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## On the braided Fourier transform on the $n$ -dimensional quantum space

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We work out in detail a theory of integrability on the braided covector Hopf algebra and the braided vector Hopf algebra of type  $A_n$  introduced by Majid. Using a braided Fourier transform very similar to the one defined by Kempf and Majid we obtain  $n$ -dimensional analogs of results by Koornwinder expressing the correspondence between products of the  $q^2$ -Gaussian  $g_{q^2}(\underline{x})$  times monomials, and products of the  $q^2$ -Gaussian  $G_{q^2}(\underline{\partial})$  times  $q^2$ -Hermite polynomials under the transform. We invert the correspondence by finding a suitable inversion, different from the one of Kempf and Majid. We show that with this transforms, whenever  $n \geq 2$ , the Plancherel measure will depend on the parity of the power series that we are transforming. © 1999 American Institute of Physics. [S0022-2488(99)00410-7]

### I. INTRODUCTION

Recently Majid (see Ref. 1 and references therein) has defined a generalization of the concept of Hopf algebra, namely braided groups. Hopf superalgebras and genuine Hopf algebras are examples of these objects, but there are more examples, associated to quantum groups, since the category of representations of a quasitriangular Hopf algebra is braided. These objects appear also in Ref. 2 with different terminology.

Kempf and Majid introduced<sup>3</sup> an integration theory for a class of braided groups arising from matrix solutions of the quantum Yang–Baxter equation as “braided covector algebras” (see also Ref. 1). They used this theory to define a formal braided Fourier transform and its inverse on these algebras. In their paper they also present the case of the braided line as an example, and the  $n$ -dimensional case in less detail.

The main problem in their theory is that it is very difficult to find an explicit integral that behaves well enough. They provide powerful general results in a theoretical way, but the description in specific cases is often hard to handle. Besides, they do not provide a definition of convergence of an integral nor do they treat in their article the case when a generalized function may be called integrable.

The purpose of this paper is to work out as far as we can the example of the  $n$ -dimensional quantum space of type  $A_n$  viewed as a braided group. We will provide different definitions of integrability, with examples and counterexamples, with respect to an integral similar to that in Ref. 3. Our definitions are based on extensions of representations of the braided covector and vector algebras. Using these facts, one can show in a more rigorous way the translation invariance of the integral proved in Ref. 3. We define different types of Fourier transforms, all based on but different from Ref. 3. One of them looks more like that in Ref. 4 since the integral does not have trivial braiding with elements in the braided group, and because the braided antipode appears in the definition. We also took inspiration from Ref. 5, where an analog of the Fourier transform for the case of the braided line is also studied, although the transform in Ref. 5 goes from an algebra to itself, while we are looking for a transform going from an algebra to its braided dual, as in Refs. 3, 4, and 6.

We find an  $n$ -dimensional analog of the correspondence between products of  $q^2$ -Gaussians

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times monomials and other  $q^2$ -Gaussians times  $q^2$ -Hermite polynomials, similar to the classical case and to the results for the braided line in Ref. 5. We give also inverses for our transforms that invert the correspondence mentioned above, similarly to what appears in Ref. 5. The main tool for this inversion formula is the symmetry between the braided vector algebra and the braided covector algebra. Kempf and Majid had already defined an inversion formula in their article, but they used properties that our integral does not have. Other inversion formulas for the braided line are to be found in Ref. 6 where the case of distributions is also treated.

One of the most interesting results is that whenever  $n \geq 2$ , there is a loss of symmetry, so that the Plancherel measure will no longer be the same in the whole space. Indeed there is an action of  $\mathbf{Z}_2^n$  associated to the parity of the power series we are working with, and the Plancherel measure will be constant only on the subspaces of power series with constant parity. Therefore, the transforms we define can also be seen as sine and cosine transform. A phenomenon of break of symmetry for  $q$ -integrals was also noted in Ref. 7, where the authors were defining a calculus associated to a  $q$ -deformed Heisenberg algebra.

Other definitions of analogs of the Fourier transform on genuine Hopf algebras, quantum spaces, or commutative algebras appeared before Ref. 5 in Refs. 8–10.

## II. NOTATION AND PRELIMINARIES

In this paper a complex algebra has, unless otherwise stated, always a unit and  $q$  is a real number in  $(0,1)$ . For a positive integer  $m$ , and for any  $q \neq 1$  we write  $[m]_q = (q^m - 1)/(q - 1)$  and  $[m]_q! = \prod_{j=1}^m [j]_q$ .

For any  $a \in \mathbf{R}$  and for any  $k \in \mathbf{Z}_{\geq 0}$ , we will put  $(a; q)_k = \prod_{l=0}^{k-1} (1 - aq^l)$ . We will also write  $(a; q)_\infty = \lim_{k \rightarrow \infty} (a; q)_k$  and for  $r$  real numbers  $a_1, \dots, a_r$  we will put  $(a_1, \dots, a_r; q)_\infty = \prod_{j=1}^r (a_j; q)_\infty$ . Finally, for  $a \geq b$  with  $a$  and  $b$  both in  $\mathbf{Z}_{\geq 0}$  we will use the  $q$ -binomial coefficient  $\begin{bmatrix} a \\ b \end{bmatrix}_q = [a]_q! / [b]_q! [a-b]_q! = (q; q)_a / (q; q)_b (q; q)_{a-b}$ .

Whenever for any capital character  $E$  we have a multi-index  $E = (e_1, \dots, e_n)$  we will put  $E_i = \sum_{j=1}^{i-1} e_j$  and  $E^i = \sum_{j=i+1}^n e_j$ . Hence  $|E| = e_i + E_i + E^i$  for every  $i$ .

We identify the set  $\{+, -\}$  with  $\mathbf{Z}_2$ , letting  $+$  correspond to  $\bar{0}$  and  $-$  correspond to  $\bar{1}$ , so that  $n$ -tuples in  $\{+, -\}^n$  can be identified with  $n$ -tuples in  $\mathbf{Z}_2^n$ . By means of this identification we define the map  $A: \mathbf{Z}^n \rightarrow \mathbf{Z}_2^n \rightarrow \{+, -\}^n$  by reducing modulo 2 first, i.e.,  $A(b) = +$  if  $b$  is even and  $-$  otherwise. We will also denote by  $B: \{+, -\}^n \rightarrow \{0, 1\}^n \subset \mathbf{Z}^n$  the map sending each “even” entry to 0 and each “odd” entry to 1.

Given an operator on the twofold tensor product of an  $n$ -dimensional vector space  $V$ , we identify this operator with the  $n^2 \times n^2$  matrix  $R$  and we denote its entries by  $R_{cd}^{ab}$  where  $a, b$  are the row entries and  $c, d$  are the column entries. For such an  $R$ , the operator acting on the  $p$ -fold tensor product of  $V$  (for  $p \geq 2$ ) as  $R$  on the  $i$ th and  $j$ th components and as the identity elsewhere will be denoted by  $R_{ij}$ . For summation we will use Einstein convention.

We recall that a braided group over a field  $K$  is an associative algebra  $A$  with multiplication  $m$  and a coassociative coalgebra with comultiplication  $\Delta$  and counit  $\varepsilon$  together with an invertible linear map  $\Psi: A \otimes A \rightarrow A \otimes A$  called braiding, and a linear map  $S: A \rightarrow A$  called braided antipode such that the following properties hold:  $\Psi(m \otimes \text{id}) = (\text{id} \otimes m)(\Psi \otimes \text{id})(\text{id} \otimes \Psi)$ ;  $\Psi(\text{id} \otimes m) = (m \otimes \text{id})(\text{id} \otimes \Psi)(\Psi \otimes \text{id})$ ;  $(\text{id} \otimes \Delta) \circ \Psi = (\Psi \otimes \text{id})(\text{id} \otimes \Psi)(\Delta \otimes \text{id})$ ;  $(\Delta \otimes \text{id}) \circ \Psi = (\text{id} \otimes \Psi)(\Psi \otimes \text{id})(\text{id} \otimes \Delta)$ ;  $1 \varepsilon = m(\text{id} \otimes S) \Delta = m(S \otimes \text{id}) \Delta$ ;  $\Delta m = (m \otimes m)(\text{id} \otimes \Psi \otimes \text{id})(\Delta \otimes \Delta)$ ;  $\varepsilon \circ m = \varepsilon \otimes \varepsilon$ ; and  $\Delta(1) = 1 \otimes 1$ . If there is no braided antipode,  $A$  is called a “braided bialgebra.”

We will work with two particular braided groups, namely the braided covector Hopf algebra  $\hat{V}(R)$  and the braided vector algebra  $V(R)$  associated to the standard matrix solution  $R$  of the quantum Yang–Baxter equation of type  $A_n$  [i.e., defining the quantum group  $\text{SL}_q(n)$ ]. For a general definition of braided covector and vector algebra, see Ref. 1, Chap. 10. One sees there that these algebras are comodule algebras for  $\text{SL}_q(n)$ , and that all possible vector spaces obtained by tensoring  $\hat{V}(R)$  and  $V(R)$  can be provided by an algebra structure such that they are again comodule algebras for  $\text{SL}_q(n)$ . The product is then defined by means of the braidings and the product in  $\hat{V}(R)$  and  $V(R)$ .

In this particular case  $\hat{V}(R)$  is the unital associative algebra generated by  $x_1, \dots, x_n$ , with relations given by  $x_i x_j = q x_j x_i$  if  $i > j$ , i.e.,  $\hat{V}(R)$  is the  $n$ -dimensional quantum space. The counit is given by  $\varepsilon(x_j) = 0$  for every  $j$ . We know that this algebra has a basis given by (increasing) ordered monomials  $x_1^{e_1} \dots x_n^{e_n}$ . The general formulas for the braiding  $\Psi$ , the comultiplication  $\Delta$ , and the antipode  $S$  in Ref. 1 reduce as follows:

$$\Psi(x_i \otimes x_j) = \begin{cases} q x_j \otimes x_i, & \text{if } i < j, \\ q^2 x_i \otimes x_i, & \text{if } i = j, \\ (q^2 - 1) x_i \otimes x_j + q x_j \otimes x_i, & \text{if } i > j, \end{cases}$$

so that, for  $i < j$ ,  $\Psi(x_i^a \otimes x_j^b) = q^{ab} x_j^b \otimes x_i^a$ ,

$$\Delta(x_1^{e_1} \dots x_n^{e_n}) = \sum_{j_1=0}^{e_1} \dots \sum_{j_n=0}^{e_n} \left( \prod_{i=1}^n \begin{bmatrix} e_i \\ j_i \end{bmatrix}_{q^2} \right) q^{\sum_{i=1}^n j_i (e_i - j_i)} x_1^{e_1 - j_1} \dots x_n^{e_n - j_n} \otimes x_1^{j_1} \dots x_n^{j_n},$$

and  $S(x_1^{e_1} \dots x_n^{e_n}) = (-1)^{|E|} q^{|E|^2 - |E|} x_1^{e_1} \dots x_n^{e_n}$ .

Here  $V(R)$  is the associative unital algebra generated by  $\partial_1, \dots, \partial_n$ , with relations given by  $\partial_i \partial_j = q \partial_j \partial_i$  if  $i < j$ . The ordered monomials provide a basis for  $V(R)$ . In this case we fix the basis given by ordered monomials with decreasing order. The braiding, counit, comultiplication, and antipode are given by

$$\Psi(\partial_i \otimes \partial_j) = \begin{cases} q \partial_j \otimes \partial_i, & \text{if } i > j, \\ q^2 \partial_i \otimes \partial_i, & \text{if } i = j, \\ (q^2 - 1) \partial_i \otimes \partial_j + q \partial_j \otimes \partial_i, & \text{if } i < j, \end{cases}$$

$\varepsilon(\partial_j) = 0$  for every  $j$ ,  $S(\partial_n^{e_n} \dots \partial_1^{e_1}) = (-1)^{|E|} q^{|E|^2 - |E|} \partial_n^{e_n} \dots \partial_1^{e_1}$ , and

$$\Delta(\partial_n^{e_n} \dots \partial_1^{e_1}) = \sum_{j_n=0}^{e_n} \dots \sum_{j_1=0}^{e_1} \left( \prod_{i=1}^n \begin{bmatrix} e_i \\ j_i \end{bmatrix}_{q^2} \right) q^{\sum_{i=1}^n j_i (e_i - j_i)} \partial_n^{e_n - j_n} \dots \partial_1^{e_1 - j_1} \otimes \partial_n^{j_n} \dots \partial_1^{j_1}.$$

By Majid's theory (see Corollary 9.2.14 and Proposition 10.3.6 in Ref. 1) we recover the braiding between  $\hat{V}(R)$  and  $V(R)$  and between  $V(R)$  and  $\hat{V}(R)$ . They are given by

$$\Psi_{V(R), \hat{V}(R)}(\partial_i \otimes x_j) = \begin{cases} q^{-1} x_j \otimes \partial_i, & \text{if } i \neq j, \\ q^{-2} x_j \otimes \partial_j + \sum_{r > j} (q^{-2} - 1) x_r \otimes \partial_r, & \text{if } i = j, \end{cases}$$

$$\Psi_{\hat{V}(R), V(R)}(x_i \otimes \partial_j) = \begin{cases} q^{-1} \partial_j \otimes x_i, & \text{if } i \neq j, \\ \sum_{r < j} (q^{-2} - 1) q^{-2(j-r)} \partial_r \otimes x_r + q^{-2} \partial_j \otimes x_j, & \text{if } i = j. \end{cases}$$

For every choice of nonzero constants  $c_j$ , for  $j = 1, \dots, n$ , there is an algebra isomorphism  $\psi$  between  $\hat{V}(R)$  and  $V(R)$  mapping  $x_j$  to  $c_j \partial_{n+1-j}$ , such that  $\Delta_{V(R)} \circ \psi = (\psi \otimes \psi) \circ \Delta_{\hat{V}(R)}$  and  $S_{V(R)} \circ \psi = S_{\hat{V}(R)}$ . In particular, for  $c_j = q^{-j + (1/2)(n-1)}$  for every  $j$ , then we also have  $\Psi_{V(R), V(R)}(\psi \otimes \psi) = \Psi_{\hat{V}(R), \hat{V}(R)}$ .

For this choice of the  $c_j$ 's we have

$$(\psi^{-1} \otimes \psi) \circ \Psi_{\hat{V}(R), V(R)} = \Psi_{V(R), V(R)} \circ (\psi \otimes \psi^{-1}),$$

$$(\psi \otimes \psi^{-1}) \circ \Psi_{V(R), \hat{V}(R)} = \Psi_{\hat{V}(R), V(R)} \circ (\psi^{-1} \otimes \psi),$$

however,  $\psi$  is not a morphism in the braided category since it is *not* true that  $(\psi \otimes \text{id}) \circ \Psi_{\hat{V}(R), V(R)} = \Psi_{\hat{V}(R), \hat{V}(R)} \circ (\text{id} \otimes \psi)$ , as one can easily see by computing the actions of the left-hand side and of the right-hand side on  $(\partial_2 \otimes x_1)$  for  $n=2$ . This has to do with the fact that  $\hat{V}(R)$  and  $V(R)$  are not dual as braided groups in the sense that there is no invariant quantum metric (see Ref. 1 and references therein).

It is also well known that there is a left action of  $V(R)$  on  $\hat{V}(R)$  where each  $\partial_j$  acts by means of braided partial differentiation with respect to  $x_j$ . In the  $A_n$  case, this turns out to be, for  $f(\underline{x}) \in \hat{V}(R)$ ,

$$\partial_j f(\underline{x}) = x_j^{-1} \left[ \frac{f(q^2 x_1, \dots, q^2 x_j, x_{j+1}, \dots, x_n) - f(q^2 x_1, \dots, q^2 x_{j-1}, x_j, \dots, x_n)}{(q^2 - 1)} \right],$$

where the inverse of  $x_j$  is only formal, and “apparent.” In particular, one has

$$\partial_j (x_1^{e_1} \cdots x_n^{e_n}) = [e_j]_{q^2} q^{E_j} x_1^{e_1} \cdots x_{j-1}^{e_{j-1}} x_j^{e_j-1} x_{j+1}^{e_{j+1}} \cdots x_n^{e_n}.$$

Formally we can repeat the same constructions with  $\hat{V}(R)^{\text{ext}}$  [resp.  $V(R)^{\text{ext}}$ ], the algebra of formal power series in the  $x_j$ 's (resp.  $\partial_j$ 's) with the given defining relations. In this case everything that we have described above is defined as in  $\hat{V}(R)$  and  $V(R)$ .

