n-COTILTING AND n-TILTING MODULES OVER RING EXTENSIONS

ALBERTO TONOLO

Abstract. When Γ is a ring extension of R, conditions are found to insure that the notion of n-tilting module and n-cotilting module pass from a left R-module V to the induced left Γ-modules $\mathrm{Tor}_i^R(\Gamma, V)$ and $\mathrm{Ext}_R^i(\Gamma, V)$, $0 \le i \le n$.

Introduction

We say that Γ is a ring extension of R if there is a ring homomorphism $\xi : R \to \Gamma$. Every left Γ-module has, through ξ, a natural structure as left R-module; conversely, given a left R-module V, $\mathrm{Hom}_R(\Gamma, V)$ and $\Gamma \otimes V$ are the two canonical ways to obtain Γ-modules. Many properties of V which involve the functor $\mathrm{Hom}_R(-, V)$ are known to be inherited by $\mathrm{Hom}_R(\Gamma, V)$: some of these, as "injective", "cogenerator" are inherited without further conditions, others, as "cotilting module", "quasi-duality module" require various conditions on V and/or $\mathrm{Hom}_R(\Gamma, V)$ (see [16]). Dually, many properties of V which involve the functor $\mathrm{Hom}_R(V, -)$ are known to be inherited by $\Gamma \otimes V$: some of these, as "projective", "generator" are inherited without further conditions, others, as "tilting module", "$*$-module" require various conditions on V and/or $\Gamma \otimes V$ (see e.g. [2], [15], [17]).

Let us concentrate on the notion of tilting and cotilting modules. Following [3], we say that a left R-module V is *n-tilting* (resp. *n-cotilting*) if it satisfies the following properties

1. $\mathrm{pd}\, V \le n$ (resp. $\mathrm{id}\, V \le n$).
2. $\mathrm{Ext}_R^i(V, V^{(\lambda)}) = 0$ (resp. $\mathrm{Ext}_R^i(V^\lambda, V) = 0$) for every cardinal λ and each $i \ge 1$.
3. There exists an exact sequence $0 \to R \to V_0 \to V_1 \to \ldots \to V_{n-1} \to V_n \to 0$ (resp. $0 \to V_n \to V_{n-1} \to \ldots \to V_1 \to V_0 \to E \to 0$ where E is an injective cogenerator of R-Mod) and V_i belongs to $\mathrm{Add}\, V$ (resp. $\mathrm{Prod}\, V$), i.e. the V_i are direct summands of arbitrary sums (resp. products) of copies of V.

Let ${}_RV$ be an n-tilting (resp. n-cotilting) module. We say that a left R-module M has *V-grade* (resp. *V-cograde*) m if $\mathrm{Ext}_R^i(V, M) = 0$ (resp. $\mathrm{Ext}_R^i(M, V) = 0$) for each $i \ne m$.

If $n = 0$, V is a progenerator (resp. injective cogenerator): in such a case each left R-module has V-grade 0 (resp. V-cograde 0).

If $n = 1$, V gives rise to a torsion theory $(\mathcal{T}, \mathcal{F})$ in the category of left R-modules, where $\mathcal{T}$ is the class of left R-modules of V-grade 0 and $\mathcal{F}$ is the class of modules of V-grade 1.

Developed originally for modules over finite dimensional algebras, simply as dual of tilting modules, the 1-cotilting modules acquired a proper independent role in the more general setting of modules over arbitrary associative rings (see [3], [5], [8], [11], [12], [21], [18]). If V is a 1-cotilting module, it gives rise to a torsion theory $(\mathcal{T}, \mathcal{F})$ in the category of left R-modules, where $\mathcal{T}$ is the class of left R-modules of V-cograde 1 and $\mathcal{F}$ is the class of modules of V-cograde 0.

The n-tilting and n-cotilting modules with $n \ge 2$ are extensively studied in [3] in relation with the theories of covers and envelopes, in [4] in connection with the finitistic dimension conjectures, in [19] and [20] in connection with the equivalence and duality theories induced by them.

Date: December 2, 2003.
Key words and phrases. n-Cotilting module, n-Tilting module, Ring extension
2000 *Mathematics Subject Classification.* Primary 16D90; Secondary 16E30.

Let Γ be a ring extension of R and E be an injective cogenerator of R-Mod. If ${}_RV$ is an n-tilting R-module and $({}_R\Gamma)^* := \mathrm{Hom}_R(\Gamma, E)$ has V-grade 0, set ${}_\Gamma U = \Gamma \otimes_R V$. In [17, Theorem 5.2], Miyashita proved, with V finitely presented n-tilting module, that if also ${}_RU$ has V-grade 0, then ${}_\Gamma U$ is tilting. But, what happens if Γ^* has V-grade > 0? and what can we say about the properties "inherited" by the left Γ-modules $\mathrm{Tor}_i^R(\Gamma, V)$, $i > 0$?

If ${}_RV$ is a 1-cotilting R-module, and ${}_R\Gamma$ has V-cograde 0, set ${}_\Gamma U = \mathrm{Hom}_R(\Gamma, V)$. In [16, Proposition 4.8], Fuller has proved that if also ${}_RU$ has V-cograde 0, then ${}_\Gamma U$ is 1-cotilting. But, what happens if Γ has V-cograde > 0? and what can we say about the properties "inherited" by the left Γ-modules $\mathrm{Ext}_R^i(\Gamma, V)$, $i > 0$?

This questions have initiated our paper.

In the first section we prove some homological lemmas; these are the key tools for the sequel.

In the second section we prove our main results. Let ${}_RV$ be a *n-tilting* module and Γ be a ring extension of R such that $({}_R\Gamma)^*$ has V-grade m. Denote by U the left Γ-module $\mathrm{Tor}_m^R(\Gamma, V)$. We prove that ${}_\Gamma U$ is $(n-m)$-tilting if and only if ${}_RU$ has V-grade m. Analogously, if ${}_RV$ is a *n-cotilting* module and Γ is a ring extension of V-cograde m, we denote by U the left Γ-module $\mathrm{Ext}_R^m(\Gamma, V)$. We prove that ${}_\Gamma U$ is $(n-m)$-cotilting if and only if ${}_RU$ has V-cograde m.

Finally, applications of our results to split extension rings are given.

