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Lyapunov Stabilizability of Controlled Diffusions via a Superoptimality Principle for Viscosity Solutions*

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Abstract. We prove optimality principles for semicontinuous bounded viscosity solutions of Hamilton–Jacobi–Bellman equations. In particular, we provide a representation formula for viscosity supersolutions as value functions of suitable obstacle control problems. This result is applied to extend the Lyapunov direct method for stability to controlled Ito stochastic differential equations. We define the appropriate concept of the Lyapunov function to study stochastic open loop stabilizability in probability and local and global asymptotic stabilizability (or asymptotic controllability). Finally, we illustrate the theory with some examples.

Key Words. Controlled degenerate diffusion, Hamilton–Jacobi–Bellman inequalities, Viscosity solutions, Dynamic programming, Superoptimality principles, Obstacle problem, Stochastic control, Stability in probability, Asymptotic stability.

AMS Classification. 49L25, 93E15, 93D05, 93D20.

1. Introduction

We consider an N-dimensional stochastic differential equation

$$dX_t = f(X_t)\,dt + \sigma(X_t)\,dW_t,$$

where W_t is a standard M-dimensional Brownian motion. Since the sixties, a stochastic Lyapunov method for the analysis of the qualitative properties of the solutions of stochastic differential equations, in analogy to the deterministic Lyapunov method, was

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developed. The main contributions are due to Has'minskii (see the monograph [20] and the references therein) and Kushner (see the monographs [25] and [27]). They introduced the notion of stability in probability and asymptotic stability in probability. This means that the probability that the trajectory leaves a given neighborhood of the equilibrium is decreasing to zero as the initial data is approaching equilibrium. If, moreover, the trajectory is asymptotically approaching equilibrium with probability decreasing to zero as the initial data is approaching equilibrium, the system is asymptotically stable in probability. Finally, if for every initial data the trajectory is asymptotically approaching equilibrium almost surely, the system is asymptotically stable in the large.

The assumption of stability in probability implies that at equilibrium both the drift and the diffusion of the stochastic systems have to vanish (for controlled systems, that there is at least one constant control $\alpha \in A$ such that $f(\cdot,\alpha)$ and $\sigma(\cdot,\alpha)$ vanish at equilibrium). So equilibrium is preserved in the presence of noise. This condition excludes linear systems with additive nonvanishing noise, nevertheless, there are classes of stochastic systems (both controlled and uncontrolled) which satisfy that condition and whose stability properties are interesting. We refer to stochastic differential systems which model population dynamics subjected to environmental noises, such as stochastic Lotka–Volterra models (see, for example, [21], [30] and the references therein), to stochastic oscillators (see [30] and the references therein), or to deterministic systems driven by a stochastic force (when one wants to stabilize only the state variable).

The stochastic analog of deterministic Lyapunov functions V are twice differentiable continuous functions, which are positive definite and proper and satisfy the infinitesimal decrease condition

$$-DV(x)\cdot f(x) - trace\left[a(x)D^2V(x)\right] \geq l(x), \tag{1}$$

with $l \geq 0$ for mere Lyapunov stability and $l > 0$ for $x \neq 0$ for asymptotic stability, where $a := \sigma\sigma^T/2$. By the Dynkin formula, this differential inequality implies that the stochastic process $V(X_t)$, where X_t is the solution of the stochastic differential equation starting from x, is a positive supermartingale. This fact translates, in the stochastic setting, the requirement on the Lyapunov function to decrease along the trajectories of the dynamical system. There is a large literature on this kind of stochastic stability: we refer to the cited monographs and to [30], see also the references therein. We also recall here the work of Florchinger [18], [19] and Deng et al. [13] on feedback stabilization for controlled stochastic differential equations by the Lyapunov function method.

In this paper we extend the Lyapunov method for stochastic differential equations essentially in two directions. First we consider controlled stochastic differential equations in $\mathbb{R}^N$,

$$dX_t = f(X_t,\alpha_t)\,dt + \sigma(X_t,\alpha_t)\,dW_t,$$

moreover, we allow the Lyapunov functions to be merely lower semicontinuous. The nonexistence of smooth Lyapunov functions is well known in the deterministic case, see [2] for stable uncontrolled systems and the surveys [35] and [2] for asymptotically stable controlled systems. Also in the stochastic case, the assumption of smoothness for Lyapunov functions is not necessary and would limit considerably the applicability of the theory and the possibility of getting a complete Lyapunov-characterization of the stabilizability in probability by means of a converse theorem. Here we give an example of an uncontrolled degenerate diffusion process that is stable in probability but does not admit a smooth Lyapunov function (Example 3 in Section 7). Kushner proved in [26] a characterization of asymptotic uniform stochastic stability (for uncontrolled systems) by means of only continuous Lyapunov functions (here, however, the infinitesimal decrease condition is not given with a differential inequality but in terms of the weak generator of the process). For stability in probability, Has'minskii provided a $\mathcal{C}^2$ Lyapunov function under the assumption of strict nondegeneracy of the diffusion: this result cannot be extended to possibly nondegenerate diffusions. Converse theorems in the controlled case appear in the Ph.D. thesis by the author [9]. In particular, we prove that the existence of a local Lyapunov function is also necessary for stability in probability. Hence we show that if the system (CSDE) in Section 2 is uniformly asymptotically stabilizable in probability then there exists a local strict Lyapunov function, which is *continuous*.

We define then a Lyapunov function for the stability in probability as a *lower semi-continuous* (*LSC*), positive definite, proper function V, continuous at 0 and satisfying in a viscosity sense the differential Hamilton–Jacobi–Bellman inequality

$$\max_{\alpha\in A}\{-DV(x)\cdot f(x,\alpha) - trace[a(x,\alpha)D^2V(x)]\} \geq l(x), \tag{2}$$

and we call it a strict Lyapunov function if $l > 0$ off 0. Our main results are the natural extensions to the controlled diffusions of the first and second Lyapunov theorems:

the existence of a local Lyapunov function implies the (*stochastic open loop*) *stabilizability in probability of* (*CSDE*); *a strict Lyapunov function implies the* (*stochastic open loop*) *asymptotic stabilizability in probability*.

The same proof provides the global versions as well: if V satisfies (2) in $\mathbb{R}^N\backslash\{0\}$ then (CSDE) is also (stochastic open loop) *Lagrange stabilizable*, i.e., has the property of uniform boundedness of trajectories, and if V is strict then the system is (stochastic open loop) asymptotically stabilizable in the large. We also give sufficient conditions for the stability of viable (controlled invariant) sets more general than an equilibrium point.

The main tool to provide such a result is a superoptimality principle for lower semicontinuous bounded viscosity supersolutions V of the Hamilton–Jacobi–Bellman equation (2). A similar approach has been exploited in the deterministic case by Barron and Jensen (see [8]) for globally asymptotically stable systems affected by disturbances and by Soravia [38], [37] for stable systems with competitive controls. Precisely we prove that

every bounded LSC viscosity supersolution V of (2) *can be represented as*

$$V(x) = \inf_{\alpha}\sup_{t\geq 0}\mathbf{E}_x\left[V(X_t^\alpha) + \int_0^t l(X_s^\alpha)\,ds\right]. \tag{3}$$

This representation formula is important on its own, since it refers to Hamilton–Jacobi–Bellman equations for which uniqueness of solutions is not expected. In particular, this formula permits us to characterize the value function

$$V(x) = \inf_{\alpha} \mathbf{E}_x \int_0^{+\infty} l(X_s^{\alpha})\, ds,$$

when it is well defined, as the minimal nonnegative LSC viscosity supersolution of (2).

Our proof adapts to the second-order case the arguments used by Soravia in [38] and [39], where he provides, in the general context of differential games, a representation formula (a superoptimality principle which holds as an equality) for supersolutions of first-order Isaacs equations. The main difficulty in the adaptation is the passage from stochastic to deterministic dynamics and the use of stochastic controls. For the definition of the classes of controls and for the compactness and measurable selection results we use, we refer mainly to the article by Haussmann and Lepeltier [22] (see also the article by El Karoui et al. [15] and the book by Stroock and Varadhan [40, Chapter 12]). For related results on the existence of optimal controls for stochastic problems we refer to the article by Kushner [28].

There is a large literature on dynamic programming and superoptimality and suboptimality principles for viscosity solutions of second-order Hamilton–Jacobi–Bellman equations, starting from the papers by Lions [29] (see also the books [16] and [23]). We recall here the recent work by Soner and Touzi on dynamic programming for stochastic target problems [32], [33]. We refer also to the paper by Swiech [41] on sub- and superoptimality principles for value functions of stochastic differential games (see also the paper by Fleming and Souganidis [17]).

In the last section we present a simple application of our Lyapunov method. We consider an asymptotically controllable deterministic system and we study under which conditions it remains stable if we add a stochastic perturbation to it. By the Lyapunov characterization of asymptotic controllability provided by Clarke et al. [11] and Rifford [31], we know that the unperturbed system admits a semiconcave Lyapunov function (except possibly at equilibrium). We obtain that the system remains stable under a small intensity condition on diffusion matrix σ, depending on the semiconcavity constant of the Lyapunov function and on the qualitative properties of the stable trajectories of the deterministic systems.

We conclude with some additional references. We recall that there are other notions of stochastic stability. Kozin introduced the exponential almost sure stability of an uncontrolled stochastic system. Stability in the mean square and p-stability were studied by means of Lyapunov functions (we refer to the monograph [20]). In the controlled case, in previous papers Bardi and the author (see [4] and [3], see also [1]) characterized by means of appropriate Lyapunov functions the almost sure stabilizability of stochastic differential equations. This is a stronger notion of stochastic stability, never verified for nondegenerate processes. Indeed, a system is almost surely stabilizable if it behaves as a deterministic stabilizable system and remains almost surely in a neighborhood of the equilibrium point. Turning to deterministic controlled systems, a complete Lyapunov characterization of the asymptotic stabilizability (called asymptotic controllability) has been proved by Sontag and Sussmann (see the articles [34] and [36] and the review paper [35]). The infinitesimal decrease condition of the Lyapunov function along the trajectories of the system is expressed in terms of Dini directional derivatives, contingent directional derivatives and proximal subgradients. There is a large literature on the stabilization of deterministic controlled system by the Lyapunov function method: we refer to the monograph [2] and to the papers [10] and [31], see also the references therein.

The paper is organized as follows. In Section 2 we introduce the stochastic control problems and recall the definitions and basic properties of the controls we use. Section 3 is devoted to the proof of the representation formula (3). Section 4 contains the definitions of stabilizability in probability, asymptotic stabilizability and Lyapunov functions; in Section 5 we apply the results in Section 3 to prove local and global versions of the Lyapunov theorems. In Section 6 we introduce the notion of a controlled attractor and we discuss the generalization of the direct Lyapunov method to the stabilization of sets. Finally, in Section 7 we present some examples illustrating the theory.