Specializing the results about the braided exponential map in Ref. 1 and references therein, one has

$$\exp(x|\partial) := \sum_{e_1, \dots, e_n \geq 0} x_1^{e_1} \cdots x_n^{e_n} \otimes \frac{\partial_n^{e_n}}{[e_n]_{q^2}!} \cdots \frac{\partial_1^{e_1}}{[e_1]_{q^2}!}.$$

By Example 10.4.16 in Ref. 1 this is equal to  $e_{q^{-2}}((1 - q^{-2}) \sum_{i=1}^n x_i \otimes \partial_i)$ , where  $e_q(z) = \sum_{k=0}^{\infty} z^k / (q; q)_k$  (see Ref. 5 for further details). It follows by straightforward computation that  $\exp(x|\partial)$  is also equal to  $E_{q^2}((1 - q^2) \sum_{i=1}^n x_i \otimes \partial_i)$ , where  $E_{q^2}(z) = \sum_{k=0}^{\infty} q^{(1/2)k(k-1)} z^k / (q; q)_k$ . Corollary 10.4.17 in Ref. 1, which appeared first in Ref. 11, gives us also a braided version of the Taylor formula. This is given by

$$\begin{aligned} f(\Delta(x_1), \dots, \Delta(x_n)) &= f(x_1 + y_1, \dots, x_n + y_n) \\ &= \sum_{e_1, \dots, e_n \geq 0} y_1^{e_1} \cdots y_n^{e_n} \left( \frac{\partial_n^{e_n}}{[e_n]_{q^2}!} \cdots \frac{\partial_1^{e_1}}{[e_1]_{q^2}!} f(x_1, \dots, x_n) \right) \\ &= \exp(y|\partial) f(\underline{x}) \end{aligned}$$

where  $y_j = x_j \otimes 1$  and  $x_i$  stands for  $1 \otimes x_i$  after the second equality sign.

### III. INTEGRATION ON $\hat{V}(R)^{\text{ext}}$

We start with the “indefinite” integral with respect to  $x_i$ . We repeat shortly the definition in Ref. 3, where the integral is viewed as an operator on  $\hat{V}(R)^{\text{ext}}$ .

*Definition 3.1:* The braided partial integral with respect to  $x_i$  acting on  $f(\underline{x}) \in \hat{V}(R)^{\text{ext}}$  is given by

$$\int_0^{x_i} f := (1 - q^2) \sum_{k=0}^{\infty} q^{2k} x_i f(q^{-2} x_1, \dots, q^{-2} x_{i-1}, q^{2k} x_i, x_{i+1}, \dots, x_n).$$

It is easy to see that the operator defined above acts as a pseudo inverse for the partial differential operator  $\partial_i$ . It is indeed only a right inverse, since it acts as a left inverse for  $\partial_i$  only on series containing  $x_i$  in every monomial of its expansion (see remark in Ref. 3, p. 6815). Each

$\int_0^{x_i}$  is a well-defined operator from  $\hat{V}(R)^{\text{ext}}$  to  $\hat{V}(R)^{\text{ext}}$  since for every basis monomial  $x_1^{e_1} \cdots x_n^{e_n}$  we can write  $\int_0^{x_i} x_1^{e_1} \cdots x_n^{e_n}$  as a monomial in the  $x_j$ 's with a coefficient that is a *convergent* series of complex numbers. Since one can read  $\int_0^{x_i} f$  as a "function" of the  $x_j$ 's, it makes sense to consider  $\int_0^{ax_i}$  for a nonzero constant  $a$ . In particular we can define  $\int_0^{(-1)^r q^{1x_i} f}$  for every integer  $r$  and  $l$  as  $(-1)^r (1-q^2)^{\sum_{k=0}^{\infty} q^{2k+l} x_i f(q^{-2} x_1, \dots, q^{-2} x_{i-1}, (-1)^r q^{2k+l} x_i, x_{i+1}, \dots, x_n)}$ .

Then, for every  $f \in \hat{V}(R)^{\text{ext}}$

$$\int_{-x_i}^{x_i} f := \int_0^{x_i} f - \int_0^{-x_i} f \quad \text{and} \quad \int_{-x_i \cdot \infty}^{x_i \cdot \infty} f := \lim_{r \rightarrow \infty} \int_{-q^{-2r} x_i}^{q^{-2r} x_i} f$$

are defined. The last definition is only formal so far, because the image of a power series is no longer a power series (coefficients might be infinite sums themselves), and we have no notion of convergence. One cannot find a convergence set because one cannot give nonzero values to noncommuting variables. This issue can be solved in different ways, so that we can give a meaning to equalities as well. The ideas here are based on the approaches of Kempf and Majid in Ref. 3 and of Koornwinder in Ref. 5 who treated the one-dimensional case. His approach was by means of extension of a suitable representation of  $\hat{V}(R)^{\text{ext}}$  and the search of a family of eigenvectors for which the integral would have a convergent eigenvalue. We approach the problem not for a single infinite integral, but for the  $n$ -dimensional integral  $I(f) := \int_{-x_n \cdot \infty}^{x_n \cdot \infty} \cdots \int_{-x_1 \cdot \infty}^{x_1 \cdot \infty} f$  which is formally

$$(1-q^2)^n \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} \sum_{\varepsilon \in \{\pm 1\}^n} q^{2|K| + \binom{n}{2}} x_1 \cdots x_n f(\varepsilon_1 q^{2k_1} x_1, \dots, \varepsilon_n q^{2k_n} x_n).$$

As we said,  $I(f)$  is not an element of  $\hat{V}(R)^{\text{ext}}$ , in general, since the coefficients with respect to the elements of the basis are not always definite. In order to define integrability, we fix an action of  $\hat{V}(R)$  on the space of power series in the  $n$  commuting variables  $z_1, \dots, z_n$  with complex coefficients. This representation corresponds with the choice of a normal form for the monomials in  $\hat{V}(R)$ . The representation, denoted by  $\triangleright$ , for monomials in the  $x_i$ 's acting on monomials in the  $z_j$ 's, is given by

$$x_1^{e_1} \cdots x_n^{e_n} \triangleright z_1^{h_1} \cdots z_n^{h_n} = q^{\sum_{i=1}^n e_i H_i} z_1^{h_1+e_1} \cdots z_n^{h_n+e_n},$$

and can be extended linearly to an action of  $\hat{V}(R)$  on formal power series in the  $z_j$ 's. We can restrict the space on which we act by taking the space  $V$  of power series which are absolutely convergent in a neighborhood of zero. This makes sense because the  $z_i$ 's commute with each other. We see that this space is invariant under the action of  $\hat{V}(R)$ . Moreover, we see that we can extend the representation of  $\hat{V}(R)$  on  $V$  to a representation of the class  $C$  given by the power series  $f$  in the  $x_i$ 's such that  $f \triangleright 1 \in V$ . Indeed one can see that

(A)  $\forall f = f(x) \in C$  and  $\forall g = g(z) \in V$  it holds that  $(f \triangleright g)(z)$  belongs to  $V$  because the associated series of absolute values is majorized by the product in  $V$  of series of absolute values associated to  $f \triangleright 1$  and  $g$ .

(B)  $\forall f$  and  $g \in C$ , their product  $fg \in C$  because  $(fg) \triangleright 1 = f \triangleright (g \triangleright 1) \in V$  by property (A).

Moreover,

(C)  $\forall f \in C$ ,  $f \triangleright 1 = 0 \Leftrightarrow f \equiv 0$ .

From now on we write  $f_{\cdot 1}$  for  $f \triangleright 1$ , for any expression  $f(x)$  for which the action on 1 makes sense. We would like to extend the representation now to the formal expressions of type  $I(f)$  for  $f \in C$ . This does not always make sense, hence we have to add further conditions. Let us take the subclass  $C'$  of  $C$  given by the series in  $C$  such that

- (a)  $f_{\cdot 1}$  can be continued analytically on  $\mathbf{R}^n + iU$  for some open neighborhood  $U$  of 0 in  $\mathbf{R}^n$ ;
- (b)  $f_{\cdot 1}$  is absolutely  $q^2$ -integrable for every  $\underline{z} \in \mathbf{R}^n$  for which every  $z_i \neq 0$ , i.e.,



$$\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \sum_{\varepsilon \in \{\pm 1\}^n} q^{2|K|} |z_1| \cdots |z_n| |f_{\cdot 1}(\varepsilon_1 q^{2k_1} z_1, \dots, \varepsilon_n q^{2k_n} z_n)| < \infty$$

for  $z$  outside the standard hyperplanes.

The class  $C'$  will be the class of integrable power series. For those series we can compute  $I(f)_{\cdot 1}$ , and this turns out to be

$$\begin{aligned} (I(f))_{\cdot 1} &= (1-q^2)^n \left[ \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} \sum_{\varepsilon} q^{2|K|+\binom{n}{2}} x_1 \cdots x_n f(\varepsilon_1 q^{2k_1} x_1, \dots, \varepsilon_n q^{2k_n} x_n) \right] \triangleright 1 \\ &= \int_{-q^{n-1}z_1 \cdot \infty}^{q^{n-1}z_1 \cdot \infty} \cdots \int_{-q^{n-i}z_i \cdot \infty}^{q^{n-i}z_i \cdot \infty} \cdots \int_{-z_n \cdot \infty}^{z_n \cdot \infty} f_{\cdot 1}(t_1, \dots, t_n) d_{q^2} t_n \cdots d_{q^2} t_1, \end{aligned}$$

where the  $q^2$ -integral in the last line is the  $q^2$ -Jackson integral in  $n$  variables obtained by iterating (8.11) in Ref. 5. It is clear that if  $f \in C'$ , then  $(I(f))_{\cdot 1}$  converges whenever  $z_j \neq 0$  for every  $j$ .

Because of (C), two objects in  $C$  are equal if and only if they act in the same way on 1. Following this philosophy, we say that two  $q^2$ -integrals of objects in  $C'$  are equal if and only if they act in the same way on 1. This will be our tool to show equalities then.

The first purpose is to show translation invariance of the operator  $I$  in a less formal way than in Ref. 3 where this appeared first. For this we need an extra assumption on the elements in  $C'$ , since we have to use Taylor's series, hence partial derivatives. We consider  $f \in C'$  satisfying the following.

(c) For some  $\eta > 0$  there exists for each  $J \in (\mathbf{Z}_{\geq 0})^n$  some constant  $C_J$  such that

$$|(D_{1,q^2}^{j_1} \cdots D_{n,q^2}^{j_n} f)_{\cdot 1}(z_1, \dots, z_n)| \leq C_J \prod_{k=1}^n (1 + |z_k|^2)^{-(1+\eta)}$$

if  $\underline{z} \in \mathbf{R}^n$ , where  $D_{j,q^2}$  denotes the standard  $q^2$ -Jackson partial derivative with respect to  $z_j$ .

For an  $f$  in  $C'$ , condition (c) implies that all the Jackson derivatives of  $f_{\cdot 1}$  are absolutely  $q^2$ -integrable for all  $\underline{z}$  in a neighborhood of 0 minus the intersection with the standard hyperplanes.

One sees immediately that  $(\partial_i f)_{\cdot 1} = (D_{i,q^2} f)_{\cdot 1}(qz_1, \dots, qz_{i-1}, z_i, \dots, z_n)$ . We show now that it makes sense to compute  $I(\partial_n^{j_n} \cdots \partial_1^{j_1} f) \triangleright 1$ , and that this is equal to zero whenever  $(j_1, \dots, j_n) \neq (0, \dots, 0)$ . We write

$$F_{\cdot 1}^J(z_1, \dots, z_n) := (\partial_n^{j_n} \cdots \partial_1^{j_1} f) \triangleright 1 = (D_{1,q^2}^{j_1} \cdots D_{n,q^2}^{j_n} (f)_{\cdot 1})(q^{j_1} z_1, \dots, q^{j_n} z_n).$$

Hence  $F_{\cdot 1}^J(\underline{z}) \in V$  if  $f$  satisfies condition (c). Moreover,  $F_{\cdot 1}^J$  is absolutely  $q^2$ -Jackson integrable if and only if for every choice of  $(h_1, \dots, h_n) \in \{\pm 1\}^n$

$$\sum_{\varepsilon \in \{\pm 1\}^n} \sum_{k_i=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} q^{2K \cdot H} |z_1 \cdots z_n| |F_{\cdot 1}^J(q^{2k_1 h_1} \varepsilon_1 z_1, \dots, q^{2k_n h_n} \varepsilon_n z_n)|$$

has a positive radius of convergence. Hence, if condition (c) holds for  $f(\underline{x})$ , and since the above sums are of the form

$$\begin{aligned} &\sum_{\varepsilon} \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} q^{2K \cdot H} |z_1 \cdots z_n| (D_{1,q^2}^{j_1} \cdots D_{n,q^2}^{j_n} f)_{\cdot 1}(q^{2k_1 h_1 + j_1} \varepsilon_1 z_1, \dots, q^{2k_n h_n + j_n} \varepsilon_n z_n) \\ &\leq C_J |z_1 \cdots z_n| \sum_{\varepsilon} \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} q^{2K \cdot H} \prod_{r=1}^n (1 + |q^{2k_r h_r} z_r|^2)^{-(1+\eta)} \end{aligned}$$

that converges since  $q \in (0,1)$ , one sees that the  $F_{\cdot 1}^J$  are  $q^2$ -integrable.

Moreover, for every  $\gamma = (\gamma_1, \dots, \gamma_n)$  with  $\gamma_j$  in  $\mathbf{R} - \{0\}$  we have

$$\begin{aligned} q^{\sum_{k=1}^n J^k} \int_{-\gamma_1 \cdot \infty}^{\gamma_1 \cdot \infty} \cdots \int_{-\gamma_n \cdot \infty}^{\gamma_n \cdot \infty} F_{\cdot 1}^J(t_1, \dots, t_n) d_{q^2} t_n \cdots d_{q^2} t_1 \\ = \int_{-q^{J^1} \gamma_1 \cdot \infty}^{q^{J^1} \gamma_1 \cdot \infty} \cdots \int_{-q^{J^n} \gamma_n \cdot \infty}^{q^{J^n} \gamma_n \cdot \infty} D_{1, q^2}^{j_1} \cdots D_{n, q^2}^{j_n} f_{\cdot 1}(t_1, \dots, t_n) d_{q^2} t_n \cdots d_{q^2} t_1 = 0. \end{aligned}$$

The proof is as in the one-dimensional case (see Ref. 5). Hence we can state the following.

**Lemma 3.2:** *Let  $f \in C'$  satisfy condition (c). Then for every  $J \neq \underline{0}$  there holds  $(I(\partial_n^{j_n} \cdots \partial_1^{j_1} f))_{\cdot 1} \equiv 0$ , so that we can conclude that  $I(\partial_n^{j_n} \cdots \partial_1^{j_1} f) = 0$ .*

*Proof:* One has

$$(I(\partial_n^{j_n} \cdots \partial_1^{j_1} f))_{\cdot 1} = \int_{-q^{n-1} z_1 \cdot \infty}^{q^{n-1} z_1 \cdot \infty} \cdots \int_{-q^{n-i} z_i \cdot \infty}^{q^{n-i} z_i \cdot \infty} \cdots \int_{-z_n \cdot \infty}^{z_n \cdot \infty} F_{\cdot 1}^J(t_1, \dots, t_n) d_{q^2} t_n \cdots d_{q^2} t_1 = 0.$$

□

**Proposition 3.3:** *Let  $f \in C'$  satisfy (c). Then  $(\text{id} \otimes I)\Delta(f) = 1 \otimes (If)$ .*

*Proof:* By the braided Taylor formula we have

$$(\text{id} \otimes I)(\Delta f) = \sum_{j_1, \dots, j_n \geq 0} \frac{y_1^{j_1} \cdots y_n^{j_n}}{[j_1]_{q^2}! \cdots [j_n]_{q^2}!} I(\partial_n^{j_n} \cdots \partial_1^{j_1} f),$$

but this is by Lemma 3.2 equal to the term with all  $j_k$ 's equal to 0. □

**Proposition 3.4:** *Let  $f(\underline{x}) \in C$  such that (c) holds for every  $\eta > 0$ . Then the statement of Proposition 3.3 holds for every polynomial  $p(\underline{x})$  times  $f(\underline{x})$ .*

*Proof:* It is not restrictive to assume that  $p(\underline{x})$  is a monomial. By property (A) there follows that also  $p(\underline{x})f(\underline{x}) \in C$ . We have to check that condition (c) holds for every element of the form  $x_1^{e_1} \cdots x_n^{e_n} f(\underline{x})$ . One sees immediately that

$$(x_1^{e_1} \cdots x_n^{e_n} f(\underline{x}))_{\cdot 1} = f_{\cdot 1}(q^{E^1} z_1, \dots, q^{E^n} z_n) z_1^{e_1} \cdots z_n^{e_n} \in V$$

if  $f(\underline{x}) \in C$ , so that  $(x_1^{e_1} \cdots x_n^{e_n} f(\underline{x}))_{\cdot 1}$  makes sense.