1. Homological Lemmas

Let Γ and R be two arbitrary associative rings with $1 \neq 0$. Consider a left Γ-module X, a left R-module Z, a R-Γ-bimodule ${}_RY_\Gamma$ and a Γ-R-bimodule ${}_\Gamma W_R$, i.e. the situations described by the symbols $({}_\Gamma X, {}_RY_\Gamma, {}_RZ)$ and $({}_\Gamma X, {}_\Gamma W_R, {}_RZ)$.

Lemma 1.1. $({}_\Gamma X, {}_RY_\Gamma, {}_RZ)$. *Let us assume* Y_Γ *or* ${}_\Gamma X$ *is flat and* $\mathrm{Ext}_R^i(Y, Z) = 0$ *for each* $0 \leq i < m$*; then*

(1) $\mathrm{Ext}_R^i(Y \otimes_\Gamma X, Z) = 0$ *for each* $0 \leq i < m$.
(2) $\mathrm{Hom}_\Gamma(X, \mathrm{Ext}_R^m(Y, Z))$ *is naturally isomorphic to* $\mathrm{Ext}_R^m(Y \otimes X, Z)$.
(3) *If* $\mathrm{Ext}_R^i(Y, Z) = 0$ *for each* $i \neq m$, *then* $\mathrm{Ext}_\Gamma^l(X, \mathrm{Ext}_R^m(Y, Z))$ *is naturally isomorphic to* $\mathrm{Ext}_R^{l+m}(Y \otimes X, Z)$ *for each* $l \geq 1$.

Proof. Consider the short exact sequence of left Γ-modules

$$(\sharp) \qquad 0 \to K \to \Gamma^{(\alpha)} \to X \to 0.$$

Since Y_Γ or ${}_\Gamma X$ is flat, then also

$$(\sharp\sharp) \qquad 0 \to Y \otimes K \to Y^{(\alpha)} \to Y \otimes X \to 0$$

is exact. If X is flat, also K is flat; hence K verifies any result we can prove for X.

1. It follows by induction from the long exact sequence obtained applying $\mathrm{Hom}_R(-, Z)$ to $(\sharp\sharp)$.

2. If $m = 0$, our claim is satisfied by the usual adjunction with the unnecessary hypothesis of flatness of Y_Γ or ${}_\Gamma X$ (the other hypothesis is empty in this case). Let $m \geq 1$. Let $0 \to Z \to I^\bullet$ be an injective resolution of Z; we denote by Z_i the i-th cosyzygy of Z with $Z_0 = Z$. Applying the functor $\mathrm{Hom}_R(Y, -)$ to the short exact sequence

$$0 \to Z_{m-1} \to I_{m-1} \to Z_m \to 0$$

we get the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Hom}_R(Y, Z_{m-1}) & \longrightarrow & \mathrm{Hom}_R(Y, I_{m-1}) & \longrightarrow & \mathrm{Hom}_R(Y, Z_m) & \longrightarrow & \mathrm{Ext}_R^1(Y, Z_{m-1}) & \longrightarrow 0 \\ & & & \downarrow\!\!\downarrow & \nearrow & & & & \\ & & & L & & & & & \end{array}$$

Applying the functor $\mathrm{Hom}_\Gamma(X,-)$ we obtain the exact sequences

(1) $0 \to \mathrm{Hom}_\Gamma(X, \mathrm{Hom}_R(Y, Z_{m-1})) \to \mathrm{Hom}_\Gamma(X, \mathrm{Hom}_R(Y, I_{m-1})) \to \mathrm{Hom}_\Gamma(X, L) \to$

$\to \mathrm{Ext}^1_\Gamma(X, \mathrm{Hom}_R(Y, Z_{m-1})) \to \mathrm{Ext}^1_\Gamma(X, \mathrm{Hom}_R(Y, I_{m-1})) \to \mathrm{Ext}^1_\Gamma(X, L) \to$
$\to \mathrm{Ext}^2_\Gamma(X, \mathrm{Hom}_R(Y, Z_{m-1}))$

(2) $0 \to \mathrm{Hom}_\Gamma(X, L) \to \mathrm{Hom}_\Gamma(X, \mathrm{Hom}_R(Y, Z_m)) \to \mathrm{Hom}_\Gamma(X, \mathrm{Ext}^1_R(Y, Z_{m-1})) \to$
$\to \mathrm{Ext}^1_\Gamma(X, L)\,.$

Let us see that

(a) $\mathrm{Ext}^j_\Gamma(X, \mathrm{Hom}_R(Y, Z_{m-1})) = 0$, $j \geq 1$, and
(b) $\mathrm{Ext}^1_\Gamma(X, L) = 0$.

(a) If $m = 1$, then $Z_{m-1} = Z_0 = Z$; since by the assumptions $\mathrm{Hom}_R(Y, Z) = 0$, we get the thesis. If $m \geq 2$, the thesis is obtained by dimension shifting using the fact that by [9, Proposition VI.5.1] and our hypothesis, for each injective left R-module ${}_RI$ and each $i \geq 1$ we have

$$\mathrm{Ext}^i_\Gamma(X, \mathrm{Hom}_R(Y, I)) \cong \mathrm{Hom}_R(\mathrm{Tor}^\Gamma_i(Y, X), I) = 0.$$

(b) Since by (a) $\mathrm{Ext}^2_\Gamma(X, \mathrm{Hom}_R(Y, Z_{m-1})) = 0$, from the exact sequence (1), we get $\mathrm{Ext}^1_\Gamma(X, L) = 0$.