2. Stochastic Control Setting

In this section we introduce the stochastic control problem and recall the definitions and basic properties of the controls we use.

We consider a controlled Ito stochastic differential equation:

$$\text{(CSDE)} \qquad \begin{cases} dX_t = f(X_t, \alpha_t)\,dt + \sigma(X_t, \alpha_t)\,dB_t, & t > 0, \\ X_0 = x. \end{cases}$$

We assume that α_t takes values in a given compact set $A \subseteq \mathbb{R}^M$, and that f and σ are continuous functions defined in $\mathbb{R}^N \times A$, taking values, respectively, in $\mathbb{R}^N$ and in the space of $N \times M$ matrices, and satisfying for all $x, y \in \mathbb{R}^N$ and all $\alpha \in A$,

$$|f(x,\alpha) - f(y,\alpha)| + \|\sigma(x,\alpha) - \sigma(y,\alpha)\| \leq C|x-y|. \tag{4}$$

We define

$$a(x,\alpha) := \tfrac{1}{2}\sigma(x,\alpha)\sigma(x,\alpha)^T$$

and assume

$$\{(a(x,\alpha),\, f(x,\alpha)) \colon \alpha \in A\} \qquad \text{is convex for all} \quad x \in \mathbb{R}^N. \tag{5}$$

We recall here the definition of admissible controls that we allow for our control problems. For precise definitions we refer to [22] and [15] (see also the references therein). Actually in these articles the problem is formulated in terms of solutions of the martingale problem, but it is also shown that there is an equivalent formulation in terms of solutions of (CSDE).

We relax the control problem by using *weak controls*, that is, admitting all the weak solutions of (CSDE). We have not assigned a priori a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ with its filtration. So when we introduce a control we mean that we are at the same time choosing also a probability space and a standard Brownian motion B_t on this space. Actually under hypothesis (4) it can be shown that the space of *strong controls* is not empty and that, under suitable assumptions on the cost functional (which are essentially the lower semicontinuity with respect to the x variable), the strong problem and the weak problem have the same value [15, Theorem 4.11].

Definition 1 (Strict Controls, Definition 2.2 of [22]). For every initial data $x \in \mathbb{R}^N$, a *strict control* is a progressively measurable A-valued process $(\alpha_t)_{t\geq 0}$ such that there exists an $\mathbb{R}^N$-valued, right continuous, almost surely continuous, progressively measurable solution X_t^α to (CSDE) (see also Definition 1.4 of [15]). We denote by $\mathcal{A}_x$ the set of strict controls for $x \in \mathbb{R}^N$.

The class of strict controls can be embedded, as in the deterministic case, in a larger class of admissible controls. We denote by $M(A)$ the set of probability measures on A endowed with the topology of weak convergence. We note that it is a separable metric space.

Definition 2 (Relaxed Controls, Definition 3.2 of [22]). For every initial data $x \in \mathbb{R}^N$, a *relaxed control* is a progressively measurable $M(A)$-valued process $(\mu_t)_{t\geq 0}$ such that there exists an $\mathbb{R}^N$-valued, right continuous, almost surely continuous, progressively measurable solution X_t^μ to (CSDE) (see also Definition 2.4 of [15]). We denote by $\mathcal{M}_x$ the set of relaxed controls for $x \in \mathbb{R}^N$.

We now choose a *canonical* probability space for our control problem. By means of this canonical space we can give a formulation of the optimization problem in a convex compact setting. The most natural canonical space for the strict control problem seems to be the space of the trajectories $X_.$ of (CSDE). It is the space of continuous functions $\mathcal{C}$ from $[0, +\infty)$ to $\mathbb{R}^N$ with its natural filtration. In order to give a control on this space, it is sufficient to specify the probability measure on $\mathcal{C}$ (which is the law of the process $X_.$) and the progressively measurable function α. Rather than working with this canonical space we consider the space of trajectories $(X_., \mu_.)$ for μ a relaxed control. Let $\mathcal{V}$ be the space of measurable functions from $[0, +\infty)$ to $M(A)$ with its canonical filtration. We denote by $M(\mathcal{V})$ the set of probability measures on $\mathcal{V}$ endowed with the stable topology (this is a topology introduced by Jacod and Memin, for a precise definition see [22, Section 3.10] and the references therein). The canonical space for the relaxed control problem will be the product space $\mathcal{C} \times \mathcal{V}$ with the product filtration. We call a relaxed control defined in this canonical space *canonic relaxed control* or *control rule* (see Definition 3.12 of [22] and also Definition 3.2 of [15]). In order to identify a canonic relaxed control, it is sufficient to specify the probability measure on the space $\mathcal{C} \times \mathcal{V}$: canonic relaxed controls can be considered as measures on the canonical space.

In the following we consider a cost functional

$$J(x, \alpha) = \sup_{t\geq 0} \mathbf{E}_x \left[V(X_t) + \int_0^t l(X_s)\, ds \right],$$

where l is a continuous, nonnegative function and V is an LSC, nonnegative function. The functional $j(x, t, \alpha) = \mathbf{E}_x[V(X_t) + \int_0^t l(X_s)\, ds]$ satisfies, for every x and t, the lower continuity assumptions required in [22] on the cost functional. Then, since the supremum of LSC maps is LSC, we get that the functional $J(x, \alpha)$ also satisfies the same lower semicontinuity assumptions. We list here the results obtained in [22] that we are going to use. The crucial assumption for all of them, besides the right choice of the class of admissible controls and the lower semicontinuity of the cost functional, is the convexity assumption (5).

The class of control rules is the class on which it is possible to formulate a dynamic programming principle and to show the existence of an optimal control. The key result is Proposition 5.2 in [22]:

for every initial data x, the class of optimal control rules admissible for x is convex and compact.

We have the following theorem stating the existence of an optimal control.

Theorem 3 (Theorem 4.7 and Corollary 4.8 of [22]). *Under the convexity assumption* (5) *and the other assumptions listed above, for every initial data $x \in \mathbb{R}^N$ there exists an optimal control rule for the control problem*

$$\inf_{\alpha} J(x, \alpha).$$

Moreover, the infimum of the cost functional computed on the class of control rules coincides with the infimum of the cost functional computed on the class of strict controls:

$$\inf_{\alpha \in \mathcal{A}_x} J(x, \alpha) = \inf_{\alpha \in \mathcal{M}_x} J(x, \alpha). \tag{6}$$

In particular, the optimal control can be chosen strict.

The two crucial properties on the control space to get a dynamic programming principle are the *stability under measurable selection* and the *stability under concatenation* (see [33]). They are satisfied by the class of control rules. We consider a measurable set-valued map from $\mathbb{R}^N$ to the space of probability measures on the canonical space $\mathcal{C} \times \mathcal{V}$, with convex compact values. Then, by a standard measurable selection theorem (see Theorem 5.3 of [15]), this map has a measurable selector. In Lemma 5.5 of [22] (see also Chapter 12 of [40] and Theorems 6.3 and 6.4 of [15]) it is proved that this measurable selector is an admissible control rule. Moreover, in Lemma 5.8 of [22] (see also Theorem 6.2 of [15]) it is shown that if we take an admissible control and then at some later stopping time we switch to an ε-optimal control from then on, the concatenated object is still admissible.

Finally, we observe that all these results remain valid if we consider, instead of the trajectories of (CSDE) in $\mathbb{R}^N$, the trajectories of this system stopped at the exit time from a given open set (see [22]).

3. Superoptimality Principles

In this section we prove a representation formula for bounded LSC viscosity supersolutions of Hamilton–Jacobi–Bellman equations. We are adapting to the second-order case the proof of optimality principles for viscosity supersolutions of first-order Hamilton-Jacobi equations given by Soravia in [38] and [39]. This requires the use of stochastic control instead of deterministic control. The representation formula is obtained by introducing a suitable sequence of obstacle problems, solved in a viscosity sense by V.

The technical core of the result is Lemma 5, which proves a suboptimality principle for the min-max value functions $L_{\lambda,k}$. The conclusion comes from a uniqueness result for viscosity solutions of such problems and from an approximation procedure. We give both global and local versions of the result.

We consider the following Hamilton–Jacobi–Bellman equation:

$$\max_{a\in A}\left\{-f(x,a)\cdot DV(x)-trace[a(x,a)D^2V(x)]\right\}-l(x)=0, \tag{7}$$

where $l\colon\ \mathbb{R}^N\to\mathbb{R}$ is a nonnegative bounded continuous function.

Theorem 4 (Representation Formula for Viscosity Supersolutions). *Consider a bounded LSC function $V\colon\ \mathbb{R}^N\to\mathbb{R}$. If V is a viscosity supersolution of the Hamilton–Jacobi–Bellman equation* (7) *in $\mathbb{R}^N$, then it can be represented as*

$$V(x)=\inf_{\alpha}\sup_{t\geq 0}\mathbf{E}_x\left[V(X_t^\alpha)+\int_0^t l(X_s^\alpha)\,ds\right], \tag{8}$$

where the infimum is taken over all strict admissible controls.

Proof. Without loss of generality, we can reduce to the case $V\geq 0$ by an appropriate translation. Since V is LSC, bounded and nonnegative, we can consider an increasing sequence of continuous, nonnegative, bounded functions V_k such that

$$V(x)=\sup_{k\geq 0}V_k(x)\qquad\text{for every}\quad x\in\mathbb{R}^N.$$

If V is continuous, we choose $V_k=V$ for every k.

Now for every $k\geq 0$, we introduce the following obstacle problem in $\mathbb{R}^N$ with unknown W and obstacle V_k:

$$\begin{aligned}&\min\{\lambda W(x)+\max_{a\in A}\left[-f(x,a)DW(x)-trace\,a(x,a)D^2W(x)\right]-l(x),\\&\qquad W(x)-V_k(x)\}=0.\end{aligned} \tag{9}$$

Obviously V is a bounded LSC viscosity supersolution of the problem (9) for every $\lambda\geq 0$ and every $k\geq 0$. For $\lambda>0$ fixed, define

$$L_{\lambda,k}(x)=\inf_{\alpha}\sup_{t\geq 0}\mathbf{E}_x\left[e^{-\lambda t}V_k(X_t^\alpha)+\int_0^t l(X_s^\alpha)e^{-\lambda s}\,ds\right].$$

The plan of the proof is the following. We define the upper semicontinuous envelope of $L_{\lambda,k}$

$$L^\star_{\lambda,k}(x):=\inf\{v(x)\mid v\geq L_{\lambda,k}\text{ in }\mathbb{R}^N,\ v\text{ continuous}\}.$$

We show that $L^\star_{\lambda,k}$ is a bounded (upper semicontinuous) viscosity subsolution of the obstacle problem (9). Then, by the comparison principle for bounded discontinuous viscosity solutions of Isaacs equations, we get that, for every $\lambda>0$, $L_{\lambda,k}(x)\leq V(x)$. From this, we can conclude, sending λ to 0 and k to $+\infty$, that V satisfies the superoptimality principle (8).