Condition (c) is on the  $q^2$ -Jackson partial derivatives on commuting variables, and it holds for  $x_1^{e_1} \cdots x_n^{e_n} f(\underline{x})$  as a consequence of the fact that for two functions  $a(\underline{z})$  and  $b(\underline{z})$  and for any  $j = 1, \dots, n$ ,

$$D_{j, q^2}(a(\underline{z})b(\underline{z})) = (D_{j, q^2}a(\underline{z}))(b(z_1, \dots, z_{j-1}, q^2 z_j, z_{j+1}, \dots, z_n)) + a(\underline{z})D_{j, q^2}(b(\underline{z})).$$

Then, by Proposition 3.3 we have the statement. □

We have a description of a class of power series in the  $x_i$ 's for which integration makes sense, although we are not able so far to make a complete classification of integrable functions. The same problem is treated in Ref. 6 for the one-dimensional case. Still, what we have is enough to allow computations in the following case.

**Example 1:** The  $q^2$ -Gaussian  $g_{q^2}(\underline{x})$  is

$$g_{q^2}(\underline{x}) := e_{q^4}(-\underline{x} \cdot \underline{x}) = e_{q^4} \left( - \sum_{j=1}^n x_j^2 \right) = \prod_{j=1}^n e_{q^4}(-x_j^2),$$



where the last equality holds because of Proposition 3.1 in Ref. 5 (this result was already in Ref. 12) and the product in the above formula is taken with increasing order on the variables. It satisfies conditions (a), (b), and (c) for every  $\eta > 0$ . This is a consequence of the one-dimensional case (see Ref. 5) and the fact that

$$(g_{q^2}(\underline{x}))_{\cdot 1}(\underline{z}) = \prod_{j=1}^n e_{q^4}(-z_j^2), \quad \partial_j(g_{q^2}(\underline{x})) = -\frac{x_j}{(1-q^2)} g_{q^2}(\underline{x})$$

and that

$$D_{q^2,1}^{j_1} \cdots D_{q^2,n}^{j_n} \left( e_{q^4} \left( -\sum_{k=1}^n z_k^2 \right) \right) = p(\underline{z}) e_{q^4} \left( -\sum_{k=1}^n z_k^2 \right),$$

where  $p(\underline{z})$  is a polynomial in the  $z_j$ 's.

It follows then that also elements of the form

$$x_1^{a_1} g_{q^2}(x_1) \cdots x_n^{a_n} g_{q^2}(x_n) = x_1^{a_1} \cdots x_n^{a_n} e_{q^4} \left( -\sum_j (q^{-A^j} x_j)^2 \right)$$

satisfy condition (c), so that for every  $p_j(x_j) \in \hat{V}(R)$  we can integrate every element of the form  $p_1(x_1) g_{q^2}(x_1) \cdots p_n(x_n) g_{q^2}(x_n)$ .

In particular, for  $f(\underline{x}) = g_{q^2}(x_1) x_1^{a_1} \cdots g_{q^2}(x_n) x_n^{a_n}$  one can compute

$$\begin{aligned} (I(f))_{\cdot 1}|_{z=\gamma} &= \int_{-q^{n-1}\gamma_1 \cdot \infty}^{q^{n-1}\gamma_1 \cdot \infty} \cdots \int_{-q^{n-i}\gamma_i \cdot \infty}^{q^{n-i}\gamma_i \cdot \infty} \cdots \int_{-\gamma_n \cdot \infty}^{\gamma_n \cdot \infty} f_{\cdot 1}(t_1, \dots, t_n) d_{q^2} t_n \cdots d_{q^2} t_1 \\ &= \prod_{j=1}^n \left( \int_{-q^{n-j}\gamma_j \cdot \infty}^{q^{n-j}\gamma_j \cdot \infty} e_{q^4}(-t_j^2) t_j^{a_j} d_{q^2} t_j \right) \\ &= \begin{cases} \prod_{j=1}^n (c_{q^2}(\gamma_j q^{n-j}) q^{-a_j^2/2} (q^2; q^4)_{a_j/2}), & \text{if } a_j \text{ even } \forall j, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where

$$c_{q^2}(\gamma) = \frac{2(1-q^2)(q^4, -q^2\gamma^2, -q^2\gamma^{-2}; q^4)_\infty}{(-\gamma^2, -q^4\gamma^{-2}, q^2; q^4)_\infty}$$

as in formula (8.15) in Ref. 5. In particular,

$$(I(f))_{\cdot 1}|_{z=\gamma} = q^{-\sum_j a_j^2/2} \prod_j (q^2; q^4)_{a_j/2} (I(g_{q^2}(\underline{x})))|_{z=\gamma}$$

if all  $a_j$ 's are even, and 0 otherwise, hence we can conclude that

$$I(f) = \begin{cases} q^{-\sum_j a_j^2/2} \prod_j (q^2; q^4)_{a_j/2} (I(g_{q^2}(\underline{x}))), & \text{if } a_j \text{ is even } \forall j, \\ 0, & \text{otherwise.} \end{cases}$$

The observation that the integral of the Gaussian times a monomial is equal to a constant times the integral of the Gaussian appeared in Refs. 1 and 3 first. In our case we have to deal with the shift and this depends on the choice of our global integral. ♠

*Remark:* The reader may wonder whether we could have chosen another realization of  $\hat{V}(R)^{\text{ext}}$  and of the integrals of elements of  $\hat{V}(R)^{\text{ext}}$  other than  $f \triangleright 1$ . Of course one might consider a different representation, or a different choice of the normal form. The advantages of a representation associated to the choice of a normal form is the fact that it is enough to test operators on 1 to state an equivalence in  $\hat{V}(R)$ . The advantage of the particular normal form that we have chosen is based on the fact that  $C$  is closed under product, hence we have a map from formal expressions in  $x_1, \dots, x_n$  to formal expressions in the  $z_1, \dots, z_n$  such that on rather big subspaces it comes exactly from an algebra homomorphism. If we had chosen another normal form, we could no longer extend the representation  $\pi$  of  $\hat{V}(R)$  on  $V$  to a representation of the subset  $S$  of  $\hat{V}(R)^{\text{ext}}$  such that  $\pi(S)(1) \subset V$ . Take, for instance,  $n=2$ , and the representation of  $\hat{V}(R)$  on  $\mathbf{R}[[z, w]]$  given by  $\pi(x_1)(f(z, w)) = zf(z, q^{-1}w)$  and  $\pi(x_2)(f(z, w)) = f(z, w)w$ . This is the representation associated to the choice of the normal form with  $x_2$  preceding  $x_1$ . ♠

Then  $a = \sum_{k=0}^{\infty} x_1^k$  and  $b = \sum_{l=0}^{\infty} x_2^l$  belong to  $S$ , but  $ab$  does not belong to  $S$  since  $\pi(\sum_{k=0}^{\infty} x_1^k)(\pi(\sum_{l=0}^{\infty} x_2^l)(1)) = \sum_{k,l=0}^{\infty} q^{-kl} z^k w^l \notin V$ .

#### IV. LATTICE INTEGRABILITY

In the previous section we saw a definition of integrable series in  $\hat{V}(R)^{\text{ext}}$ . Unfortunately, the above method fails for another analog of the Gaussian we would like to  $q^2$ -integrate, namely the  $q^2$ -Gaussian

$$G_{q^2}(\underline{x}) := E_{q^4}(-\underline{x} \cdot \underline{x}) = E_{q^4}\left(-\sum_{j=1}^n x_j^2\right) = E_{q^4}(-x_n^2) \cdots E_{q^4}(-x_1^2).$$

In Ref. 5, Sec. 9, it is also shown that  $G_{q^2}(\underline{x})$  does not satisfy condition (c), nor condition (b) for  $n=1$ . On the other hand, it is also shown there that for a given choice of a  $q^2$ -lattice of the form  $\{\pm q^{2k} \gamma | k \in \mathbf{Z}\}$ , namely for  $\gamma=1$ ,  $(I(G_{q^2}(x_1)))_{|z_1=1}$  is absolutely convergent. Hence one can introduce a weaker version of integrability in  $\hat{V}(R)^{\text{ext}}$ , which we will call “lattice integrability,” requiring for an  $f(\underline{x})$  such that  $f_{|1}$  is entire that there is a  $q^2$ -lattice  $L(\gamma) = \{(\pm \gamma_1 q^{2k_1}, \dots, \pm \gamma_n q^{2k_n}) | K \in \mathbf{Z}^n\}$  in  $\mathbf{R}_{\neq 0}^n$  such that the expression  $(I(f))_{|z \in L(\gamma)}$  is absolutely convergent. Of course if a generalized function is  $q^2$ -integrable, then it is lattice integrable for every choice of a lattice. One can easily see that the power series  $E_{q^4}(-x_1^2) \cdots E_{q^4}(-x_n^2)$  is lattice integrable for  $\gamma = (q^{n-1}, \dots, q^{n-j}, \dots, 1)$ . Unfortunately, this power series is not the  $q^2$ -Gaussian  $G_{q^2}(\underline{x})$  that we wanted to integrate, for  $n \geq 2$ . Besides, we can show that already for  $n=2$ ,  $G_{q^2}(\underline{x})$  is not lattice integrable although it is entire.

(Counter)example 1: Let us consider  $G_{q^2}(\underline{x})$  for  $n=2$ . We write for simplicity  $x_1=x$  and  $x_2=y$ , and  $z_1=z$ ,  $z_2=w$ . Then

$$G_{q^2}(\underline{x}) = \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l} q^{2k^2+2l^2-2k-2l} y^{2l} x^{2k}}{(q^4; q^4)_l (q^4; q^4)_k} = \sum_{l=0}^{\infty} \frac{(-1)^l q^{2l^2-2l} E_{q^4}(-(q^{2l}x)^2) y^{2l}}{(q^4; q^4)_l},$$

hence

$$\begin{aligned} G_{q^2}(\underline{x})_{|1} &= \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l} q^{2k^2+2l^2-2k-2l} q^{4kl} z^{2k} w^{2l}}{(q^4; q^4)_l (q^4; q^4)_k} \\ &= \sum_{l=0}^{\infty} \frac{(-1)^l q^{2l^2-2l} E_{q^4}(-(q^{2l}z)^2) w^{2l}}{(q^4; q^4)_l} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{2k^2-2k} E_{q^4}(-(q^{2k}w)^2) z^{2k}}{(q^4; q^4)_k}, \end{aligned}$$

which is entire since it is majorized by  $E_{q^4}(|w|^2)E_{q^4}(|z|^2)$ . Now we wonder whether this expression is lattice integrable or not. In order to have that, we would need that for some  $\gamma = (\gamma_1, \gamma_2)$ ,

$$\int_{-q\gamma_1 \cdot \infty}^{q\gamma_1 \cdot \infty} \int_{-\gamma_2 \cdot \infty}^{\gamma_2 \cdot \infty} |(G_{q^2})_{\cdot 1}(t_1, t_2)| d_{q^2} t_1 d_{q^2} t_2 < \infty.$$

For this we would need that

$$\sum_{h_1=-\infty}^{\infty} \sum_{h_2=-\infty}^{\infty} q^{2|H|} \left| \sum_{l=0}^{\infty} \frac{(-1)^l q^{2l^2-2l} E_{q^4}(-(q^{2l+2h_1} q \gamma_1)^2) (q^{2h_2} \gamma_2)^{2l}}{(q^4; q^4)_l} \right| < \infty,$$

therefore we have to look at the limit for  $h_j \rightarrow -\infty$  of the summands. Clearly by the discussion in Sec. 9 of Ref. 5, we see that we would need to have  $\gamma_1 = q$ . With a similar reasoning we see that  $\gamma_2$  must be equal to 1. Now, for general  $z$  and  $w$  we have

$$\begin{aligned} (G_{q^2}(\underline{x})_{\cdot 1})(z, w) &= \sum_{l=0}^{\infty} \frac{(-1)^l q^{2l^2-2l} w^{2l} (q^{4l} z^2; q^4)_{\infty}}{(q^4; q^4)_l} \\ &= (z^2; q^4)_{\infty} \sum_{l=0}^{\infty} \frac{(-1)^l q^{2l^2-2l} w^{2l}}{(q^4; q^4)_l (z^2; q^4)_l} \\ &= (z^2; q^4)_{\infty} \phi_1(0; z^2; q^4, w^2), \end{aligned}$$

which is the  $q^4$  version of the  $q$ -Bessel function described in Ref. 9. For  $(z, w) = (q^{2-2r}, q^{2s})$  with  $r \geq 0$  and  $s$  any integer, we have, by the estimates (2.6) and the following estimates in Ref. 9, that

$$|(G_{q^2}(\underline{x})_{\cdot 1})(q^{2-2r}, q^{2s})| = q^{2r(r-1)} q^{4rs} (q^{4r+4}; q^4)_{\infty} |\phi_1(0; q^{4r+4}; q^4, q^{4r+4s})|.$$

For  $r \rightarrow \infty$  and  $s = -r$  this behaves like  $q^{2r(r-1)-4r^2} \rightarrow \infty$ . Hence  $G_{q^2}(\underline{x})$  is not lattice integrable. ♠

*Remarks:* An analog of the symmetry (2.2) for  $q$ -Bessel functions in Ref. 9 holds in our case, namely,

$$E_{q^4}(-x_1^2)_{\cdot 1} \phi_1(0; x_1^2; q^4, x_2^2) = G_{q^2}(\underline{x}) = {}_1\phi_1(0; x_2^2; q^4, x_1^2) E_{q^4}(-x_2^2),$$

once we agree that in  ${}_1\phi_1$  every time we have a product of type  $x_2^1/(x_1^2; q^4)_l$ , the terms in  $x_1$  have to be taken *before* the terms in  $x_2$ . Hence, in general, one has

$$E_q(-x_1 - x_2) = E_q(-x_1) {}_1\phi_1(0; x_1; q, x_2) = {}_1\phi_1(0; x_2; q, x_1) E_q(-x_2)$$

with the above meaning for  ${}_1\phi_1$  in noncommuting variables.

Another equality involving a  ${}_1\phi_1$  and exponentials in  $q$ -commuting variables is obtained by writing  $E_q(-x_1)E_q(-x_2)$  as

$$(x_2; q)_{\infty} \sum_{l=0}^{\infty} \frac{(-1)^l q^{(1/2)(l^2-l)} (q^{-1} x_2; q)_l x_1^l}{(q; q)_l}$$

and using (3.12) in Ref. 5 with  $x_1 = -y$  and  $x_2 = -x$ . Then one obtains

$$\sum_{l=0}^{\infty} \frac{(-1)^l q^{(1/2)(l^2-l)} x_1^l (x_2; q)_l}{(q; q)_l} = E_q(x_1 x_2) E_q(-x_1),$$

where the sum on the left-hand side can be considered as a  ${}_1\phi_1$  in noncommuting variables once assumed that  $x_1$  always precedes  $x_2$  in products. These facts were pointed out to me by T. Koornwinder. ♠

On the other hand, one can easily check that for every  $A=(a_1, \dots, a_n)$  in  $\mathbf{R}_{>0}^n$  and every  $E=(e_1, \dots, e_n)$ , then  $x_1^{e_1} \cdots x_n^{e_n} E_{q^4}(- (a_1 x_1)^2) \cdots E_{q^4}(- (a_n x_n)^2)$  is lattice integrable in the  $q^2$ -lattice generated by  $\gamma$  where  $\gamma_j = a_j^{-1} q^{n-j+E^j}$ . Unfortunately, lattice integrability carries a lot of technical work with it whenever one wants to prove anything like translation invariance, for instance. This is a consequence of the fact that, in order to state that the integral of  $\partial_n^{e_n} \cdots \partial_1^{e_1} f(\underline{x})$  is zero, one needs to keep track of the lattice in which this series is integrable, which in general is not the same as the lattice in which  $f(\underline{x})$  is integrable, unless  $e_j$  is even for every  $j$ . Indeed, consider  $n=2$  and  $f(\underline{x}) = E_{q^4}(-x_1^2) E_{q^4}(-x_2^2)$ . Then,  $f(\underline{x})$  is integrable for  $(\gamma_1, \gamma_2) = (q, 1)$  while  $\partial_2(f(\underline{x})) = -E_{q^4}(-q^2 x_1^2) [x_2 / (1 - q^2)] E_{q^4}(-q^4 x_2^2)$  is integrable for  $(\gamma_1, \gamma_2) = (1, 1)$ . However, since “morally” the integral of a function which is odd in a variable is zero, we might as well define the integral of every odd function to be zero by changing the definition of the integral. Namely, we define the new integral  $I'$  to be the integral of the even part of the series  $f(\underline{x})$ . We formalize this definition.

