | (0.0030) |
| $\alpha_1$ | 0.1 | 0.1009 | 0.1023 | 0.1031 | 0.9976 | 0.1009 |
| | | (0.0245) | (0.0223) | (0.0218) | (0.0216) | (0.0211) |
| $\beta_1$ | 0.7 | 0.6856 | 0.6888 | 0.6888 | 0.6889 | 0.6890 |
| | | (0.0859) | (0.0747) | (0.0723) | (0.0745) | (0.0718) |
| Skew Parameter | 1.1783 | - | - | 1.1774 | - | 1.1774 |
| | | | | (0.0284) | | (0.0283) |
| Kurt Parameter | -1.4002 | - | 5.4745 | 5.6529 | -1.3882 | -1.3608 |
| | | | (0.5310) | (0.5543) | (0.2228) | (0.2220) |
| Skewness | 0.5 | 0 | 0 | 0.6053 | 0 | 0.4995 |
| Kurtosis | 5.0 | 3 | 7.1183 | 7.7293 | 4.8167 | 4.9960 |

Table 2: *Results of the Monte Carlo experiment with $T = 2000, 3000$. Data Generating Process: $y_t = \epsilon_t = \sqrt{h_t} z_t$, $h_t = 0.01 + 0.1\epsilon_{t-1}^2 + 0.7h_{t-1}$ and $z_t$ SGSH with skewness and kurtosis, respectively, 0.5 and 5. Standard deviations of the estimated parameters are reported in brackets.*

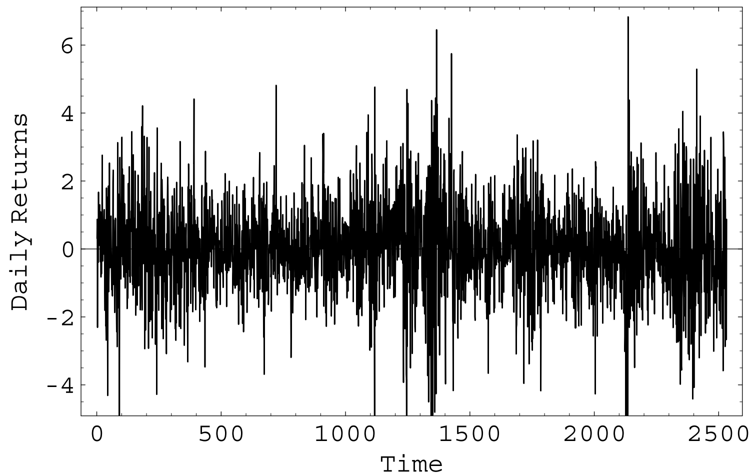


Figure 4: *Daily log returns of the MIBTEL index from July 1993 to April 2003*

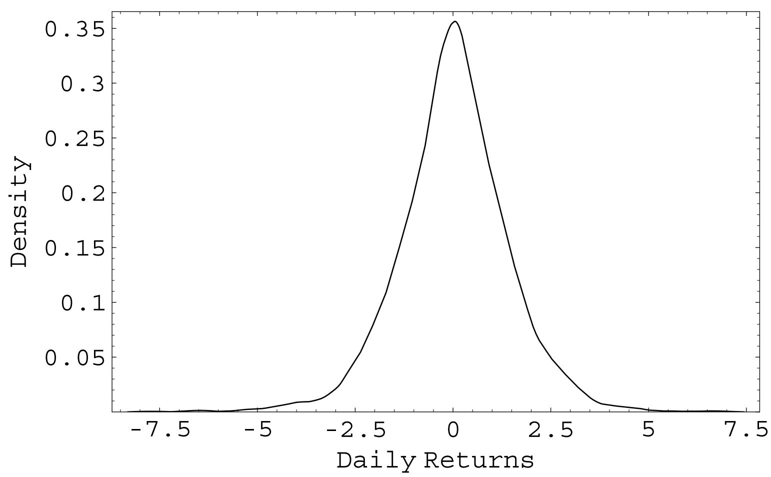


Figure 5: *Nonparametric density of MIBTEL data*

| | |
|---|---|
| Number of observations | 2532 |
| Mean | 0.0226 |
| Standard deviation | 1.3952 |
| Skewness | -0.1424 |
| Kurtosis | 4.9989 |
| J-B normality test statistic | 429.749 |
| Minimum | -7.7075 |
| Maximum | 6.8318 |
| Q(12) | 24.7652 |
| Q(24) | 51.1403 |
| Q(36) | 61.5989 |

Table 3: *Summary statistics of the daily log returns of the MIBTEL index (July, 1993-April, 2003)*

and Engle, 1993) defined as

$$\begin{aligned} y_t =& \mu + \rho y_{t-1} + \epsilon_t, \\ \epsilon_t =& \sigma_t z_t, \\ \sigma_t^\delta =& \omega + \alpha_1 (|\epsilon_{t-1}| - \phi\, \epsilon_{t-1})^\delta + \beta_1 \sigma_{t-1}^\delta, \end{aligned} \tag{4.1}$$

where $\omega > 0$, $\delta \geq 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and $-1 < \phi < 1$. This model is quite interesting since it includes seven ARCH extensions as special cases and couples the flexibility of the varying exponent $\delta$ with the skewness coefficient $\phi$ which accounts for the "leverage effect". Following Ding, Granger and Engle (1993), a stationary solution of (4.1) is given by

$$\mathsf{E}(\sigma_t^\delta) = \frac{\omega}{1 - \alpha_1 E(|z| - \phi z)^\delta - \beta_1}$$

and it depends on the density of $z$. Such a solution exists if $\alpha_1 \mathsf{E}(|z| - \phi z)^\delta + \beta_1 < 1$ and it can be seen as a measure of volatility persistence.

