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# UNIQUENESS RESULTS FOR SECOND-ORDER BELLMAN–ISAACS EQUATIONS UNDER QUADRATIC GROWTH ASSUMPTIONS AND APPLICATIONS*

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**Abstract.** In this paper, we prove a comparison result between semicontinuous viscosity sub- and supersolutions growing at most quadratically of second-order degenerate parabolic Hamilton–Jacobi–Bellman and Isaacs equations. As an application, we characterize the value function of a finite horizon stochastic control problem with unbounded controls as the unique viscosity solution of the corresponding dynamic programming equation.

**Key words.** degenerate parabolic equations, nonlinear Hamilton–Jacobi equations, nonlinear Isaacs equations, viscosity solutions, unbounded solutions, maximum principle, linear quadratic problems.

**AMS subject classifications.** 35K65, 49L99, 49L25, 35B50, 49N10

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## 1. Introduction.

In this paper, we are interested in the second-order equation

$$\begin{cases} \dfrac{\partial w}{\partial t} + H(x,t,Dw,D^2w) + G(x,t,Dw,D^2w) = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ w(x,0) = \psi(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 1$, $T > 0$, the unknown $w$ is a real-valued function defined in $\mathbb{R}^N \times [0,T]$, $Dw$ and $D^2w$ denote, respectively, its gradient and Hessian matrix, and $\psi$ is a given initial condition. The Hamiltonians $H, G : \mathbb{R}^N \times [0,T] \times \mathbb{R}^N \times \mathcal{S}_N(\mathbb{R}) \to \mathbb{R}$ are continuous in all their variables and have the form

$$H(x,t,p,X) = \inf_{\alpha \in A} \left\{ \langle b(x,t,\alpha), p \rangle + \ell(x,t,\alpha) - \mathrm{Tr}\left[\sigma(x,t,\alpha)\sigma^T(x,t,\alpha)X\right] \right\} \tag{1.2}$$

and

$$G(x,t,p,X) = \sup_{\beta \in B} \left\{ -\langle g(x,t,\beta), p \rangle - f(x,t,\beta) - \mathrm{Tr}\left[c(x,t,\beta)c^T(x,t,\beta)X\right] \right\}. \tag{1.3}$$

This kind of equation is of particular interest for applications since it relies on differential game theory (Isaacs equations) or on deterministic and stochastic control problems when either $H \equiv 0$ or $G \equiv 0$ (Hamilton–Jacobi–Bellman equations).

Notations and precise assumptions are given in section 2 but we point out that we allow one of the control set $A$ or $B$ to be unbounded and the solutions to (1.1) may have quadratic growth. Our model case is the well-known stochastic linear quadratic

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problem. We refer to Bensoussan [10], Fleming and Rishel [18], Fleming and Soner [19], Øksendal [38], Yong and Zhou [41], and the references therein for an overview and to Examples 2.2 and 3.1 below. This problem can be described as follows. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space and let $(W_t)_t$ be an $\mathcal{F}_t$-adapted standard $M$-Brownian motion. The control set is $A = \mathbb{R}^k$ for some $k > 0$ and we consider the linear stochastic differential equation

$$\begin{cases} dX_s = [A(s)X_s + B(s)\alpha_s]ds + [C(s)X_s + D(s)]dW_s & \text{for } t \leq s \leq T, \\ X_t = x, \end{cases}$$

where $\alpha_s \in \mathcal{A}_t$, the set of $A$-valued $\mathcal{F}_t$-progressively measurable controls and the adapted process $X_s$ is the solution. The linear quadratic problem consists in minimizing the quadratic cost

$$V(x,t) = \inf_{\alpha_s \in \mathcal{A}_t} \mathrm{E}\left\{ \int_t^T [\langle X_s, Q(s)X_s\rangle + R|\alpha_s|^2]\, ds + \langle X_T, SX_T\rangle \right\}, \tag{1.4}$$

where $A(\cdot), B(\cdot), C(\cdot), D(\cdot), Q(\cdot)$, and $S$ are deterministic matrix-valued functions of suitable size and $R > 0$ to simplify the presentation. The Hamilton–Jacobi equation associated to this problem is

(1.5)
$$\begin{cases} -\dfrac{\partial w}{\partial t} - \langle A(t)x, Dw\rangle - \langle x, Q(t)x\rangle + \dfrac{1}{4R}|B(t)^T Dw|^2 - \mathrm{Tr}\left[a(x,t)D^2 w\right] = 0, \\ w(x,T) = \langle x, Sx\rangle, \end{cases}$$

where $a(x,t) = (C(t)x + D(t))(C(t)x + D(t))^T/2$. Note that this equation is of type (1.1) (with $G \equiv 0$) since

$$\frac{1}{4R}|B(t)^T Dw|^2 = \sup_{\alpha\in\mathbb{R}^k}\Big\{ -\langle B(t)\alpha, Dw\rangle - R|\alpha|^2 \Big\}. \tag{1.6}$$

In this paper, we are concerned with two issues about this problem.

The first question relies on the partial differential equation (1.5). We note that the quadratic cost with unbounded controls leads to a quadratic term with respect to the gradient variable. From the terminal condition, we expect the solutions have quadratic growth. Moreover, the diffusion matrix may be degenerate. Therefore, we cannot hope to obtain smooth solutions in general. We need to consider weak solutions, namely, viscosity solutions. (We refer the reader who is not familiar with this notion of solutions to Crandall, Ishii and Lions [15], Fleming and Soner [19], Bardi and Capuzzo Dolcetta [3], and Barles [6], and all the references therein). We obtain the existence of a unique continuous viscosity solution for (1.5) and for a large class of equations of type (1.1) (see Theorem 2.1 and Corollary 2.1).

We point out that the results obtained in this paper are beyond the classical comparison results for viscosity solutions (see, e.g., [15]) because of the growth of both the solutions and the Hamiltonians. In fact, most of the comparison results in the literature require that either the solutions are uniformly continuous or the Hamiltonian is uniformly continuous with respect to the gradient uniformly in the $x$ variable. (In our case this amounts to assuming that both controls sets are compact.) Let us mention that uniqueness and existence problems for a class of first-order Hamiltonians corresponding to unbounded control sets and under assumptions including deterministic linear quadratic problems have been addressed by several authors; see, e.g., the book by Bensoussan [10], the papers of Alvarez [2], Bardi and Da Lio [4], Cannarsa and Da Prato [12], and Rampazzo and Sartori [40] in the case of convex operators, and the papers of Da Lio and McEneaney [16] and Ishii [22] for more general operators. As for second-order Hamiltonians under quadratic growth assumptions, Ito [23] obtained the existence of locally Lipschitz solutions to particular equations of the form (1.1) under more regularity conditions on the data, by establishing a priori estimates on the solutions, whereas Crandall and Lions in [14] proved a uniqueness result for very particular operators depending only on the Hessian matrix of the solution. Kobylanski [26] studied equations with the same kind of quadratic nonlinearity in the gradient as ours, but her existence and uniqueness results hold in the class of bounded viscosity solutions. Finally, one can find existence and uniqueness results for viscosity solutions which may have a quadratic growth in [7] for quasilinear degenerate parabolic equations.

The second question we deal with in this paper concerns the link between the control problem (1.4) and equation (1.5). The rigorous connection between the Hamilton–Jacobi–Bellman and optimal control is usually performed by means of a principle of optimality. For deterministic control problems which lead to a first-order Hamilton–Jacobi equation, see Bardi and Capuzzo Dolcetta [3] and Barles [6]; for the connections between stochastic control problems and second-order Hamilton–Jacobi–Bellmann equations see Fleming and Rishel [18], Krylov [27], Lions [31, 32, 33], Fleming and Soner [19], Yong and Zhou [41, Theorem 3.3], and the references therein.

However, for stochastic differential equations with unbounded controls as in stochastic linear quadratic problems, additional difficulties arise. Some results in this direction were obtained for infinite horizon problems, in the deterministic case by Barles [5] and in the stochastic case by Alvarez [1]. In this paper, we characterize the value function (1.4) as the unique solution of (1.5). Actually, our results apply for a larger class of unbounded stochastic control problems

$$V(x,t) = \inf_{\alpha_s \in \mathcal{A}_t} \mathrm{E}\left\{ \int_t^T \ell(X_s, s, \alpha_s)\, ds + \psi(X_T) \right\}, \tag{1.7}$$

where the process $X_s$ is governed by

$$\begin{cases} dX_s = b(X_s, s, \alpha_s)ds + \sigma(X_s, s, \alpha_s)dW_s, \\ X_t = x, \end{cases} \tag{1.8}$$

where $A$ is a possibly unbounded subset of a normed linear space and all the data are continuous with the following restricted growths: $b$ grows at most linearly with respect to both the control and the state, $\sigma$ grows at most linearly with respect to the state and is bounded in the control variable, $\psi$ can have a quadratic growth, and $\ell$ grows at most quadratically with respect to both the control and the state with a coercitivity assumption

$$\ell(x, \alpha, t) \geq \frac{\nu}{2}|\alpha|^2 - C(1 + |x|^2). \tag{1.9}$$

In this case, the Hamilton–Jacobi equation looks like (1.1) with $G \equiv 0$. (See section 3 for details.) Because of the unbounded framework, the use of an optimality principle to establish the connection between the control problem and the equation is more delicate than usual. Thus we follow another strategy which consists in comparing directly the value function with the unique solution of the Hamilton–Jacobi equation as long as this latter exists (see Theorem 3.1).

It is worth noticing that, surprisingly, for the general stochastic linear problem, it is not even clear how to give sense to the partial differential equation associated to the stochastic control problem! For instance, consider again the above linear quadratic problem where $\sigma(x, t, \alpha) = C(t)x + D(t)$ is replaced by $C(t)x + D(t)\alpha$ which depends now both on the state and the control. Taking $A, C, Q \equiv 0$ and $B, R, D \equiv 1$ to simplify, the Hamiltonian in (1.5) becomes

$$\sup_{\alpha \in \mathbb{R}^k} \left\{ -\langle \alpha, Dw \rangle - |\alpha|^2 - \frac{|\alpha|^2}{2} \Delta w \right\}, \tag{1.10}$$

which is $+\infty$ as soon as $\Delta w \leq -2$. The connection between the control problem and the equation in this case was already investigated (see Yong and Zhou [41] and the references therein). The results need a priori knowledges about the value function. (The value function and its derivatives are supposed to remain in the domain of the Hamiltonian.) We do not consider the case when $\sigma$ is unbounded in the control variable in this paper; this is the aim of a future work. When finishing this paper, we learned that Krylov [28] succeeded in treating the general stochastic linear regulator. But his assumptions are designed to solve this latter problem (the data are supposed to be polynomials of degree 1 or degree 2 in $(x, \alpha)$) and the proofs rely heavily on the particular form of the data.

Another important example of equations of type (1.1) where concave and convex Hamiltonians appear is the first-order equation

$$\frac{\partial w}{\partial t} + \min_{\alpha \in \mathbb{R}^k} \left\{ \frac{\gamma^2}{2} |\alpha|^2 - \langle \sigma(x)\alpha, Dw \rangle \right\} + \max_{\beta \in B} \{ -\langle g(x, \beta), Dw \rangle - f(x, \beta) \} = 0 \tag{1.11}$$

in $\mathbb{R}^N \times (0, T)$. This kind of equation is related to the so-called $H_\infty$-Robust control problem. This problem can be seen as a deterministic differential game. See Example 3.3 and McEneaney [34, 35, 36], Nagai [37], and the references therein for details.

Finally, we point out that one of the main fields of application of these types of equations and problems is mathematical finance; see, e.g., Lamberton and Lapeyre [29], Fleming and Soner [19], Øksendal [38], and the references therein for an introduction. For recent papers which deal with equations we are interested in, see Pham [39] and Benth and Karlsen [11] (see Example 3.2).

Let us now describe how the paper is organized.

Section 2 is devoted to the study of (1.1). More precisely, we prove a uniqueness result for (1.1) in the set of continuous functions growing at most quadratically in the state variable under the assumption that either $A$ or $B$ is an unbounded control set, the functions $b, g$ and $\ell, f$ grow, respectively, at most linearly and quadratically with respect to both the control and the state. Instead the functions $\sigma, c$ are assumed to grow at most linearly with respect to the state and bounded in the control variable.