By the definition and the boundness of V_k, we get that $L_{\lambda,k}$ is bounded. We want to prove that its upper semicontinuous envelope $L^{\star}_{\lambda,k}$ is a viscosity subsolution of the obstacle problem (9). To get this result it is sufficient to check that $L^{\star}_{\lambda,k}$ is a viscosity subsolution of the Hamilton–Jacobi–Bellman equation

$$\lambda L(x) + \max_{a\in A}\{-f(x,a)\cdot DL(x) - trace[a(x,a)D^2L(x)]\} - l(x) = 0 \tag{10}$$

at the points x where $L^{\star}_{\lambda,k}(x) > V_k(x)$. This result can be obtained by standard methods in the theory of viscosity solutions if we prove a local suboptimality principle for $L^{\star}_{\lambda,k}$ on such points x (see [12] and [16]).

First we need the following technical lemma whose proof we postpone to the end.

Lemma 5. *If $L^{\star}_{\lambda,k}(x) > V_k(x)$ then there exists a sequence $x_n \to x$ with $L_{\lambda,k}(x_n) \to L^{\star}_{\lambda,k}(x)$ and $L_{\lambda,k}(x_n) > V_k(X_n)$ for which there exists $\varepsilon > 0$ such that*

$$L_{\lambda,k}(x_n) \le \inf_{\alpha} \mathbf{E}_{x_n}\left[e^{-\lambda t} L^{\star}_{\lambda,k}(X^{\alpha}_n(t)) + \int_0^t l(X^{\alpha}_n(s))e^{-\lambda s}\,ds\right] \tag{11}$$

for $t \le \varepsilon$ and $|x_n - x| \le \varepsilon$.

This is a local suboptimality principle. This inequality, by a standard argument in the theory of viscosity solution (for the detailed argument see, for example, [6], see also [16]), implies that $L^{\star}_{\lambda,k}$ is a viscosity subsolution of (7) at the points x such that $L^{\star}_{\lambda,k}(x) > V_k(x)$. From this we deduce that $L^{\star}_{\lambda,k}$ is a viscosity subsolution of (9).

By the comparison principle obtained for Isaacs operators and bounded discontinuous viscosity solutions by Ishii in Theorem 7.3 of [24], we get that $V(x) \ge L^{\star}_{\lambda,k}(x)$ for every $\lambda > 0$ and $k \ge 0$. In particular, for $T > 0$ and k fixed, we get

$$\begin{aligned} V(x) &\ge \lim_{\lambda\to 0} L_{\lambda,k}(x) \ge \lim_{\lambda\to 0} \inf_{\alpha} \sup_{t\in[0,T]} \mathbf{E}_x\left[e^{-\lambda t} V_k(X^{\alpha}_t) + \int_0^t l(X^{\alpha}_s)e^{-\lambda s}\,ds\right] \\ &\ge \lim_{\lambda\to 0} e^{-\lambda T} \inf_{\alpha} \sup_{t\in[0,T]} \mathbf{E}_x\left[V_k(X^{\alpha}_t) + \int_0^t l(X^{\alpha}_s)\,ds\right]. \end{aligned}$$

Therefore, for every $T > 0$ and $k \ge 0$,

$$V(x) \ge \inf_{\alpha} \sup_{t\in[0,T]} \mathbf{E}_x\left[V_k(X^{\alpha}_t) + \int_0^t l(X^{\alpha}_s)\,ds\right].$$

Now we want to pass to the limit for $k \to +\infty$. For x fixed and every $k \ge 0$ we consider an admissible control α_k such that

$$V(x) + \frac{1}{k} \ge \sup_{t\in[0,T]} \mathbf{E}_x\left[V_k(X^{\alpha_k}_t) + \int_0^t l(X^{\alpha_k}_s)\,ds\right]. \tag{12}$$

By the definitions recalled in Section 2 we can associate to each couple $(X^{\alpha_k}_{\cdot}, \alpha_k)$ a control rule P_k. By the compactness of the space of control rules, we can extract a subsequence of control rules, which we continue to denote by P_k, that converges to some control rule P in the stable topology. This control rule is the measure of a trajectory of (CSDE) $(X_\cdot, \mu)$ driven by a relaxed control μ. Since the convergence in the stable topology implies in particular the weak convergence of the measures P_k to the measure P, we get immediately that for every $t \in [0, T]$,

$$\lim_{k\to+\infty} \mathbf{E}_x \int_0^t l(X_s^{\alpha_k})\, ds = \mathbf{E}_x \int_0^t l(X_s)\, ds,$$

where the expected value on the left- and right-hand sides is computed, respectively, using the measures P_k and P.

Recalling now that $V = \sup_k V_k$, where V_k are continuous functions, it is easy to show that V can be obtained as

$$V(x) = \liminf_{k\to+\infty, y\to x} V_k(y) := \sup_\delta \inf \left\{ V_k(y) \mid |x-y| \le \delta, k \ge \frac{1}{\delta} \right\}.$$

Moreover, since convergence in the stable topology also implies convergence in probability of $X_t^{\alpha_k}$ to X_t, we get that, for $t \in [0, T]$ fixed, we can extract a subsequence $X_t^{\alpha_k}$ which converges to X_t almost surely with respect to the measure P. Then, along this subsequence,

$$\liminf_{k\to+\infty} V_k(X_t^{\alpha_k}) \ge \liminf_{k\to+\infty, y\to X_t} V_k(y) \ge V(X_t), \qquad P \text{ almost surely.}$$

By the Fatou lemma and the definition of stable convergence we deduce that, for each $t \in [0, T]$, along a subsequence,

$$\liminf_{k\to+\infty} \mathbf{E} V_k(X_t^{\alpha_k}) \ge \mathbf{E} V(X_t),$$

where the expected value is computed respectively using the measures P_k and P.

To summarize, for every $t \in [0, T]$ we get, from (12),

$$V(x) \ge \lim_{k\to+\infty} \mathbf{E}_x \left[V_k(X_t^{\alpha_k}) + \int_0^t l(X_s^{\alpha_k})\, ds \right] \ge \mathbf{E}_x \left[V(X_t) + \int_0^t l(X_s)\, ds \right].$$

So for every $T > 0$ there exists a control rule for which

$$V(x) \ge \sup_{t\in[0,T]} \mathbf{E}_x \left[V(X_t) + \int_0^t l(X_s)\, ds \right].$$

Now, by statement (6) in Theorem 3, we obtain

$$V(x) \ge \inf_\alpha \sup_{t\in[0,T]} \mathbf{E}_x \left[V(X_t^\alpha) + \int_0^t l(X_s^\alpha)\, ds \right]. \tag{13}$$

Now it remains only to let $T \to +\infty$. For $\varepsilon > 0$, consider an $\varepsilon/2$ optimal control α for (13): in particular, it gives $V(x) + \varepsilon/2 \ge \mathbf{E}_x[V(X_T^\alpha) + \int_0^T l(X_s^\alpha)\, ds]$. We consider now the set-valued map $x \to R(x) := \{\varepsilon/2^2$ optimal control rules γ for (13)$\}$. By the results recalled in Section 2 (see [22]), we can extract a measurable selection of this map. So the map $\omega \to X_T^\alpha(\omega) \to R(X_T^\alpha(\omega))$ also has a measurable selection, which we denote by β. We observe that, by the properties of control rules recalled in Section 2, β is an admissible control rule. We get that

$$\begin{aligned}\mathbf{E}_x V(X_T^\alpha) + \frac{\varepsilon}{2^2} &\geq \mathbf{E}_x \sup_{t\in[T,2T]} \mathbf{E}_{X_T^\alpha}\left[V(X_t^\beta) + \int_T^t l(X_s^\beta)\,ds\right] \\ &\geq \sup_{t\in[T,2T]} \mathbf{E}_x\left[V(X_t^\beta) + \int_T^t l(X_s^\beta)\,ds\right].\end{aligned}$$

Moreover, the control rule obtained concatenating this selected control and α is still an admissible control rule. Therefore we obtain

$$\begin{aligned}V(x) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} &\geq \mathbf{E}_x\left[V(X_T^\alpha) + \frac{\varepsilon}{2^2} + \int_0^T l(X_s^\alpha)\,ds\right] \\ &\geq \sup_{t\in[T,2T]} \mathbf{E}_x\left[V(X_t^\beta) + \int_0^T l(X_s^\alpha)\,ds + \int_T^t l(X_s^\beta)\,ds\right].\end{aligned}$$

Now we consider an $\varepsilon/2^3$ optimal control γ for $V(X_{2T}^\beta)$ and conclude as above that

$$V(x) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \frac{\varepsilon}{2^3} \geq \sup_{t\in[2T,3T]} \mathbf{E}_x\left[V(X_t^\gamma) + \int_0^{2T} l(X_s^\beta)\,ds + \int_T^t l(X_s^\gamma)\,ds\right].$$

We can then proceed recursively and conclude by induction that we can construct an admissible control rule P such that

$$V(x) + \varepsilon \geq \sup_{t\geq 0} \mathbf{E}_x\left[V(X_t) + \int_0^t l(X_s)\,ds\right].$$

By statement (6) in Theorem 3 and recalling that ε is arbitrary, we obtain

$$V(x) \geq \inf_\alpha \sup_{t\geq 0} \mathbf{E}_x\left[V(X_t) + \int_0^t l(X_s)\,ds\right] \geq V(x), \tag{14}$$

which is the desired formula. □

We give here the proof of the technical Lemma 5

Proof of Lemma 5. If the statement were not true, for every sequence $x_m \to x$ with $L_{\lambda,k}(x_m) \to L^\star_{\lambda,k}(x)$ and $L_{\lambda,k}(x_m) > V_k(x_m)$, for every $n > 0$ we could find $t_n \leq 1/n$ and x_{m_n} with $|x_{m_n} - x| \leq 1/n$ such that

$$L_{\lambda,k}(x_{m_n}) > \inf_\beta \mathbf{E}_{x_{m_n}}\left[e^{-\lambda t_n} L^\star_{\lambda,k}(X^\beta_{m_n}(t_n)) + \int_0^{t_n} l(X^\beta_{m_n}(s))e^{-\lambda s}\,ds\right]. \tag{15}$$