Now composing the exact sequences (1) and (2) we obtain the following commutative diagram with exact rows, where the vertical arrows are natural isomorphisms

$$\begin{array}{ccccccc}
\mathrm{Hom}_\Gamma(X, \mathrm{Hom}_R(Y, I_{m-1})) & \to & \mathrm{Hom}_\Gamma(X, \mathrm{Hom}_R(Y, Z_m)) & \to & \mathrm{Hom}_\Gamma(X, \mathrm{Ext}^1_R(Y, Z_{m-1})) & \to & 0 \\
\downarrow \cong & & \downarrow \cong & & & & \\
\mathrm{Hom}_R(Y \otimes X, I_{m-1}) & \longrightarrow & \mathrm{Hom}_R(Y \otimes X, Z_m) & \longrightarrow & \mathrm{Ext}^1_R(Y \otimes X, Z_{m-1}) & \longrightarrow & 0
\end{array}$$

Thus we obtain the natural isomorphisms

$$\mathrm{Hom}_\Gamma(X, \mathrm{Ext}^m_R(Y, Z)) = \mathrm{Hom}_\Gamma(X, \mathrm{Ext}^m_R(Y, Z_0)) \cong \mathrm{Hom}_\Gamma(X, \mathrm{Ext}^1_R(Y, Z_{m-1})) \cong$$
$$\cong \mathrm{Ext}^1_R(Y \otimes X, Z_{m-1}) \cong \mathrm{Ext}^m_R(Y \otimes X, Z)\,.$$

3. Applying the functors $\mathrm{Hom}_\Gamma(-, \mathrm{Ext}^m_R(Y, Z))$ and $\mathrm{Ext}^m_R(-, Z)$ to the exact sequences ($\sharp$) and ($\sharp\sharp$) at the beginning of the proof, by (2) we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{Hom}_\Gamma(\Gamma^{(\alpha)}, \mathrm{Ext}^m_R(Y, Z)) & \longrightarrow & \mathrm{Hom}_\Gamma(K, \mathrm{Ext}^m_R(Y, Z)) & \longrightarrow & \mathrm{Ext}^1_\Gamma(X, \mathrm{Ext}^m_R(Y, Z)) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & & & \\
\mathrm{Ext}^m_R(Y^{(\alpha)}, Z) & \longrightarrow & \mathrm{Ext}^m_R(Y \otimes K, Z) & \longrightarrow & \mathrm{Ext}^{m+1}_R(Y \otimes X, Z) & \longrightarrow & 0;
\end{array}$$

then $\mathrm{Ext}^1_\Gamma(X, \mathrm{Ext}^m_R(Y, Z)) \cong \mathrm{Ext}^{m+1}_R(Y \otimes X, Z)$. Assume (3) holds for $l = j$; using the diagram

$$\begin{array}{ccc}
\mathrm{Ext}^j_\Gamma(K, \mathrm{Ext}^m_R(Y, Z)) & \xrightarrow{\cong} & \mathrm{Ext}^{j+1}_\Gamma(X, \mathrm{Ext}^m_R(Y, Z)) \\
\downarrow & & \\
\mathrm{Ext}^{m+j}_R(Y \otimes K, Z) & \xrightarrow{\cong} & \mathrm{Ext}^{m+j+1}_R(Y \otimes X, Z)
\end{array}$$

we get (3) for $j + 1$. Now we conclude by induction. ■

Analogously, dualizing the proof of the above lemma, it can be proved the following

Lemma 1.2. $({}_\Gamma X, {}_\Gamma W_R, {}_R Z)$. *Let us assume ${}_\Gamma W$ is projective or ${}_\Gamma X$ is injective, and* $\mathrm{Tor}^R_i(W, Z) = 0$ *for each* $0 \leq i < m$*; then*

(1) $\mathrm{Ext}^i_R(Z, \mathrm{Hom}_\Gamma(W, X)) = 0$ *for each* $0 \le i < m$.
(2) $\mathrm{Hom}_\Gamma(\mathrm{Tor}^R_m(W, Z), X)$ *is naturally isomorphic to* $\mathrm{Ext}^m_R(Z, \mathrm{Hom}_\Gamma(W, X))$.
(3) *If* $\mathrm{Tor}^R_i(W, Z) = 0$ *for each* $i \ne m$, *then* $\mathrm{Ext}^l_\Gamma(\mathrm{Tor}^R_m(W, Z), X)$ *is naturally isomorphic to* $\mathrm{Ext}^{l+m}_R(Z, \mathrm{Hom}_\Gamma(W, X))$ *for each* $l \ge 1$.

Given a left R-module H, and a natural number l we denote by ${}^{\perp_{>l}}H$ the class of all left R-modules M such that $\mathrm{Ext}^{l+i}_R(M, H) = 0$ for each $i > 0$, and by ${}^{\top_{>l}}H$ the class of all right R-modules N such that $\mathrm{Tor}^R_{l+i}(N, H) = 0$ for each $i > 0$. The following technical lemmas will be useful in the proof of our main theorems.

Lemma 1.3. *Let* ${}_RZ$ *and* ${}_RE$ *be left* R*-modules with* E *injective. Suppose to have a long exact sequence*

$$0 \to Z_n \overset{\phi_n}{\to} Z_{n-1} \overset{\phi_{n-1}}{\to} \dots \to Z_1 \overset{\phi_1}{\to} Z_0 \overset{\phi_0}{\to} E \to 0$$

with $Z_i \le^\oplus Z^{\alpha_i}$, $0 \le i \le n$, *for suitable cardinals* α_i. *Denote by* K_i *the kernel of* ϕ_i *and* $K_{-1} = E$.

(1) *If* $Y \in \mathrm{Ker}\,\mathrm{Ext}^l_R(-, Z)$, *then* $Y \in \mathrm{Ker}\,\mathrm{Ext}^l_R(-, Z_j)$ *for each* $0 \le j \le n$, *and* $l \ge 0$.
(2) *If* $Y \in {}^{\perp_{>l}}Z$, *then* $Y \in {}^{\perp_{>l}}K_j$ *for each* $0 \le j \le n-1$.

Assume $\mathrm{Ext}^i_R(Y, Z) = 0$ *for each* $i \ne m \ge 0$. *Then*

(3) $\mathrm{Ext}^i_R(Y, K_j) = 0$ *for* $i > m$ *and* j *arbitrary, or* $i < m$ *and* $i \ne j+1$.
(4) $\mathrm{Ext}^{i-1}_R(Y, K_{i-2}) \cong \mathrm{Hom}_R(Y, E)$ *for each* $i \le m$.

Proof. 1. $\mathrm{Ext}^l_R(Y, Z_j) \le^\oplus \mathrm{Ext}^l_R(Y, Z^{\alpha_j}) = (\mathrm{Ext}^l_R(Y, Z))^{\alpha_j} = 0$.