One of the main tools within the theory of viscosity solutions to obtain a uniqueness result is to show a comparison result between viscosity upper semicontinuous subsolutions and lower semicontinuous supersolutions to (1.1); see Theorem 2.1. Indeed, the existence and the uniqueness (Corollary 2.1) follow as a by-product of the comparison result and Perron's method of Ishii [21]. However, under our general assumptions one cannot expect the existence of a solution for all times, as Example 2.2 shows.

The method we use in proving the comparison Theorem 2.1 is similar in spirit to the one applied by Ishii in [22] in the case of first-order Hamilton–Jacobi equations and it is based on a kind of linearization procedure of the equation. Roughly speaking, it consists of three main steps: (1) one computes the equations satisfied by $\Psi_\mu = U - \mu V$ ($U, V$ being, respectively, the sub- and supersolution of the original pde and $0 < \mu < 1$ a parameter); (2) for all $R > 0$ one constructs a strict supersolution $\chi_R^\mu$ of the linearized equation such that $\chi_R^\mu(x,t) \to 0$ as $R \to \infty$; (3) one shows that $\Psi_\mu(x,t) \le \chi_R^\mu(x,t)$ and then one concludes by letting first $R \to \infty$ and then $\mu \to 1$.

In section 3, we give applications to finite horizon stochastic control problems previously mentioned and we provide some examples.

In section 4, we deal with particular cases where both controls are unbounded but $H$ (or $G$) is "predominant" in $H + G$ (see Remark 2.2 and Theorem 4.1). For instance, we are able to deal with equations of the form

$$\frac{\partial w}{\partial t} - \frac{|\Sigma_1(x)Dw|^2}{2} + \frac{|\Sigma_2(x)Dw|^2}{2} = 0 \text{ in } \mathbb{R}^N \times (0,T),$$

where $\Sigma_1$, $\Sigma_2$ are $N \times k$ matrices, which corresponds to the case $\alpha, \beta \in \mathbb{R}^k$, $\sigma \equiv c \equiv 0$, $b(x,t,\alpha) = \Sigma_1(x)\alpha$, $g(x,t,\beta) = \Sigma_2(x)\beta$, and $\ell(x,t,\alpha) = |\alpha|^2/2$, $f(x,t,\beta) = |\beta|^2/2$ in (1.2) and (1.3). The comparison result applies if either $(\Sigma_1\Sigma_1^T)(x) > (\Sigma_2\Sigma_2^T)(x)$ or $(\Sigma_1\Sigma_1^T)(x) < (\Sigma_2\Sigma_2^T)(x)$.

When neither $H$ nor $G$ is predominant, the problem seems to be very difficult and our only result takes place in dimension $N = 1$: we have comparison for

$$\frac{\partial w}{\partial t} + h(x,t)|Dw|^2 = 0 \quad \text{in } \mathbb{R}^N \times (0,T), \tag{1.12}$$

where the function $h$ may change sign (see (A5) for details). Finally, we point out that assumptions and proofs in section 4 essentially differ from those of Theorem 2.1 (see Remark 4.1).

**2. Comparison result for the Hamilton–Jacobi equation (1.1).** In order to give precise assumptions on (1.1) and (1.2), (1.3), we need to introduce some notation. For all integers $N, M \ge 1$ we denote by $\mathcal{M}_{N,M}(\mathbb{R})$ (respectively, $\mathcal{S}_N(\mathbb{R})$, $\mathcal{S}_N^+(\mathbb{R})$) the set of real $N \times M$ matrices (respectively, real symmetric matrices, real symmetric nonnegative $N \times N$ matrices). All the norms which appear in the sequel are denoted by $|\cdot|$. The standard Euclidean inner product in $\mathbb{R}^N$ is written $\langle \cdot, \cdot \rangle$. We recall that a modulus of continuity $m : \mathbb{R} \to \mathbb{R}^+$ is a nondecreasing continuous function such that $m(0) = 0$. Finally, $B(x,r) = \{y \in \mathbb{R}^N : |x - y| < r\}$ is the open ball of center $x$ and radius $r > 0$.

We list the basic assumptions on $H$, $G$ and $\psi$. We assume that there exist positive constants $\bar{C}$ and $\nu$ such that

**(A1)** (assumptions on $H$ given by (1.2)):

(i) $A$ is a subset of a separable complete normed space. The main point here is the possible unboundedness of $A$. Therefore, to emphasize this property in what follows, we take $A = \mathbb{R}^k$ for some $k \ge 1$ (see Remark 2.1 above);

(ii) $b \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R}^k; \mathbb{R}^N)$ satisfying for $x, y \in \mathbb{R}^N, t \in [0,T], \alpha \in \mathbb{R}^k$,

$$\begin{aligned} |b(x,t,\alpha) - b(y,t,\alpha)| &\le \bar{C}(1 + |\alpha|)|x - y|, \\ |b(x,t,\alpha)| &\le \bar{C}(1 + |x| + |\alpha|); \end{aligned}$$

(iii) $\ell \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R}^k; \mathbb{R})$ satisfying for $x \in \mathbb{R}^N, t \in [0,T], \alpha \in \mathbb{R}^k$,

$$\bar{C}(1+|x|^2+|\alpha|^2) \geq \ell(x,t,\alpha) \geq \frac{\nu}{2}|\alpha|^2+\ell_0(x,t,\alpha) \text{ with } \ell_0(x,t,\alpha) \geq -\bar{C}(1+|x|^2),$$

and for every $R > 0$, there exists a modulus of continuity $m_R$ such that for all $x, y \in B(0,R), t \in [0,T], \alpha \in \mathbb{R}^k$,

$$|\ell(x,t,\alpha) - \ell(y,t,\alpha)| \leq (1+|\alpha|^2)\, m_R(|x-y|); \tag{2.1}$$

(iv) $\sigma \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R}^k; \mathcal{M}_{N,M}(\mathbb{R}))$ is locally Lipschitz continuous with respect to $x$ uniformly for $(t,\alpha) \in [0,T] \times \mathbb{R}^k$ and satisfies for every $x \in \mathbb{R}^N,\ t \in [0,T],\ \alpha \in \mathbb{R}^k$,

$$|\sigma(x,t,\alpha)| \leq \bar{C}(1+|x|).$$

**(A2)** (assumptions on $G$ given by (1.3)):

(i) $B$ is a bounded subset of a normed space;

(ii) $g \in C(\mathbb{R}^N \times [0,T] \times B; \mathbb{R}^N)$ is locally Lipschitz continuous with respect to $x$ uniformly for $(t,\beta) \in [0,T] \times B$ and satisfies for every $x \in \mathbb{R}^N,\ t \in [0,T],\ \beta \in B$,

$$|g(x,t,\beta)| \leq \bar{C}(1+|x|);$$

(iii) $f \in C(\mathbb{R}^N \times [0,T] \times B; \mathbb{R})$ is locally uniformly continuous with respect to $x$ uniformly in $(t,\beta) \in [0,T] \times B$ and satisfies for every $x \in \mathbb{R}^N,\ t \in [0,T],\ \beta \in B$,

$$|f(x,t,\beta)| \leq \bar{C}(1+|x|^2);$$

(iv) $c \in C(\mathbb{R}^N \times [0,T] \times B; \mathcal{M}_{N,M}(\mathbb{R}))$ is locally Lipschitz continuous with respect to $x$ uniformly for $(t,\beta) \in [0,T] \times B$ and satisfies for every $x \in \mathbb{R}^N,\ t \in [0,T],\ \beta \in B$,

$$|c(x,t,\beta)| \leq \bar{C}(1+|x|).$$

**(A3)** (assumptions on the initial condition $\psi$):
$\psi \in C(\mathbb{R}^N; \mathbb{R})$ and

$$|\psi(x)| \leq \bar{C}(1+|x|^2) \quad \text{for every } x \in \mathbb{R}^N.$$

*Remark* 2.1. (i) Concerning (A1)(i), we choose to take $A = \mathbb{R}^k$ in this section to emphasize the possible unboundedness of $A$ in the notation. Indeed, the calculations when $A$ is any subset of a complete separable normed space are the same and are based on the following inequality: for every $\rho > 0, \gamma \in \mathbb{R}$,

$$\inf_{\alpha \in A} \left\{\rho|\alpha|^2 + \gamma|\alpha|\right\} = \inf_{\alpha \in A} \left\{ \left( \sqrt{\rho}|\alpha|^2 + \frac{\gamma}{2\sqrt{\rho}} \right)^2 - \frac{\gamma^2}{4\rho} \right\} \geq -\frac{\gamma^2}{4\rho}. \tag{2.2}$$

(ii) Note that with respect to the gradient variable, $H$ is a concave function and $G$ is a convex function. Under Assumptions (A1) and (A2), classical computations show that $H$ and $G$ are continuous in all their variables.

*Example* 2.1. The typical case we have in mind is when $H$ is quadratic in the gradient variable, for instance, $A = \mathbb{R}^k$, $\ell(x,t,\alpha) = |\alpha|^2/2$, $\sigma \equiv 0$, and $b(x,t,\alpha) =$

$a(x)\alpha$, where $a \in C(\mathbb{R}^N; \mathcal{M}_{N,k}(\mathbb{R}))$ is locally Lipschitz continuous and bounded for all $x \in \mathbb{R}^N$. It leads to

$$H(x,p) = \inf_{\alpha\in\mathbb{R}^k} \left\{ \langle a(x)\alpha, p\rangle + \frac{|\alpha|^2}{2} \right\} = -\frac{|a(x)^T p|^2}{2}. \tag{2.3}$$

This particular example is treated both in Ishii [22] and in Da Lio and McEneaney [16] in the case of first-order Hamilton–Jacobi–Bellman equations under more restrictive assumptions than ours. In particular, $a$ has to be a nonsingular matrix in [22]. See also section 4 for some further comments.

For any $O \subseteq \mathbb{R}^K$, we denote by $USC(O)$ the set of upper semicontinuous functions in $O$ and by $LSC(O)$ the set of lower semicontinuous functions in $O$.

The main result of this section is the following theorem.

THEOREM 2.1. *Assume* (A1)–(A3). *Let $U \in USC(\mathbb{R}^N \times [0,T])$ be a viscosity subsolution of* (1.1) *and $V \in LSC(\mathbb{R}^N \times [0,T])$ be a viscosity supersolution of* (1.1). *Suppose that $U$ and $V$ have quadratic growth, i.e., there exists $\hat{C} > 0$ such that for all $x \in \mathbb{R}^N$, $t \in [0,T]$,*

$$|U(x,t)|, \, |V(x,t)| \le \hat{C}(1+|x|^2). \tag{2.4}$$

*Then $U \le V$ in $\mathbb{R}^N \times [0,T]$.*

The question of the existence of a continuous solution to (1.1) is not completely obvious. In the framework of viscosity solutions, existence is usually obtained as a consequence of the comparison principle by means of Perron's method as soon as we can build a sub- and a supersolution to the problem. Here, the comparison principle is proved in the class of functions satisfying the quadratic growth condition (2.4). Therefore, to perform the above program of existence, we need to be able to build quadratic sub- and supersolutions to (1.1). In general one can expect to build such sub- and supersolutions only for a short time. (See the following lemma and Corollary 2.1.) In Example 2.2, we see that solutions may not exist for all time.