Let  $f(\underline{x})$  be any formal power series in the  $x_j$ 's. We want to decompose it in  $2^n$  series depending on the parity with respect to each variable. Let

$$\Pi_j^\pm : \hat{V}(R)^{\text{ext}} \rightarrow \hat{V}(R)^{\text{ext}} f(\underline{x}) \mapsto \frac{1}{2} (f(\underline{x}) \pm f(x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_n))$$

for every  $j$  and for any choice of  $\pm$ . This makes sense formally, and makes sense even concretely for the series in  $C$ . Clearly those operators commute; they are projections on the space of power series that are even (resp. odd) in the  $j$ th variable, so that  $\Pi_j^+ \Pi_j^- = 0$  for every  $j$ . We define then for every choice of  $\beta$  in  $\{\pm\}^n$  the operators  $\Pi_\beta : \hat{V}(R) \rightarrow \hat{V}(R)$  as  $(\Pi_1^{\beta_1}) \circ \cdots \circ (\Pi_n^{\beta_n})$ . They are all projections on their image  $E_\beta$ , and clearly the decomposition of the space of power series in the  $x_i$ 's descends to a decomposition of the space  $C$  in  $2^n$  spaces that we will call  $C_\beta$ . We also write  $V^\beta := C_\beta \triangleright 1$ . We will denote  $\Pi_{(+, \dots, +)}$  by  $\Pi_0$  for simplicity.

In particular,  $\Pi_0 f(\underline{x}) = 2^{-n} \sum_{\varepsilon \in \{\pm 1\}^n} f(\varepsilon_1 x_1, \dots, \varepsilon_n x_n)$  is even in every variable, and we define the integral  $I'$  to be the composition  $I \circ \Pi_0$ .

*Remarks:* Since we work in characteristic zero,  $I'f$  is also formally equal to

$$\int_0^{x_n \cdot \infty} \cdots \int_0^{x_1 \cdot \infty} \sum_{\varepsilon \in \{\pm 1\}^n} f(\varepsilon_1 x_1, \dots, \varepsilon_n x_n).$$

Clearly the class of  $I'$  integrable series is bigger than the class of  $I$  integrable series, since all odd series are integrable and their integral is zero. Since  $I'$  integrability of a series  $f(\underline{x})$  coincides with  $I$  integrability of  $\Pi_0 f(\underline{x})$ , if  $f(\underline{x})$  is a series which is even in all the variables, then  $f(\underline{x})$  is  $I$  integrable  $\Leftrightarrow f(\underline{x})$  is  $I'$  integrable since  $f(\underline{x}) = \Pi_0 f(\underline{x})$ . ♠

One can also introduce lattice  $I'$  integrability. Again, for series in  $C_{(+, \dots, +)}$ , lattice integrability and lattice  $I'$  integrability trivially coincide, and for a generic  $f(\underline{x})$ , lattice  $I'$  integrability trivially coincides with lattice  $I$  integrability of  $\Pi_0 f(\underline{x})$  in the same lattice.

We can provide generalizations of Lemma 3.2, and Propositions 3.3 and 3.4 by introducing condition (c') for an  $f$  such that  $\Pi_0 f \in C'$ :

(c') For some  $\eta > 0$ , there exists for each  $J = (j_1, \dots, j_n) \in \mathbf{Z}_{\geq 0}^n$  and  $\beta \in \{\pm\}^n$  such that  $j_k$  is even (resp. odd) if  $\beta_k = +$  (resp.  $-$ ), some constant  $K_J$  such that

$$|(D_{1,q^2}^{j_1} \cdots D_{n,q^2}^{j_n} (\Pi_\beta f)_{\cdot 1})(z_1, \dots, z_n)| \leq K_J \prod_{k=1}^n (1 + |z_k|^2)^{-(1+\eta)}$$

if  $\underline{z} \in \mathbf{R}^n$ .

Then we have the following Lemma

**Lemma 4.1:** Let  $f \in C'$  satisfy condition (c'). Then for every  $J \neq \emptyset$  there holds  $(I'(\partial_n^{j_n} \cdots \partial_1^{j_1} f)) \triangleright 1 \equiv 0$ , so that we can conclude that  $I'(\partial_n^{j_n} \cdots \partial_1^{j_1} f) = 0$ .

*Proof:*  $I'(\partial_n^{j_n} \cdots \partial_1^{j_1} f) = I(\partial_n^{j_n} \cdots \partial_1^{j_1} \Pi_\beta f)$  for  $\beta$  related to  $J$  as in condition (c').  $\square$

**Proposition 4.2:** Let  $f \in C'$  satisfy (c'). Then  $(\text{id} \otimes I')\Delta(f) = 1 \otimes (I'f)$ .  $\square$

**Proposition 4.3:** Let  $f(\underline{x}) \in C$  such that (c') holds for every  $\eta > 0$ . Then the statement of Proposition 3.2 holds for every polynomial  $p(\underline{x})$  times  $f(\underline{x})$ .  $\square$

We also have another invariance property that is analogous to the classical property (for  $n = 1$ ):

$$\int_{-\infty}^{\infty} \frac{1}{2} (f(x) + f(-x)) dx = \int_{-\infty}^{\infty} \frac{1}{2} (f(x+y) + f(x-y)) dx.$$

**Proposition 4.4:** Let  $f(\underline{x}) \in C'$  satisfy (c'). Then  $(\text{id} \otimes I)(\Pi_0 \otimes \text{id})\Delta(f) = 1 \otimes (I'f)$ . If  $f(\underline{x})$  satisfies condition (c') for every  $\eta > 0$ , then the statement is true for every series of the form  $x_1^{e_1} \cdots x_n^{e_n} f(\underline{x})$ .

*Proof:* The proof uses Taylor's formula with summation only on even  $j_k$ 's.  $\square$

Observe that for even  $j_k$ 's

$$\begin{aligned} (I(\partial_n^{j_n} \cdots \partial_1^{j_1} f))_{.1} &= (I'(\partial_n^{j_n} \cdots \partial_1^{j_1} f))_{.1} \\ &= q^{-\sum_k j_k} \int_{-q^{n-1}z_1}^{q^{n-1}z_1} \cdots \int_{-z_n}^{z_n} D_{1,q^2}^{j_1} \cdots D_{n,q^2}^{j_n} f_{.1}(t_1, \dots, t_n) d_{q^2} t_n \cdots d_{q^2} t_1, \end{aligned}$$

hence the proposition above is interesting also because it can be proved for lattice integrability with simple changes in the hypothesis and in the proof. This reads as follows. Let  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbf{R}^n$ . We define the following spaces,

$$C(\gamma) = \{f(\underline{x}) \in \hat{V}(R)^{\text{ext}} | f_{.1}|_{z=\gamma} \text{ is absolutely convergent}\}$$

and  $C(q^{2K}, \gamma)$  as the space of  $f(\underline{x}) \in C(\gamma)$  such that  $f_{.1}$  can be continued analytically on a domain containing the  $q^2$ -lattice  $L(\gamma)$  generated by  $\gamma$ . Clearly  $C(\gamma)$  is closed with respect to the multiplication, hence it acts on the space  $V_\gamma$  of power series in commuting variables  $z_1, \dots, z_n$  that are absolutely convergent for  $z = \gamma$ , hence on a polydisc with polyradius  $(|\gamma_1|, \dots, |\gamma_n|)$ . Let  $f(\underline{x})$  be a series in  $C(q^{2K}, \gamma)$  for a given  $\gamma$ . Then it makes sense to investigate  $I'(f(\underline{x}))_{.1}$  at  $z_j = q^{n-j}\gamma_j$  and if this expression is absolutely convergent, then we say that  $f(\underline{x})$  is lattice integrable. Actually, we would only need  $\Pi_0(f) \in C(q^{2K}, \gamma)$ , but, since we want to compute integrals of products, we keep the restriction on  $f(\underline{x})$ .

Consider now the lattice version of condition (c):

(c'') Let  $f(\underline{x}) \in C(q^{2K}, \gamma)$  be such that for every  $J$  with even entries  $j_1, \dots, j_n$ , the Jackson partial derivatives  $D_{1,q^2}^{j_1} \cdots D_{n,q^2}^{j_n} f_{.1}$  exist on the lattice  $L(\gamma)$ , and are such that

$$|(D_{1,q^2}^{j_1} \cdots D_{n,q^2}^{j_n} f_{.1})(q^{\pm 2k_1}\gamma_1, \dots, q^{\pm 2k_n}\gamma_n)| = O(q^{2(1+\eta)K_-})$$

for  $k_i \rightarrow \infty$ , for some  $\eta > 0$ , where  $K_-$  is the sum of the  $k_j$ 's appearing with the minus sign.

We introduce the equivalence relation  $\sim_\gamma$  between two expressions  $f(\underline{x})$  and  $g(\underline{x})$  belonging to  $C(q^{2K}, \gamma)$  as follows:

$$f(\underline{x}) \sim_\gamma g(\underline{x}) \Leftrightarrow f_{.1}(\underline{z}) = g_{.1}(\underline{z}), \forall \underline{z} \in L(\gamma).$$

**Proposition 4.5:** Let  $f(\underline{x})$  satisfy condition (c') for a given  $\gamma$ , and let  $\gamma'$  denote the  $n$ -tuple  $(q^{n-1}\gamma_1, \dots, q^{n-j}\gamma_j, \dots, \gamma_n)$ . Then we have the following.

(i) For every  $J$  such that every  $j_k$  is even,  $I'(\partial_n^{j_n} \cdots \partial_1^{j_1} f) \sim_{\gamma'} 0$ .

(ii)  $(\text{id} \otimes I)(\Pi_0 \otimes \text{id})\Delta(f) \sim_\gamma 1 \otimes (I'f)$ .

Moreover, if  $f(\underline{x})$  satisfies condition (c'') for every  $\eta$  and for every  $J \in \mathbf{Z}_{\geq 0}^n$ , then for every monomial  $x_1^{e_1} \cdots x_n^{e_n}$  we have the following.

- (iii)  $I'(\partial_n^{2j_n} \dots \partial_1^{2j_1}(x_1^{e_1} \dots x_n^{e_n} f(\underline{x}))) \sim_{\gamma''} 0$  for every  $J$ , where  $\gamma_j'' = q^{n-j+E^j} \gamma_j$ .  
 (iv)  $(\text{id} \otimes I)(\Pi_0 \otimes \text{id}) \Delta(x_1^{e_1} \dots x_n^{e_n} f(\underline{x})) \sim_{\gamma''} 1 \otimes I'(x_1^{e_1} \dots x_n^{e_n} f(\underline{x}))$  where  $\gamma''$  is as above.

*Proof:* Statements (i) and (ii) are clear by the remark after the proof of Proposition 4.4. In order to prove (iii) we recall that

$$q^{4\sum_k J^k} I'(\partial_n^{2j_n} \dots \partial_1^{2j_1}(x_1^{e_1} \dots x_n^{e_n} f(\underline{x})))_{.1} \\ = \int_{-q^{n-1}z_1 \cdot \infty}^{q^{n-1}z_1 \cdot \infty} \dots \int_{-z_n \cdot \infty}^{z_n \cdot \infty} D_{1,q^2}^{2j_1} \dots D_{n,q^2}^{2j_n}(t_1^{e_1} \dots t_n^{e_n} f_{.1}(q^{E^1} t_1, \dots, q^{E^n} t_n)) d_{q^2} t,$$

hence for  $z = \gamma''$  this expression converges, and it converges to zero. By invariance under  $q^2$ -shifts of the Jackson integral we get the statement. Statement (iv) follows from statement (iii).  $\square$

*Remark:* Observe that in the proof of (iii) in Proposition 4.5 the lattice in which we compute the equality depends only on the parity of the  $e_j$ 's and that it is enough to be able to keep under control the partial Jackson derivatives of  $(P_{\beta} f)_{.1}$  with  $\beta_j = +$  (resp.  $-$ ) if  $e_j$  is even (resp. odd).  $\spadesuit$

One may check that  $E_{q^4}(-x_1^2) \dots E_{q^4}(-x_n^2)$  satisfies all conditions of Proposition 4.5. Computations are left to the reader.

## V. LATTICE ORDER INTEGRABILITY

We are still left with the problem that the  $q^2$ -Gaussian  $G_{q^2}(\underline{x})$  is not lattice integrable, even with respect to  $I'$ . We have to weaken again our condition and introduce the concept of *lattice order integrability*. To simplify notation, we use analogs of  $I'$  instead of  $I$ . What we do is repeatedly apply a one-dimensional integral with respect to a noncommutative variable, say  $x_j$ . If this expression “has a meaning” (i.e., this expression applied to 1 converges after evaluation at  $z_j = \gamma_j$ ), then we will identify it with a power series in noncommuting variables, in one variable less, and we are allowed to go further and repeat the procedure. Namely:

*Definition 5.1:* A formal power series  $f(\underline{x}) \in C$  is said to be *lattice order integrable* (l.o. integrable) if there is an ordering of  $1, \dots, n$ , denoted by the corresponding permutation  $\sigma \in S_n$ , and an  $n$ -tuple  $\gamma \in \mathbf{R}_{>0}^n$  such that for every  $j \in \{1, \dots, n\}$  the expression  $\int_{\sigma(j)} (I_{\sigma(j-1)} \dots I_{\sigma(1)} f)$  is entire, where  $\int_{\sigma(k)} g$  and  $I_{\sigma(k)} g$  are defined inductively as follows. For a formal power series  $f$  in  $\{x_1, \dots, x_n\} - \{x_{\sigma(1)}, \dots, x_{\sigma(k-1)}\}$ ,  $\int_{\sigma(k)} f$  is the formal expression in the commuting variables  $\{z_1, \dots, z_n\} - \{z_{\sigma(1)}, \dots, z_{\sigma(k)}\}$  defined as

$$\left( \int_{\sigma(k)} f \right) (\underline{z}) := \left( \int_{-x_{\sigma(k)} \cdot \infty}^{x_{\sigma(k)} \cdot \infty} \prod_0 f \right) \Big|_{z_{\sigma(k)} = \gamma_{\sigma(k)}}.$$

If  $\int_{\sigma(k)} f$  is entire,  $I_{\sigma(k)} f$  will denote the unique power series in the noncommuting indeterminates  $\{x_1, \dots, x_n\} - \{x_{\sigma(1)}, \dots, x_{\sigma(k)}\}$  such that  $(\int_{\sigma(1)} f) = (I_{\sigma(1)} f)_{.1}$ . If  $f(\underline{x})$  is l.o. integrable, we define the constant  $I''_{(\sigma, \gamma)} f := \int_{\sigma(n)} (I_{\sigma(n-1)} \dots I_{\sigma(1)} f)$  to be the *lattice order integral* of  $f(\underline{x})$  associated to the order  $\sigma$  and the lattice  $L(\gamma)$ .

Clearly there is quite a difference between  $I$  and  $I''_{(\sigma, \gamma)}$  since  $I$  maps formal power series to formal expressions in  $x_1, \dots, x_n$  while  $I''_{(\sigma, \gamma)}$  maps l.o. integrable power series to constants. We will see later what the relation is between the two maps, on the space where they are both defined. We will also see in the examples that even if a power series is l.o. integrable for every order, it could still not be lattice integrable.

Observe that by definition of  $\int_I$ , power series that are odd in some variables are automatically defined to be l.o. integrable and that the integral will be zero for every choice of  $\sigma$ . For this reason, we will only investigate lattice order integrability for even power series. We can state a few results about lattice order integrability.

*Proposition 5.2:* Let  $f(\underline{x})$  be an even element of  $\hat{V}(R)^{\text{ext}}$  such that, for some  $\tau \in S_n$ , and for some power series in one indeterminate  $f_1, \dots, f_n$ , we can write  $f(\underline{x}) = f_{\rho(1)}(x_{\rho(1)}) \dots$



$f\rho(n)(x\rho(n))$  where  $\rho=\tau^{-1}$ . If  $f(\underline{x})$  is l.o. integrable and every  $\int_l f \neq 0$ , then each  $f_j$  (viewed as a power series in one variable) is entire and lattice integrable. Conversely, if each  $f_j$  (viewed as a power series in one variable) is entire and lattice integrable, then  $f(\underline{x})$  is l.o. integrable. In this case,  $f(\underline{x})$  is lattice order integrable for every order  $\sigma$  and a suitable lattice depending on  $\sigma$ . Moreover, one has

$$I''_{(\sigma,\gamma)}(f(\underline{x})) = q^{l(\sigma)+l(\tau)} \prod_{j=1}^n \int_{-\gamma_j \cdot \infty}^{\gamma_j \cdot \infty} (f_j)_{\cdot 1}(t_j) d_{q^2} t_j,$$

where  $l$  denotes the usual length of a permutation.