We reduce to this x_{m_n} subsequence which we denote by x_n for simplicity. By definition of $L_{\lambda,k}$, for every $\varepsilon_n > 0$ and every control α, there exists $T(\varepsilon_n, \alpha)$ such that

$$L_{\lambda,k}(x_n) - \varepsilon_n < \mathbf{E}_{x_n}\left[e^{-\lambda T(\varepsilon_n,\alpha)} V_k(X_n^\alpha(T(\varepsilon_n, \alpha))) + \int_0^{T(\varepsilon_n,\alpha)} l(X_n^\alpha(s))e^{-\lambda s} ds\right]. \quad (16)$$

By inequality (15), we can choose a sequence $\varepsilon_n \to 0$ and controls β_n for which

$$L_{\lambda,k}(x_n) - 2\varepsilon_n \geq \mathbf{E}_{x_n}\left[e^{-\lambda t_n} L^\star_{\lambda,k}(X_n^{\beta_n}(t_n)) + \int_0^{t_n} l(X_n^{\beta_n}(s))e^{-\lambda s}\, ds\right]. \quad (17)$$

Therefore for every control α we obtain, from inequalities (16) and (17),

$$\begin{aligned}
&\mathbf{E}_{x_n}\left[e^{-\lambda t_n} L^\star_{\lambda,k}(X_n^{\beta_n}(t_n)) + \int_0^{t_n} l(X_n^{\beta_n}(s))e^{-\lambda s}\, ds\right] + \varepsilon_n \\
&\quad \leq L_{\lambda,k}(x_n) - \varepsilon_n \\
&\quad < \mathbf{E}_{x_n}\left[e^{-\lambda T(\varepsilon_n,\alpha)} V_k(X_n^\alpha(T(\varepsilon_n, \alpha))) + \int_0^{T(\varepsilon_n,\alpha)} l(X_n^\alpha(s))e^{-\lambda s}\, ds\right].
\end{aligned}$$

We now claim that for every n there exists an α such that $T(\varepsilon_n, \alpha) \leq t_n$. Assume by contradiction that there exists N such that, for every α admissible, $T(\varepsilon_N, \alpha) > t_N$, in particular, $T(\varepsilon_N, \alpha_N) > t_N$ for every control α_N which for $t \leq t_N$ coincides with β_N: by the previous inequality we get

$$\begin{aligned}
&\mathbf{E}_{x_N} L^\star_{\lambda,k}(X_N^{\beta_N}(t_N)) + \varepsilon_N \\
&\quad < \mathbf{E}_{x_N}\Bigg[e^{-\lambda(T(\varepsilon_N,\alpha_N)-t_N)} V_k(X_N^{\alpha_N}(T(\varepsilon_N, \alpha_N))) \\
&\qquad\quad + \int_{t_N}^{T(\varepsilon_N,\alpha_N)} l(X_N^{\alpha_N}(s))e^{-\lambda(s-t_N)}\, ds\Bigg].
\end{aligned} \quad (18)$$

We consider the set-valued map

$$\begin{aligned}
x \to R(x) := \Bigg\{&\gamma \text{ control rule} \,\Big|\, L_{\lambda,k}(x) - \frac{\varepsilon_N}{2} \\
&\geq \sup_{t\geq 0} \mathbf{E}_x \left[e^{-\lambda t} V_k(X_t^\gamma) + \int_0^t l(X_s^\gamma)e^{-\lambda s}\, ds\right]\Bigg\}.
\end{aligned}$$

By the results recalled in Section 2 (see [22]), we can extract a measurable selection of this map. So the map $\omega \to X_N^{\beta_N}(t_N)(\omega) \to R(X_N^{\beta_N}(t_N)(\omega))$ also has a measurable selection, which we denote by γ_N. We observe that, by the properties of control rules recalled in Section 2, γ_N is an admissible control rule and the concatenated control of γ_N with β_N is still an admissible control rule P_N which is the measure associated to the couple $(X_N(.), \mu_N)$. This gives

$$\mathbf{E}_{x_N} L_{\lambda,k}(X_N^{\beta_N}(t_N)) - \frac{\varepsilon_N}{2} \geq \sup_{t\geq t_N} \mathbf{E}_{x_N}\left[e^{-\lambda t} V_k(X_N(t)) + \int_{t_N}^t l(X_N(s))e^{-\lambda s}\, ds\right]. \quad (19)$$

Recalling that $L^\star_{\lambda,k} \geq L_{\lambda,k}$, concatenating inequalities (18) and (19) we obtain

$$\sup_{t\geq t_N} \mathbf{E}_{x_N}\left[e^{-\lambda t}V_k(X_N(t)) + \int_{t_N}^{t} l(X_N(s))e^{-\lambda s}\,ds\right] + \frac{\varepsilon_N}{2}$$
$$\leq \mathbf{E}_{x_N}\left[e^{-\lambda T(\varepsilon_N,\mu_N)}V_k(X_N(T(\varepsilon_N,\mu_N))) + \int_{t_N}^{T(\varepsilon_N,\mu_N)} l(X_N(s))e^{-\lambda s}\,ds\right].$$

This is not possible if, as we assumed, $T(\varepsilon_N, \mu_N) \geq t_N$: then we get a contradiction. Therefore there exists for every n an admissible control rule P_n such that $T(\varepsilon_n, \mu_n) \leq t_n$: choosing $\alpha = \mu_n$ in inequality (16) we get

$$L_{\lambda,k}(x_n) - \varepsilon_n \leq \mathbf{E}_{x_n}\left[e^{-\lambda T(\varepsilon_n,\mu_n)}V_k(X_n(T(\varepsilon_n,\mu_n))) + \int_0^{T(\varepsilon_n,\mu_n)} l(X_n(s))e^{-\lambda s}\right].$$

For every n we say $A_n = \{\omega \mid X_n(T(\varepsilon_n, \mu_n)) \in B(x_n, (1/\sqrt[4]{n})\}$ and $B_n = \Omega\backslash A_n$. Since for every n the trajectory is a Markov process and the drift and the diffusion of this control problem are equi-Lipschitz and equi-bounded in the compacts with respect to the control it is possible to show (we refer to pp. 284–285 of [14] for the proof) that $P_n(B_n) \leq K(1+|x_n|^2)^{3/2}(1/n)^{3/8}$ where K depends on the constant C in (4). Therefore we get

$$L_{\lambda,k}(x_n) - \frac{\varepsilon_n}{2} < \int_{A_n}\left[V_k(X_n(T(\varepsilon_n,\mu_n))) + \int_0^{T(\varepsilon_n,\mu_n)} l(X_n(s))e^{-\lambda s}\,ds\right]dP_n$$
$$+ \int_{B_n}\left[V_k(X_n(T(\varepsilon_n,\mu_n))) + \int_0^{T(\varepsilon_n,\mu_n)} l(X_n(s))e^{-\lambda s}ds\right]dP_n$$
$$\leq \left[\sup_{B(x,2/\sqrt{n})} V_k(y) + \sup_{B(x,2/\sqrt{n})} l(y)t_n\right] + K\left(1+|x_n|^2\right)^{3/2}\left(\frac{1}{n}\right)^{3/8}.$$

From this, since V_k is continuous and $t_n \leq 1/n$, letting $n \to +\infty$, we deduce

$$L^\star_{\lambda,k}(x) \leq V_k(x)$$

in contradiction with our assumption. □

Remark. The previous result can be proved in more general situations: consider a bounded, nonnegative, LSC viscosity supersolution $V\colon \mathbb{R}^N \to \mathbb{R}$ of

$$\max_{a\in A}\left\{-f(x,a)\cdot DV(x) - trace[a(x,a)D^2V(x)]\right\} + k(x)V(x) \geq l(x),$$

where $k\colon \mathbb{R}^N \to \mathbb{R}$ is a Lipschitz continuous nonnegative function. The proof of Theorem 4 applies directly and we obtain the representation formula

$$V(x) = \inf_\alpha \sup_{t\geq 0} \mathbf{E}_x\left[V(X_t^\alpha)e^{-\int_0^t k(X_s^\alpha)\,ds} + \int_0^t l(X_s^\alpha)e^{-\int_0^s k(X_u^\alpha)du}\,ds\right].$$

We can also prove a localized version of Theorem 4.

Corollary 6. *Consider an open set $\mathcal{O} \subseteq \mathbb{R}^N$. For every $\delta > 0$, consider the set $\mathcal{O}_\delta := \{x \in \mathcal{O} \mid d(x, \partial\mathcal{O}) > \delta\}$ and denote by τ_δ^α the stopping time at which the sample function of the process X_t^α reaches the boundary $\partial\mathcal{O}_\delta$: we denote by $\tau_\delta^\alpha(t)$ the minimum between τ_δ^α and t. Assume that $V\colon \overline{\mathcal{O}} \to \mathbb{R}$ is a bounded nonnegative function. If V is an LSC viscosity supersolution of the Hamilton–Jacobi–Bellman equation (7) in $\mathcal{O}$, then it can be represented, for every δ, $x \in \mathcal{O}_\delta$, as*

$$V(x) = \inf_\alpha \sup_{t\geq 0} \mathbf{E}_x \left[V(X^\alpha_{\tau_\delta^\alpha(t)}) + \int_0^{\tau_\delta^\alpha(t)} l(X_s^\alpha)\, ds \right].$$

Proof. We fix $\delta > 0$ and a smooth cut off function $0 \leq \xi \leq 1$ such that $\xi(x) = 0$ for $x \in \mathbb{R}^N \setminus \mathcal{O}$ and $\xi(x) = 1$ for $x \in \mathcal{O}_\delta$. We consider the stochastic controlled differential equation in $\mathbb{R}^N$:

$$(\mathrm{CSDE})' \qquad \begin{cases} dX_t = f(X_t, \alpha_t)\xi^2(X_t)\, dt + \sigma(X_t, \alpha_t)\xi(X_t)\, dB_t, & t > 0, \\ X_0 = x. \end{cases}$$

Observe that for $x \in \mathcal{O}_\delta$, the solution $(X')^\alpha$ to (CSDE)$'$ coincides a.s. with the solution X^α to (CSDE) up to time τ_δ^α. We define the process $X^\alpha_{\tau_\delta^\alpha(t)}$ obtained by stopping the process $(X')_t^\alpha$ at the instant it reaches the boundary of $\mathcal{O}_\delta$: it has an Ito stochastic differential and is still a strong Markov process with continuous trajectories (see, for example, Lemma 3.3.1 of [20] and the references therein).