LEMMA 2.1. *Assume* (A1)–(A3). *If $K \ge \bar{C}+1$ and $\rho$ are large enough, then $\underline{u}(x,t) = -K\mathrm{e}^{\rho t}(1+|x|^2)$ is a viscosity subsolution of* (1.1) *in $\mathbb{R}^N \times [0,T]$ and there exists $0 < \tau \le T$ such that $\overline{u}(x,t) = K\mathrm{e}^{\rho t}(1+|x|^2)$ is a viscosity supersolution of* (1.1) *in $\mathbb{R}^N \times [0,\tau]$.*

*Proof of Lemma* 2.1. We only verify that $\overline{u}$ is a supersolution (the proof that $\underline{u}$ is a subsolution being similar and simpler). Since $K \ge \bar{C}+1$, we have $\overline{u}(x,0) = K(1+|x|^2) \ge \psi(x)$. Moreover, since $\overline{u}$ is smooth and using (A1), (A2), and (2.2), we have

$$\begin{aligned}
&\frac{\partial \overline{u}}{\partial t} + H(x,t,D\overline{u},D^2\overline{u}) + G(x,t,D\overline{u},D^2\overline{u}) \\
&= K\rho\mathrm{e}^{\rho t}(1+|x|^2) + H(x,t,2K\mathrm{e}^{\rho t}x, 2K\mathrm{e}^{\rho t}Id) + G(x,t,2K\mathrm{e}^{\rho t}x, 2K\mathrm{e}^{\rho t}Id) \\
&\ge K\rho\mathrm{e}^{\rho t}(1+|x|^2) + \inf_{\alpha\in\mathbb{R}^k} \Big\{ -\bar{C}(1+|x|+|\alpha|)\} 2K\mathrm{e}^{\rho t}|x| + \frac{\nu}{2}|\alpha|^2 \\
&\qquad\qquad - \bar{C}(1+|x|^2) - \bar{C}^2(1+|x|)^2 2K\mathrm{e}^{\rho t} \Big\} \\
&\qquad + \sup_{\beta\in B}\{-\bar{C}(1+|x|)2K\mathrm{e}^{\rho t}|x| - \bar{C}(1+|x|^2) - \bar{C}^2(1+|x|)^2 2K\mathrm{e}^{\rho t}\} \\
&\ge K\rho\mathrm{e}^{\rho t}(1+|x|^2) - K(10\bar{C}+12\bar{C}^2)\mathrm{e}^{\rho t}(1+|x|^2) + \inf_{\alpha\in\mathbb{R}^k}\Big\{-2K\bar{C}\mathrm{e}^{\rho t}|x||\alpha| + \frac{\nu}{2}|\alpha|^2\Big\} \\
&\ge \left[\rho - 10\bar{C} - 12\bar{C}^2 - \frac{2\bar{C}^2K\mathrm{e}^{\rho t}}{\nu}\right] K\mathrm{e}^{\rho t}(1+|x|^2).
\end{aligned}$$

We notice that if $t\rho \le 1$ and $\rho > 0$ is large enough, then the quantity between the brackets is nonnegative. Hence the result follows with $0 < \tau = 1/\rho$. □

As explained above, Theorem 2.1 together with Perron's method implies the following result. We omit its proof since it is standard.

COROLLARY 2.1. *Assume* (A1)–(A3). *Then there is* $\tau > 0$ *such that there exists a unique continuous viscosity solution of* (1.1) *in* $\mathbb{R}^N \times [0,\tau]$ *satisfying the growth condition* (2.4).

*Remark* 2.2. (i) For global existence results under further regularity assumptions on the data see [23]. For a case where blowup in finite time occurs, see Example 2.2.

(ii) Theorem 2.1 and Corollary 2.1 hold when replacing "inf" by "sup" and "$\ell$" by "$-\ell$" in $H$ or/and "sup" by "inf" and "$f$" by "$-f$" in $G$. To adapt the proofs, one can use the change of function $w' := -w$ or to consider $\mu U - V$ instead of $U - \mu V$ in the proof of Theorem 2.1. Therefore it is possible to deal either with unbounded controls in the sup in order to have a convex quadratic Hamiltonian or to deal with unbounded control in the inf in order to have a concave quadratic Hamiltonian.

(iii) Up to replace $t$ by $T-t$, all our results hold for

$$\begin{cases} -\dfrac{\partial w}{\partial t} + H(x,t,Dw,D^2w) + G(x,t,Dw,D^2w) = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ w(x,T) = \psi(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{2.5}$$

This latter equation with terminal condition is the one which arises usually in control theory; see Example 2.2 and section 3.

(iv) In this section, we are not able to consider the case when both $\alpha$ in $H$ and $\beta$ in $G$ are unbounded controls. Roughly speaking, a reason is that unbounded controls lead to quadratic Hamiltonians. When both controls are unbounded, we then obtain two quadratic-type Hamiltonians, a concave and a convex one. Let us explain the difficulty on a model case where

$$\begin{aligned} H(x,p) &= \inf_{\alpha \in \mathbb{R}^k} \left\{ \langle a_1(x)\alpha, p\rangle + \frac{|\alpha|^2}{2} \right\} = -\frac{|a_1(x)^T p|^2}{2}, \\ G(x,p) &= \sup_{\beta \in \mathbb{R}^k} \left\{ \langle a_2(x)\beta, p\rangle - \frac{|\beta|^2}{2} \right\} = \frac{|a_2(x)^T p|^2}{2}, \end{aligned}$$

where $a_1, a_2 \in C(\mathbb{R}^N; \mathcal{M}_{N,k}(\mathbb{R}))$ are locally Lipschitz continuous and bounded. The difficulty to treat such a case is related to our strategy of proof which relies on a kind of linearization procedure (see Lemma 2.2 and its proof). In this simple case, this linearization uses in a crucial way the convex inequality

$$\frac{|p|^2}{\mu} - |q|^2 \ge -\frac{|p-q|^2}{1-\mu} \quad \text{for all } p, q \in \mathbb{R}^N \text{ and } 0 < \mu < 1,$$

which does not work at the same time for a concave and a convex Hamiltonian. Of course, in this simple case, there are alternative ways to solve the problem: we have

$$H(x,p) + G(x,p) = \frac{1}{2}\langle (a_1 + a_2)(a_2 - a_1)^T p, p\rangle, \tag{2.6}$$

then we can apply Theorem 2.1 to $H$ given by 2.6 to add some assumptions on $a_1$ or $a_2$ (for instance, $(a_1 + a_2)(a_2 - a_1)^T$ is a nonnegative symmetric matrix with a locally Lipschitz squareroot). In section 4, we provide another approach to solving such equations (see Theorem 4.1 and Remark 4.1).

*Example* 2.2 (a deterministic linear quadratic control problem). Linear quadratic control problems (see also section 3 for stochastic linear quadratic control problems) are the typical examples we have in mind since they lead to Hamilton–Jacobi equations with quadratic terms. On the other hand, the value function can blow up in finite time. Consider the control problem (in dimension 1 for sake of simplicity)

$$\begin{cases} dX_s = \alpha_s \, ds, \quad s \in [t,T], \ 0 \le t \le T, \\ X_t = x \in \mathbb{R}, \end{cases}$$

where the control $\alpha \in \mathcal{A}_t := L^2([t,T];\mathbb{R})$ and the value function is given by

$$V(x,t) = \inf_{\alpha \in \mathcal{A}_t} \left\{ \rho \int_t^T (|\alpha_s|^2 + |X_s|^2) \, ds - |X_T|^2 \right\} \quad \text{for some } \rho > 0.$$

Then the value function, when it is finite, is the unique viscosity solution of the Hamilton–Jacobi equation of type (1.1) which reads

$$\begin{cases} -w_t + \frac{1}{4\rho}|w_x|^2 = \rho x^2 \qquad \text{in } \mathbb{R} \times (0,T), \\ \qquad w(x,T) = -x^2. \end{cases} \tag{2.7}$$

(See Theorem 3.1 for a proof of this result.) Looking for a solution $w$ under the form $w(x,t) = \varphi(t)x^2$, we obtain that $\varphi$ is a solution of the differential equation

$$-\varphi' + \frac{\varphi^2}{\rho} = \rho \ \text{ in } (0,T), \qquad \varphi(T) = -1. \tag{2.8}$$

We distinguish two cases.

*Case* 1. If $\rho \ge 1$, then the solution of (2.8) is defined in the whole interval $[0,T]$ for all $T > 0$ and is given by

$$\varphi(t) = \rho \, \frac{(\rho-1)\mathrm{e}^{2(T-t)} - (\rho+1)}{(\rho-1)\mathrm{e}^{2(T-t)} + \rho + 1}, \tag{2.9}$$

which is a function decreasing from $\varphi(0)$ to $-1$. Therefore (2.7) admits a unique viscosity solution in $\mathbb{R} \times [0,T]$ which is the value function of the control problem

$$V(x,t) = \varphi(t)x^2.$$

Note that if $\rho > 1$ and $T > \ln((\rho+1)/(\rho-1))/2$, then $\varphi(0) > 0$. It follows that the value function satisfies (2.4) but is neither bounded from above nor bounded from below.

*Case* 2. If $0 < \rho < 1$ and $T > \ln((1+\rho)/(1-\rho))/2$, then the solution of (2.8) is given by (2.9) in $(\bar\tau, T]$, where

$$\bar\tau := T - \frac{1}{2} \ln \left( \frac{1+\rho}{1-\rho} \right),$$

and blows up at $t = \bar\tau$. Therefore we have existence for (2.7) only in $\mathbb{R} \times (\bar\tau, T]$.

*Proof of Theorem* 2.1. We divide the proof of the theorem into two steps.

*Step* 1. We first assume that $\ell_0 \ge 0$, $f \le 0$, and $\psi \le 0$ in (A1), (A2), and (A3).

The proof is based on the two following lemmas, whose proofs are postponed.

LEMMA 2.2 (linearization of (1.1)). *Let $0 < \mu < 1$ and set $\Psi = U - \mu V$. Suppose $\ell_0 \geq 0$, $f \leq 0$ and $\psi \leq 0$. Then $\Psi$ is a USC viscosity subsolution of*

$$\mathcal{L}[w] := \frac{\partial w}{\partial t} - \frac{\bar{C}^2}{2\nu(1-\mu)}|Dw|^2 - 2\bar{C}(1+|x|)|Dw|$$

$$- \sup_{\alpha\in\mathbb{R}^k} \operatorname{Tr}\left[\sigma(x,t,\alpha)\sigma^T(x,t,\alpha)D^2 w\right] - \sup_{\beta\in B} \operatorname{Tr}\left[c(x,t,\beta)c^T(x,t,\beta)D^2 w\right] = 0 \tag{2.10}$$

*in $\mathbb{R}^N \times (0,T)$, with the initial condition*

$$w(\cdot,0) \leq (1-\mu)\psi \leq 0. \tag{2.11}$$

LEMMA 2.3. *Consider the parabolic problem*

$$\begin{cases} \varphi_t - r^2\varphi_{rr} - r\varphi_r = 0 & \text{in } [0,+\infty)\times(0,T], \\ \varphi(r,0) = \varphi_R(r) & \text{in } [0,+\infty), \end{cases} \tag{2.12}$$

*where $\varphi_R(r) = \max\{0, r-R\}$ for some $R > 0$. Then (2.12) has a unique solution $\varphi \in C([0,+\infty)\times[0,T])\cap C^\infty([0,+\infty)\times(0,T])$ such that for all $t \in (0,T]$, $\varphi(\cdot,t)$ is positive, nondecreasing, and convex in $[0,+\infty)$. Moreover, for every $(r,t) \in [0,+\infty)\times(0,T]$,*

$$\varphi(r,t) \geq \varphi_R(r), \quad 0 \leq \varphi_r(r,t) \leq \mathrm{e}^T \quad \text{and} \quad \varphi(r,t) \underset{R\to+\infty}{\longrightarrow} 0. \tag{2.13}$$

Let $\Phi(x,t) = \varphi(C(1+|x|^2)\mathrm{e}^{Lt}, Mt) + \eta t$, where $\varphi$ is given by Lemma 2.3, $L, M, \eta$ are positive constants to be determined, and $C > \max\{\bar{C}, \hat{C}\}$, where $\bar{C}$ and $\hat{C}$ are the constants appearing in the assumptions.

*Claim.* We can choose the constants $L$ and $M$ such that $\Phi$ is a strict supersolution of (2.10) at least in $\mathbb{R}^N \times (0,\tau]$ for small $\tau$.

To prove the claim, we have to show that $\mathcal{L}[\Phi] > 0$ in $\mathbb{R}^N \times (0,\tau]$ for some $\tau > 0$. The function $\Phi \in C(\mathbb{R}^N \times [0,T]) \cap C^\infty(\mathbb{R}^N \times (0,T])$ and we have, for $t > 0$,

$$\Phi_t = \eta + LC(1+|x|^2)\mathrm{e}^{Lt}\varphi_r + M\varphi_t, \quad D\Phi = 2Cx\mathrm{e}^{Lt}\varphi_r,$$
$$\text{and } D^2\Phi = 2C\,Id\,\mathrm{e}^{Lt}\varphi_r + 4C^2\mathrm{e}^{2Lt}\varphi_{rr}\,x\otimes x.$$

Using (A1) and (A2), for all $(x,t) \in \mathbb{R}^N \times (0,T]$, we get

$$\mathcal{L}[\Phi] \geq \eta + LC(1+|x|^2)\mathrm{e}^{Lt}\varphi_r + M\varphi_t - \frac{\bar{C}^2}{2\nu(1-\mu)}|2Cx\mathrm{e}^{Lt}\varphi_r|^2$$
$$- 2\bar{C}(1+|x|)|2Cx\mathrm{e}^{Lt}\varphi_r| - 2\bar{C}^2(1+|x|)^2|2C\,Id\,\mathrm{e}^{Lt}\varphi_r + 4C^2\mathrm{e}^{2Lt}x\otimes x\,\varphi_{rr}|.$$

Setting $r = C(1+|x|^2)\mathrm{e}^{Lt}$ and since $C > \bar{C}$, we obtain

$$\mathcal{L}[\Phi] \geq \eta + M\varphi_t - 16C^2r^2\varphi_{rr} - 8C(C+1)r\varphi_r + \left(L - \frac{2C^3\mathrm{e}^{Lt}\varphi_r}{\nu(1-\mu)}\right)r\varphi_r.$$

Our aim is to fix the parameters $M$ and $L$ in order to make $\mathcal{L}[\Phi]$ positive.