2. By (1), $Y \in {}^{\perp_{>l}}Z_j$ for each $0 \le j \le n$. Then, for each $i \ge 1$ we have $\mathrm{Ext}^{l+i}_R(Y, K_j) \cong$

$$\cong \mathrm{Ext}^{l+i+1}_R(Y, K_{j+1}) \cong \dots \cong \mathrm{Ext}^{l+i+n-j-1}_R(Y, K_{n-1}) = \mathrm{Ext}^{l+i+n-j-1}_R(Y, Z_n) = 0.$$

3 and 4. By (2), $\mathrm{Ext}^i_R(Y, K_j) = 0$ for each j and each $i > m$. Let $i < m$ (and hence $m \ge 1$). For $i > j+1$ we have $\mathrm{Ext}^i_R(Y, K_j) \cong \mathrm{Ext}^{i-j-1}_R(Y, K_{-1}) = \mathrm{Ext}^{i-j-1}_R(Y, E) = 0$. For $i \le j+1$ we have $\mathrm{Ext}^i_R(Y, K_j) \cong \mathrm{Hom}_R(Y, K_{j-i})$. If $i < j+1$ we have $0 \to \mathrm{Hom}_R(Y, K_{j-i}) \to \mathrm{Hom}_R(Y, Z_{j-i}) = 0$; otherwise for $i = j+1$ we get $\mathrm{Ext}^{j+1}_R(Y, K_j) \cong \mathrm{Hom}_R(Y, K_{-1}) = \mathrm{Hom}_R(Y, E)$. ■

Arguing in a dual way, it can be proved the following:

Lemma 1.4. *Let* ${}_RZ$ *and* ${}_RP$ *be left* R*-modules with* P *projective. Suppose to have a long exact sequence*

$$0 \to P \overset{\psi_0}{\to} Z_0 \overset{\psi_1}{\to} Z_1 \to \dots \to Z_{n-1} \overset{\psi_n}{\to} Z_n \to 0$$

with $Z_i \le^\oplus Z^{(\alpha_i)}$, $0 \le i \le n$, *for suitable cardinals* α_i. *Denote by* C_i *the cokernel of* ψ_i.

(1) *If* $W \in \mathrm{Ker}\,\mathrm{Tor}^R_l(-, Z)$, *then* $W \in \mathrm{Ker}\,\mathrm{Tor}^R_l(-, Z_j)$ *for each* $0 \le j \le n$, *and* $l \ge 0$.
(2) *If* $W \in {}^{\top_{>l}}Z$, *then* $W \in {}^{\top_{>l}}C_j$ *for each* $0 \le j \le n-1$.

Assume $\mathrm{Tor}^R_i(W, Z) = 0$ *for each* $i \ne m \ge 0$. *Then*

(3) $\mathrm{Tor}^R_i(W, C_j) = 0$ *for* $i > m$ *and* j *arbitrary, or* $i < m$ *and* $i \ne j+1$.
(4) $\mathrm{Tor}^R_{i-1}(W, C_{i-2}) \cong W \otimes_R P$ *for each* $i \le m$.

2. The Main Theorem

Let V be a left R-module. We denote by $\mathrm{Prod}\,V$ the class of all direct summands of arbitrary products of copies of V, and by $\mathrm{Add}\,V$ the class of all direct summands of arbitrary sums of copies of V.

The left R-module V is n*-cotilting* if it satisfies the following properties

C1. $\mathrm{id}\,V \le n$.
C2. $\mathrm{Ext}^i_R(V^\lambda, V) = 0$ for every cardinal λ and each $i \ge 1$.

C3. There exists an exact sequence

$$0 \to V_n \to V_{n-1} \to \dots \to V_1 \to V_0 \to E \to 0$$

where E is an injective cogenerator of R-Mod and V_i belongs to $\operatorname{Prod} V$.

The left R-module V is *n-tilting* if it satisfies the following properties

T1. $\operatorname{pd} V \leq n$.
T2. $\operatorname{Ext}^i_R(V, V^{(\lambda)}) = 0$ for every cardinal λ and each $i \geq 1$.
T3. There exists an exact sequence

$$0 \to R \to V_0 \to V_1 \to \dots \to V_{n-1} \to V_n \to 0$$

where V_i belongs to $\operatorname{Add} V$.

The module ${}_RV$ is said to be *partial n-cotilting* (resp. *partial n-tilting*) if it satisfies properties C1 and C2 (resp. T1 and T2).

It is easy to check that any n-cotilting (resp. n-tilting) module ${}_RV$ satisfies the following property

C3'. For a left R-module M, $\operatorname{Ext}^i_R(M, V) = 0$ for each $i \geq 0$ implies $M = 0$

(T3'. For a left R-module M, $\operatorname{Ext}^i_R(V, M) = 0$ for each $i \geq 0$ implies $M = 0$).

A partial 1-cotilting (resp. 1-tilting) module satisfying property C3' (resp. T3') is 1-cotilting (resp. 1-tilting) (see [5, Proposition 2.3] and [10, Theorem 3]). If $n \geq 2$ there exist examples (see [14]) of partial n-cotilting (resp. n-tilting) modules satisfying property C3' (resp. T3'), which are not n-cotilting (resp. n-tilting).

Definition 2.1. Let V be a left R-module.

(1) We say that a module N_R has *V-cograde m* if $\operatorname{Ext}^i_R(L, V) = 0$ for $i \neq m$.
(2) We say that a module ${}_RL$ has *V-grade m* if $\operatorname{Ext}^i_R(V, L) = 0$ for $i \neq m$.

Observe that if V is n-cotilting (n-tilting) the V-cograde (V-grade) of a module $\neq 0$, if it exists, is uniquely determined.

In the following theorems we will analyze how the conditions C1–C3, C3' and T1–T3, T3' are inherited by ring extensions of R.

Theorem 2.2. *Let ${}_RV$ be a left R-module and Γ be a ring extension of R with ${}_R\Gamma$ having V-cograde m. Denote by U the Γ-module $\operatorname{Ext}^m_R(\Gamma, V)$.*

(1) *If $\operatorname{id}{}_RV \leq n$, then $\operatorname{id}{}_\Gamma U \leq n - m$.*
(2) *If ${}_RV$ satisfies property C3', then also ${}_\Gamma U$ satisfies property C3'.*
(3) *The following are equivalent*
  (a) $\operatorname{Ext}^{m+i}_R(U, V) = 0$ *for each $i \geq 1$;*
  (b) ${}_RU$ *has V-cograde m;*
  (c) $\operatorname{Ext}^i_\Gamma(U, U) = 0$ *for each $i \geq 1$.*

Let us assume one of the three equivalent properties in (3); then

(4) *If ${}_RV$ satisfies property C2, then also ${}_\Gamma U$ satisfies property C2.*
(5) *If ${}_RV$ satisfies property C3, then also ${}_\Gamma U$ satisfies property C3.*

Proof. 1. Let M be an arbitrary left Γ-module. By Lemma 1.1, (3), we have

$$\operatorname{Ext}^l_\Gamma(M, U) \cong \operatorname{Ext}^{l+m}_R(M, V);$$

therefore the injective dimension of U is less than or equal to $n - m$.