*Proof:* ( $\Rightarrow$ ) Suppose that  $f(\underline{x})$  is as in the hypothesis, and that each  $f_j$  is entire and lattice integrable for a given  $\tilde{\gamma}_j$ . We write  $f_j(x_j) = \sum_k c_{jk} x_j^k$  for every  $j$ . For an  $n$ -tuple  $K$  and for  $p \in \{1, \dots, n\}$ , we will also write

$$K_{\tau,p} := \sum_{\substack{j > p \\ \tau(j) < \tau(p)}} k_j \quad \text{and} \quad K^{\tau,p} := \sum_{\substack{j < p \\ \tau(j) > \tau(p)}} k_j.$$

We fix a  $\sigma$ . Then for  $\sigma(1) = l$  and  $\gamma_l = \tilde{\gamma}_l$  one has

$$\begin{aligned} & \left( \int_l f \right) (z_1, \dots, z_{l-1}, z_{l+1}, \dots, z_n) \\ &= 2(1-q^2) \sum_{h=-\infty}^{\infty} q^{2h} \gamma_l \sum_{K'} c_{1k_1} \cdots \hat{c}_{lk_n} \cdots c_{nk_n} (q^{-1}z_1)^{k_1} \cdots (q^{-1}z_{l-1})^{k_{l-1}} \\ & \quad \times z_{l+1}^{k_{l+1}} \cdots z_n^{k_n} q^{\sum_{j \neq l} k_j K'_{\tau,j}} f_l(q^{2h+K'_{\tau,p}+K'^{\tau,p}\gamma_l}), \end{aligned}$$

where  $K'$  is the  $(n-1)$ -tuple obtained by  $K$  by deleting  $\sigma(1) = l$ , and

$$K'_{\tau,p} := \sum_{\substack{l \neq j > p \\ \tau(j) < \tau(p)}} k_j \quad \text{and} \quad K^{\tau,p} := \sum_{\substack{l \neq j < p \\ \tau(j) > \tau(p)}} k_j.$$

The last equality holds because

$$\begin{aligned} \sum_p k_p K_{\tau,p} &= k_l K_{\tau,l} + \sum_{p > l} k_p K_{\tau,p} + \sum_{p < l} k_p K_{\tau,p} \\ &= k_l K_{\tau,l} + \sum_{p > l} k_p K'_{\tau,p} + \sum_{p < l} k_p K'_{\tau,p} + \sum_{\substack{p < l \\ \tau(l) < \tau(p)}} k_p k_l \\ &= k_l (K_{\tau,l} + K^{\tau,l}) + \sum_{p \neq l} k_p K'_{\tau,p}. \end{aligned}$$

By convergence of the  $q^2$ -Jackson integral of  $f_{l \cdot 1}$ , together with the fact that the other  $f_k$ 's are entire and the fact that  $K'_{\tau,p} + K'^{\tau,p}$  is an even number because the  $f_j$ 's are even, one can invert the order of summation in the above sum, using dominated convergence. One gets

$$\begin{aligned}
& \left( \int_l f \right) (z_1, \dots, z_{l-1}, z_{l+1}, \dots, z_n) \\
&= \sum_{K'} q^{\sum_{j \neq l} k_j K'_{\tau, j} c_{1k_1} \dots c_{nk_n}} (q^{-1} z_1)^{k_1} \dots (q^{-1} z_{l-1})^{k_{l-1}} z_{l+1}^{k_{l+1}} \dots z_n^{k_n} \\
& \quad \times q^{-K'_{\tau, p} - K'_{\tau, p}} \int_{-q^{K'_{\tau, p} + K'_{\tau, p}} \gamma_{l, \infty}}^{q^{K'_{\tau, p} + K'_{\tau, p}} \gamma_{l, \infty}} (f_l)_{\cdot 1}(t_l) d_q 2t_l.
\end{aligned}$$

The above power series is entire since all the  $f_j$ 's are, and one finds that

$$I_{\sigma(1)} f = \left( \int_{-\gamma_{l, \infty}}^{\gamma_{l, \infty}} (f_l)_{\cdot 1}(t_l) d_q 2t_l \right) \prod_{\substack{j \\ \rho(j) \neq l}} f_{\rho(j)}(q^{\eta_{\rho(j)} + \theta_{\rho(j)} x_{\rho(j)}})$$

with

$$\eta_k = \begin{cases} -1, & \text{if } k < \sigma(1), \\ 0, & \text{if } k > \sigma(1), \end{cases} \quad \theta_k = \begin{cases} -1, & \text{if } k < l \text{ and } \tau(k) > \tau(l), \\ 1, & \text{if } k > l \text{ and } \tau(k) < \tau(l), \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we are again in the hypothesis of the proposition, but in the case  $(n-1)$ . Since the statement in one dimension is obvious, lattice order integrability is proved, considering a shifted lattice. For every new step we make, the argument of the  $f_j$  that still has to be integrated will be shifted by powers of  $q$ . If one goes through computations, one finds that the exponential of  $q$  in the shift of the argument of  $f_r$  with  $r = \sigma(s)$  is

$$-\sum_s(\sigma, \tau) = -[\#\{j < s \mid \sigma(s) < \sigma(j)\} + \#\{j < s \mid (\sigma(j) - \sigma(s))(\tau\sigma(j) - \tau\sigma(s)) < 0\}],$$

hence the right lattice to integrate is the one defined by  $\gamma_{\sigma(s)} = \tilde{\gamma}_{\sigma(s)} q^{\sum_s(\sigma, \tau)}$ . In this setting, the integral will be the product of the  $q^2$ -Jackson integrals of the  $f_{j, \cdot 1}$ 's multiplied by a power of  $q$  with exponent

$$\sum_{s=1}^n \sum_s(\sigma, \tau) = l(\sigma) + \sum_{s=1}^n \#\{j < s \mid (\sigma(j) - \sigma(s))(\tau\sigma(j) - \tau\sigma(s)) < 0\} = l(\sigma) + l(\tau),$$

since the second term in the sum is equal to the cardinality of

$$\{j, s \mid j < s\} \cap (\{j, s \mid \sigma(j) < \sigma(s), \tau\sigma(j) > \tau\sigma(s)\} \cup \{j, s \mid \sigma(s) < \sigma(j), \tau\sigma(s) > \tau\sigma(j)\}).$$

For the converse of the statement, one sees that if  $f(\underline{x})$  can be written as a product of one-dimensional power series, those series have to be entire, and if there is a  $\sigma$  such that  $f(\underline{x})$  is lattice order integrable, this means that  $f_{\sigma(1)}$  is lattice integrable on  $q^{2k_1} \gamma_{\sigma(1)}$ , and so on, for the following  $f_j$ 's, with shifted argument. By the  $\Rightarrow$  part, we see that lattice order integrability has to hold for every  $\sigma'$ .  $\square$

*Example 1:* By the above proposition, for every  $a_j \neq 0$  and for every  $e_j \in \mathbf{Z}_{\geq 0}$  the formal power series

$$f(\underline{x}) = x_n^{e_n} E_{q^4}(-a_n^2 x_n^2) \dots x_1^{e_1} E_{q^4}(-a_1^2 x_1^2)$$

is l.o. integrable for every  $\sigma$  and  $\gamma_{\sigma(k)} = a_{\sigma(k)}^{-1} q^{(k-1) + \#\{j < k \mid \sigma(k) < \sigma(j)\}}$ , since in this case  $\tau(k) = n - k + 1$ . In particular,  $G_{q^2}(q^2 \underline{x})$  and all products of type  $G_{q^2}(q^2 \underline{x}) x_1^{e_1} \dots x_n^{e_n}$  are l.o. integrable for every choice of the order  $\sigma$ . One has

$$I''_{(\sigma, \gamma)} f = \begin{cases} b_{q^2}^n \left[ \prod_j (q^2; q^4)_{f_j} \right] q^{2n+2|E|} q^{l(\sigma)} \prod_{j=1}^n a_j^{-1-e_j} & \text{if } e_j = 2f_j \text{ for every } j, \\ 0, & \text{otherwise,} \end{cases}$$

where  $b_{q^2} = (1 - q^2)(q^2, -q^2, -1; q^2)_\infty$  and the result follows by Ref. 5. In particular we observe that the result depends on the choice of  $\sigma$  only in a straightforward way and that  $L(\gamma)$  does not depend on  $E$ , but only on  $\sigma$  and the  $a_j$ 's. Therefore one may consider the relation between  $I''_{(\sigma, \gamma)} f$  and  $I''_{(\sigma, \gamma)}(E_{q^4}(-\sum_k a_k^2 x_k^2))$ . One immediately sees that if all the  $e_j$ 's are even,

$$I''_{(\sigma, \gamma)}(f) = \frac{(\prod_j (q^2; q^4)_{f_j} q^{2|E|})}{(\prod_j a_j^{e_j})} I''_{(\sigma, \gamma)} \left( E_{q^4} \left( -\sum_k a_k^2 x_k^2 \right) \right).$$

We say in this case (and whenever an equivalence of integrals  $I''_{(\sigma, \gamma)}$  holds, with the same  $\sigma$  and  $\gamma$  on both sides) that  $I(x_n^{e_n} E_{q^4}(-a_n^2 x_n^2) \cdots x_1^{e_1} E_{q^4}(-a_1^2 x_1^2))$  is “weakly equivalent” to  $I(E_{q^4}(-\sum_k a_k^2 x_k^2))$ . In particular, one has weak equivalence of the expression  $I(x_n^{e_n} \times E_{q^4}(-q^2 x_n^2) \cdots x_1^{e_1} E_{q^4}(-q^2 x_1^2))$  and the expression  $\prod_j (q^2; q^4)_{f_j} I(G_{q^2}(q^2 \underline{x}))$ . We also want to point out that the above  $f(\underline{x})$  is an example of the fact that one can have l.o. integrability for every order and still not have lattice integrability. ♠

*Properties and Remarks:*

- (a) It is easy to check that if  $f(\underline{x})$  is l.o. integrable for the order  $\sigma$  and the lattice  $L(\gamma)$ . Then, for every  $n$ -tuple of nonzero real numbers  $(a_1, \dots, a_n)$ , the power series  $f_A(\underline{x}) = f(a_1 x_1, \dots, a_n x_n)$  is also l.o. integrable for the same order  $\sigma$  and for  $\gamma$  replaced by  $\tilde{\gamma}$  where  $\tilde{\gamma}_j = a_j^{-1} \gamma_j$  for every  $j$ . Then one has equivalence of the numbers  $I''_{(\sigma, \gamma)}(f) = (a_1 \cdots a_n) I''_{(\sigma, \tilde{\gamma})}(f_A)$ .
- (b) It is also obvious that if  $f(\underline{x})$  is l.o. integrable for  $\sigma$  and  $\gamma$ . Then the resulting  $I''_{(\sigma, \gamma)}(f)$  is invariant under shifts of each  $\gamma_j$  by even powers of  $q$ .
- (c) In the definition of lattice order integrability the requirement on the  $\int_{\sigma(k)} f$ 's to be entire for every  $k$  can be weakened to analyticity. The weaker version of the definition is left to the reader. ♠

□

By the discussion above, one can conclude that for well-behaved series [by this we mean series satisfying condition (c), (c'), etc.] also the integral  $I''_{(\sigma, \gamma)}$  is invariant under translation.

*Remarks:* The whole construction of lattice order integrability may look artificial, and it may seem to be a definition that is useful only in a noncommutative setting. However, this is not the case. One can define a similar concept of integrability also for power series in commutative variables.

## VI. THE BRAIDED FOURIER TRANSFORM

Now we have all ingredients for the introduction of braided Fourier transforms on a subspace of  $\hat{V}(R)^{\text{ext}}$ . We introduce two transforms, related to each other by a shift in the arguments and the application of the antipode to one of them. As we already said, the first time that a Fourier transform for this kind of algebra appeared was in Ref. 3, from which we took inspiration. One of the goals of this section is to provide  $n$ -dimensional analogs to formulas (8.19)–(8.21), hence to Theorem 8.1 in Ref. 5. The difference with Ref. 5 is that in our version, the algebra  $\hat{V}(R)^{\text{ext}} \otimes V(R)^{\text{ext}}$  has the braided product  $(m \otimes m)(\text{id} \otimes \Psi \otimes \text{id})$  instead of the ordinary one, although in normal form his formulas and ours for  $n = 1$  coincide. The difference with Ref. 3 lies mainly in the fact that our integral is not bosonic [i.e., it does not have trivial braiding with elements of the algebras  $\hat{V}(R)^{\text{ext}}$  and  $V(R)^{\text{ext}}$ ]. The use of the antipode appears also in Ref. 4 where the case of finite-dimensional braided groups is treated. The following transforms behave nicely with respect

to a convolution product and with respect to the action of  $V(R)$  on  $\hat{V}(R)^{\text{ext}}$ . They also respect various classical properties of the Fourier transform. These facts are developed in Refs. 3 and 13.

We say that an element  $f(\underline{x})$  of  $\hat{V}(R)^{\text{ext}}$  is of class  $\mathcal{I}$  if  $f(\underline{x})x_1^{e_1}\cdots x_n^{e_n}$  is  $I'$  integrable for every monomial  $x_1^{e_1}\cdots x_n^{e_n}$ . We say that it is of class  $\mathcal{I}_{(\sigma,\gamma)}$  if for every monomial  $x_1^{e_1}\cdots x_n^{e_n}$ , the power series  $f(\underline{x})x_1^{e_1}\cdots x_n^{e_n}$  is lattice order integrable for  $\sigma$  and  $\gamma$ .

Again, we do not provide a complete classification of  $\mathcal{I}$ , but we give a class for which this makes sense, which is big enough to reach our goal. Indeed, power series satisfying condition (c) of Sec. III belong to  $\mathcal{I}$ , hence products of  $e_{q^2}(-x_j^2)$  and polynomials belong to  $\mathcal{I}$  provided that for every  $j \in \{1, \dots, n\}$ , the one-dimensional  $q^2$ -Gaussian  $e_{q^2}(-x_j^2)$  appears in the product.