We first choose $M > 16C^2 + 8C$. Since $\varphi$ is a solution of (2.12) (Lemma 2.3), we obtain

$$\mathcal{L}[\Phi] > \eta + \left(L - \frac{2C^3\mathrm{e}^{Lt}\varphi_r}{\nu(1-\mu)}\right)r\varphi_r. \tag{2.14}$$

Then, taking $L > \frac{2C^3 e^{T+1}}{\nu(1-\mu)}$, we get $\mathcal{L}[\Phi] > \eta > 0$ for all $x \in \mathbb{R}^N$ and $t \in (0,\tau]$, where $\tau = 1/L$. This proves the claim.

We continue by considering

$$\max_{\mathbb{R}^N \times [0,\tau]} \{\Psi - \Phi\}, \tag{2.15}$$

where $\Psi$ is the function defined in Lemma 2.2 which is a viscosity subsolution of (2.10) and $\Phi$ is the strict supersolution of (2.10) in $\mathbb{R}^N \times (0,\tau]$ we built above.

From (2.13), we have $\Phi(x,t) \geq C(1+|x|^2) > \bar{C}(1+|x|^2) \geq \Psi(x,t)$ for $|x| \geq R$. It follows that the maximum (2.15) is achieved at a point $(\bar{x},\bar{t}) \in \mathbb{R}^N \times [0,\tau]$. We claim that $\bar{t} = 0$. Indeed, suppose by contradiction that $\bar{t} > 0$. Then since $\Psi$ is a viscosity subsolution of (2.10), by taking $\Phi$ as a test function, we would have $\mathcal{L}[\Phi](\bar{x},\bar{t}) \leq 0$, which contradicts the fact that $\Phi$ is a strict supersolution.

Thus, for all $(x,t) \in \mathbb{R}^N \times [0,\tau]$,

$$\Psi(x,t) - \Phi(x,t) \leq \Psi(\bar{x},0) - \Phi(\bar{x},0) \leq (1-\mu)\psi(\bar{x}),$$

where the last inequality follows from (2.11) and the fact that $\Phi \geq 0$. Since we assumed that $\psi$ is nonpositive, for every $(x,t) \in \mathbb{R}^N \times [0,\tau]$, we have $\Psi(x,t) \leq \Phi(x,t)$. Letting $\eta$ go to 0 and $R$ to $+\infty$, we get by Lemma 2.3, $\Psi \leq 0$ in $\mathbb{R}^N \times [0,\tau]$.

By a step-by-step argument, we prove that $\Psi \leq 0$ in $\mathbb{R}^N \times [0,T]$. Therefore $\Psi = U - \mu V \leq 0$ in $\mathbb{R}^N \times [0,T]$. Letting $\mu$ go to 1, we obtain $U \leq V$ as well, which concludes Step 1.

*Step* 2. *The general case.* The idea is to reduce to the first case by a suitable change of function (see Ishii [22]). Suppose that $w$ is a solution of (1.1). Then, a straightforward computation shows that $\bar{w}(x,t) = w(x,t) - C(1+|x|^2)e^{\rho t}$ for $C > \bar{C}, \hat{C}$ and $\rho > 0$ is a solution of

$$\begin{cases} \bar{w}_t + \inf_{\alpha \in \mathbb{R}^k} \left\{ \langle b(x,t,\alpha), D\bar{w} \rangle + \bar{\ell}(x,t,\alpha) - \mathrm{Tr}\left[\sigma(x,t,\alpha)\sigma^T(x,t,\alpha)D^2\bar{w}\right] \right\} \\ \quad + \sup_{\beta \in B} \left\{ -\langle g(x,t,\beta), D\bar{w} \rangle - \bar{f}(x,t,\beta) - \mathrm{Tr}\left[c(x,t,\beta)c^T(x,t,\beta)D^2\bar{w}\right] \right\} = 0, \\ \bar{w}(x,0) = \psi(x) - C(1+|x|^2), \end{cases} \tag{2.16}$$

where

$$\begin{aligned} \bar{\ell}(x,t,\alpha) &= \ell(x,t,\alpha) + 2Ce^{\rho t}\langle b(x,t,\alpha), x \rangle - 2Ce^{\rho t}\,\mathrm{Tr}\left[\sigma(x,t,\alpha)\sigma^T(x,t,\alpha)\right] \\ &\qquad + \frac{1}{2}C\rho e^{\rho t}(1+|x|^2), \\ \bar{f}(x,t,\beta) &= f(x,t,\beta) + 2Ce^{\rho t}\langle g(x,t,\beta), x \rangle + 2Ce^{\rho t}\,\mathrm{Tr}\left[c(x,t,\beta)c^T(x,t,\beta)\right] \\ &\qquad - \frac{1}{2}C\rho e^{\rho t}(1+|x|^2). \end{aligned}$$

We observe that $\bar{\ell}$ and $\bar{f}$ still satisfy, respectively, assumptions (A1)(iii) and (A2)(iii). Moreover from (A3), we can choose $C > \bar{C}$ in order that $\bar{\psi} \leq 0$.

Next we show that if $\rho > 0$ is chosen in a suitable way, then $\bar{\ell}(x,t,\alpha) \geq \bar{\nu}|\alpha|^2/2$ for all $(x,t,\alpha) \in \mathbb{R}^N \times [0,1/\rho] \times \mathbb{R}^k$ and $f(x,t,\beta) \leq 0$ for all $(x,t,\beta) \in \mathbb{R}^N \times [0,1/\rho] \times B$.

Indeed, for all $(x,t,\alpha) \in \mathbb{R}^N \times [0,1/\rho] \times \mathbb{R}^k$ by (A1) we have

$$\begin{aligned}\bar\ell(x,t,\alpha) &\geq \frac{\nu}{2}|\alpha|^2 - \bar C(1+|x|^2) - 2C\bar C e^{\rho t}(1+|x|+|\alpha|)|x| - 2C\bar C^2 e^{\rho t}(1+|x|)^2\\ &\qquad + \frac{1}{2}C\rho e^{\rho t}(1+|x|^2)\\ &\geq \frac{\nu}{2}|\alpha|^2 - (\bar C + 4C\bar C + 4C\bar C^2)e^{\rho t}(1+|x|^2) - 2C\bar C e^{\rho t}|\alpha||x|\\ &\qquad + \frac{1}{2}C\rho e^{\rho t}(1+|x|^2).\end{aligned}$$

But

$$2C\bar C e^{\rho t}|\alpha||x| \leq \frac{\nu}{4}|\alpha|^2 + \frac{16C^2\bar C^2 e^{2\rho t}}{\nu}|x|^2\,.$$

Therefore, by choosing

$$\rho > 2\frac{\bar C}{C} + 8\bar C + 8\bar C^2 + \frac{32C\bar C^2 e}{\nu}, \tag{2.17}$$

we have

$$\bar\ell(x,t,\alpha) \geq \frac{\nu}{4}|\alpha|^2 \quad \text{for all } (x,t,\alpha) \in \mathbb{R}^N \times [0,1/\rho] \times \mathbb{R}^k,$$

which is the desired estimate with $\bar\nu = \nu/2 > 0$ and $\ell_0 \equiv 0$.

The next step consists in choosing $\rho$ such that $\bar f \leq 0$. Using (A2), the same kind of calculation as above shows that taking

$$\rho > 8(\bar C + \bar C^2) + 2 \tag{2.18}$$

ensures $\bar f \leq 0$.

Finally, if we choose $C > \bar C$ and $\rho$ as the maximum of the two quantities appearing in (2.17) and (2.18), we are in the framework of Step 1 in $\mathbb{R}^N \times [0,\rho]$. Setting $\bar U = U - C(1+|x|^2)e^{\rho t}$ and $\bar V = V - C(1+|x|^2)e^{\rho t}$, from Step 1 we get $\bar U \leq \bar V$ in $\mathbb{R}^N \times [0,1/\rho]$; thus $U \leq V$ in $\mathbb{R}^N \times [0,1/\rho]$. Then by a step-by-step argument we obtain the comparison in $\mathbb{R}^N \times [0,T]$. □

*Remark* 2.3. A key fact in the proof to build a strict supersolution of (2.10) is to use a function which is the solution of the auxiliary—and simpler—pde (2.12). This idea comes from mathematical finance to deal with equations related to the Blake and Scholes formula. See, for instance, Lamberton and Lapeyre [29] and Barles et al. [8].

We turn to the proof of Lemmas 2.2 and 2.3.

*Proof of Lemma* 2.2. For $0 < \mu < 1$, let $\tilde V = \mu V$ and $\Psi = U - \tilde V$. We divide the proof into steps.

*Step* 1. *A new equation for* $\tilde V$. It is not difficult to see that if $V$ is a supersolution of (1.1), then $\tilde V$ is a supersolution of

$$\begin{aligned}&\tilde V_t + \mu\, H\left(x,t,\frac{D\tilde V}{\mu},\frac{D^2\tilde V}{\mu}\right) + \mu\, G\left(x,t,\frac{D\tilde V}{\mu},\frac{D^2\tilde V}{\mu}\right) \geq 0 \quad \text{in } \mathbb{R}^N \times (0,T),\\ &\tilde V(x,0) \geq \mu\,\psi(x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathbb{R}^N.\end{aligned} \tag{2.19}$$

*Step* 2. *Viscosity inequalities for* $U$ *and* $\tilde V$. This step is classical in viscosity theory. Let $\varphi \in C^2(\mathbb{R}^N \times (0,T])$ and $(\bar x,\bar t) \in \mathbb{R}^N \times (0,T]$ be a local maximum of $\Psi - \varphi$. We can assume that this maximum is strict in the same ball $\overline{B}(\bar{x},r)\times[\bar{t}-r,\bar{t}+r]$ (see [6] or [3]). Let

$$\Theta(x,y,t)=\varphi(x,t)+\frac{|x-y|^2}{\varepsilon^2}$$

and consider

$$M_\varepsilon := \max_{x,y\in\overline{B}(\bar{x},r),\, t\in[\bar{t}-r,\bar{t}+r]}\{U(x,t)-\tilde{V}(y,t)-\Theta(x,y,t)\}.$$

This maximum is achieved at a point $(x_\varepsilon,y_\varepsilon,t_\varepsilon)$ and, since the maximum is strict, we know [6], [3] that

$$\frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon^2}\to 0 \quad \text{as } \varepsilon\to 0$$

and

$$M_\varepsilon = U(x_\varepsilon,t_\varepsilon)-\tilde{V}(y_\varepsilon,t_\varepsilon)-\Theta(x_\varepsilon,y_\varepsilon,t_\varepsilon)\longrightarrow U(\bar{x},\bar{t})-\tilde{V}(\bar{x},\bar{t})-\varphi(\bar{x},\bar{t})=\Psi(\bar{x},\bar{t})-\varphi(\bar{x},\bar{t}).$$

This means that at the limit $\varepsilon\to 0$, we obtain some information on $\Psi-\varphi$ at $(\bar{x},\bar{t})$ which will provide the new equation for $\Psi$. Before that, we can take $\Theta$ as a test function to use the fact that $U$ is a subsolution and $\tilde{V}$ a supersolution. Indeed, $(x,t)\in\overline{B}(\bar{x},r)\times[\bar{t}-r,\bar{t}+r]\mapsto U(x,t)-\tilde{V}(y_\varepsilon,t)-\Theta(x,y_\varepsilon,t)$ achieves its maximum at $(x_\varepsilon,t_\varepsilon)$ and $(y,t)\in\overline{B}(\bar{x},r)\times[\bar{t}-r,\bar{t}+r]\mapsto -U(x_\varepsilon,t)+\tilde{V}(y,t)+\Theta(x_\varepsilon,y,t)$ achieves its minimum at $(y_\varepsilon,t_\varepsilon)$. Thus, by Theorem 8.3 in the user's guide [15], for every $\rho>0$, there exist $a_1,a_2\in\mathbb{R}$ and $X,Y\in\mathcal{S}_N$ such that