2. Let M be a left Γ-module satisfying $\operatorname{Ext}^i_\Gamma(M, U) = 0$ for each $i \geq 0$. By Lemma 1.1 we have that $\operatorname{Ext}^i_R(M, V) = 0$ for each $i \geq 0$; therefore, since ${}_RV$ satisfies property C3', we conclude $M = 0$.

3. (a) and (b) are equivalent by Lemma 1.1, (1), while (b) and (c) are equivalent by Lemma 1.1, (3).

4. Again by Lemma 1.1, (3), for each $i \geq 1$ and each cardinal α, we have $\mathrm{Ext}^i_\Gamma(U^\alpha, U) = \mathrm{Ext}^{m+i}_R(U^\alpha, V)$. Since by hypothesis $\mathrm{Ext}^{m+i}_R(U, V) = 0$ for each $i > 0$, we have, following the notation of [6], that $U \in {}^{\perp_{>m}}V =: \mathcal{X}_{m+1}$; therefore, since by [6, Lemma 3.5] $\mathcal{X}_{m+1}$ is closed under products, we get $\mathrm{Ext}^{m+i}_R(U^\alpha, V) = 0$ for each $i > 0$.

5. Let E be an injective cogenerator of the category R-Mod. By [16, Proposition 2.1], $\mathrm{Hom}_R(\Gamma, E)$ is an injective cogenerator of Γ-Mod. By hypothesis we have an exact sequence

$$(*) \qquad 0 \to V_n \overset{\phi_n}{\to} V_{n-1} \overset{\phi_{n-1}}{\to} \dots \to V_1 \overset{\phi_1}{\to} V_0 \overset{\phi_0}{\to} E \to 0$$

with $V_i \in \mathrm{Prod}\, V$, $0 \leq i \leq n$. Let α_i be cardinals such that $V_i \leq^\oplus V^{\alpha_i}$.

We introduce the following notation:

- $K_i = \mathrm{Ker}\, \phi_i$, $0 \leq i \leq n-1$ and $K_{-1} = E$ in R-Mod,
- $U_i := \mathrm{Ext}^m_R(\Gamma, V_i)$, $0 \leq i \leq n$, in Γ-Mod.

The Γ-modules $U_i := \mathrm{Ext}^m_R(\Gamma, V_i)$ belongs to $\mathrm{Prod}\, U$:

$$U_i \leq^\oplus \mathrm{Ext}^m_R(\Gamma, V^{\alpha_i}) = U^{\alpha_i}.$$

Case $m = 0$. Since by Lemma 1.3, (3), we have $\mathrm{Ext}^1_R(\Gamma, K_j) = 0$ for each $0 \leq j \leq n-1$, applying the functor $\mathrm{Hom}_R(\Gamma, -)$ to the V-resolution $(*)$ of the injective cogenerator E, we get the long exact sequence

$$0 \to U_n = \mathrm{Hom}_R(\Gamma, V_n) \to \dots \to U_0 = \mathrm{Hom}_R(\Gamma, V_0) \to \mathrm{Hom}_R(\Gamma, E) \to 0,$$

obtaining the wanted U-resolution of the Γ-injective cogenerator $\mathrm{Hom}_R(\Gamma, E)$.

Case $m \geq 1$. By Lemma 1.3, (1), $\mathrm{Ext}^l_R(\Gamma, V_i) = 0$ for each $l \neq m$ and $0 \leq i \leq n$. Therefore applying $\mathrm{Hom}_R(\Gamma, -)$ to the short exact sequence

$$0 \to K_i \to V_i \to K_{i-1} \to 0, \quad \text{with} \quad 0 \leq i \leq n-1$$

we get

$$\mathrm{Hom}_R(\Gamma, E) = \mathrm{Hom}_R(\Gamma, K_{-1}) \cong \mathrm{Ext}^{m-1}_R(\Gamma, K_{m-2})$$

and the exact sequences

$$0 \to \mathrm{Ext}^m_R(\Gamma, K_0) \to \mathrm{Ext}^m_R(\Gamma, V_0) \to 0$$

$$0 \to \mathrm{Ext}^m_R(\Gamma, K_1) \to \mathrm{Ext}^m_R(\Gamma, V_1) \to \mathrm{Ext}^m_R(\Gamma, K_0) \to 0$$

$$\dots$$

$$0 \to \mathrm{Ext}^m_R(\Gamma, K_{m-2}) \to \mathrm{Ext}^m_R(\Gamma, V_{m-2}) \to \mathrm{Ext}^m_R(\Gamma, K_{m-3}) \to 0$$

$$0 \to \mathrm{Ext}^{m-1}_R(\Gamma, K_{m-2}) \to \mathrm{Ext}^m_R(\Gamma, K_{m-1}) \to \mathrm{Ext}^m_R(\Gamma, V_{m-1}) \to \mathrm{Ext}^m_R(\Gamma, K_{m-2}) \to 0$$

$$0 \to \mathrm{Ext}^m_R(\Gamma, K_m) \to \mathrm{Ext}^m_R(\Gamma, V_m) \to \mathrm{Ext}^m_R(\Gamma, K_{m-1}) \to 0$$

$$\dots$$

$$0 \to \mathrm{Ext}^m_R(\Gamma, K_{n-1}) = \mathrm{Ext}^m_R(\Gamma, V_n) \to \mathrm{Ext}^m_R(\Gamma, V_{n-1}) \to \mathrm{Ext}^m_R(\Gamma, K_{n-2}) \to 0.$$

Thus, we achieve the solid part of the following diagram with exact rows and columns; the dashed part is obtained by a pullback construction.