We extend V outside $\mathcal{O}$ as a bounded nonnegative LSC function that we continue to denote by V. So it is immediate to show that V is a viscosity supersolution in $\mathbb{R}^N$ of the equation

$$\max_{a\in A}\{-f(x,a)\xi^2(x)\cdot DV(x) - trace[a(x,a)\xi^2(x)D^2V(x)]\} - l(x)\xi^2(x) = 0. \quad (20)$$

We can apply Theorem 4 to V. Indeed, it is sufficient to define

$$L_{\lambda,k}(x) = \begin{cases} \inf_\alpha \sup_{t\geq 0} \mathbf{E}_x \Big[e^{-\lambda\tau_\delta^\alpha(t)} V_k(X^\alpha_{\tau_\delta^\alpha(t)}) \\ \qquad + \int_0^t l(X^\alpha_{\tau_\delta^\alpha(s)})\xi^2(X^\alpha_{\tau_\delta^\alpha(s)}) e^{-\lambda\tau_\delta^\alpha(t)}\, ds \Big] & \text{in } \overline{\mathcal{O}_\delta}, \\ V_k(x) & \text{in } \mathbb{R}^N \setminus \overline{\mathcal{O}_\delta}. \end{cases}$$

We can repeat the proof in Theorem 4 (all the results in [22] also hold for the stopped process Y^α) and we get that $L^\star_{\lambda,k}$ is a viscosity supersolution of the obstacle problem (9) in $\mathbb{R}^N$. So again repeating the same arguments of Theorem 4 we get that V satisfies the following representation formula for $x \in \mathcal{O}_\delta$:

$$V(x) = \inf_\alpha \sup_{t\geq 0} \mathbf{E}_x \left[V(X^\alpha_{\tau_\delta^\alpha(t)}) + \int_0^{\tau_\delta^\alpha(t)} l(X_s^\alpha)\, ds \right]. \qquad \square$$

Remark (Minimal Nonnegative Solution). These representation formulas for viscosity solutions are interesting on their own, as we have pointed out in the Introduction: indeed they apply to Hamilton–Jacobi–Bellman equations for which there are no comparison principles and then no uniqueness of solutions. We consider the following Hamilton–Jacobi–Bellman equation in $\mathbb{R}^N$:

$$\max_{a\in A}\left\{-f(x,a)\cdot DV(x) - trace[a(x,a)D^2V(x)]\right\} = l(x)$$

with $l \geq 0$: since the constant function $U \equiv 0$ is always a subsolution, it is interesting to characterize the minimal nonnegative supersolution.

From a control point of view, the natural solution seems to be the value function of the infinite horizon control problem with running cost l,

$$V_\infty(x) = \inf_\alpha \mathbf{E}_x \int_0^{+\infty} l(X_s^\alpha)\,ds.$$

If V_∞ is well defined and bounded, then it is possible to show that it is LSC, by an argument based on the properties of the class of admissible relaxed controls. Moreover, by standard methods in the theory of viscosity solutions (see [17] and [12]), it is possible to show that V_∞ is a viscosity supersolution of the previous Hamilton–Jacobi–Bellman equation. In this case an easy application of the previous theorems gives that every bounded, nonnegative, viscosity supersolution V of the Hamilton–Jacobi–Bellman equation in $\mathbb{R}^N$ satisfies

$$V(x) \geq V_\infty(x),$$

therefore V_∞ is the minimal nonnegative viscosity supersolution of the equation.

Remark (Representation Formula for Viscosity Subsolutions). The counterpart of Theorem 4 for viscosity subsolutions is straightforward from classical suboptimality principles: let $U\colon \mathbb{R}^N \to \mathbb{R}$ be an upper semicontinuous bounded viscosity subsolution of the Hamilton–Jacobi–Bellman equation

$$\max_{a\in A}\left\{-f(x,a)\cdot DU(x) - trace[a(x,a)D^2U(x)]\right\} \leq l(x),$$

then the function U can be represented as

$$U(x) = \inf_\alpha \inf_{t\geq 0} \mathbf{E}_x \left[U(X_t^\alpha) + \int_0^t l(X_s^\alpha)\,ds\right].$$

4. Stability in Probability and Lyapunov Functions

We begin this section with the notion of both Lyapunov and asymptotic stability in probability. They were introduced by Has'minskii and Kushner (see [20] and [25]) in the case of uncontrolled stochastic differential equations. We present their natural extension to the case of controlled diffusions.

Definition 7 (Stabilizability in Probability). The controlled system (CSDE) is (*stochastic open loop*) *stabilizable in probability* at the origin if for all $\varepsilon, k > 0$ there exists $\delta > 0$ such that for every $|x| \leq \delta$ there exists a control $\overline{\alpha}_{.} \in \mathcal{A}_x$ such that the corresponding trajectory $\overline{X}_{.}$ verifies

$$\mathbf{P}_x \left(\sup_{t \geq 0} |\overline{X}_t| \geq k \right) \leq \varepsilon.$$

This is equivalent to assuming that, for every positive k,

$$\lim_{x \to 0} \inf_{\alpha} \mathbf{P}_x \left(\sup_{t \geq 0} |X_t^{\alpha}| \geq k \right) = 0.$$

The system is (*stochastic open loop*) *Lagrange stabilizable in probability*, or it has the property of *uniform boundedness of trajectories*, if for each $\varepsilon > 0$, $R > 0$ there is $S > 0$ such that, for any initial point x with $|x| \leq R$,

$$\inf_{\alpha} \mathbf{P}_x \left(\sup_{t \geq 0} |X_t^{\alpha}| \geq S \right) \leq \varepsilon.$$

This is equivalent to assuming that, for every $R > 0$,

$$\lim_{S \to +\infty} \sup_{|x| \leq R} \inf_{\alpha} \mathbf{P}_x \left(\sup_{t \geq 0} |X_t^{\alpha}| \geq S \right) = 0.$$

Remark. Stabilizability in probability implies that the origin is a *controlled equilibrium* of (CSDE), i.e.,

$$\exists\, \overline{\alpha} \in A, \qquad f(0, \overline{\alpha}) = 0, \qquad \sigma(0, \overline{\alpha}) = 0.$$

In fact, the definition gives for any $\varepsilon > 0$, for $k > 0$ fixed, an admissible control such that the corresponding trajectory starting at the origin satisfies $\mathbf{P}(\sup_{t \geq 0} |X_t| \geq k) \leq \varepsilon$ so

$$\mathbf{E}_0 \int_0^{+\infty} l(|X_t|) e^{-\lambda t}\, dt \leq \frac{\varepsilon}{\lambda}$$

for any $\lambda > 0$ and any real function l such that $0 \leq l(r) \leq 1$ for any r and $l(r) = 0$ for $r \leq k$. Then $\inf_{\alpha_{.} \in \mathcal{A}_0} \mathbf{E}_0 \int_0^{+\infty} l(|X_t|) e^{-\lambda t}\, dt = 0$. Theorem 3 implies that the inf is attained: therefore for any $k > 0$ there is a minimizing control which produces a trajectory satisfying a.s. $|X_t| \leq k$ for all $t \geq 0$. So $\inf_{\alpha_{.} \in \mathcal{A}_0} \mathbf{E}_0 \int_0^{+\infty} |X_t| e^{-\lambda t}\, dt = 0$ for any $\lambda > 0$. Again Theorem 3 implies that the inf is attained, and the minimizing control produces a trajectory satisfying a.s. $|X_t| = 0$ for all $t \geq 0$. The conclusion follows from standard properties of stochastic differential equations.

Regarding Lagrange stabilizability, we observe that, using standard properties of diffusions under regularity assumptions (4), it is possible to prove (see [14] and [20]) that, for every fixed $T > 0$ and $R > 0$,

$$\lim_{S\to+\infty} \sup_{|x|\le R} \inf_{\alpha} \mathbf{P}_x \left(\sup_{0\le t\le T} |X_t^\alpha| \ge S \right) = 0.$$

Nevertheless, Lagrange stabilizability is a stronger condition since it requires that

$$\lim_{S\to+\infty} \sup_{|x|\le R} \inf_{\alpha} \sup_{T\ge 0} \mathbf{P}_x \left(\sup_{0\le t\le T} |X_t^\alpha| \ge S \right) = 0.$$

Controlled diffusion is said to be asymptotically stabilizable (or controllable) in probability if the equilibrium point is not only stabilizable but also an attracting point for the system, locally around the equilibrium point.

Definition 8 (Asymptotic Stabilizability in Probability). The controlled system is locally *asymptotically stabilizable in probability* (or *asymptotically controllable in probability*) at the origin if for all $\varepsilon, k > 0$ there exists $\delta > 0$ such that for every $|x| \le \delta$ there exists a control $\overline{\alpha}_\cdot \in \mathcal{A}_x$ such that the corresponding trajectory $\overline{X}_\cdot$ verifies

$$\mathbf{P}_x \left(\sup_{t\ge 0} |\overline{X}_t| \ge k \right) \le \varepsilon \quad \text{and} \quad \mathbf{P}_x \left(\limsup_{t\to+\infty} |\overline{X}_t| > 0 \right) \le \varepsilon.$$

This is equivalent to assuming that, for all $k > 0$,

$$\lim_{x\to 0} \inf_{\alpha} \left[\mathbf{P}_x \left(\sup_{t\ge 0} |X_t^\alpha| \ge k \right) + \mathbf{P}_x \left(\limsup_{t\to+\infty} |X_t^\alpha| > 0 \right) \right] = 0.$$

There is a global version of the previous stability notion:

Definition 9 (Asymptotic Stabilizability in the Large). The controlled system is *asymptotic stabilizable in the large* at the origin if it is Lyapunov stabilizable in probability around the equilibrium and, for every $x \in \mathbb{R}^N$,

$$\inf_{\alpha} \mathbf{P}_x \left(\limsup_{t\to+\infty} |X_t^\alpha| > 0 \right) = 0.$$

This means that for every $\varepsilon > 0$ and for every initial data x we can choose an admissible control in $\mathcal{A}_x$ which drives the trajectory to the equilibrium with probability greater than $1 - \varepsilon$.

Next we give the appropriate definition of control Lyapunov functions for the study of the stochastic stabilities defined above.

Definition 10 (Lyapunov Function). Let $\mathcal{O} \subseteq \mathbb{R}^N$ be a bounded open set containing the origin. A function $V\colon \mathcal{O} \to \mathbb{R}$ is a *local Lyapunov function* for (CSDE) if it satisfies the following conditions:

(i) it is LSC and continuous at the origin;
(ii) it is *positive definite*, i.e., $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$;
(iii) it is bounded;
(iv) it is a viscosity supersolution of the equation

$$\max_{\alpha \in A}\{-DV(x) \cdot f(x,\alpha) - trace[a(x,\alpha)D^2V(x)]\} = 0 \qquad \text{in } \mathcal{O}. \tag{21}$$

We introduce the notion of a strict Lyapunov function in both the local and global setting.