$(a_1,D_x\Theta(x_\varepsilon,y_\varepsilon,t_\varepsilon),X)\in\bar{\mathcal{P}}^{2,+}(U)(x_\varepsilon,t_\varepsilon),(a_2,-D_y\Theta(x_\varepsilon,y_\varepsilon,t_\varepsilon),Y)\in\bar{\mathcal{P}}^{2,-}(\tilde{V})(y_\varepsilon,t_\varepsilon),$

$a_1-a_2=\Theta_t(x_\varepsilon,y_\varepsilon,t_\varepsilon)=\varphi_t(x_\varepsilon,t_\varepsilon)$ and

$$-\left(\frac{1}{\rho}+|M|\right)I\le\begin{pmatrix} X & 0\\ 0 & -Y\end{pmatrix}\le M+\rho M^2,\quad \text{where } M=D^2\Theta(x_\varepsilon,y_\varepsilon,t_\varepsilon). \tag{2.20}$$

Setting $p_\varepsilon=2\dfrac{x_\varepsilon-y_\varepsilon}{\varepsilon^2}$, we have

$$D_x\Theta(x_\varepsilon,y_\varepsilon,t_\varepsilon)=p_\varepsilon+D\varphi(x_\varepsilon,t_\varepsilon)\quad\text{and}\quad D_y\Theta(x_\varepsilon,y_\varepsilon,t_\varepsilon)=-p_\varepsilon,$$

and

$$M=\begin{pmatrix} D^2\varphi(x_\varepsilon,t_\varepsilon)+2I/\varepsilon^2 & -2I/\varepsilon^2\\ -2I/\varepsilon^2 & 2I/\varepsilon^2\end{pmatrix}.$$

Thus, from (2.20), it follows

$$\langle Xp,p\rangle-\langle Yq,q\rangle\le\langle D^2\varphi(x_\varepsilon,t_\varepsilon)p,p\rangle+\frac{2}{\varepsilon^2}|p-q|^2+m\left(\frac{\rho}{\varepsilon^4}\right), \tag{2.21}$$

where $m$ is a modulus of continuity which is independent of $\rho$ and $\varepsilon$. In what follows, $m$ will always denote a generic modulus of continuity independent of $\rho$ and $\varepsilon$.

Writing the subsolution viscosity inequality for $U$ and the supersolution inequality for $\tilde{V}$ by means of the semijets and subtracting the inequalities, we obtain

$$
\begin{aligned}
&\varphi_t(x_\varepsilon,t_\varepsilon) + H\big(x_\varepsilon,t_\varepsilon,D\varphi(x_\varepsilon,t_\varepsilon)+p_\varepsilon,X\big) - \mu\, H\left(y_\varepsilon,t_\varepsilon,\frac{p_\varepsilon}{\mu},\frac{Y}{\mu}\right)\\
&\qquad + G\big(x_\varepsilon,t_\varepsilon,D\varphi(x_\varepsilon,t_\varepsilon)+p_\varepsilon,X\big) - \mu\, G\left(y_\varepsilon,t_\varepsilon,\frac{p_\varepsilon}{\mu},\frac{Y}{\mu}\right) \le 0.
\end{aligned}
\tag{2.22}
$$

*Step* 3. *Estimate of* $\mathcal{G} := G\big(x_\varepsilon,t_\varepsilon,D\varphi(x_\varepsilon,t_\varepsilon)+p_\varepsilon,X\big) - \mu\, G\big(y_\varepsilon,t_\varepsilon,\frac{p_\varepsilon}{\mu},\frac{Y}{\mu}\big)$. For simplicity, we set

$$
c(x_\varepsilon,t_\varepsilon,\beta)=c_x \quad \text{and} \quad c(y_\varepsilon,t_\varepsilon,\beta)=c_y.
$$

We have

$$
\begin{aligned}
\mathcal{G} &= \sup_{\beta\in B}\big\{-\langle g(x_\varepsilon,t_\varepsilon,\beta),D\varphi(x_\varepsilon,t_\varepsilon)+p_\varepsilon\rangle - f(x_\varepsilon,t_\varepsilon,\beta) - \mathrm{Tr}\left[c_xc_x^TX\right]\big\}\\
&\quad - \sup_{\beta\in B}\big\{-\langle g(y_\varepsilon,t_\varepsilon,\beta),p_\varepsilon\rangle - \mu f(y_\varepsilon,t_\varepsilon,\beta) - \mathrm{Tr}\left[c_yc_y^TY\right]\big\}\\
&\ge \inf_{\beta\in B}\{\langle g(y_\varepsilon,t_\varepsilon,\beta)-g(x_\varepsilon,t_\varepsilon,\beta),p_\varepsilon\rangle - \langle g(x_\varepsilon,t_\varepsilon,\beta),D\varphi(x_\varepsilon,t_\varepsilon)\rangle\\
&\qquad -(1-\mu)f(y_\varepsilon,t_\varepsilon,\beta)+f(y_\varepsilon,t_\varepsilon,\beta)-f(x_\varepsilon,t_\varepsilon,\beta) - \mathrm{Tr}\left[c_xc_x^TX - c_yc_y^TY\right]\big\}.
\end{aligned}
$$

From (A2), if $L_{g,r}$ is the Lipschitz constant of $g$ in $\overline{B}(\bar{x},r)\times[\bar{t}-r,\bar{t}+r]$, then we have

$$
\langle g(y_\varepsilon,t_\varepsilon,\beta)-g(x_\varepsilon,t_\varepsilon,\beta),p_\varepsilon\rangle \le L_{g,r}|y_\varepsilon-x_\varepsilon||p_\varepsilon| \le 2L_{g,r}\frac{|y_\varepsilon-x_\varepsilon|^2}{\varepsilon^2} = m(\varepsilon)
$$

and

$$
-\langle g(x_\varepsilon,t_\varepsilon,\beta),D\varphi(x_\varepsilon,t_\varepsilon)\rangle \ge -\bar{C}(1+|x_\varepsilon|)|D\varphi(x_\varepsilon,t_\varepsilon)|\,.
$$

By assumption, $f\le 0$ thus $-(1-\mu)f(y_\varepsilon,t_\varepsilon,\beta)\ge 0$. Again from (A2) it follows that

$$
f(y_\varepsilon,t_\varepsilon,\beta)-f(x_\varepsilon,t_\varepsilon,\beta) \ge -m(|y_\varepsilon-x_\varepsilon|).
$$

Let us denote by $(e_i)_{1\le i\le N}$ the canonical basis of $\mathbb{R}^N$. By using (2.21), we obtain

$$
\begin{aligned}
\mathrm{Tr}\left[c_xc_x^TX - c_yc_y^TY\right] &= \sum_{i=1}^N \langle Xc_xe_i,c_xe_i\rangle - \langle Yc_ye_i,c_ye_i\rangle\\
&\le \mathrm{Tr}\left[c_xc_x^TD^2\varphi(x_\varepsilon,t_\varepsilon)\right] + \frac{2}{\varepsilon^2}|c_x-c_y|^2 + m\left(\frac{\rho}{\varepsilon^4}\right)\\
&\le \mathrm{Tr}\left[c_xc_x^TD^2\varphi(x_\varepsilon,t_\varepsilon)\right] + 2L_{c,r}^2\frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon^2} + m\left(\frac{\rho}{\varepsilon^4}\right)\\
&\le \mathrm{Tr}\left[c_xc_x^TD^2\varphi(x_\varepsilon,t_\varepsilon)\right] + m(\varepsilon) + m\left(\frac{\rho}{\varepsilon^4}\right),
\end{aligned}
$$

where $L_{c,r}$ is a Lipschitz constant for $c$ in $\bar{B}(x,r)$. Hence, since all the moduli are independent of $\varepsilon$, $\rho$ and the control, we have

$$
\begin{aligned}
\mathcal{G} &\ge -\bar{C}(1+|x_\varepsilon|)|D\varphi(x_\varepsilon,t_\varepsilon)| + \inf_{\beta\in B}\big\{-\mathrm{Tr}\left[c(x_\varepsilon,t_\varepsilon,\beta)c(x_\varepsilon,t_\varepsilon,\beta)^TD^2\varphi(x_\varepsilon,t_\varepsilon)\right]\big\}\\
&\quad + m(\varepsilon) + m\left(\frac{\rho}{\varepsilon^4}\right).
\end{aligned}
$$

*Step* 4. *Estimate of* $\mathcal{H} := H\big(x_\varepsilon, t_\varepsilon, D\varphi(x_\varepsilon,t_\varepsilon)+p_\varepsilon, X\big) - \mu\, H\big(y_\varepsilon, t_\varepsilon, \frac{p_\varepsilon}{\mu}, \frac{Y}{\mu}\big)$. With the same notation as in Step 3, we have

$$\mathcal{H} \geq \inf_{\alpha\in\mathbb{R}^k} \{\langle b(x_\varepsilon,t_\varepsilon,\alpha) - b(y_\varepsilon,t_\varepsilon,\alpha), p_\varepsilon\rangle + \langle b(x_\varepsilon,t_\varepsilon,\alpha), D\varphi(x_\varepsilon,t_\varepsilon)\rangle \\ + (1-\mu)\ell(y_\varepsilon,t_\varepsilon,\alpha) + \ell(x_\varepsilon,t_\varepsilon,\alpha) - \ell(y_\varepsilon,t_\varepsilon,\alpha) - \mathrm{Tr}\left[\sigma_x\sigma_x^T X - \sigma_y\sigma_y^T Y\right]\}.$$

From (A1) these estimates follow:

$$\langle b(x_\varepsilon,t_\varepsilon,\alpha) - b(y_\varepsilon,t_\varepsilon,\alpha), p_\varepsilon\rangle \geq -\bar{C}(1+|\alpha|)|x_\varepsilon - y_\varepsilon||p_\varepsilon| \geq -\bar{C}|\alpha|m(\varepsilon) + m(\varepsilon),$$

$$\langle b(x_\varepsilon,t_\varepsilon,\alpha), D\varphi(x_\varepsilon,t_\varepsilon)\rangle \geq -\bar{C}(1+|x_\varepsilon|)|D\varphi(x_\varepsilon,t_\varepsilon)| - \bar{C}|\alpha||D\varphi(x_\varepsilon,t_\varepsilon)|,$$

$$\ell(x_\varepsilon,t_\varepsilon,\alpha) - \ell(y_\varepsilon,t_\varepsilon,\alpha) \geq -(1+|\alpha|^2)m_r(|x_\varepsilon - y_\varepsilon|) \geq -|\alpha|^2 m(\varepsilon) + m(\varepsilon),$$

$$(1-\mu)\ell(y_\varepsilon,t_\varepsilon,\alpha) \geq (1-\mu)\left(\frac{\nu}{2}|\alpha|^2 + \ell_0(x,t,\alpha)\right) \geq \frac{\nu(1-\mu)}{2}|\alpha|^2,$$

where the last inequality follows from the fact that by assumption, $\ell_0(x,t,\alpha)\geq 0$.