$$\begin{array}{ccccccccccccc}
 & & & & & & & & 0 & & 0 & & \\
 & & & & & & & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & U_n & \overset{\xi_n}{\dashrightarrow} & \cdots & \overset{\xi_{m+2}}{\dashrightarrow} & U_{m+1} & \overset{\xi_{m+1}}{\dashrightarrow} & P & \dashrightarrow & \mathrm{Hom}_R(\Gamma,E) & \dashrightarrow & 0 \\
 & & \Vert & & & & \Vert & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_n & \longrightarrow & \cdots & \longrightarrow & U_{m+1} & \longrightarrow & U_m & \longrightarrow & \mathrm{Ext}_R^m(\Gamma,K_{m-1}) & \longrightarrow & 0 \\
 & & & & & & & & \downarrow{\scriptstyle \xi_m} & & \downarrow & & \\
 & & & & & & & & U_{m-1} & = & U_{m-1} & & \\
 & & & & & & & & \downarrow{\scriptstyle \xi_{m-1}} & & \downarrow & & \\
 & & & & & & & & \vdots & & \vdots & & \\
 & & & & & & & & \downarrow{\scriptstyle \xi_1} & & \downarrow & & \\
 & & & & & & & & U_0 & = & U_0 & & \\
 & & & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & & & 0 & & 0 & &
\end{array}$$

To see that the exact sequence

$$0 \to U_n \to ... \to U_{m+1} \to P \to \mathrm{Hom}_R(\Gamma, E) \to 0$$

is the wanted U-presentation in Γ-Mod of the injective cogenerator $\mathrm{Hom}_R(\Gamma, E)$, we have to prove that P belongs to $\mathrm{Prod}\, U$. Denote by C_{i-1} the image of ξ_i, $1 \leq i \leq n$. We will show in several steps that $\mathrm{Ext}^1_\Gamma(C_{m-1}, P) = 0$ to conclude that P is a direct summand of U_m, and hence $P \in \mathrm{Prod}\, U$.

- $\mathrm{Ext}^i_\Gamma(U_r, U_s) = 0$ for each $i \geq 1$, and $0 \leq r, s \leq n$.

We have $\mathrm{Ext}^i_\Gamma(U_r, U_s) \leq^{\oplus} \mathrm{Ext}^i_\Gamma(U^{\alpha_r}, U^{\alpha_s}) = [\mathrm{Ext}^i_\Gamma(U^{\alpha_r}, U)]^{\alpha_s}$. Since ${}_RU$ has V-cograde m, by Lemma 1.1, (3), $\mathrm{Ext}^i_\Gamma(U^{\alpha_r}, U) \cong \mathrm{Ext}^{i+m}_R(U^{\alpha_r}, V) = 0$ since ${}^{\perp_{>m}}V$ is closed under products (see [6, Proposition 3.5]).

- $\mathrm{Ext}^i_\Gamma(C_{m-1}, U_{m+j}) = 0$ for each $i, j \geq 1$.

We have for $l \geq 1$: $0 = \mathrm{Ext}^i_\Gamma(U_{m-l}, U_{m+j}) \to \mathrm{Ext}^i_\Gamma(C_{m-l}, U_{m+j}) \to$

$$\to \mathrm{Ext}^{i+1}_\Gamma(C_{m-l-1}, U_{m+j}) \to \mathrm{Ext}^{i+1}_\Gamma(U_{m-l}, U_{m+j}) = 0;$$

therefore $\mathrm{Ext}^i_\Gamma(C_{m-1}, U_{m+j}) \cong \mathrm{Ext}^{i+m-1}_\Gamma(C_0, U_{m+j}) = \mathrm{Ext}^{i+m-1}_\Gamma(U_0, U_{m+j}) = 0$.

- $\mathrm{Ext}^1_\Gamma(C_{m-1}, C_m) = 0$ for each $i \geq 1$.

We have for each $i, j \geq 1$: $0 = \mathrm{Ext}^i_\Gamma(C_{m-1}, U_{m+j}) \to \mathrm{Ext}^i_\Gamma(C_{m-1}, C_{m+j-1}) \to$

$$\to \mathrm{Ext}^{i+1}_\Gamma(C_{m-1}, C_{m+j}) \to \mathrm{Ext}^{i+1}_\Gamma(C_{m-1}, U_{m+j}) = 0;$$

therefore $\mathrm{Ext}^1_\Gamma(C_{m-1}, C_m) \cong \mathrm{Ext}^{n-m}_\Gamma(C_{m-1}, C_{n-1}) = \mathrm{Ext}^{n-m}_\Gamma(C_{m-1}, U_n) = 0$.

- $\mathrm{Ext}^1_\Gamma(C_{m-1}, P) = 0$.

We have

$$0 = \mathrm{Ext}^1_\Gamma(C_{m-1}, C_m) \to \mathrm{Ext}^1_\Gamma(C_{m-1}, P) \to \mathrm{Ext}^1_\Gamma(C_{m-1}, \mathrm{Hom}_R(\Gamma, E)) = 0. \blacksquare$$

Let us consider now the tilting case. Let us fix an injective cogenerator E of R-Mod. Given an R-R-bimodule ${}_RH_R$, we denote by H^* the left R-module $\mathrm{Hom}_R(H, E)$. The following proposition together Lemmas 1.2, 1.4 will permit us to get the "tilting version" of Theorem 2.2, only dualizing its proof.

Proposition 2.3. *Let $R \xrightarrow{\xi} \Gamma$ be a ring homomorphism. The following are equivalent:*

(1) *Any injective cogenerator of* Γ-Mod*, endowed with its structure as left R-module induced by ξ, has V-grade m.*
(2) $\operatorname{Tor}_i^R(\Gamma, V) = 0$ *for each* $i \neq m$.
(3) $({}_R\Gamma)^*$ *has V-grade m.*

Proof. Let Θ and E be injective cogenerators of Γ-Mod and R-Mod, respectively. By [9, Proposition VI.5.1], $\operatorname{Ext}_R^i(V, \Theta) = \operatorname{Ext}_R^i(V, \operatorname{Hom}_\Gamma(\Gamma, \Theta)) = 0$ if and only if $\operatorname{Tor}_i^R(\Gamma, V) = 0$ if and only if $\operatorname{Hom}_R(\operatorname{Tor}_i^R(\Gamma, V), E) = 0$ if and only if $\operatorname{Ext}_R^i(V, ({}_R\Gamma)^*) = 0$. ∎

Thus we get

Theorem 2.4. *Let ${}_RV$ be a left R-module and Γ be a ring extension of R. Assume $({}_R\Gamma)^*$ has V-grade m and denote by U the Γ-module* $\operatorname{Tor}_m^R(\Gamma, V)$.