Definition 11 (Local Strict Lyapunov Function). Let $\mathcal{O} \subseteq \mathbb{R}^N$ be a bounded open set containing the origin. A function $V\colon \mathcal{O} \to \mathbb{R}$ is a *local strict Lyapunov function* for (CSDE) if it satisfies conditions (i)–(iii) in the previous definition and

(iv′) it is a viscosity supersolution of the equation

$$\max_{\alpha \in A}\left\{-DV(x) \cdot f(x,\alpha) - trace\left[a(x,\alpha)D^2V(x)\right]\right\} = l(x) \qquad \text{in } \mathcal{O}, \tag{22}$$

where $l\colon \mathcal{O} \to \mathbb{R}$ is a positive definite, bounded and uniformly continuous function.

Definition 12 (Global Strict Lyapunov Function). Let $\mathcal{O} \subseteq \mathbb{R}^N$ be an open set containing the origin. A function $V\colon \mathcal{O} \to \mathbb{R}$ is a *global strict Lyapunov function* for (CSDE) if it satisfies the following conditions:

(i) it is LSC and continuous at the origin;
(ii) it is *positive definite*, i.e., $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$;
(iii) it is *proper*, i.e., $\lim_{x\to\partial\mathcal{O}} V(x) = +\infty$, or, equivalently, its level sets $\{x \mid V(x) \leq \mu\}$ are bounded;
(iv) it is a viscosity supersolution of the equation

$$\max_{\alpha \in A}\left\{-DV(x) \cdot f(x,\alpha) - trace\left[a(x,\alpha)D^2V(x)\right]\right\} = l(x) \qquad \text{in } \mathcal{O}, \tag{23}$$

where $l\colon \mathcal{O} \to \mathbb{R}$ is a positive definite uniformly continuous function.

5. Direct Lyapunov Theorems

In this section we develop a direct Lyapunov method for the study of stabilizability in probability of controlled diffusions both in the local and global setting. For the uncontrolled case, the extension of the Lyapunov second method to the case of stochastic systems is due to Has'minskii and Kushner independently (see the monographs [20] and [25], and see also the references therein for earlier related results).

The main tool of the proof of the Lyapunov theorems is the representation formula for viscosity solutions obtained in Section 3.

Theorem 13 (Stabilizability in Probability). *Assume conditions* (4) *and* (5) *and the existence of a local Lyapunov function* V *in the open set* $\mathcal{O}$. *Then*:

(i) *the system is stabilizable in probability,*
(ii) *if in addition the Lyapunov function is global*, *then the system is also Lagrange stabilizable in probability.*

Proof. We start by proving (i). We fix $k > 0$ such that $B_k \subset \mathcal{O}$. We fix $\varepsilon > 0$ and define $\eta = \varepsilon \min_{|y|\geq k} V(y)$. We denote by $\tau_k^\alpha(x)$ the first exit time of the trajectory X_t^α from the open ball B_k centered at the origin with radius k. By the continuity at the origin of V we can find $\theta > 0$ such that if $|x| \leq \theta$ then $V(x) \leq \eta/2$. The superoptimality principle in Corollary 6 gives, for $|x| \leq \theta \wedge k$,

$$\eta/2 \geq V(x) = \inf_\alpha \sup_{t\geq 0} \mathbf{E}_x V(X_{t\wedge\tau_k^\alpha(x)}).$$

We now choose an $\eta/2$ optimal control $\overline{\alpha} \in \mathcal{A}_x$ for the previous control problem, we denote by $\overline{X}_t$ the corresponding trajectory, stopped at the exit time from $\mathcal{O}$, and we get for every $t \geq 0$,

$$\eta \geq \mathbf{E}_x V(\overline{X}_{t\wedge\tau_k^{\overline{\alpha}}}) \geq \int_{\{\sup_{0\leq s\leq t} |\overline{X}_s|\geq k\}} V(\overline{X}_{\tau_k^{\overline{\alpha}}}) \geq \mathbf{P}\left(\sup_{0\leq s\leq t} |\overline{X}_s| \geq k\right) \min_{|y|\geq k} V(y).$$

As $t \to +\infty$, we obtain the following bound on the probability that the trajectory $\overline{X}_t$ leaves the ball B_k:

$$\mathbf{P}\left(\sup_{t\geq 0} |\overline{X}_t| \geq k\right) \leq \frac{\eta}{\min_{|y|\geq k} V(y)} = \varepsilon.$$

This proves the stabilizability in probability.

We pass now to (ii). Repeating the argument above we get that, for every $k > 0$,

$$\inf_\alpha \mathbf{P}\left(\sup_{t\geq 0} |X_t^\alpha| \geq k\right) \leq \frac{V(x)}{\min_{|y|\geq k} V(y)}.$$

This implies the Lagrange stabilizability: indeed, given $R > 0$ and $\varepsilon > 0$, we choose k such that

$$\frac{\max_{|y|\leq R} V(y)}{\min_{|y|\geq k} V(y)} \leq \varepsilon. \qquad \square$$

In case the system admits a strict Lyapunov function we prove that there exists a control which not only stabilizes diffusion in probability but also drives it asymptotically to the equilibrium. We obtain the result using standard martingale inequalities; in the uncontrolled case, a similar proof of asymptotic stability has been given in [13] (see also [30]).

Theorem 14 (Asymptotic Stabilizability). *Assume conditions* (5) *and* (4) *and the existence of a local strict Lyapunov function in an open set* $\mathcal{O}$. *Then*

(i) *the system* (*CSDE*) *is locally asymptotically stabilizable in probability*;
(ii) *if the strict Lyapunov function is global*, *then the system* (*CSDE*) *is asymptotically stabilizable in the large*.

Proof. We start by proving (i). For every $k > 0$, such that $B_k \subset \mathcal{O}$, we get, by Corollary 6, that the function V satisfies, for $x \in B_k$, the following superoptimality principle:

$$V(x) = \inf_{\alpha} \sup_{t \geq 0} \mathbf{E}_x \left[V(X^{\alpha}_{\tau^{\alpha}_k(t)}) + \int_0^{\tau^{\alpha}_k(t)} l(X^{\alpha}_s)\, ds \right], \tag{24}$$

where the trajectories are stopped at the exit time from B_k. By Theorem 3 there exists an optimal control $\overline{\alpha} \in \mathcal{A}_x$ for this value problem. We indicate by $\overline{X}_{\cdot}$ the corresponding trajectory and by $\overline{\tau}$ the exit time from the open ball B_k. Repeating the proof of Theorem 13 we get the stabilizability in probability:

$$\mathbf{P}_x(\overline{\tau} < +\infty) = \mathbf{P}_x\left(\sup_{t \geq 0} |\overline{X}_t| \geq k\right) \leq \frac{V(x)}{\min_{|y| \geq k} V(y)}.$$

We denote $B(x) = \{\omega \mid |\overline{X}_t(\omega)| \leq k, \forall t \geq 0\}$. By the previous estimate $\mathbf{P}_x(B(x)) = \mathbf{P}_x\,(\overline{\tau} = +\infty) \geq 1 - V(x)/(\min_{|y| \geq k} V(y))$.

We claim that $l(\overline{X}_t(\omega)) \to 0$ as $t \to +\infty$ for almost all $\omega \in B(x)$, from this, using the positive definiteness of the function l, we can deduce that

$$\mathbf{P}_x\left(\limsup_{t \to +\infty} |\overline{X}_t| > 0\right) \leq \frac{V(x)}{\min_{|y| \geq k} V(y)}$$

which gives, by continuity at the origin of the function V, the asymptotic stabilizability in probability.

We assume by contradiction that the claim is not true: then there exists $\varepsilon > 0$, a subset $\Omega_\varepsilon \subseteq B(x)$ with $\mathbf{P}(\Omega_\varepsilon) > 0$, and for every $\omega \in \Omega_\varepsilon$ a sequence $t_n(\omega) \to +\infty$ such that $l(\overline{X}_{t_n}(\omega)) > \varepsilon$. We define

$$F(k) := \max_{|x| \leq k, \alpha \in A} |f(x, \alpha)|, \qquad \Sigma(k) = \max_{|x| \leq k, \alpha \in A} \|\sigma(x, \alpha)\|.$$

We indicate by $\overline{\tau}(s)$ the minimum between $\overline{\tau}$ and s and compute, for $t \geq 0$ fixed,

$$\begin{aligned}
&\mathbf{E}\left\{\sup_{t \leq s \leq t+h} |\overline{X}_{\overline{\tau}(s)} - \overline{X}_{\overline{\tau}(t)}|^2\right\} \\
&\quad = \mathbf{E}\left\{\sup_{t \leq s \leq t+h} \left| \int_{\overline{\tau}(t)}^{\overline{\tau}(s)} f(\overline{X}_u, \overline{\alpha}_u)\, du + \int_{\overline{\tau}(t)}^{\overline{\tau}(s)} \sigma(\overline{X}_u, \overline{\alpha}_u)\, dB_u \right|^2 \right\}
\end{aligned}$$

$$\leq 2\mathbf{E}\left\{\sup_{t\leq s\leq t+h}\left|\int_{\overline{\tau}(t)}^{\overline{\tau}(s)} f(\overline{X}_u,\overline{\alpha}_u)\,du\right|^2\right\}$$

$$+\,2\mathbf{E}\left\{\sup_{t\leq s\leq t+h}\left|\int_{\overline{\tau}(t)}^{\overline{\tau}(s)} \sigma(\overline{X}_u,\overline{\alpha}_u)\,dB_u\right|^2\right\}$$

$$\leq 2F^2(k)h^2+2\mathbf{E}\left\{\sup_{t\leq s\leq t+h}\left|\int_{\overline{\tau}(t)}^{\overline{\tau}(s)} \sigma(\overline{X}_u,\overline{\alpha}_u)\,dB_u\right|^2\right\}.$$

By Theorem 3.4 in [14] (the process $|\int_t^s \sigma(\overline{X}_u,\overline{\alpha}_u)\,dB_u|$ is a positive semimartingale) we get

$$\mathbf{E}\left\{\sup_{t\leq s\leq t+h}|\overline{X}_{\overline{\tau}(s)}-\overline{X}_{\overline{\tau}(t)}|^2\right\}$$

$$\leq 2F^2(k)h^2+8\sup_{t\leq s\leq t+h}\mathbf{E}\left\{\left|\int_{\overline{\tau}(t)}^{\overline{\tau}(s)} \sigma(\overline{X}_u,\overline{\alpha}_u)\,dB_u\right|^2\right\}$$

$$\leq 2F^2(k)h^2+8\sup_{t\leq s\leq t+h}\mathbf{E}\left\{\int_{\overline{\tau}(t)}^{\overline{\tau}(s)} \left|\sigma(\overline{X}_u,\overline{\alpha}_u)\right|^2\,du\right\}\leq 2F^2(k)h^2+8\Sigma^2(k)h.$$