By proceeding exactly as in Step 3 one can show that

$$\mathrm{Tr}\left[\sigma_x\sigma_x^T X - \sigma_y\sigma_y^T Y\right] \leq \mathrm{Tr}\left[\sigma_x\sigma_x^T D^2\varphi(x_\varepsilon,t_\varepsilon)\right] + m(\varepsilon) + m\left(\frac{\rho}{\varepsilon^4}\right), \tag{2.23}$$

where $m$ is independent of $\alpha$. Thus, using (2.2), we have

$$\begin{aligned}\mathcal{H} &\geq \inf_{\alpha\in\mathbb{R}^k}\left\{\left(\frac{\nu(1-\mu)}{2} + m(\varepsilon)\right)|\alpha|^2 - \bar{C}(|D\varphi(x_\varepsilon,t_\varepsilon)| + m(\varepsilon))|\alpha|\right\} \\ &\quad + \inf_{\alpha\in\mathbb{R}^k}\left\{-\mathrm{Tr}\left[\sigma(x_\varepsilon,t_\varepsilon,\alpha)\sigma(x_\varepsilon,t_\varepsilon,\alpha)^T D^2\varphi(x_\varepsilon,t_\varepsilon)\right]\right\} \\ &\quad - \bar{C}(1+|x_\varepsilon|)|D\varphi(x_\varepsilon,t_\varepsilon)| + m(\varepsilon) + m\left(\frac{\rho}{\varepsilon^4}\right) \\ &\geq -\frac{(\bar{C}|D\varphi(x_\varepsilon,t_\varepsilon)| + m(\varepsilon))^2}{2\nu(1-\mu)+m(\varepsilon)} - \bar{C}(1+|x_\varepsilon|)|D\varphi(x_\varepsilon,t_\varepsilon)| \\ &\quad + \inf_{\alpha\in\mathbb{R}^k}\left\{-\mathrm{Tr}\left[\sigma(x_\varepsilon,t_\varepsilon,\alpha)\sigma(x_\varepsilon,t_\varepsilon,\alpha)^T D^2\varphi(x_\varepsilon,t_\varepsilon)\right]\right\} + m(\varepsilon) + m\left(\frac{\rho}{\varepsilon^4}\right).\end{aligned} \tag{2.24}$$

*Step* 5. Finally, from (2.22), (2.23), and (2.24), letting first $\rho$ go to 0 and then sending $\varepsilon$ to 0, we obtain

$$\begin{aligned}\mathcal{L}[\varphi](\bar{x},\bar{t}) = \varphi_t(\bar{x},\bar{t}) &- \frac{\bar{C}^2}{2\nu(1-\mu)}|D\varphi(\bar{x},\bar{t})|^2 - 2\bar{C}(1+|\bar{x}|)|D\varphi(\bar{x},\bar{t})| \\ &+ \inf_{\alpha\in\mathbb{R}^k}\left\{-\mathrm{Tr}\left[\sigma(\bar{x},\bar{t},\alpha)\sigma(\bar{x},\bar{t},\alpha)^T D^2\varphi(\bar{x},\bar{t})\right]\right\} \\ &+ \inf_{\beta\in B}\left\{-\mathrm{Tr}\left[c(\bar{x},\bar{t},\beta)c(\bar{x},\bar{t},\beta)^T D^2\varphi(\bar{x},\bar{t})\right]\right\} \leq 0,\end{aligned}$$

which means exactly that $\Psi$ is a subsolution of (2.10). □

*Proof of Lemma* 2.3. Set $\chi(s,t) = \varphi(\mathrm{e}^s,t)$ for $(s,t)\in\mathbb{R}\times[0,+\infty)$. A straightforward calculation shows that $\varphi$ satisfies (2.12) if and only if $\chi$ is a solution of the heat equation

$$\begin{cases} \chi_t - \chi_{ss} = 0 & \text{in } \mathbb{R}\times(0,T), \\ \chi(s,0) = \varphi_R(\mathrm{e}^s) & \text{in } \mathbb{R}. \end{cases} \tag{2.25}$$

Since the initial data satisfy the growth estimate $|\chi(s,0)| < \mathrm{e}^{s^2}$, by classical results on the heat equation (John [24], Evans [17]), we know there exists a unique classical solution $\chi \in C(\mathbb{R}\times[0,T])\times C^\infty(\mathbb{R}\times(0,T])$ of (2.25). It is given by the representation formula: for every $(s,t)\in\mathbb{R}\times[0,T]$,

$$\chi(s,t) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} \mathrm{e}^{-\frac{(s-y)^2}{4t}}\varphi_R(\mathrm{e}^y)\,dy = \frac{1}{\sqrt{4\pi t}}\int_{\log R}^{+\infty} \mathrm{e}^{-\frac{(s-y)^2}{4t}}(\mathrm{e}^y - R)\,dy. \tag{2.26}$$

From the above formula, it follows $\chi(s,t) > 0$ for all $(s,t)\in\mathbb{R}\times(0,T]$. Let $h>0$. We have $\varphi_R(\mathrm{e}^s)\le\varphi_R(\mathrm{e}^{s+h})$ for all $s\in\mathbb{R}$. Since $\chi(\cdot+h,\cdot)$ is a solution of (2.25) with initial data $\varphi_R(\mathrm{e}^{s+h})$, by the maximum principle, we obtain $\chi(s,t)\le\chi(s+h,t)$. This proves that $\chi$ is nondecreasing with respect to $s$. It follows that $\varphi(r,t)=\chi(\log r,t)$ is the unique solution of (2.12) and $\varphi\in C([0,+\infty)\times[0,T])\cap C^\infty((0,+\infty)\times(0,T])$ is positive and nondecreasing. Moreover, if the initial data are convex, we know that the solution of a quasi-linear equation like (2.12) is convex in the space variable for every time (see, e.g., Giga et al. [20]). Thus $\varphi(\cdot,t)$ is convex in $[0,+\infty)$ for all $t\in[0,T]$.

It remains to prove the estimates (2.13). Noticing that $\varphi_R(\mathrm{e}^s)\le\mathrm{e}^s$ and that $(s,t)\mapsto\varphi_R(\mathrm{e}^s)$ and $(s,t)\mapsto\mathrm{e}^{s+t}$ are respectively sub- and supersolution (in the viscosity sense, for example) of (2.25), by the maximum principle, we obtain $\varphi_R(\mathrm{e}^s)\le\chi(s,t)\le\mathrm{e}^{s+t}\le\mathrm{e}^{s+T}$ for $(s,t)\in\mathbb{R}\times[0,T]$. It follows

$$\varphi_R(r)\le\varphi(r,t)\le\mathrm{e}^T r \quad \text{for } (r,t)\in[0,+\infty]\times[0,T]. \tag{2.27}$$

This gives the first estimate. To prove the second estimate, we note that $\varphi(\cdot,t)$ is a convex nondecreasing function satisfying (2.27). It follows $0\le\varphi_r(r,t)\le\mathrm{e}^T$ for $(r,t)\in[0,+\infty]\times(0,T]$. The last assertion is obvious, using the dominated convergence theorem in (2.26). It completes the proof of the lemma. ◻

**3. Applications.** This section is divided into two parts. In the first part we consider a finite horizon unbounded stochastic control problem and we characterize the value function as the unique viscosity solution of the corresponding dynamic programming equation, which is a particular case of (1.1). In the second part we list some concrete examples of model cases to which the results of section 1 can be applied.

**3.1. Unbounded stochastic control problems.** Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},P)$ be a filtered probability space, $W_t$ be an $\mathcal{F}_t$-adapted standard $M$-Brownian motion such that $W_0=0$ a.s., and let $A$ be a subset of a separable normed space (possibly unbounded). We consider a *finite horizon unbounded stochastic control problem* for controlled diffusion processes $X_s^{t,x}$ whose dynamic is governed by a stochastic differential equation of the form

$$\begin{cases} dX_s^{t,x} = b(X_s^{t,x},s,\alpha_s)ds + \sigma(X_s^{t,x},s,\alpha_s)dW_s, & s\in(t,T),\ 0\le t\le T,\\ X_t^{t,x} = x\in\mathbb{R}^N, \end{cases} \tag{3.1}$$

where the control $\alpha_s\in A$, $b\colon\mathbb{R}^N\times\mathbb{R}\times A\to\mathbb{R}^N$ is a continuous vector field and $\sigma$ is a continuous real $N\times M$ matrix. The payoff to be minimized is

$$J(t,x,\alpha) = \mathrm{E}_{tx}\left\{\int_t^T \ell(X_s^{t,x},s,\alpha_s)\,ds + \psi(X_T^{t,x})\right\},$$

where $\mathrm{E}_{tx}$ is the expectation with respect to the event $X_t^{t,x}=x$, the functions $\ell:\mathbb{R}^N\times[0,T]\times A\to\mathbb{R}$ and $\psi:\mathbb{R}^N\to\mathbb{R}$ are continuous, $\alpha_s\in\mathcal{A}_t$, the set of $A$-valued

$\mathcal{F}_t$-progressively measurable controls such that

$$\mathrm{E}_{tx}\left(\int_t^T |\alpha_s|^2\,ds\right) < +\infty, \tag{3.2}$$

and $X_s^{t,x}$ is the solution of (3.1). The value function is defined by

$$V(x,t) = \inf_{\alpha_s \in \mathcal{A}_t} J(t,x,\alpha_s). \tag{3.3}$$

At least formally, the dynamic programming equation associated to this control problem is

(3.4)
$$-\frac{\partial w}{\partial t} + \sup_{\alpha\in A}\left\{-\langle b(x,t,\alpha), Dw\rangle - \ell(x,t,\alpha) - \frac{1}{2}\mathrm{Tr}\left(\sigma(x,t,\alpha)\sigma(x,t,\alpha)^T D^2 w\right)\right\} = 0$$

in $\mathbb{R}^N \times (0,T)$, with the terminal value condition $w(x,T) = \psi(x)$.

Our main goal is to characterize the value function $V$ as the unique continuous viscosity solution of (3.4) with the terminal value condition $V(x,T) = \psi(x)$. We recall that the fact that the value function is a viscosity solution of (3.4) is in general obtained by a direct use of the dynamic programming principle. Since we are in an unbounded control framework, the proof of the dynamic programming principle is rather delicate, and thus we follow another strategy which consists in comparing directly $V$ with the unique viscosity solution $U$ of (3.4) obtained by Corollary 2.1 when this latter exists.

We make the following assumptions on the data.

**(S0)** $A$ is a subset (possibly unbounded) of a separable complete normed space.

**(S1)** $b \in C(\mathbb{R}^N \times [0,T] \times A; \mathbb{R}^N)$ and there exists $\bar{C} > 0$ such that for all $x, y \in \mathbb{R}^N$, $t \in [0,T]$, and $\alpha \in A$ we have

$$|b(x,t,\alpha) - b(y,t,\alpha)| \le \bar{C}|x-y|,$$
$$|b(x,t,\alpha)| \le \bar{C}(1+|x|+|\alpha|);$$

**(S2)** $\sigma \in C(\mathbb{R}^N \times [0,T] \times A; \mathcal{M}_{N,M})$ and there exists $\bar{C} > 0$ such that for all $x, y \in \mathbb{R}^N$, $t \in [0,T]$ and $\alpha \in A$ we have

$$|\sigma(x,t,\alpha) - \sigma(y,t,\alpha)| \le \bar{C}|x-y|,$$
$$|\sigma(x,t,\alpha)| \le \bar{C}(1+|x|).$$

Moreover we assume that $\ell$ and $\psi$ satisfy, respectively, (A1)(iii) and (A3).

We first observe that under the current assumptions (S1) and (S2) on $b$ and $\sigma$, for any control $\alpha \in \mathcal{A}_t$ satifying (3.2) and any random variable $Z$ such that $\mathrm{E}[Z] < \infty$, there exists a unique strong solution of the stochastic differential equation (3.1) which satisfies

$$\mathrm{E}\left\{\sup_{t\le s\le T} |X_s^{t,Z}|^2\right\} < \infty$$

(see, e.g., Appendix D in [19]). Moreover, we have better estimates on the trajectories of (3.1).