(1) *If* $\operatorname{pd}{}_RV \leq n$, *then* $\operatorname{pd}{}_\Gamma U \leq n - m$.
(2) *If ${}_RV$ satisfies property T3', then also ${}_\Gamma U$ satisfies property T3'.*
(3) *The following are equivalent*
    (a) $\operatorname{Ext}_R^{m+i}(V, U) = 0$ *for each* $i \geq 1$;
    (b) *U has V-grade m;*
    (c) $\operatorname{Ext}_\Gamma^i(U, U) = 0$ *for each* $i \geq 1$.

Let us assume one of the three equivalent properties in (3); then

(4) *If ${}_RV$ satisfies property T2, then also ${}_\Gamma U$ satisfies property T2.*
(5) *If ${}_RV$ satisfies property T3, then also ${}_\Gamma U$ satisfies property T3.*

Resuming, we have the following generalizations of [16, Proposition 4.8] and part of [17, Theorem 5.2]:

Corollary 2.5. (1) *Let ${}_RV$ be an n-(partial) cotilting module. Let Γ be a ring extension of R with ${}_R\Gamma$ having V-cograde m, and* ${}_\Gamma U = \operatorname{Ext}_R^m(\Gamma, V)$. *Then ${}_\Gamma U$ is an $n-m$-(partial) cotilting module if and only if ${}_RU$ has V-cograde m.*
(2) *Let ${}_RV$ be an n-(partial) tilting module. Let Γ be a ring extension of R with $({}_R\Gamma)^*$ having V-grade m, and* ${}_\Gamma U = \operatorname{Tor}_m^R(\Gamma, V)$. *Then ${}_\Gamma U$ is an $n-m$-(partial) tilting module if and only if ${}_RU$ has V-grade m.*

Proof. It is sufficient to prove that if ${}_\Gamma U$ is $n - m$-(partial) cotilting, then ${}_RU$ has V-cograde m. But this follows easily by the equivalence of (b) and (c) in Theorem 2.2, (3). Analogously for the tilting case. ∎

If $\Gamma = R \ltimes Q$ is a split extension of R by Q, of course Γ and R are ring extensions of each other via the ring homomorphisms $R \to \Gamma$, $r \mapsto (r, o)$ and $\Gamma \to R$, $(r, q) \mapsto r$. Since R is a direct summand of ${}_R\Gamma$, if Γ has V-cograde m, then necessarily $m = 0$. Analogously, since R is a direct summand of Γ_R, by Proposition 2.3 if $({}_R\Gamma)^*$ has V-grade m, then necessarily $m = 0$.

Proposition 2.6. *Let ${}_RV$ be a left R-module and $\Gamma = R \ltimes Q$ be a split extension of R by Q. Assume ${}_RQ$ has V-cograde* 0. *Then* ${}_\Gamma U = \operatorname{Hom}_R(\Gamma, V)$ *is an n-(partial) cotilting module if and only if ${}_RV$ is an n-(partial) cotilting module and the left R-module* $\operatorname{Hom}_R(Q, V)$ *has V-cograde* 0.

Proof. Assume ${}_RV$ is n-(partial) cotilting and the left R-module $\operatorname{Hom}_R(Q, V)$ has V-cograde 0. The ring Γ is isomorphic as left R-module to $R \oplus Q$. Therefore by hypothesis ${}_R\Gamma$ and the left R-module

$$\operatorname{Hom}_R(\Gamma, V) \cong \operatorname{Hom}_R(R \oplus Q, V) = {}_RV \oplus \operatorname{Hom}_R(Q, V)$$

have V-cograde 0. Then by Corollary 2.5 we get ${}_\Gamma U$ is a n-(partial) cotilting module. Conversely, assume ${}_\Gamma U$ n-(partial) cotilting. First of all we have

$$\mathrm{Hom}_\Gamma(R, {}_\Gamma U) = \mathrm{Hom}_\Gamma(R, \mathrm{Hom}_R(\Gamma, V)) \cong \mathrm{Hom}_R(\Gamma \otimes_\Gamma R, V) \cong \mathrm{Hom}_R(R, V) \cong V.$$

To apply Corollary 2.5 we have to prove that ${}_\Gamma V$ and ${}_\Gamma R$ has U-cograde 0. By [16, Lemma 2.2] ${}_\Gamma U = \mathrm{Hom}_R(\Gamma, V) \cong {}_\Gamma V \oplus \mathrm{Hom}_R(Q_\Gamma, V)$ as left Γ-modules. Therefore, by property C2 of n-(partial) cotilting modules, we have $\mathrm{Ext}^i_\Gamma(V, U) = 0$ for each $i > 0$ and the wanted condition $0 = \mathrm{Ext}^i_\Gamma(\mathrm{Hom}_R(Q, V), V) = \mathrm{Ext}^i_R(\mathrm{Hom}_R(Q, V), V)$. Finally, by Lemma 1.1 we get

$$\mathrm{Ext}^i_\Gamma(R, U) \cong \mathrm{Ext}^i_R(R, V) = 0.$$

Then we conclude that ${}_R V$ is a n-(partial) cotilting module. ∎

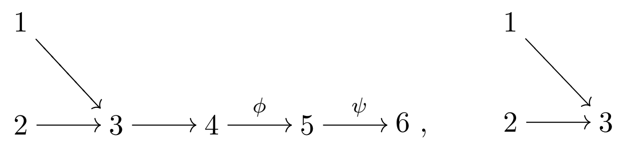
Arguing in a dual way we can prove the "tilting" version of the above proposition.