Then Chebyshev inequality gives, for every $t\geq 0$ fixed,

$$\mathbf{P}\left(\sup_{t\leq s\leq t+h}|\overline{X}_{\overline{\tau}(s)}-\overline{X}_{\overline{\tau}(t)}|>r\right)\leq\frac{2F^2(k)h^2+8\Sigma^2(k)h}{r^2}. \tag{25}$$

Since l is continuous, we can fix δ such that $|l(x)-l(y)|\leq\varepsilon/2$ if $|x-y|\leq\delta$ and $|x|,|y|\leq k$: we compute

$$\mathbf{P}\left(\sup_{t\leq s\leq t+h}|l(\overline{X}_{\overline{\tau}(s)})-l(\overline{X}_{\overline{\tau}(t)})|\leq\frac{\varepsilon}{2}\right)$$

$$\geq\mathbf{P}\left(\sup_{t\leq s\leq t+h}|\overline{X}_{\overline{\tau}(s)}-\overline{X}_{\overline{\tau}(t)}|\leq\delta\right)$$

$$\geq 1-\frac{2F^2(k)h^2+8\Sigma^2(k)h}{\delta^2}.$$

We choose h such that $0<(2F^2(k)h^2+8\Sigma^2(k)h)/\delta^2\leq\mathbf{P}_x(\Omega_\varepsilon)-r$ for some $r>0$ so that, for every $t\geq 0$,

$$\mathbf{P}\left(\sup_{t\leq s\leq t+h}|l(\overline{X}_{\overline{\tau}(s)})-l(\overline{X}_{\overline{\tau}(t)})|\leq\frac{\varepsilon}{2}\right)\geq 1+r-\mathbf{P}_x(\Omega_\varepsilon). \tag{26}$$

From (24), letting $t\to+\infty$, we get

$$V(x)\geq\int_{B(x)}\int_0^{+\infty} l(\overline{X}_s)\,ds\geq\int_{\Omega_\varepsilon}\int_0^{+\infty} l(\overline{X}_s)\,ds\geq\int_{\Omega_\varepsilon}\sum_n\int_{t_n}^{t_n+h} l(\overline{X}_s)\,ds$$

$$\geq \int_{\Omega_\varepsilon} \sum_n h \inf_{[t_n(\omega), t_n(\omega)+h]} l(\overline{X}_t(\omega)) \geq h \sum_n \int_{\Omega_\varepsilon} \inf_{[t_n(\omega), t_n(\omega)+h]} l(\overline{X}_t(\omega))$$

$$\geq h \frac{\varepsilon}{2} \sum_n \mathbf{P}\left[\left(\sup_{t_n \leq s \leq t_n+h} |l(\overline{X}_s) - l(\overline{X}_{t_n})| \leq \frac{\varepsilon}{2}\right) \cap \Omega_\varepsilon\right] \geq h \sum_n \frac{\varepsilon}{2} r = +\infty,$$

where the last inequalities are obtained using the strong Markov property of the process $\overline{X}_{\tau_K(\cdot)}$. This gives a contradiction: then, for every $\varepsilon > 0$, $\mathbf{P}(\Omega_\varepsilon) = 0$. We have proved that $l(\overline{X}_t) \to 0$ as $t \to +\infty$ for almost all $\omega \in B(x)$, now the positive definiteness of l implies that

$$\mathbf{P}_x \left\{\lim_{t\to+\infty} |\overline{X}_t| = 0\right\} = \mathbf{P}_x(B_x) \geq 1 - \frac{V(x)}{\min_{|y|\geq k} V(y)}.$$

We now prove statement (ii). If $\mathcal{O}$ coincides with the whole space, arguing as above, we get that for every $k > 0$ and $x \in B_k$ there exists a strict control α^k such that the corresponding trajectory X^k verifies

$$\mathbf{P}_x \left(\limsup_{t\to+\infty} l(X_t^k) > 0\right) \leq \frac{V(x)}{\min_{|y|\geq k} V(y)}.$$

Using the properness of the function V, by letting $k \to +\infty$, we get that, for every $x \in \mathcal{O}$,

$$\inf_\alpha \mathbf{P}_x \left(\limsup_{t\to+\infty} l(X_t^\alpha) > 0\right) = 0 \tag{27}$$

which gives, by the positive definiteness of the function l, asymptotic stabilizability in the large. □

Remark (Uniform Asymptotic Stabilizability in Probability). The existence of a Lyapunov function implies stronger asymptotic stability of the system, which we call uniform asymptotic stabilizability. Moreover, we will show in a forthcoming paper that uniform asymptotic stabilizability can be completely characterized in terms of strict Lyapunov functions.

The system (CSDE) is *uniformly asymptotically stabilizable in probability* in $\mathcal{O}$ if for every $x \in \mathcal{O}$ there exists $\overline{\alpha} \in \mathcal{A}_x$ such that, for every $k > 0$,

$$\lim_{x\to 0} \mathbf{P}\left(\sup_{t\geq 0} |\overline{X}_t| \geq k\right) = 0,$$

$$\sup_{x\in\mathcal{O}} T_x^{\overline{\alpha}}(\mathcal{O}\backslash B_k) < +\infty,$$

where $T_x^{\overline{\alpha}}(\mathcal{O}\backslash B_k)$ is the expected time spent by the trajectory $\overline{X}$ in the set $\mathcal{O}\backslash B_k$.

The fact that the existence of a Lyapunov function implies uniform asymptotic stabilizability follows very easily from the representation formula for the function V and the positive definiteness of the function l:

$$V(x) \geq \mathbf{E}_x \int_0^{\overline{\tau}} l(\overline{X}_s)\, ds \geq T_x^{\overline{\alpha}}(\mathcal{O} \backslash B_r) \inf_{y \in \mathcal{O} \backslash B_r} l(y),$$

which implies

$$\sup_{x \in \mathcal{O}} T_x^{\overline{\alpha}}(\mathcal{O} \backslash B_r) \leq \frac{\sup_{x \in \mathcal{O}} V(x)}{\inf_{y \in \mathcal{O} \backslash B_r} l(y)} = \frac{\|V\|_\infty}{\inf_{y \in \mathcal{O} \backslash B_r} l(y)}.$$

The proof of the fact that uniform asymptotic stability implies asymptotic stability (in particular that for every initial data there exists a control asymptotically driving the trajectory to the origin almost surely) is an argument based on the continuity properties of the trajectories of (CSDE) of the type (25) we proved in Theorem 14.

6. Attractors

Next we extend the results in Section 4 to study the stabilizability of general closed sets $M \subseteq \mathbb{R}^N$. We denote by $d(x, M)$ the distance between a point $x \in \mathbb{R}^N$ and the set M.

We recall that a closed set M is *viable* with respect to a stochastic controlled dynamical system if for every $x \in M$ there exists an admissible control such that the corresponding trajectory remains almost surely in M.

Definition 15 (Stabilizability in Probability at M). A closed set M is *stabilizable in probability* for (CSDE) if for every $k > 0$ there exists $\delta > 0$ such that, for every x at distance less than δ from M, there exists an admissible control $\overline{\alpha}$ such that the corresponding trajectory $\overline{X}_.$ verifies

$$\mathbf{P}_x \left(\sup_{t \geq 0} d(\overline{X}_t, M) \geq k \right) \leq \varepsilon.$$

Remark. We observe that if M is stabilizable in probability according to the previous definition, then in particular it is viable. In fact, for every $\varepsilon > 0$ fixed, the definition gives that, for $x \in M$, $\inf_\alpha \mathbf{E}_x \int_0^{+\infty} e^{-\lambda t} k_\varepsilon(X_t)\, dt = 0$ for any $\lambda > 0$ and any smooth function k_ε which is nonnegative, bounded and null on the points at distance less than ε from M. By Theorem 3, the infimum is attained, therefore for every $\varepsilon > 0$ there is a control $\overline{\alpha} \in \mathcal{A}_x$ whose corresponding trajectory stays almost surely at distance less than ε from M: in particular, for every $\lambda > 0$, $\inf_\alpha \mathbf{E}_x \int_0^{+\infty} e^{-\lambda t} |X_t|\, dt = 0$. Therefore, again by Theorem 3, there exists, for every $x \in M$, a minimizing control whose corresponding trajectory stays in M almost surely for all $t \geq 0$.

A geometric characterization of the viability of closed sets with respect to a stochastic differential controlled equation has been given in [5] (see also the references therein). According to this characterization, we note that the fact that the set M is stabilizable in probability implies that the diffusion has to degenerate on its boundary: for every

$x \in \partial M$ there exists $\alpha \in A$ such that $\sigma(x, \alpha) \cdot p = 0$ for every p generalized normal vector to M at x.

We introduce the notion of controlled attractiveness: it coincides, when the system is uncontrolled, with the standard notion of pathwise forward attractiveness (see [20]).

Definition 16 (Controlled Attractor). The set M is a *controlled attractor* for the system (CSDE) in the open set $\mathcal{O} \subseteq \mathbb{R}^N$ if for every initial data $x \in \mathcal{O}$ then

$$\inf_{\alpha} \mathbf{P}_x \left(\limsup_{t \to +\infty} d(X_t^\alpha, M) > 0 \right) = 0.$$

This means that for every $\varepsilon > 0$ there exists $\overline{\alpha} \in \mathcal{A}_x$ such that the corresponding trajectory asymptotically approaches the set M with probability at least $1 - \varepsilon$.

The set $\mathcal{O}$ is called the *domain of attraction* for M: if it coincides with $\mathbb{R}^N$ the set M is a *global* attractor.

Remark. We consider a function $V\colon \mathbb{R}^N \to \mathbb{R}$ which satisfies the conditions in Definition 12 of strict global Lyapunov function with the only difference that the function l is assumed only nonnegative definite. The proof of Theorem 14 can be repeated in this case: we obtain that, for every $x \in \mathbb{R}^N$,

$$\inf_{\alpha} \mathbf{P}_x \left(\limsup_{t \to +\infty} l(X_t^\alpha) > 0 \right) = 0. \tag{28}$$

We introduce the set $\mathcal{L} := \{y \mid l(y) = 0\}$. From (28) we get that, for every $x \in \mathbb{R}^N$,

$$\inf_{\alpha} \mathbf{P}_x \left(\limsup_{t \to +\infty} d(X_t^\alpha, \mathcal{L}) > 0 \right) = 0,$$

which means that $\mathcal{L}$ is a controlled global attractor for the system. Results of this kind for uncontrolled diffusion processes can be found in [30] and [13]. The earlier paper of Kushner [27] also studies a stochastic version of the La Salle invariance principle, namely, that the omega limit set of the process is an invariant subset of $\mathcal{L}$, in a suitable sense.