Lemma 3.1. *Assume* (S0), (S1), *and* (S2). *For every* $(x,t) \in \mathbb{R}^N \times [0,T]$ *and every* $\alpha_s \in \mathcal{A}_t$, *the solution* $X_s^{t,x}$ *of* (3.1) *corresponding to* $\alpha_s$ *satisfies the following properties:*

(i) *there exists a constant $C > 0$ such that*

$$\mathrm{E}_{tx}\left\{\sup_{t\le s\le T}|X_s^{t,x}|^2\right\} \le \left(|x|^2 + C(T-t) + C\mathrm{E}_{tx}\int_t^T |\alpha_s|^2\,ds\right)\mathrm{e}^{C(T-t)}\ ; \tag{3.5}$$

(ii) *there exists $C_{x,\alpha} > 0$ which depends on $x$ and on the control $\alpha_s$ such that for all $s, s' \in [t,T]$,*

$$\mathrm{E}_{tx}\{|X_s^{t,x} - X_{s'}^{t,x}|\} \le C_{x,\alpha}|s-s'|^{1/2}. \tag{3.6}$$

*In particular for all $\tau \in [t,T]$ we have*

$$\mathrm{E}_{tx}\left\{\sup_{t\le s\le \tau}|X_s^{t,x} - x|\right\} \le C_{x,\alpha}|\tau|^{1/2}. \tag{3.7}$$

*Proof of Lemma* 3.1. We start by proving (i). Let us take an increasing sequence of $C^2$ functions $\varphi_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $R > 0$, $\varphi_R'(r) = 0$ if $r > 2R$, $\varphi_R''(r) \le 0$ and $\varphi_R(r) \uparrow r$, $\varphi_R'(r) \uparrow 1$, as $R \to +\infty$. By applying Ito's formula to the process $\varphi_R(|X_s^{t,x}|^2)$, for a.e. $t \le \tau \le T$, we have (dropping the argument of $\varphi_R$ and its derivatives)

$$\begin{aligned}\varphi_R(|X_\tau^{t,x}|^2) = \varphi_R(|x|^2) &+ \int_t^\tau 2\varphi_R'\langle X_s^{t,x}, b(X_s^{t,x}, s, \alpha_s)\rangle\,ds \\ &+ \int_t^\tau \left(\varphi_R'\mathrm{Tr}[\sigma\sigma^T(X_s^{t,x}, s, \alpha_s)] + 2\varphi_R''|\sigma^T(X_s^{t,x}, s, \alpha_s)X_s^{t,x}|^2\right)\,ds \\ &+ \int_t^\tau 2\varphi_R'\langle X_s^{t,x}, \sigma(X_s^{t,x}, s, \alpha_s)dW_s\rangle.\end{aligned} \tag{3.8}$$

By using the current assumptions on $b$ and $\sigma$ the following estimate holds:

$$\begin{aligned}&\int_t^\tau \left(2\varphi_R'\langle X_s^{t,x}, b(X_s^{t,x}, s, \alpha_s)\rangle + \varphi_R'\mathrm{Tr}[\sigma\sigma^T(X_s^{t,x}, s, \alpha_s)]\right. \\ &\qquad \left. + \,2\,\varphi_R''|\sigma^T(X_s^{t,x}, s, \alpha_s)X_s^{t,x}|^2\right)\,ds \\ &\le 2C\int_t^\tau \varphi_R'|X_s^{t,x}|(1 + |X_s^{t,x}| + |\alpha_s|)\,ds + C\int_t^\tau \varphi_R'(1 + |X_s^{t,x}|^2)\,ds \quad \text{a.s.,}\end{aligned}$$

where the constant $C$ depends on neither the control $\alpha_s$ nor on $R$. Moreover, we observe that since $\varphi_R' = 0$ for $t > 2R$ we have

$$\mathrm{E}_{tx}\left\{\int_t^\tau |\varphi_R'\langle X_s^{t,x}, \sigma(X_s, s, \alpha_s)\rangle|^2\,ds\right\} < +\infty,$$

hence the expectation of the stochastic integral is zero. By taking the expectation in (3.9) and applying Fubini's theorem we obtain

$$\mathrm{E}_{tx}\{\varphi_R(|X_\tau^{t,x}|^2)\} \le \varphi_R(|x|^2) + C\int_t^\tau \mathrm{E}_{tx}\{\varphi_R'[2 + 5|X_s^{t,x}|^2 + |\alpha_s|^2]\}\,ds.$$

Since $\varphi_R, \varphi_R'$ are increasing sequences, we can apply Levi's theorem. Therefore by letting $R \to \infty$ we obtain, for every $t \le \tau \le T$,

$$\begin{aligned}\mathrm{E}_{tx}\{|X_\tau^{t,x}|^2\} &\le |x|^2 + C\int_t^\tau \mathrm{E}_{tx}\{2 + 5|X_s^{t,x}|^2 + |\alpha_s|^2\}\,ds \\ &\le |x|^2 + 5C\int_t^\tau \mathrm{E}_{tx}|X_s^{t,x}|^2\,ds + C\mathrm{E}_{tx}\left\{\int_t^\tau |\alpha_s|^2\,ds\right\} + 2C(T-t).\end{aligned}$$

Applying Gronwall's inequality we obtain

$$\mathrm{E}_{tx}\{|X^{t,x}_\tau|^2\} \le \left(|x|^2 + 2C(T-t) + C\,\mathrm{E}_{tx}\left\{\int_t^T |\alpha_s|^2\,ds\right\}\right)\mathrm{e}^{5C(T-t)}.$$

We conclude by Doob's maximal inequality (see, e.g., [25]).

The proof of (ii) is an extension of the one in the Appendix D in [19] and we leave it to the reader. □

PROPOSITION 3.1. *Assume* (S0), (S1), (S2), (A1)(iii) *for* $\ell$ *and* (A3) *for* $\psi$. *Then there exists* $0 \le \tau < T$ *such that the value function* $V$ *is finite and satisfies the quadratic growth condition* (2.4) *in* $\mathbb{R}^N \times [\tau, T]$.

*Proof of Proposition* 3.1. We aim to show that if the constants $\rho, C > 0$ are large enough, then there exists $\tau > 0$ depending on $\rho$ such that $|V(x,t)| \le C(1+|x|^2)\mathrm{e}^{\rho(T-t)}$ for all $(x,t) \in \mathbb{R}^N \times [\tau, T]$. The upper estimate is obtained by majorizing directly the value function with the cost functional corresponding to a constant control and using the estimates of the trajectories in Lemma 3.1. The most difficult is to prove the estimate from below since $V$ is defined by an infimum. To this purpose we take any control $\alpha_s \in \mathcal{A}_t$. By applying Ito's formula to the process $(1+|X^{t,x}_s|^2)\mathrm{e}^{\rho(t-s)}$, $X^{t,x}_s$ being the trajectory corresponding to $\alpha_s$, we have the following estimate:

$$\begin{aligned} d[(1+|X^{t,x}_s|^2)\mathrm{e}^{\rho(T-s)}] &= -\rho\mathrm{e}^{\rho(T-s)}(1+|X^{t,x}_s|^2)ds + \mathrm{e}^{\rho(T-s)}\mathrm{Tr}\,(\sigma\sigma^T(X^{t,x}_s,s,\alpha_s))ds \\ &\quad + 2\mathrm{e}^{\rho(T-s)}\langle X^{t,x}_s, b(X^{t,x}_s,s,\alpha_s)ds + \sigma(X^{t,x}_s,s,\alpha_s)dW_s\rangle. \end{aligned} \tag{3.9}$$

Integrating both sides of (3.9) from $t$ to $T$ and taking the expectation we get

$$\begin{aligned} \mathrm{E}_{tx}\{1+|X^{t,x}_T|^2\} - (1+|x|^2)\mathrm{e}^{\rho(T-t)} = \mathrm{E}_{tx}\Bigg\{\int_t^T \Big(& -\rho(1+|X^{t,x}_s|^2) \\ & + 2\langle X^{t,x}_s, b(X^{t,x}_s,s,\alpha_s)\rangle \\ & + \mathrm{Tr}(\sigma\sigma^T(X_s,s,\alpha_s)\Big)\mathrm{e}^{\rho(T-s)}\,ds\Bigg\}. \end{aligned} \tag{3.10}$$

We notice that in the above estimate we supposed that the expectation of the stochastic integral is zero. This is false in general but we can overcome such a difficulty by an approximation argument which is similar to the one used in the proof of Lemma 3.1. Now for any $\varepsilon$-optimal control $\alpha_s$ for $V(x,t)$, by using (3.10), we get

$$\begin{aligned} & V(x,t) + C(1+|x|^2)\mathrm{e}^{\rho(T-t)} + \varepsilon \\ & \quad \ge \mathrm{E}_{tx}\Bigg\{\int_t^T \Big(\ell(X^{t,x}_s,s,\alpha_s) - 2C\mathrm{e}^{\rho(T-s)}\langle X^{t,x}_s, b(X^{t,x}_s,s,\alpha_s)\rangle \\ & \qquad - C\mathrm{e}^{\rho(T-s)}\mathrm{Tr}(\sigma\sigma^T(X^{t,x}_s,s,\alpha_s)) + \rho\mathrm{e}^{\rho(T-s)}C(1+|X_s|^2)\Big)\,ds \\ & \qquad + \psi(X^{t,x}_T) + C(1+|X^{t,x}_T|^2)\Bigg\} \\ & \quad = \mathrm{E}_{tx}\left\{\int_t^T \bar{\ell}(X^{t,x}_s,s,\alpha_s)\,ds + \bar{\psi}(X^{t,x}_T)\right\}, \end{aligned}$$

where $\bar{\ell}(x,t,\alpha) := \ell(x,t,\alpha) - 2Ce^{\rho(T-t)}\langle b(x,t,\alpha),x\rangle - 2Ce^{\rho(T-t)}\mathrm{Tr}\,a(x,t,\alpha) + C\rho e^{\rho(T-t)}(1+|x|^2)$ and $\bar{\psi}(x) := \psi(x) + C(1+|x|^2)$. By analogous arguments as those used in section 2 one can see that for $\rho, C > 0$ large enough there is $\tau > 0$ such that $\bar{\ell}$ and $\bar{\psi}$ are nonnegative in $\mathbb{R}^N \times [\tau, T]$. Thus we can conclude since $\varepsilon$ is arbitrary. ∎

*Remark* 3.1. If $\ell, \psi$ are bounded from below, namely, they satisfy, for some $C > 0$, the two conditions $\ell(x,t,\alpha) \geq \nu|\alpha|^2 - C$ and $\psi(x) \geq -C$, then $V$ is finite and satisfies the growth condition (2.4) in $\mathbb{R}^N \times [0,T]$ (i.e., for all time).

Next we prove that $V$ is the unique viscosity solution of (3.4). We start with the following proposition.

PROPOSITION 3.2. *Under the assumptions of Proposition* 3.1 *we have*

(i) (*superoptimality principle*) *for all* $t \in (\tau, T]$ *and* $0 < h \leq T - t$ *and for all stopping time* $t \leq \theta \leq T$, *we have*

(3.11)
$$V(x,t) \geq \inf_{\alpha_s \in \mathcal{A}_t} \mathrm{E}_{tx} \left\{ \int_t^{(t+h)\wedge\theta} \ell(X_s^{t,x}, s, \alpha_s)\, ds + V_*(X_{(t+h)\wedge\theta}^{t,x}, (t+h)\wedge\theta) \right\};$$

(ii) *the function* $V$ *is a supersolution of* (3.4) *in* $\mathbb{R}^N \times [\tau, T]$.

*Proof of Proposition* 3.2. The proof of (i) is a standard routine and we refer the reader, for instance, to [41]. The opposite inequality is more delicate; see Krylov [27, 28].