Table 4 presents the results of the maximum likelihood estimation of the parameters of the considered model under the various assumptions on innovations. In brackets, parametric bootstrap standard errors are reported (cf. Efron and Tibshirani, 1993, p. 306). In the table we present also (see the last four rows) the value of the averaged loglikelihood function at maximum, the measure of volatility persistence, skewness and kurtosis, for each specification of the model innovations. Note that

i) The volatility of the examined financial series requires the flexibility of the APARCH model, because it is evident that the estimates of the skewness coefficient ($\phi$) and of the power parameter ($\delta$) suggest that the usual GARCH model is not appropriate;

ii) $AR(1) - APARCH(1,1)$ seems adequate to describe the dynamics of the first two moments of the returns for the period of interest;

| | normal | TST | STST | GSH | SGSH |
|---|---|---|---|---|---|
| $\mu$ | 0.0126 | 0.0147 | 0.0106 | 0.0152 | 0.0108 |
| | (0.0238) | (0.0237) | (0.0247) | (0.0234) | (0.0248) |
| $\rho$ | 0.0495 | 0.0362 | 0.0349 | 0.0353 | 0.0340 |
| | (0.0197) | (0.0201) | (0.0202) | (0.0198) | (0.0203) |
| $\omega$ | 0.0927 | 0.0962 | 0.0962 | 0.0967 | 0.0966 |
| | (0.0212) | (0.0234) | (0.0227) | (0.0231) | (0.0226) |
| $\alpha_1$ | 0.0827 | 0.0962 | 0.0962 | 0.0967 | 0.0966 |
| | (0.0345) | (0.0294) | (0.0274) | (0.0283) | (0.0297) |
| $\phi$ | 0.1626 | 0.1972 | 0.1940 | 0.1988 | 0.1953 |
| | (0.0545) | (0.0655) | (0.0624) | (0.0638) | (0.0651) |
| $\beta_1$ | 0.8311 | 0.8410 | 0.8414 | 0.8406 | 0.8411 |
| | (0.0266) | (0.0243) | (0.0239) | (0.0241) | (0.0237) |
| $\delta$ | 2.6820 | 2.4519 | 2.4565 | 2.4623 | 2.4670 |
| | (0.5550) | (0.5284) | (0.5265) | (0.5207) | (0.5325) |
| Skew Parameter | - | - | 0.9791 | - | 0.9788 |
| | | | (0.0286) | | (0.0282) |
| Kurt Parameter | - | 11.3880 | 11.3244 | 1.2912 | 1.2877 |
| | | (2.8372) | (1.8896) | (0.4041) | (0.4387) |
| Averaged LogLik | -1.6719 | -1.6653 | -1.6652 | -1.6656 | -1.6655 |
| $\alpha_1 \mathrm{E}(\lvert z\rvert - \phi z)^\delta + \beta_1$ | 0.9634 | 0.9689 | 0.9706 | 0.9705 | 0.9722 |
| Skewness | 0 | 0 | -0.0486 | - | -0.0504 |
| Kurtosis | 3 | 3.8121 | 3.8214 | 3.8532 | 3.8571 |

Table 4: *Estimation results.* Averaged LogLik *refers to the averaged loglikelihood value at maximum.*

iii) the stationarity condition of the APARCH model is satisfied for the five distributions, because $\alpha_1 \mathsf{E}(|z| - \phi z)^\delta + \beta_1 < 1$ at the maximum of loglikelihood functions;

iv) as expected, the model with nonnormal innovations interprets better the features of the series, as can be seen from the value of averaged loglikelihood function;

v) the skewness coefficients of the unconditional distributions STST and SGSH considerably reduce with respect to the skewness coefficient shown in tab. 3, therefore it seems that the skewness of the APARCH model, which characterizes the conditional variance, is sufficient to explain the global skewness of the series.

Finally, it is important to underline again that the estimates of the parameters of the model with Student-$t$ or generalized secant hyperbolic innovations, both symmetric and skewed, are very similar, confirming what we have seen in the preceding section.

## 5 Conclusions

In this paper we have introduced the generalized secant hyperbolic distribution in the GARCH context, also with a reparameterization of the density following the method proposed by Fernández and Steel (1998) in order to account for skewness. Using Monte Carlo simulations, we have seen that the maximum likelihood estimate of the parameters of a $GARCH(1,1)$ model with generalized secant hyperbolic innovations is similar to the one obtained with Student-$t$ innovations, confirming the results by Vaughan (2002) in a different context. The same result has been obtained with an application to the MIBTEL index of a $AR(1) - APARCH(1,1)$ model. The advantage of using in a GARCH context the generalized secant hyperbolic for the innovations with respect to the others distributions with heavy tails, in particular the Student-$t$, consists in the fact that the moments of this distribution exist and can be represented in explicit form for all values of the shape parameters and can be therefore easily used to account for the high skewness and kurtosis characterizing financial series.

## Notes

[1] The results have been obtained using *MathStatica* (Rose and Smith, 2002), a software package of the symbolic software *Mathematica* (Wolfram, 1999).

[2] See at the web page http://functions.wolfram.com/10.08.02.0001.01

[3] See at the web page http://functions.wolfram.com/10.08.07.0001.01

[4] The results of simulations and of the application shown in the next section have been obtained using a FORTRAN 90 code implemented on a 2300 MHz PC Intel on Windows 2000; the code uses the random number generator and the optimization routines of the maximum likelihood estimates of the NAG Fortran library.

[5] The bandwidth of the nonparametric density has been obtained with the Sheather and Jones method (1991).

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