We can generalize the notion of a control Lyapunov function in order to study the attractiveness and the stabilizability of a set M.

Definition 17 (Control M-Lyapunov Function). Let $M \subseteq \mathbb{R}^N$ be a closed set and the $\mathcal{O} \subseteq \mathbb{R}^N$ be an open set containing M. A function $V\colon \mathcal{O} \to [0, +\infty)$ is a *control M-Lyapunov function* for (CSDE) if it satisfies

(i) it is LSC and continuous at every $x \in \partial M$;
(ii) it is *M-positive definite*, i.e., $V(x) > 0$ for $x \notin M$ and $V(x) = 0$ for $x \in M$;
(iii) V is *M-proper*, i.e., its level sets $\{x \mid V(x) \leq \mu\}$ are bounded;

(iv) it is a viscosity supersolution of the equation

$$\max_{\alpha\in A}\left\{-DV(x)\cdot f(x,\alpha)-trace\left[a(x,\alpha)D^2V(x)\right]\right\}\geq l(x)\qquad x\in\mathcal{O}.$$

If $l(x)\geq 0$ then V is a control M-Lyapunov function, and if $l(x)$ is an M-positive definite, Lipschitz continuous bounded function then V is a strict control Lyapunov function.

We can therefore prove very similar results for the case of a set M as for the case of an equilibrium point.

Theorem 18. *If the system* (*CSDE*) *admits a control M-Lyapunov function V then the system is M stabilizable*; *if moreover the function V is a strict control M-Lyapunov function then the set M is a controlled attractor for the system with a domain of attraction equal to $\mathcal{O}$.*

Proof. In order to prove this result, one can repeat the proofs given in Theorems 13 and 14 since the function V satisfies a superoptimality principle and the level sets of V are a local basis of neighborhoods of M. □

7. Examples

In this section we present some very simple examples illustrating the theory.

The first example is about *stochastic perturbations of stabilizable systems*. We apply the Lyapunov theorems to show that an asymptotically controllable deterministic dynamical system continues to be stabilizable or asymptotically stabilizable in probability if we perturb it with a white noise of small enough intensity. The idea to prove it relies on the fact that if the stochastic perturbation is small enough, then a Lyapunov function for the deterministic systems remains a Lyapunov function also for the stochastic one.

Example 1. We consider a deterministic controlled system in $\mathbb{R}^N$:

$$\dot{X}_t = f(X_t,\alpha_t),\tag{29}$$

where $f(x,a)$ is a Lipschitz continuous, locally bounded function in x uniformly with respect to a and the control α is a measurable function taking values in a compact space A. We assume that the system is globally asymptotically (stochastic open loop) stabilizable at the origin, i.e., asymptotically controllable in the terminology of deterministic systems [35], [36]. By the converse Lyapunov theorem [34], [36], there exists a continuous control Lyapunov function for the system, i.e., for some positive definite continuous function L, there exists a proper, positive definite function V satisfying, in $\mathbb{R}^N$,

$$\max_{\alpha\in A}\{-f(x,\alpha)\cdot DV\}\geq L(x)\tag{30}$$

in the viscosity sense. Moreover, we can choose the function V to be semiconcave away from the origin as proved by Rifford in [31]. This means that for every $\delta>0$ there exists a semiconcavity constant $C_\delta > 0$ such that the function

$$V(x) - \frac{C_\delta}{2}|x|^2$$

is concave in $\mathbb{R}^N \backslash B_\delta$. The semiconcavity constant C_δ is an upper bound on the second derivatives of the function (to be intended in the sense of distributions). In particular, by the definition of semiconcavity, we get that for every $|x| > \delta$ and for every $(p, X) \in \mathcal{J}^{2,-}V(x) := \{(q, Y) \in \mathbb{R}^N \times S(N)$: for $y \to x$, $V(y) \geq V(x) + q \cdot (y - x) + \frac{1}{2}(y - x) \cdot Y(y - x) + o(|y - x|^2)\}$ (see [12]), then

$$C_\delta \mathbf{I}_N - X \geq 0. \tag{31}$$

We study under which conditions the system continues to be asymptotically or Lyapunov stabilizable if we perturb it with a white noise. We consider the perturbed system

$$dX_t = f(X_t, \alpha_t)\,dt + \sigma(X_t, \alpha_t)\,dB_t,$$

where $(B_t)_t$ is an M-dimensional white noise and the function $\sigma(x, a)$ is Lipschitz continuous in x uniformly with respect to a and takes values in the space of the $N \times M$ dimensional matrices with entries in $\mathbb{R}$.

By the semiconcavity inequality (31) and by (30), we get, for every $|x| > \delta$, and for every $(p, X) \in \mathcal{J}^{2,-}V(x)$,

$$\begin{aligned}
&\max_{\alpha \in A}\{-f(x, \alpha) \cdot p - \mathit{trace}\; a(x, \alpha)X\} \\
&\quad \geq \max_{\alpha \in A}\{-f(x, \alpha) \cdot p\} - \max_{\alpha \in A}\{\mathit{trace}\; a(x, \alpha)X\} \\
&\quad \geq L(x) - C_\delta \max_{\alpha \in A}\{\mathit{trace}\; a(x, \alpha)\}.
\end{aligned}$$

Therefore, if the diffusion σ satisfies a small intensity condition

$$\mathit{trace}\; a(x, \alpha) \leq \frac{L(x)}{C_\delta}, \qquad \forall \alpha \in A, \quad \forall\, |x| > \delta,$$

we can conclude that the function V is a control Lyapunov function for the stochastic system and then, according to Theorem 13, the system is both Lyapunov and Lagrange stabilizable in probability.

If, moreover, for every $\delta > 0$,

$$\mathit{trace}\; a(x, \alpha) < \frac{L(x)}{C_\delta}, \qquad \forall \alpha \in A, \quad \forall\, |x| > \delta,$$

it is possible to construct a positive definite, Lipschitz continuous function l such that V is a viscosity supersolution of

$$\max_{a \in A}\left\{-f(x, a) \cdot DV(x) - \mathit{trace}\; a(x, \alpha) D^2 V(x)\right\} \geq l(x)$$

and then, by Theorem 14, the system is asymptotically stabilizable in the large at equilibrium.

A similar result can be obtained in the case of local asymptotically controllable systems.

In the next example we give conditions on a radial function to be a Lyapunov function for stability in probability.

Example 2. In this example we consider as candidate Lyapunov function for the general controlled system (CSDE), the function $V(x) = |x|^\gamma$ for some positive $\gamma > 0$ and study under which conditions the system is stabilizable.

We compute

$$DV(x) = \gamma |x|^{\gamma-2} x, \qquad D^2 V(x) = \gamma |x|^{\gamma-2}\mathbf{I} + \gamma(\gamma-2)|x|^{\gamma-4}(x \cdot x^T).$$

Therefore

$$\begin{aligned}
&\max_{a\in A}\left\{-f(x,a)\cdot DV(x) - \mathit{trace}\; a(x,\alpha) D^2 V(x)\right\}\\
&\quad = \gamma|x|^{\gamma-2}\max_{a\in A}\{-f(x,a)\cdot x\\
&\qquad\qquad - \mathit{trace}\; a(x,\alpha) - (\gamma-2)|x|^{-2}\,\mathit{trace}\, a(x,\alpha)(x\cdot x^T))\big\}\\
&\quad = \gamma|x|^{\gamma-2}\max_{a\in A}\{-f(x,a)\cdot x - \mathit{trace}\, a(x,\alpha) - (\gamma-2)|x|^{-2}|\sigma(x,a)^T\cdot x|^2\}.
\end{aligned}$$

If $\gamma \le 2$, this gives that V is a Lyapunov function for the system if for every x there exists $\alpha \in A$ such that $f(x,\alpha)\cdot x + \mathit{trace}\; a(x,\alpha) \le 0$. We can observe that, since $\mathit{trace}\; a(x,\alpha) \ge 0$ for every α, the radial component of the drift f must be everywhere nonpositive, for some $\alpha \in A$. In particular, it must be negative to compensate for the destabilizing role of the diffusion, whenever $\mathit{trace}\; a(x,\alpha)$ is nonnull.

We end with an example of an uncontrolled system that does not have a smooth Lyapunov function but has an LSC Lyapunov function. Our example is a variant of a deterministic one by Krasovskii, see [2].

Example 3. Consider the stochastic system in $\mathbb{R}^3$:

$$(S) \qquad \begin{cases} dX_t = (1+f(Z_t))\,Y_t\,dt,\\ dY_t = [-(1+f(Z_t))X_t + Y_t(X_t^2+Y_t^2)^3\sin^2(\pi/(X_t^2+Y_t^2))]\,dt,\\ dZ_t = bZ_t\,dt + \sigma Z_t dB_t,\end{cases}$$

where B_t is a one-dimensional Brownian motion, b and σ are nonzero constants such that $\sigma^2 - 2b > 0$ and f satisfies $f(0) = 0$ and the hypotheses for the existence and uniqueness of solutions to (S). The system is stable in probability around the equilibrium point $(0,0,0)$. We observe that if we choose as initial data $(x_0, y_0, 0)$ the trajectory is a.s. $(X_t, Y_t, 0)$ for $t \geq 0$ where (X_t, Y_t) satisfies

$$(RS) \quad \begin{cases} \dot{X}_t = Y_t, \\ \dot{Y}_t = -X_t + Y_t(X_t^2 + Y_t^2)^3 \sin^2(\pi/(X_t^2 + Y_t^2)). \end{cases}$$

Then if $V(\cdot, \cdot, \cdot)$ is a Lyapunov function for the system (S) then $V(\cdot, \cdot, 0)$ has to be a Lyapunov function for (RS). The system (RS) does not admit a continuous Lyapunov function: the circles $\{x^2 + y^2 = 1/n\}$ are limit cycles and between two circles the trajectories go from the internal towards the external circle (see [2]). Every Lyapunov function for (RS) is then LSC and coincides with

$$W_c(x, y) := c_n \qquad \text{for} \quad \frac{1}{n} < x^2 + y^2 \leq \frac{1}{n-1}$$

for some sequence c of positive numbers c_n decreasing to 0 as $n \to +\infty$.

We now construct a Lyapunov function for the system (S). We choose k such that $0 < k < 1 - (2b/\sigma^2)$ and fix a sequence c as above: it is easy to verify that the function $V(x, y, z) = W_c(x, y) + z^k$ is a local Lyapunov function for the system (S) on $\{(x, y, z)\colon\ x^2 + y^2 < 1\ z \in (-1, 1)\}$ according to Definition 10.

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