Proposition 2.7. *Let ${}_R V$ be a left R-module and $\Gamma = R \ltimes Q$ be a split extension of R by Q. Assume ${}_R Q^*$ has V-grade 0. Then ${}_\Gamma U = \Gamma \otimes V$ is an n-(partial) tilting module if and only if ${}_R V$ is an n-(partial) tilting module and the left R-module $Q \otimes_R V$ has V-grade 0.*

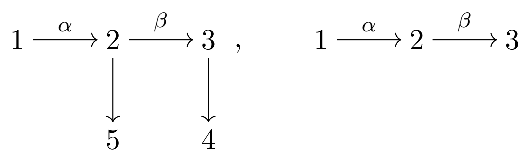
Example 2.8. (1) The abelian group $C = \oplus_{p \neq q} \mathbb{Z}_{p^\infty} \oplus \mathbb{Q} \oplus \mathbb{J}_q$ is 1-cotilting (see [13, Proposition 2.15]). Consider the ring homomorphism $\mathbb{Z} \to \mathbb{Z}/q^n\mathbb{Z}$. Clearly the abelian group $\mathbb{Z}/q^n\mathbb{Z}$ has C-cograde 1; moreover $\mathrm{Ext}^1_\mathbb{Z}(\mathbb{Z}/q^n\mathbb{Z}, C) \cong \mathbb{Z}/q^n\mathbb{Z}$. Therefore $\mathrm{Ext}^1_\mathbb{Z}(\mathbb{Z}/q^n\mathbb{Z}, C) \cong \mathbb{Z}/q^n\mathbb{Z}$ is a 0-cotilting $\mathbb{Z}/q^n\mathbb{Z}$-module, i.e. an injective cogenerator.

(2) The abelian group $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ is 1-tilting. Consider the ring homomorphism $\mathbb{Z} \to \mathbb{Z}/q^n\mathbb{Z}$. Clearly the abelian group $(\mathbb{Z}/q^n\mathbb{Z})^* \cong \mathbb{Z}/q^n\mathbb{Z}$ has $\mathbb{Q}$-grade 1; moreover $\mathrm{Ext}^1_\mathbb{Z}(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}, \mathbb{Z}/q^n\mathbb{Z}) \cong \mathbb{Z}/q^n\mathbb{Z}$. Therefore $\mathrm{Ext}^1_\mathbb{Z}(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}, \mathbb{Z}/q^n\mathbb{Z}) \cong \mathbb{Z}/q^n\mathbb{Z}$ is a 0-tilting $\mathbb{Z}/q^n\mathbb{Z}$-module, i.e. a projective generator.

(3) Let k be an algebraically closed field. Denote by R and Γ the k-algebras of finite representation type given by the quivers



with relation $\psi \circ \phi = 0$, and by S and Δ the k-algebras of finite representation type given by the quivers



with relation $\beta \circ \alpha = 0$. Clearly there exist ring homomorphisms $R \to \Gamma$ and $S \to \Delta$. Since R and S have global and injective dimension two, the regular modules ${}_R R$ and ${}_S S$ are 2-cotilting modules (see [14, Lemma 1]). It is easy to see that ${}_R\Gamma$ and ${}_R U := \mathrm{Ext}^1_R(\Gamma, R) = 2 \oplus 2 \oplus 2 \oplus 1 \oplus 1 \oplus \begin{smallmatrix} 1\,2 \\ 3 \end{smallmatrix}$, and ${}_S\Delta$ and ${}_S W := \mathrm{Ext}^1_S(\Delta, S) = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ have R-cograde and S-cograde 1, respectively. Then ${}_\Gamma U$ and ${}_\Delta W$ are 1-cotilting modules; ${}_\Gamma U$ is actually an injective cogenerator of Γ-Mod, while $\mathrm{id}\, {}_\Delta W = 1$.

Dualizing these, one get analogous examples for the tilting case.

References

[1] F. D. Anderson and K. R. Fuller. *Rings and categories of modules*, 2nd ed., GTM **13**, Springer 1992.
[2] I. Assem and N. Marmaridis. *Tilting modules over split-by-nilpotent extensions*, Comm. Algebra **26**, (1998), 1547–1555.
[3] L. Angeleri Hügel and F.U. Coelho. *Infinitely generated tilting modules of finite projective dimension*, Forum Math. **13**, (2001), 239–250.
[4] L. Angeleri Hügel and J. Trlifaj. *Tilting theory and the finitistic dimension conjecture*, to appear in Trans. Amer. Math. Soc.
[5] L. Angeleri Hügel, A. Tonolo and J. Trlifaj. *Tilting preenvelopes and cotilting precovers*, Algebras and Representation Theory **4**, (2001), 155–170.
[6] S. Bazzoni. *n-Cotilting and n-Tilting Modules*, submitted.
[7] W. Bruns and J. Herzog. *Cohen-Macaulay rings*, Cambridge studies in advanced mathematics **39**, Cambridge University Press, 1993.
[8] R. R. Colby and K. R. Fuller. *Tilting, cotilting and serially tilted rings*, Comm. Algebra **18**, (1990), 1585–1615.
[9] H. Cartan and S. Eilenberg. *Homological Algebra*, Princeton University Press, Princeton N.J., 1956.
[10] R. Colpi. *Tilting modules and $*$-modules*, Comm. Algebra **21** (1993), 1095–1102.
[11] R. Colpi, G. D'Este and A. Tonolo. *Quasi-tilting modules and counter equivalences*, J.Algebra **191** (1997), 461–494.
[12] R. Colpi and K. R. Fuller. *Cotilting modules and bimodules*, Pacific J. Math. **192(2)** (2000), 275–291.
[13] R. Colpi, A. Tonolo and J. Trlifaj. *Partial cotilting modules and the lattices induced by them*, Comm. Algebra **25(10)**, (1997), 3225-3237.
[14] G. D'Este. *On tilting and cotilting-type modules*, preprint.
[15] K. R. Fuller. *$*$-modules over ring extensions*, Comm. Algebra **25**, (1997), 2839–2860.
[16] K. R. Fuller. *Ring Extensions and Duality*, Contemporary Math. **259**, (2000), 213–222.
[17] Y. Miyashita. *Tilting modules of finite projective dimension*, Math. Z. **193**, (1986), 113–146.
[18] A. Tonolo. *Generalizing Morita duality: A homological approach*, J. Algebra, **232**, (2000), 282–298.
[19] A. Tonolo. *Tilting modules of finite projective dimension: sequentially static and costatic modules*, J. of Algebra and its Applications, **1(3)**, (2002), 295–305.
[20] A. Tonolo. *Sequentially reflexive modules*, preprint.
[21] J. Trlifay. *Cotorsion theories induced by tilting and cotilting modules*, (Perth, 2000), 285–300, Contemp. Math., 273, Amer. Math. Soc., Providence, RI, 2001.

Dipartimento di Matematica Pura ed Applicata, Università di Padova - Italy
E-mail address: tonolo@math.unipd.it