**Definition 6.1:** The braided Fourier transforms  $F$  and  $F_S$  are defined on the class  $\mathcal{I}$  and they have images in  $\hat{V}(R)^{\text{ext}} \otimes V(R)^{\text{ext}}$ . They are given by

$$F := (I' \otimes \text{id})(m \otimes \text{id}) \left( \text{id} \otimes \exp \left( x \left| \frac{i}{(1-q^2)} (\partial_1, \dots, q^{-j+1} \partial_j, \dots, q^{-n+1} \partial_n) \right. \right) \right),$$

$$F_S := (I' \otimes S)(m \otimes \text{id}) \left( \text{id} \otimes \exp \left( x \left| \frac{iq^2}{(1-q^2)} (q^{n-1} \partial_1, \dots, q^{n-j} \partial_j, \dots, \partial_n) \right. \right) \right).$$

For an  $f(\underline{x}) \in \mathcal{I} \cap C_\beta$  one has that

$$F(f(\underline{x})) := (I' \otimes \text{id}) \left( f(\underline{x}) E_{q^2} \left( i \sum_{j=1}^n x_j \otimes q^{-(j-1)} \partial_j \right) \right)$$

$$= \sum_{\substack{e_1, \dots, e_n \\ A(e_j) = \beta_j}} I(f(\underline{x}) x_1^{e_1} \cdots x_n^{e_n}) \otimes \frac{i^{|E|} q^{-\sum_j E_j}}{\prod_{j=1}^n (q^2; q^2)_{e_j}} \partial_n^{e_n} \cdots \partial_1^{e_1}$$

and

$$F_S(f(\underline{x})) := (I' \otimes S) \left( f(\underline{x}) E_{q^2} \left( iq^2 \sum_{j=1}^n x_j \otimes q^{(n-j)} \partial_j \right) \right)$$

$$= \sum_{\substack{e_1, \dots, e_n \\ A(e_j) = \beta_j}} I(f(\underline{x}) x_1^{e_1} \cdots x_n^{e_n}) \otimes \frac{(-i)^{|E|} q^{|E|^2 - \sum_j e_j E_j + |E| + \sum_j E_j}}{\prod_{j=1}^n (q^2; q^2)_{e_j}} \partial_1^{e_1} \cdots \partial_n^{e_n},$$

where  $A(e_j) = +$  (resp.  $-$ ) if  $e_j$  is even (resp. odd). Here we used that  $S(\partial_n^{e_n} \cdots \partial_1^{e_1}) = (-1)^{|E|} q^{|E|^2 - |E| - \sum_j e_j E_j} \partial_1^{e_1} \cdots \partial_n^{e_n}$ . It is clear that the second components in the tensor product of  $F(f(\underline{x}))$  and of  $F_S(f(\underline{x}))$  will also have parity  $\beta$ . In order to provide formulas analogous to (8.21) and (8.19) in Ref. 5, we need to compute  $F_S$  for

$$M(\underline{x}, A) = e_{q^4}(-x_1^2) x_1^{a_1} \cdots e_{q^4}(-x_n^2) x_n^{a_n} = x_1^{a_1} \cdots x_n^{a_n} e_{q^4} \left( - \sum_j (q^{-A_j} x_j)^2 \right)$$

and for  $H(\underline{x}, A) = e_{q^4}(-x_1^2) \tilde{h}_{a_1}(x_1; q^2) \cdots e_{q^4}(-x_n^2) \tilde{h}_{a_n}(x_n; q^2)$  where the  $\tilde{h}_{a_j}$ 's are the discrete  $q$ -Hermite II polynomials (see Ref. 14 and references therein) that are defined by

$$\tilde{h}_l(z; q) := z^l {}_2\phi_1(q^{-n}, q^{-n+1}; 0; q^2, -q^2 z^{-2}) = (q; q)_l \sum_{k=0}^{[l/2]} \frac{(-1)^k q^{-2kl + k(2k+1)} z^{l-2k}}{(q^2; q^2)_k (q; q)_{l-2k}}.$$

Both  $M(\underline{x}, A)$  and  $H(\underline{x}, A)$  satisfy condition (c) of Sec. III, so that the transform is defined on both series. We first compute the transform  $F_S$  on a generic  $f(\underline{x})$ . In order to give a meaning to the

transform we apply the realization map  $\pi_\gamma$  sending a power series  $g(\underline{x})$  to  $g_{\cdot 1}(z)$ , followed by evaluation at  $z = \gamma$ , to the first component of  $F_S(f(\underline{x}))$ . By the assumption that  $f(\underline{x}) \in \mathcal{I}$  we know that this is well defined so that  $(\pi_\gamma \otimes \text{id})F_S(f(\underline{x}))$  is a genuine power series in the noncommuting  $\partial_j$ 's. By the computations in the previous section, one obtains, for an  $f(\underline{x}) \in C_\beta \cap \mathcal{I}$ ,

$$(\pi_\gamma \otimes \text{id})(F_S(f(\underline{x}))) = \sum_{\substack{e_1, \dots, e_n \\ A(e_j) = \beta_j}} \frac{(-i)^{|E|} q^{\sum_j (e_j^2 + e_j)}}{\prod_{k=1}^n (q^2; q^2)_{e_k}} \\ \times \left( \int_{-\gamma_1 q^{n-1+E_{1,\infty}}}^{\gamma_1 q^{n-1+E_{1,\infty}}} \dots \int_{-\gamma_n q^{E_{n,\infty}}}^{\gamma_n q^{E_{n,\infty}}} f_{\cdot 1}(\underline{t}) t_1^{e_1} \dots t_n^{e_n} d_{q^2} \underline{t} \right) \partial_1^{e_1} \dots \partial_n^{e_n}.$$

By invariance of the  $q^2$ -integral we see that the integration bounds do not depend on  $E$ , but only on the parity of its components, hence they only depend on  $\beta$ . In particular, for  $f(\underline{x}) = f_1(x_1) \dots f_n(x_n) \in \mathcal{I} \cap C_\beta$  one has that

$$(\pi_\gamma \otimes \text{id})F_S(f(\underline{x})) = \prod_{k=1}^n \left[ \sum_{A(e_k) = \beta_k} \frac{(-i)^{e_k} q^{e_k^2 + e_k} \partial_k^{e_k}}{(q^2; q^2)_{e_k}} \left( \int_{-q^{B(\beta)_k + n - k} \gamma_k \cdot \infty}^{q^{B(\beta)_k + n - k} \gamma_k \cdot \infty} [(f_k)_{\cdot 1}(t_k)] t_k^{e_k} d_{q^2} t_k \right) \right],$$

where the product is taken in *increasing* order and  $B(\beta) = (b(\beta)_1, \dots, b(\beta)_n)$  is the  $n$ -tuple  $\{0, 1\}^n$  such that the  $k$ th entry is 0 (resp. 1) if  $\beta_k$  is even (resp. odd) and  $B(\beta)_k = \sum_{j=1}^{k-1} b(\beta)_j$  as usual.

Hence we come to an  $n$ -dimensional version of formula (8.21) in Ref. 5. Let  $M(\underline{x}, A)$  as above. We remind the reader that in this case  $\beta_j = +$  (resp.  $-$ ) if  $a_j$  is even (resp. odd). Then we use the one-dimensional case to obtain our result. Indeed, expanding in power series the left-hand side of (8.21) and using  $q^2$  instead of  $q$  one has

$$c_{q^2}(\gamma) q^{-a^2 - a} i^a h_a(t; q^2) E_{q^4}(-q^4 t^2) = \sum_{\substack{k \geq 0 \\ k+a \text{ even}}} \frac{i^k q^{k^2 + k}}{(q^2; q^2)_k} \left( \int_{-\gamma \cdot \infty}^{\gamma \cdot \infty} x^{a+k} e_{q^4}(-x^2) d_{q^2} x \right) t^k \\ = \sum_{\substack{k \geq 0 \\ k+a \text{ even}}} \frac{i^k q^{k^2 + k}}{(q^2; q^2)_k} c_{q^2}(\gamma) q^{-(a+k)^2/2} (q^2; q^4)_{(a+k)/2} t^k.$$

Using the above formula we obtain

$$(\pi_\gamma \otimes \text{id})F_S(M(\underline{x}, A)) = \prod_{j=1}^n \left[ c_{q^2}(q^{n-j+B(\beta)_j} \gamma_j) \right. \\ \times \sum_{\substack{e_j \geq 0 \\ e_j + a_j \text{ even}}} \left( \frac{(-i)^{e_j} q^{e_j^2 + e_j}}{(q^2; q^2)_{e_j}} q^{-(e_j + a_j)^2/2} (q^2; q^4)_{e_j + a_j/2} \right) \partial_j^{e_j} \Bigg] \\ = (-i)^{|A|} \left[ \prod_{j=1}^n c_{q^2}(q^{n-j+B(\beta)_j} \gamma_j) \right] q^{\sum_j a_j(1-a_j)} \\ \times \prod_{k=1}^n [E_{q^4}(-q^4 \partial_k^2) h_{a_k}(\partial_k; q^2)], \quad (1)$$

where

(i) the product is taken in *increasing* order, and

- (ii)  $h_l(z; q^2)$  is the discrete  $q^2$ -Hermite  $I$  polynomial of degree  $l$  (see Ref. 14 and references therein) and is defined as

$$h_l(z; q^2) := z^l \phi_0(q^{-2l}, q^{2-2l}; q^4, q^{4l-2} z^{-2}) = (q^2; q^2)_l \sum_{k=0}^{[1/2]} \frac{(-1)^k q^{2k(k-1)} z^{l-2k}}{(q^4; q^4)_k (q^2; q^2)_{l-2k}}.$$

We observe that the only part of  $(\pi_\gamma \otimes \text{id})F_S(M(\underline{x}, A))$  involving the  $\gamma_j$ 's is the coefficient, equal to  $[\int_{-q^{B(\beta)_{nX_n} \cdot \infty}}^{q^{B(\beta)_{nX_n} \cdot \infty}} \cdots \int_{-q^{B(\beta)_{1X_1} \cdot \infty}}^{q^{B(\beta)_{1X_1} \cdot \infty}} g_{q^2}] \cdot 1|_{z=\gamma}$ , i.e., it is a shifted integral of the Gaussian  $g_{q^2}(\underline{x})$  where the shift only depends on the parity of the function  $M(\underline{x}, A)$ , i.e., only on the parity of the  $a_j$ 's. So, we conclude that

$$F_S(e_{q^4}(-x_1^2)x_1^{a_1} \cdots e_{q^4}(-x_n^2)x_n^{a_n}) = \left[ \int_{-q^{B(\beta)_{nX_n} \cdot \infty}}^{q^{B(\beta)_{nX_n} \cdot \infty}} \cdots \int_{-q^{B(\beta)_{1X_1} \cdot \infty}}^{q^{B(\beta)_{1X_1} \cdot \infty}} g_{q^2} \right] \\ \otimes (-i)^{|A|} q^{\sum_j a_j(1-a_j)} \prod_{k=1}^n [E_{q^4}(-q^4 \partial_k^2) h_{a_k}(\partial_k; q^2)]. \quad (2)$$

The above result gives the analog of the classical reciprocity between Gaussians times a monomial and rescaled Gaussians times a Hermite polynomial under the Fourier transform in  $\mathbf{R}^n$ . From the above result we derive an analog of formula (8.19) in Ref. 5, for  $H(\underline{x}, A)$  defined above.  $H(\underline{x}, A)$  is also contained in one of the subspaces  $C_\beta$  since each  $\tilde{h}_a(x_j; q^2)$  has constant parity. We obtain

$$F_S(e_{q^4}(-x_1^2) \tilde{h}_{a_1}(x_1; q^2) \cdots e_{q^4}(-x_n^2) \tilde{h}_{a_n}(x_n; q^2)) \\ = \left[ \int_{-q^{B(\beta)_{nX_n} \cdot \infty}}^{q^{B(\beta)_{nX_n} \cdot \infty}} \cdots \int_{-q^{B(\beta)_{1X_1} \cdot \infty}}^{q^{B(\beta)_{1X_1} \cdot \infty}} g_{q^2} \right] \\ \otimes \prod_{k=1}^n \left[ (q^2; q^2)_{a_k} \sum_{s_k=0}^{[1/2a_k]} \frac{(-i)^{a_k} q^{-\frac{2}{s_k} + a_k}}{(q^4; q^4)_{s_k} (q^2; q^2)_{a_k - 2s_k}} h_{a_k - 2s_k}(\partial_k; q^2) E_{q^4}(-q^4 \partial_k^2) \right] \\ = \left[ \int_{-q^{B(\beta)_{nX_n} \cdot \infty}}^{q^{B(\beta)_{nX_n} \cdot \infty}} \cdots \int_{-q^{B(\beta)_{1X_1} \cdot \infty}}^{q^{B(\beta)_{1X_1} \cdot \infty}} g_{q^2} \right] (-i)^{|A|} q^{\sum_k (a_k - a_k^2)} \otimes \prod_{j=1}^n (\partial_j^{a_j} E_{q^4}(-q^4 \partial_j^2)), \quad (3)$$

where the last equality follows from (10) in Ref. 5 and the product is taken in increasing order.

Observe that for well-behaved functions, and for an  $n$ -tuple  $A = (a_1, \dots, a_n)$  of nonzero real numbers, the braided Fourier transform of  $f(a_1 x_1, \dots, a_n x_n)$  can be obtained by the braided Fourier transform of  $f(\underline{x})$ . More precisely,  $(\pi_\gamma \otimes \text{id})F_S(f(a_1 x_1, \dots, a_n x_n)) = (\prod_j a_j)^{-1} (\pi_{\tilde{\gamma}} \otimes \text{id}) \times (F_S(f(\underline{x}))) (a_1^{-1} \partial_1, \dots, a_n^{-1} \partial_n)$ , where  $\tilde{\gamma}$  denotes the  $n$ -tuple obtained by  $\gamma$  multiplying each component  $\gamma_j$  by  $a_j$ . Clearly, similar results holds for  $(\pi_\gamma \otimes \text{id})F$ , hence they hold for  $F_S$  and for  $F$ .

Now we want to compute the braided Fourier transform for monomials or polynomials times a  $q^2$ -Gaussian of type  $G_{q^2}(\underline{x})$ . We cannot use the same definition since  $G_{q^2}(\underline{x})$  is not even lattice integrable. Therefore, we introduce a weaker notion of braided Fourier transform.

**Definition 6.2:** The “weak” braided Fourier transforms  $F''(\sigma, \gamma)$  and  $F_S''(\sigma, \gamma)$  are defined on the class  $\mathcal{I}_{(\sigma, \gamma)}$ . They map this class to  $V(R)^{\text{ext}}$  and they are defined as

$$F''(\sigma, \gamma) := (I''_{(\sigma, \gamma)} \otimes \text{id})(m \otimes \text{id}) \left( \text{id} \otimes \exp \left( x \left| \frac{i}{(1-q^2)} (\partial_1, \dots, q^{1-j} \partial_j, \dots, q^{1-n} \partial_n) \right. \right) \right), \\ F_S''(\sigma, \gamma) := (I''_{(\sigma, \gamma)} \otimes S)(m \otimes \text{id}) \left( \text{id} \otimes \exp \left( x \left| \frac{i q^2}{(1-q^2)} (q^{n-1} \partial_1, \dots, q^{n-j} \partial_j, \dots, \partial_n) \right. \right) \right).$$



We will use this new notion in order to derive an  $n$ -dimensional version of (8.20) in Ref. 5. Namely, we will derive the transform  $F''(\sigma, \gamma)$  of the formal power series  $N(\underline{x}, A) = E_{q^4}(-q^4 x_n^2) x_n^{a_n} \cdots E_{q^4}(-q^4 x_1^2) x_1^{a_1}$  for given positive integers  $a_1, \dots, a_n$ . One has to compute  $I''_{(\sigma, \gamma)}(N(\underline{x}, A) x_1^{e_1} \cdots x_n^{e_n})$  for every  $E$  with each  $e_j \equiv a_j \pmod{2}$ , for some fixed  $\sigma$  and  $\gamma$ . This makes sense by the computations in Sec. V because  $N(\underline{x}, A) x_1^{e_1} \cdots x_n^{e_n}$  is equal to

$$q^{-\sum_j E^j(e_j + a_j)} E_{q^4}(-q^{4-2E^n} x_n^2) x_n^{e_j + a_j} \cdots E_{q^4}(-q^{4-2E^1} x_1^2) x_1^{e_1 + a_1},$$

which is l.o. integrable for every  $\sigma$ , with  $\gamma_{\sigma(k)} = q^{(k-1) + A^{\sigma(k)} + \#\{j < k | \sigma(j) > \sigma(k)\}}$ . In particular, since we showed that the resulting integrals differ only by a factor  $q^{l(\sigma)}$ , we compute it only for  $\sigma = \text{id}$  and  $\gamma_k = q^{k-1+A^k}$ . Then, denoting by  $N'(\underline{x}, A)$  the power series obtained by  $N(\underline{x}, A)$  by multiplying the argument of the  $E_{q^4}(-x_j^2)$  by  $q^{-E^j}$  for every  $j$ , one has

$$F''(\sigma, \gamma)(N(\underline{x}, A)) = \sum_{\substack{e_1, \dots, e_n \\ e_j + a_j = 2h_j}} \frac{i^{|E|} q^{-\sum_j E^j - \sum_j E^j(e_j + a_j)}}{\prod_j (q^2; q^2)_{e_j}} (I''_{(\text{id}, \gamma)}(N'(\underline{x}, A + E))) \partial_n^{e_n} \cdots \partial_1^{e_1}.$$

Hence

$$\begin{aligned} F''(\sigma, \gamma)(N(\underline{x}, A)) &= (-1)^{|A|} I''_{(\text{id}, \tilde{\gamma})} \left( E_{q^4} \left( - \sum_j q^4 x_j^2 \right) \right) \prod_{j=n}^1 \left[ \sum_{e_j + a_j = 2h_j} \frac{(-i)^{e_j} (q^2; q^4)_{h_j}}{\prod_j (q^2; q^2)_{e_j}} \partial_j^{e_j} \right] \\ &= q^{\sum_j (a_j^2 - a_j)} i^{|A|} b_{q^2}^n q^{\binom{n}{2}} \left( \prod_{j=n}^1 \tilde{h}_{a_j}(\partial_j; q^2) e_{q^4}(-\partial_j^2) \right). \end{aligned} \quad (4)$$

Here  $\tilde{\gamma}$  denotes the  $n$ -tuple such that  $\tilde{\gamma}_k = q^{A^k} \gamma_k$  and the product is taken in decreasing order. For the last equality we used (9.15) and (8.20) in Ref. 5.

Using the definition of the  $h_{a_j}$ 's, formula (4) above, and formula (8.17) in Ref. 5 one also gets the following result:

$$\begin{aligned} F''(\sigma, \gamma)(E_{q^4}(-q^4 x_n^2) h_{a_n}(x_n; q^2) \cdots E_{q^4}(-q^4 x_1^2) h_{a_1}(x_1; q^2)) \\ = i^{|A|} q^{\sum_j (a_j^2 - a_j)} I''_{(\sigma, \tilde{\gamma})} \left( E_{q^4} \left( -q^4 \sum_j x_j^2 \right) \right) \prod_{j=n}^1 \partial_j^{a_j} e_{q^4}(-\partial_j^2), \end{aligned} \quad (5)$$

where the product is taken in *decreasing* order and  $\tilde{\gamma}$  is given by  $\tilde{\gamma}_k = q^{A^k} \gamma_k$  for every  $k$  as before.

## VII. INTEGRAL ON $V(R)^{\text{ext}}$ AND INVERSE TRANSFORM

We provide now an inverse for the braided Fourier transforms, at least on the subspaces of the image of  $\mathcal{I}$  and  $\mathcal{I}_{(\sigma, \gamma)}$ . In order to do this we need also the integral on  $V(R)^{\text{ext}}$ . Since there is a symmetry between  $\hat{V}(R)^{\text{ext}}$  and  $V(R)^{\text{ext}}$ , one can simply repeat the definitions and computations keeping in mind that whenever we had a left action involving  $\hat{V}(R)^{\text{ext}}$ , we will need a right action in the case of  $V(R)^{\text{ext}}$ . We will only provide the necessary formulas, while the properties and the proofs of similar statements as those of Secs. III–V are left to the reader. We observe that all the results in this Section can be achieved both by direct computation or by using the symmetry  $\psi: \hat{V}(R)^{\text{ext}} \rightarrow V(R)^{\text{ext}}$  defined in Sec. II.

Similarly as for  $\hat{V}(R)^{\text{ext}}$ ,  $\hat{V}(R)^{\text{ext}}$  acts on the right on  $V(R)^{\text{ext}}$  by braided partial differentiation. For a monomial  $\partial_n^{e_n} \cdots \partial_1^{e_1}$ , and for  $j \in \{1, \dots, n\}$  we have

$$(\partial_n^{e_n} \cdots \partial_1^{e_1}) \leftarrow x_j = [e_j]_{q^2} \partial_n^{e_n} \cdots \partial_{j+1}^{e_{j+1}} \partial_j^{e_j-1} (q \partial_{j-1})^{e_{j-1}} \cdots (q \partial_1)^{e_1}.$$

There holds a right version of Taylor's formula, namely,

$$\Delta(g(\partial)) = g(\Delta(\partial)) = g(\partial) \leftarrow \left( \sum_{e_1, \dots, e_n \geq 0} \frac{x_1^{e_1} \cdots x_n^{e_n} \otimes \partial_n^{e_n} \cdots \partial_1^{e_1}}{[e_1]_{q^2}! \cdots [e_n]_{q^2}!} \right).$$

The (indefinite)  $q^2$ -integral acting from the right is

$$g(\partial) \leftarrow \int_0^{\partial_i} := (1 - q^2) \sum_{k=0}^{\infty} g(\partial_n, \dots, q^{2k} \partial_i, q^{-2} \partial_{i-1}, \dots, q^{-2} \partial_1) q^{2k} \partial_i,$$

and again as in the case of  $\hat{V}(R)^{\text{ext}}$ , one can define  $\int_0^{a\partial_i}$  for a nonzero constant  $a$ . The global integral is formally obtained as the limit for all  $r_j \rightarrow \infty$  of

$$g(\partial) \leftarrow \int_{-q^{-2r_1}\partial_1}^{q^{-2r_1}\partial_1} \cdots \int_{-q^{-2r_n}\partial_n}^{q^{-2r_n}\partial_n},$$

where

$$g(\partial) \leftarrow \int_{-a\partial_j}^{a\partial_j} := \left( g(\partial) \leftarrow \int_0^{a\partial_j} \right) - \left( g(\partial) \leftarrow \int_0^{-a\partial_j} \right)$$

for every  $a$ . Hence we have formally

$$\begin{aligned} g(\partial) \leftarrow \int^{\partial} &:= g(\partial) \leftarrow \int_{-\partial_1 \cdot \infty}^{\partial_1 \cdot \infty} \cdots \int_{-\partial_n \cdot \infty}^{\partial_n \cdot \infty} \\ &= (1 - q^2)^n \sum_{\varepsilon \in \{\pm 1\}^n} \sum_{k_n = -\infty}^{\infty} \cdots \sum_{k_1 = -\infty}^{\infty} g(\varepsilon_n q^{2k_n} \partial_n, \dots, \varepsilon_1 q^{2k_1} \partial_1) q^{2|K|} \partial_1 \cdots \partial_n. \end{aligned}$$

As in Sec. III, one can define an action of  $V(R)$  on the power series in the  $n$  commuting indeterminates  $z_1, \dots, z_n$ , in order to give a meaning to the integral. One can use this action to define integrability, lattice integrability, and lattice order integrability as in Secs. IV and V, and we leave this to the reader. The action will be the right regular action after the choice of a normal form. It is denoted by  $\triangleleft$  and it is defined on monomials as

$$(z_1^{k_1} \cdots z_n^{k_n}) \triangleleft \partial_n^{e_n} \cdots \partial_1^{e_1} := q^{\sum_j K_j e_j} z_1^{e_1 + k_1} \cdots z_n^{e_n + k_n} = z_1^{k_1} (q^{K_1} z_1)^{e_1} \cdots z_n^{k_n} (q^{K_n} z_n)^{e_n}.$$

For simplicity we will denote  $1 \triangleleft g(\partial) := {}_1.g(\underline{\partial})$  for every expression  $g(\partial)$  for which the action on 1 makes sense.

As in Sec. IV we construct projections  $P_j^{\pm}$  defined on  $V(R)$  and  $V(R)^{\text{ext}}$  for every  $j = 1, \dots, n$  and every choice of  $+$  or  $-$  as follows:

$$P_j^{\pm} : V(R)^{\text{ext}} \rightarrow V(R)^{\text{ext}} g(\partial) \mapsto \frac{1}{2} [g(\partial) \pm g(\dots, -\partial_j, \dots)].$$

Again, the  $P_j^{\pm}$ 's commute with each other, they are projections on the subspaces of  $V(R)^{\text{ext}}$  consisting of even (resp. odd) elements with respect to the  $j$ th variable, and  $P_j^+ P_j^- = 0$  for every  $j$ . Then, for every  $\beta \in \{-, +\}^n$  we define  $P_{\beta}$  as the composite  $P_1^{\beta_1} \circ \cdots \circ P_n^{\beta_n}$ . The  $P_{\beta}$ 's are all projections on their image  $G_{\beta}$  and clearly the decomposition  $V(R)^{\text{ext}} = \bigoplus_{\beta} G_{\beta}$  corresponds to the decomposition of  $\mathbb{C}[[z]]$  in series that are either odd or even in each variable after applying the action on 1. We write  $P_0$  for  $P_{(+, \dots, +)}$ . We can define again the integral  $J'$  defined by  $J'g := (P_0 g) \leftarrow \int^{\partial}$ .

In particular one can check that for a  $g(\partial)$  for which this makes sense one has

$$1. \left( (P_0(g(\partial))) \leftarrow \int^{\partial} \right) = \int_{-z_1 q^{n-1} \cdot \infty}^{z_1 q^{n-1} \cdot \infty} \cdots \int_{-z_n q^{n-n} \cdot \infty}^{z_n q^{n-n} \cdot \infty} 1. (P_0(g))(\underline{t}) d_{q^2} \underline{t}$$

and

$$1. (g \leftarrow D_1^{j_1} \cdots D_n^{j_n}) = (D_{q^2}^J(1 \cdot g))(q^{J^1} z_1, \dots, q^{J^n} z_n),$$

We also observe that under the “symmetry”  $\psi: \hat{V}(R)^{\text{ext}} \rightarrow V(R)^{\text{ext}}$  mapping  $x_j$  to  $q^{-j+(1/2)(n-1)} \partial_{n-j+1}$  one has

$$\psi(I f(x_1, \dots, x_n)) = q^{-n} \left[ (\psi(f(\underline{x}))(\underline{\partial})) \leftarrow \int_{-q^{-n+1} \partial_1 \cdot \infty}^{q^{-n+1} \partial_1 \cdot \infty} \cdots \int_{-q^{-n+2j-1} \partial_j \cdot \infty}^{q^{-n+2j-1} \partial_j \cdot \infty} \cdots \int_{-q^{n-1} \partial_n \cdot \infty}^{q^{n-1} \partial_n \cdot \infty} \right]. \quad (6)$$

One defines the integral  $J''_{(\sigma, \gamma)}$  for l.o. integrable power series as the right-handed version of  $I''_{(\sigma, \gamma)}$ . We will need  $J''_{(\sigma, \gamma)}(E_{q^4}(-q^4 \partial_1^2) \partial_1^{a_1}, \dots, E_{q^4}(-q^4 \partial_n^2) \partial_n^{a_n})$ . One checks as for the case of  $\hat{V}(R)^{\text{ext}}$  that the integrand is actually l.o. integrable for every  $\sigma$  for a suitable choice of  $\gamma$ , and that the results differ only by a factor  $q^{l(\sigma)}$ . In particular, for  $\sigma = \text{id}$  one needs  $\gamma_k = q^{(k-1)+A^k}$  and one gets

$$J''_{(\sigma, \gamma)}(E_{q^4}(-q^4 \partial_1^2) \partial_1^{a_1}, \dots, E_{q^4}(-q^4 \partial_n^2) \partial_n^{a_n}) = \begin{cases} q^{\binom{n}{2}} b_{q^2}^n \prod_j (q^2; q^4)_{b_j}, & \text{if } a_j = 2b_j \forall j, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$J''_{(\sigma, \gamma)}(E_{q^4}(-q^4 \partial_1^2) \partial_1^{a_1}, \dots, E_{q^4}(-q^4 \partial_n^2) \partial_n^{a_n}) = \left( J''_{(\sigma, \gamma)} \left( E_{q^4} \left( -q^4 \sum_j \partial_j^2 \right) \right) \right) \prod_j (q^2; q^2)_{b_j}$$

if  $a_j = 2b_j$  for every  $j$ .

Properties like right invariance of the integral, nullity of the integral of the partial derivative of a power series, etc., can be proved as in Sec. III–V.

We introduce now an inversion formula for  $F_S$  and  $F$  and their weak analogs. We will use the symmetry between  $\hat{V}(R)^{\text{ext}}$  and  $V(R)^{\text{ext}}$  and the results at the end of the previous section in order to provide an analog of Theorem 8.1 in Ref. 5. An inversion formula for the braided Fourier transform is to be found in Ref. 3, but the authors had the hypothesis that the integral is “bosonic,” i.e., it has a trivial braiding with  $\hat{V}(R)$ , or  $V(R)$  for  $n \geq 2$ , which is not our case as the reader can easily check (see also Ref. 15 for a few remarks about this property of the integral). Moreover, the element  $vol$  in Ref. 3 is not necessarily convergent.

We say that a power series  $g(\partial)$  in  $V(R)^{\text{ext}}$  is of class  $\mathcal{J}$  if every monomial times  $g(\partial)$  is  $J'$  integrable. This is possible, for instance, if  $g(\partial)$  satisfies conditions similar to condition (c) of Sec. III. We say that  $g(\partial)$  is of class  $\mathcal{J}_{(\sigma, \gamma)}$  if there is an order  $\sigma$  and a lattice  $L(\gamma)$  such that every monomial times  $g(\partial)$  is lattice order integrable for  $\sigma$  and  $\gamma$ .

**Definition 7.1:** We define the linear maps  $G, G_S: \mathcal{J} \rightarrow \hat{V}(R)^{\text{ext}} \otimes V(R)^{\text{ext}}$  by

$$G := \left[ (\text{id} \otimes m) \left( \exp \left( x \left| \frac{i}{(1-q^2)} (\partial_1, \dots, q^{-j+1} \partial_j, \dots, q^{-n+1} \partial_n) \right. \right) \otimes \text{id} \right) \right] \leftarrow (\text{id} \otimes J'),$$

$$G_S := \left[ (S \otimes m) \left( \exp \left( x \left| \frac{i q^2}{(1-q^2)} (q^{n-1} \partial_1, \dots, q^{n-j} \partial_j, \dots, \partial_n) \right. \right) \otimes \text{id} \right) \right] \leftarrow (\text{id} \otimes J').$$

For instance, for  $G_S$ , the transform of a given  $g(\partial) \in \mathcal{J}$  of fixed parity  $\beta$  will be

$$\begin{aligned}
G_S(g(\partial)) &= \left( (S \otimes \text{id}) \left( E_{q^2} \left( i q^2 \sum_j q^{(n-j)} x_j \otimes \partial_j \right) \right) g(\partial) \right) \leftarrow (\text{id} \otimes J') \\
&= \sum_{\substack{e_1, \dots, e_n \\ A(e_j) = \beta_j}} \frac{(-i)^{|E|} q^{|E|^2 + |E| \sum_j (n-j) e_j}}{\Pi_j(q^2; q^2)_{e_j}} x_1^{e_1} \cdots x_n^{e_n} \otimes (\partial_n^{e_n} \cdots \partial_1^{e_1} g(\partial) \leftarrow (\text{id} \otimes J')).
\end{aligned}$$

If  $\tau: V(R)^{\text{ext}} \otimes \hat{V}(R)^{\text{ext}} \rightarrow \hat{V}(R)^{\text{ext}} \otimes V(R)^{\text{ext}}$  denotes the usual flip operator, putting  $c_j = q^{-j+(1/2)(n-1)}$  for every  $j=1, \dots, n$  we observe after some computations that, for  $f(\underline{x}) = f_1(x_1) \cdots f_n(x_n)$  with  $f_j$  of parity  $\beta_j$ , there holds

$$\begin{aligned}
\tau(\psi \otimes \psi^{-1}) F_S(f(\underline{x})) &= \sum_{\substack{e_1, \dots, e_n \\ A(e_j) = \beta_j}} \frac{(-i)^{|E|} q^{|E|^2 + |E| \sum_j E_j}}{\Pi_j(q^2; q^2)_{e_j}} (c_1^{-1} x_1)^{e_1} \cdots (c_j^{-1} x_j)^{e_{n-j+1}} \cdots (c_n^{-1} x_n)^{e_1} \\
&\quad \otimes [\partial_n^{e_1} \cdots \partial_1^{e_n} f_1(\partial_n) \cdots f_n(\partial_1)] \leftarrow \int_{-q^{-n+1+E_n-E_1} c_n \partial_1 \cdot \infty}^{q^{-n+1+E_n-E_1} c_n \partial_1 \cdot \infty} \cdots \int_{-q^{n-1+E_1-E^n} c_1 \partial_n \cdot \infty}^{q^{n-1+E_1-E^n} c_1 \partial_n \cdot \infty}.
\end{aligned}$$

Hence, we see that the formal expression of  $\tau(\psi \otimes \psi^{-1}) F_S(f(\underline{x}))$  coincides with the formal expression of

$$G_S(f_1(\partial_n) \cdots f_n(\partial_1)) ((c_1^{-1} x_1, \dots, c_n^{-1} x_n) \otimes (q^{n-1+E_1-E^1} c_1 \partial_n, \dots, q^{E_n-E^n+1-n} c_n \partial_1)).$$

In the same way one shows that the formal expression of  $\tau(\psi \otimes \psi^{-1}) F(f(\underline{x}))$  coincides with

$$G(f_1(\partial_n) \cdots f_n(\partial_1)) ((c_1^{-1} x_1, \dots, c_n^{-1} x_n) \otimes (q^{n-1+E_1-E^1} c_1 \partial_n, \dots, q^{E_n-E^n+1-n} c_n \partial_1)).$$

We can use the symmetry between  $F_S$  and  $G_S$ , together with formula (2), in order to compute  $G_S(\partial_n^{a_n} e_{q^4}(-\partial_n^2) \cdots \partial_1^{a_1} e_{q^4}(-\partial_1^2))$  for given positive integers  $a_1, \dots, a_n$  of parity, respectively,  $\beta_1, \dots, \beta_n$ . This symmetry tells us that

$$G_S(e_{q^4}(-\partial_n^2) \partial_n^{a_n} \cdots e_{q^4}(-\partial_1^2) \partial_1^{a_1}) = (L_\psi \otimes L'_{\psi, \beta}) [\tau(\psi \otimes \psi^{-1}) \tau F_S(e_{q^4}(-x_1^2) x_1^{a_1} \cdots e_{q^4}(-x_n^2) x_n^{a_n})],$$

where  $L_\psi$  is the shift operator mapping  $x_j$  to  $c_j x_j$  and  $L'_{\psi, \beta}$  is the shift operator mapping  $\partial_j$  to  $q^{n-2j+1+B(\beta)^j-B(\beta)_j} c_{n-j+1}^{-1} \partial_j$ . Then the above expression is equal to

$$\begin{aligned}
&(-i)^{|A|} q^{\sum_j (a_j - a_j^2)} \prod_{j=n}^1 [E_{q^4}(-q^4 x_j^2) h_{a_j}(x_j; q^2)] \otimes L'_{\psi, \beta} \psi \int_{-q^{B(\beta)^1} x_n \cdot \infty}^{q^{B(\beta)^1} x_n \cdot \infty} \cdots \int_{-q^{B(\beta)^n} x_1 \cdot \infty}^{q^{B(\beta)^n} x_1 \cdot \infty} g_{q^2} \\
&= (-i)^{|A|} q^{\sum_j (a_j - a_j^2)} \prod_{j=n}^1 [E_{q^4}(-q^4 x_j^2) h_{a_j}(x_j; q^2)] \\
&\quad \otimes q^{-n} L'_{\psi, \beta} \left( g_{q^2} (c_1 \partial_n, \dots, c_n \partial_1) \leftarrow \int_{-q^{B(\beta)^1-n+1} \partial_1 \cdot \infty}^{q^{B(\beta)^1-n+1} \partial_1 \cdot \infty} \cdots \int_{-q^{B(\beta)^n+n-1} \partial_n \cdot \infty}^{q^{B(\beta)^n+n-1} \partial_n \cdot \infty} \right).
\end{aligned}$$

Hence we can conclude that

$$\begin{aligned}
G_S(e_{q^4}(-\partial_n^2) \partial_n^{a_n} \cdots e_{q^4}(-\partial_1^2) \partial_1^{a_1}) &= (-i)^{|A|} q^{\sum_j (a_j - a_j^2)} \prod_{j=n}^1 [E_{q^4}(-q^4 x_j^2) h_{a_j}(x_j; q^2)] \\
&\quad \otimes \left( g_{q^2} \leftarrow \int_{-q^{B(\beta)^1} \partial_1 \cdot \infty}^{q^{B(\beta)^1} \partial_1 \cdot \infty} \cdots \int_{-q^{B(\beta)^n} \partial_n \cdot \infty}^{q^{B(\beta)^n} \partial_n \cdot \infty} \right), \tag{7}
\end{aligned}$$

where the second component of the tensor product clearly depends only on the parity of the  $a_j$ 's. Formula (7) can also be obtained by direct computation. Using the definition of the  $\tilde{h}_m(z; q^2)$ , with the same relation as before between the  $a_j$ 's and the  $\beta_j$ 's, one obtains

$$G_S(\tilde{h}_{a_n}(\partial_n; q^2) e_{q^4(-\partial_n^2)} \cdots \tilde{h}_{a_1}(\partial_1; q^2) e_{q^4(-\partial_1^2)}) \\ = (-i)^{|A|} q^{\sum_j (a_j - a_j^2)} \prod_{j=n}^1 [E_{q^4}(-q^4 x_j^2) x_j^{a_j}] \otimes \left( g_{q^2} \leftarrow \int_{-q^{B(\beta)_1 \partial_1 \cdot \infty}}^{q^{B(\beta)_1 \partial_1 \cdot \infty}} \cdots \int_{-q^{B(\beta)_n \partial_n \cdot \infty}}^{q^{B(\beta)_n \partial_n \cdot \infty}} \right), \quad (8)$$

which is the  $V(R)^{\text{ext}}$  version of (3). By these results we can conclude the following:

**Proposition 7.2:** Let  $\beta = (\beta_1, \dots, \beta_n) \in \{\pm 1\}^n$ ,  $\sigma$  be any order, and  $\gamma$  be the  $n$ -tuple with components given by  $\gamma_{\sigma(k)} = q^{B(\beta)^{\sigma(k)} + k - 1 + \#\{j < k | \sigma(j) > \sigma(k)\}}$ . Then on power series in  $\hat{V}(R)^{\text{ext}}$  of type  $E_{q^4}(-q^4 x_n^2) p_n(x_n) \cdots E_{q^4}(-q^4 x_1^2) p_1(x_1)$ , where the  $p_i$ 's are polynomials of fixed parity  $\beta_1, \dots, \beta_n$ , there holds

$$G_S \circ F''(\sigma, \gamma) = \text{id} \otimes (I''_{(\sigma, \tilde{\gamma})} G_{q^2}(q^2 \underline{x})) \left( g_{q^2} \leftarrow \int_{-q^{B(\beta)_1 \partial_1 \cdot \infty}}^{q^{B(\beta)_1 \partial_1 \cdot \infty}} \cdots \int_{-q^{B(\beta)_n \partial_n \cdot \infty}}^{q^{B(\beta)_n \partial_n \cdot \infty}} \right),$$

where  $\tilde{\gamma}$  is such that  $\tilde{\gamma}_k = \gamma_k q^{B(\beta)_k}$  for every  $k$ . Therefore for power series in  $V(R)^{\text{ext}}$  of the form  $w_n(\partial_n) e_{q^4(-\partial_n^2)} \cdots w_1(\partial_1) e_{q^4(-\partial_1^2)}$ , where the  $w_j$ 's are polynomials of fixed parity  $\beta_1, \dots, \beta_n$ , one has

$$(F''(\sigma, \gamma) \otimes \text{id}) G_S = \text{id} \otimes (I''_{(\sigma, \tilde{\gamma})} G_{q^2}(q^2 \underline{x})) \left( g_{q^2} \leftarrow \int_{-q^{B(\beta)_1 \partial_1 \cdot \infty}}^{q^{B(\beta)_1 \partial_1 \cdot \infty}} \cdots \int_{-q^{B(\beta)_n \partial_n \cdot \infty}}^{q^{B(\beta)_n \partial_n \cdot \infty}} \right)$$

with  $\tilde{\gamma}$  as before.

*Proof:* It follows by (5) and (7).  $\square$

A slightly more general version of this proposition holds by considering a proper  $\gamma$  and  $q^2$ -Gaussians where the argument is multiplied by a nonzero constant.

We also want to consider another inverse transform, the weak transform inverting (2) and (3).

**Definition 7.3:** The ‘‘weak’’ transforms  $G''(\sigma, \gamma)$  and  $G_S''(\sigma, \gamma)$  map  $\mathcal{J}_{(\sigma, \gamma)}$  to  $\hat{V}(R)^{\text{ext}}$  and are defined as

$$G''(\sigma, \gamma) := \left[ (\text{id} \otimes m) \left( \exp \left( x \left| \frac{i}{(1-q^2)} (\partial_1, \dots, q^{-j+1} \partial_j, \dots, q^{-n+1} \partial_n) \right. \right) \otimes \text{id} \right) \right] \leftarrow (\text{id} \otimes J''_{(\sigma, \gamma)}),$$

$$G_S''(\sigma, \gamma) := \left[ (S \otimes m) \left( \exp \left( x \left| \frac{i q^2}{(1-q^2)} (q^{n-1} \partial_1, \dots, q^{n-j} \partial_j, \dots, \partial_n) \right. \right) \otimes \text{id} \right) \right] \leftarrow (\text{id} \otimes J''_{(\sigma, \gamma)}).$$

As in formulas (4) and (5), one finds, for  $\sigma = \text{id}$  and  $\gamma_k = q^{A^k + k - 1}$  (and similarly for different  $\sigma$ 's),

$$G''_{(\text{id}, \gamma)}(E_{q^4}(-q^4 \partial_1^2) \partial_1^{a_1} \cdots E_{q^4}(-q^4 \partial_n^2) \partial_n^{a_n}) = q^{\sum_j (a_j^2 - a_j)} i^{|A|} \left( \prod_{j=1}^n \tilde{h}_{a_j}(x_j; q^2) e_{q^4(-x_j^2)} \right) \\ \times [J''_{(\text{id}, \tilde{\gamma})}(G_{q^2}(q^2 \underline{\partial}))] \quad (9)$$

and

$$G''_{(\text{id}, \gamma)}(E_{q^4}(-q^4 \partial_1^2) h_{a_1}(\partial_1) \cdots E_{q^4}(-q^4 \partial_n^2) h_{a_n}(\partial_n)) \\ = q^{\sum_j (a_j^2 - a_j)} i^{|A|} \left( \prod_{j=1}^n x_j^{a_j} e_{q^4(-x_j^2)} \right) [J''_{(\text{id}, \tilde{\gamma})}(G_{q^2}(q^2 \underline{\partial}))], \quad (10)$$

where in both formulas  $\tilde{\gamma}_k = q^{k-1}$  for every  $k$  and the product is taken in increasing order. These formulas can be obtained by using the symmetry or by direct computation. One has the second inversion property.

**Proposition 7.4:** Let  $\beta = (\beta_1, \dots, \beta_n) \in \{\pm 1\}^n$ ,  $\sigma$  be any order, and  $\gamma$  be the  $n$ -tuple with components given by  $\gamma_{\sigma(k)} = q^{B(\beta)^{\sigma(k)} + k - 1 + \#\{j < k | \sigma(j) > \sigma(k)\}}$ . Then on power series in  $V(R)^{\text{ext}}$  of type  $E_{q^4}(-q^4 \partial_1^2) p_1(\partial_1) \cdots E_{q^4}(-q^4 \partial_n^2) p_n(\partial_n)$ , where the  $p_j$ 's are polynomials of fixed parity  $\beta_1, \dots, \beta_n$ , there holds

$$F_S \circ G''(\sigma, \gamma) = \left[ \int_{-q^{B(\beta)_{nX_n} \cdot \infty}}^{q^{B(\beta)_{nX_n} \cdot \infty}} \cdots \int_{-q^{B(\beta)_{1X_1} \cdot \infty}}^{q^{B(\beta)_{1X_1} \cdot \infty}} g_{q^2} \right] (J''_{(\sigma, \tilde{\gamma})}(G_{q^2}(q^2 \partial))) \otimes \text{id},$$

where  $\tilde{\gamma}_{\sigma(k)} = \gamma_{\sigma(k)} q^{B(\beta)^{\sigma(k)}}$ . Therefore for power series in  $\hat{V}(R)^{\text{ext}}$  having the form  $w_1(x_1) \times e_{q^4}(-x_1^2) \cdots w_n(x_n) e_{q^4}(-x_n^2)$  where the  $w_j$ 's are polynomials of fixed parity  $\beta_1, \dots, \beta_n$ , one has

$$(\text{id} \otimes G''(\sigma, \gamma)) F_S = \left[ \int_{-q^{B(\beta)_{nX_n} \cdot \infty}}^{q^{B(\beta)_{nX_n} \cdot \infty}} \cdots \int_{-q^{B(\beta)_{1X_1} \cdot \infty}}^{q^{B(\beta)_{1X_1} \cdot \infty}} g_{q^2} \right] (J''_{(\sigma, \tilde{\gamma})}(G_{q^2}(q^2 \partial))) \otimes \text{id}$$

with  $\tilde{\gamma}$  as before.

*Proof:* It follows by (2), (3), (9), and (10).  $\square$

One observes that in this case the Plancherel measure is always a product of integrals of  $q^2$ -Gaussians, but the integration bounds depend on the parity of the power series. So these transforms could also be seen as sine and cosine transforms (see for this also Ref. 9 where the  $q$ -sine and  $q$ -cosine transforms in commuting variables are defined).

*Remark:* The break in symmetry appearing in  $q^2$ -integration is a phenomenon that has recently been observed in Ref. 7. Sometimes this lack of symmetry can be avoided, for instance if the generalized function  $f(\underline{x})$  that we want to integrate (and transform) is lattice integrable in  $L(\gamma)$  and  $L(q\gamma)$ . In this case we could replace  $q^2$ -integration by  $q$ -integration since  $\int_{-\gamma \cdot \infty}^{\gamma \cdot \infty} f_{\cdot 1}(t) d_q t = \int_{-\gamma \cdot \infty}^{\gamma \cdot \infty} f_{\cdot 1}(t) d_q t + \int_{-q\gamma \cdot \infty}^{q\gamma \cdot \infty} f_{\cdot 1}(t) d_q t$ . The new defined integral will be again invariant under translation. The  $q$ -integral of a  $q^2$ -Gaussians  $g_{q^2}(\underline{x})$  times a monomial will be similar to the  $q^2$ -integral of the same expression. Using the  $q$ -integral in the definition of  $(\pi_\gamma \otimes \text{id}) F_S$  will provide results similar to formula (1) but with  $\prod_j c_{q^2}(q^{n-j+B(\beta)_j} \gamma_j)$  replaced by  $\prod_j (c_{q^2}(\gamma_j) + c_{q^2}(q\gamma_j))$ . The result will be therefore independent of the parity of the  $a_j$ 's. However, this approach cannot be used for  $G_{q^2}(\underline{x})$ , since we have seen that there is only one  $q^2$ -lattice for which  $G_{q^2}(\underline{x})$  is lattice order integrable.  $\spadesuit$

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<sup>1</sup>S. Majid, *Foundations of Quantum Group Theory* (Cambridge U. P., Cambridge, England, 1995).

<sup>2</sup>M. Hashimoto and T. Hayashi, "Quantum multilinear algebra," *Tohoku Math. J.* **44**, 471–521 (1992).

<sup>3</sup>A. Kempf and S. Majid, "Algebraic  $q$ -integration and Fourier theory on quantum and braided spaces," *J. Math. Phys.* **35**, 6802–6837 (1994).

<sup>4</sup>C. Chrissomalakos, "Remarks on quantum integration," *Commun. Math. Phys.* **184**, 1–25 (1997).

<sup>5</sup>T. H. Koornwinder, "Special functions and  $q$ -commuting variables," in *Special Functions,  $q$ -series and Related Topics*, edited by M. E. H. Ismail, D. R. Masson, and M. Rahman, Fields Institute Communications 14, AMS (1997), pp. 131–166.

<sup>6</sup>M. Olshanetsky and V. Rogov, "The  $q$ -Fourier transform of  $q$ -distributions," IHES preprint q-alg 9712055 (1997).

<sup>7</sup>B. L. Cerchiai, R. Hinterding, J. Madore, and J. Wess, "A Calculus Based on a  $q$ -deformed Heisenberg Algebra," *QA/9809160* (1998).

<sup>8</sup>T. Masuda, K. Mimachi, Y. Nakagami, M. Noumi, and K. Ueno, "Representations of the quantum group  $SU_q(2)$  and the little  $q$ -Jacobi polynomials," *J. Funct. Anal.* **99**, 357–387 (1991).



- <sup>9</sup>T. H. Koornwinder and R. F. Swarttouw, "On  $q$ -analogues of the Fourier and Hankel transforms," Trans. Am. Math. Soc. **333**, 445–461 (1992).
- <sup>10</sup>V. V. Lyubashenko and S. Majid, "Braided groups and quantum Fourier transform," J. Alg. **166**, 506–528 (1994).
- <sup>11</sup>S. Majid, "Free braided differential calculus, braided binomials theorem and the braided exponential map," J. Math. Phys. **34**, 4843–4856 (1993).
- <sup>12</sup>M. P. Schützenberger, "Une interprétation de certaines solutions de l'équation fonctionnelle:  $F(x+y)=F(x)F(y)$ ," C. R. Acad. Sci. Paris **236**, 352–353 (1953).
- <sup>13</sup>G. Carnovale and T. Koornwinder, "A  $q$ -analogue of convolution on the line" (preprint).
- <sup>14</sup>R. Koekoek and R. Swarttouw, "The Askey-scheme of hypergeometric orthogonal polynomials and its  $q$ -analogue," Revised Version of Report 94-05 Delft University of Technology, Faculty TWI (1996).
- <sup>15</sup>C. Chrysomalakos and B. Zumino, "Translations, integrals and Fourier transforms in the quantum plane," in *Salam-festschrift*, Proceedings of the "Conference on Highlights of Particles and condensed matter physics," edited by A. Ali, J. Ellis, and S. Randjbar-Daemi (ICTP, Trieste, Italy, 1993).