We turn to the proof of (ii), showing that the superoptimality principle implies that $V_*$ is a viscosity supersolution of (3.4). Let $\phi \in C^2(\mathbb{R}^N \times [0,T])$ and $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (\tau, T)$ be a local minimum of $V_* - \phi$. We can assume that $V_*(\bar{x},\bar{t}) = \phi(\bar{x},\bar{t})$ and that the maximum is strict, i.e., $V_*(x,t) > \phi(x,t)$ for all $(x,t) \in \bar{B}(\bar{x},\varepsilon) \times [\bar{t}-\varepsilon, \bar{t}+\varepsilon]$ with $(x,t) \neq (\bar{x},\bar{t})$ (see [3] or [6]). We assume by contradiction that there exists $\delta_\varepsilon > 0$ such that for all $(x,t) \in \bar{B}(\bar{x},\varepsilon) \times [\bar{t}-\varepsilon, \bar{t}+\varepsilon]$, we have

(3.12)
$$-\phi_t(x,t) + \sup_{\alpha \in A}\left\{ -\langle b(x,t,\alpha), D\phi(x,t)\rangle - \ell(x,t,\alpha) - \mathrm{Tr}\left[\frac{1}{2}\sigma\sigma^T(x,t,\alpha)D^2\phi(x,t)\right]\right\} \leq -\delta_\varepsilon.$$

Since $(\bar{x},\bar{t})$ is a strict minimum of $V_* - \phi$, it follows that there exists $\eta_\varepsilon$ such that

$$V_*(x,t) \geq \phi(x,t) + \eta_\varepsilon \quad \text{for all } (x,t) \in \partial B(\bar{x},\varepsilon) \times [\bar{t}-\varepsilon, \bar{t}+\varepsilon]. \tag{3.13}$$

From now on, we fix $0 < h < \varepsilon/2$ such that $h\delta_\varepsilon < \eta_\varepsilon$. Let us denote by $\tau_{t,x}$ the exit time of the trajectory $X_s^{t,x}$ from the ball $B(\bar{x},\varepsilon)$. We first observe that by the continuity of the trajectory (see Lemma 3.1), we have $\tau_{t,x} > t$ for all $(x,t) \in B(\bar{x},\varepsilon) \times [0,T)$. For every $(x,t) \in B(\bar{x},\varepsilon) \times (\bar{t}-\varepsilon/2, \bar{t}+\varepsilon/2)$, there exists a control $\alpha_s \in \mathcal{A}_t$ such that

$$V(x,t) + \frac{\delta_\varepsilon h}{2} \geq \mathrm{E}_{tx}\left\{ \int_t^{(t+h)\wedge\tau_{t,x}} \ell(X_s^{t,x}, s, \alpha_s)\, ds + V_*(X_{(t+h)\wedge\tau_{t,x}}^{t,x}, (t+h)\wedge\tau_{t,x}) \right\}.$$

Since $V_* \geq \phi$ in $\bar{B}(\bar{x},\varepsilon) \times [\bar{t}-\varepsilon, \bar{t}+\varepsilon]$, if $\tau_{t,x} < t+h$, then, from (3.13), we have

$$V_*(X_{(t+h)\wedge\tau_{t,x}}^{t,x}, (t+h)\wedge\tau_{t,x}) \geq \phi(X_{\tau_{t,x}}^{t,x}, \tau_{t,x}) + \eta_\varepsilon.$$

Therefore the following estimate holds:

$$
\begin{aligned}
V(x,t)+\frac{\delta_\varepsilon h}{2} &\geq \mathrm{E}_{tx}\left\{\left[\int_t^{\tau_{t,x}} \ell(X_s^{t,x},s,\alpha_s)\,ds+\phi(X_{\tau_{t,x}}^{t,x},\tau_{t,x})+\eta_\varepsilon\right]1_{\{\tau_{t,x}<t+h\}}\right\}\\
&\quad+\mathrm{E}_{tx}\left\{\left[\int_t^{t+h} \ell(X_s^{t,x},s,\alpha_s)\,ds+\phi(X_{t+h}^{t,x},t+h)\right]1_{\{\tau_{t,x}\geq t+h\}}\right\}\\
&\geq \mathrm{E}_{tx}\Big\{[I(\tau_{t,x})+\eta_\varepsilon]1_{\{\tau_{t,x}<t+h\}}\Big\}+\mathrm{E}_{tx}\Big\{I(t+h)1_{\{\tau_{t,x}\geq t+h\}}\Big\}, \qquad (3.14)
\end{aligned}
$$

where for all $\tau'>0$

$$
I(\tau')=\int_t^{\tau'} \ell(X_s^{t,x},s,\alpha_s)\,ds+\phi(X_{\tau'}^{t,x},\tau').
$$

Applying Ito's formula to the process $\phi(X_{\tau'}^{t,x},\tau')$, we obtain

$$
\begin{aligned}
I(\tau')=\int_t^{\tau'} &\Big(\ell(X_s^{t,x},s,\alpha_s)+\phi_t(X_s^{t,x},s)+\langle D\phi(X_s^{t,x},s),b(X_s^{t,x},s,\alpha_s)\rangle\\
&+\frac{1}{2}\mathrm{Tr}\,[\sigma\sigma^T(X_s^{t,x},s,\alpha_s)D^2\phi]\Big)\,ds\\
&+\phi(x,t)+\int_t^{\tau'}\langle D\phi(X_s^{t,x},s),\sigma(X_s^{t,x},s,\alpha_s)dW_s\rangle \quad \text{a.s.}
\end{aligned}
$$

Note that the expectation of the above stochastic integral is zero for $\tau'\in[t,t+\varepsilon]$.

Now we can estimate the two last terms in (3.14). For the first term, we have

$$
\begin{aligned}
&\mathrm{E}_{tx}\Big\{[I(\tau_{t,x})+\eta_\varepsilon]1_{\{\tau_{t,x}<t+h\}}\Big\}\\
&\geq -\mathrm{E}_{tx}\left\{\left[\int_t^{\tau_{t,x}}\Big(-\phi_t(X_s^{t,x},s)+\sup_{\alpha\in A}\{-\ell(X_s^{t,x},s,\alpha)-\langle D\phi(X_s^{t,x},s),b(X_s^{t,x},s,\alpha)\rangle\right.\right.\\
&\qquad\left.\left.-\frac{1}{2}\mathrm{Tr}\,[\sigma\sigma^T(X_s^{t,x},s,\alpha)D^2\phi]\}\Big)\,ds-\phi(x,t)-\eta_\varepsilon\right]1_{\{\tau_{t,x}<t+h\}}\right\}.
\end{aligned}
$$

Since $X_s^{t,x}\in B(\bar{x},\varepsilon)$ when $s\leq\tau_{t,x}$ and since $t+h<\bar{t}+\varepsilon$, from (3.12), we get

$$
\begin{aligned}
\mathrm{E}_{tx}\Big\{[I(\tau_{t,x})+\eta_\varepsilon]1_{\{\tau_{t,x}<t+h\}}\Big\} &\geq -\mathrm{E}_{tx}\left\{\left[\int_t^{\tau_{t,x}}(-\delta_\varepsilon)\,ds-\phi(x,t)-\eta_\varepsilon\right]1_{\{\tau_{t,x}<t+h\}}\right\}\\
&\geq \delta_\varepsilon \mathrm{E}_{tx}\left[(\tau_{t,x}-t)1_{\{\tau_{t,x}<t+h\}}\right]\\
&\quad+(\eta_\varepsilon+\phi(x,t))P(\{\tau_{t,x}<t+h\})\\
&\geq(\eta_\varepsilon+\phi(x,t))P(\{\tau_{t,x}<t+h\}). \qquad (3.15)
\end{aligned}
$$

For the second term, we proceed in the same way, noting that if $\tau_{t,x}\geq t+h$, then for all $t\leq s\leq t+h$, $X_s^{t,x}\in B(\bar{x},\varepsilon)$, and it allows us to apply (3.12). More precisely we have

(3.16)

$$
\begin{aligned}
&\mathrm{E}_{tx}\Big\{I(t+h)1_{\{\tau_{t,x}\geq t+h\}}\Big\}\\
&\quad \geq -\mathrm{E}_{tx}\left\{\left[\int_t^{t+h}\Big(-\phi_t(X_s^{t,x},s)+\sup_{\alpha\in A}\{-\ell(X_s^{t,x},s,\alpha)-\langle D\phi(X_s^{t,x},s),b(X_s^{t,x},s,\alpha)\rangle\right.\right.\\
&\qquad \left.\left.-\frac{1}{2}\mathrm{Tr}\,[\sigma\sigma^T(X_s^{t,x},s,\alpha)D^2\phi]\}\Big)\,ds-\phi(x,t)\right]1_{\{\tau_{t,x}\geq t+h\}}\right\}\\
&\quad \geq (\delta_\varepsilon h+\phi(x,t))P(\{\tau_{t,x}\geq t+h\}).
\end{aligned}
$$

Combining (3.14), (3.15), and (3.16), we get

$$
\begin{aligned}
V(x,t)+\frac{\delta_\varepsilon h}{2}\geq{}& \eta_\varepsilon P(\{\tau_{t,x}<t+h\})+\delta_\varepsilon h\,P(\{\tau_{t,x}\geq t+h\})\\
&+\phi(x,t)\left[P(\{\tau_{t,x}<t+h\})+P(\{\tau_{t,x}\geq t+h\})\right].
\end{aligned}
$$

Since $\eta_\varepsilon>\delta_\varepsilon h$ and $P(\{\tau_{t,x}<t+h\})+P(\{\tau_{t,x}\geq t+h\})=1$, we get

$$
V(x,t)\geq\phi(x,t)+\frac{\delta_\varepsilon h}{2}.
$$

The above inequality is valid for all $(x,t)\in B(\bar{x},\varepsilon)\times(\bar{t}-\varepsilon/2,\bar{t}+\varepsilon/2)$, and thus we have

$$
\liminf_{(x,t)\to(\bar{x},\bar{t})}V(x,t)=V_*(\bar{x},\bar{t})\geq\phi(\bar{x},\bar{t})+\frac{\delta_\varepsilon h}{2},
$$

which is a contradiction with the choice of $\phi$. ◻

THEOREM 3.1. *Under the assumptions of Proposition* 3.1*, the function $V$ is the unique continuous viscosity solution of* (3.4) *in* $\mathbb{R}^N\times[\tau,T]$.

*Proof of Theorem* 3.1. Let $U$ be the unique solution of (3.4) in $\mathbb{R}^N\times[\tau,T]$ such that $U(x,T)=\psi(x)$ given by Theorem 2.1. Our goal is to prove that $V\equiv U$ in $\mathbb{R}^N\times[\tau,T]$. The inequality $U\leq V_*$ follows by combining Proposition 3.2 and Theorem 2.1. To show that $V^*\leq U$ in $\mathbb{R}^N\times[\tau,T]$, we proceed as follows.

*Step* 1. We consider the functions $\widetilde{V}(x,t):=V(x,t)-C(1+|x|^2)\mathrm{e}^{\rho(T-t)}$ and $\widetilde{U}(x,t):=U(x,t)-C(1+|x|^2)\mathrm{e}^{\rho(T-t)}$. As it is proved in Step 2 of the proof of Theorem 2.1, $\widetilde{U}$ is the unique solution of

(3.17)

$$
\begin{cases}
-w_t+\sup\limits_{\alpha\in A}\left\{-\dfrac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,t,\alpha)D^2w)-\langle b(x,t,\alpha),Dw\rangle-\bar{\ell}(x,t,\alpha)\right\}=0 \text{ in } \mathbb{R}^N\times(\tau,T),\\
w(x,T)=\bar{\psi}(x),
\end{cases}
$$

where $\bar{\ell}(x,t,\alpha) := \ell(x,t,\alpha)+2C\mathrm{e}^{\rho(T-t)}\langle b(x,t,\alpha),x\rangle+C\mathrm{e}^{\rho(T-t)}\mathrm{Tr}\,\sigma\sigma^T(x,t,\alpha) -C\rho\mathrm{e}^{\rho(T-t)}(1+|x|^2)$ and $\bar{\psi}(x):=\psi(x)-C(1+|x|^2)$.

*Step* 2. Claim: for all $(x,t)\in\mathbb{R}^N\times[\tau,T]$, $\widetilde{V}$ satisfies

$$
\widetilde{V}(x,t)\leq\inf_{\alpha_s\in\mathcal{A}_t}\mathrm{E}_{tx}\left\{\int_t^T\bar{\ell}(X_s,s,\alpha_s)\,ds+\bar{\psi}(X_t)\right\}. \tag{3.18}
$$

To prove the claim let us take any $\alpha_s \in \mathcal{A}_t$. Arguing exactly as in the proof of Proposition 3.1, from (3.10), we have

$$\begin{aligned}\widetilde{V}(x,t) \leq \mathrm{E}_{tx}\Bigg\{\int_t^T \Big(\ell(X_s^{x,t},s,\alpha_s) + 2\mathrm{e}^{\rho(T-s)}\langle X_s^{x,t}, b(X_s^{x,t},t,\alpha_s)\rangle \\ + \mathrm{e}^{\rho(T-s)}\,\mathrm{Tr}(\sigma\sigma^T(X_s^{x,t},s,\alpha_s)) - \rho\mathrm{e}^{\rho(T-s)}(1+|X_s^{x,t}|^2)\Big)\,ds + \psi(X_T^{x,t}) \\ - C(1+|X_T^{x,t}|^2)\Bigg\} \\ = \mathrm{E}_{tx}\left\{\int_t^T \bar{\ell}(X_s^{x,t},s,\alpha_s)\,ds + \bar{\psi}(X_T^{x,t})\right\}.\end{aligned}$$

Since $\alpha_s$ is arbitrary we get (3.18) and prove the claim.

*Step* 3. Choose $C, \rho > 0$ large enough so that for all $(x,t,\alpha) \in \mathbb{R}^N \times [\tau, T] \times A$, we have

$$-\widetilde{C}\mathrm{e}^{2\rho(T-t)}(1+|x|^2) + \frac{\nu}{4}|\alpha|^2 \leq \bar{\ell}(x,t,\alpha) \leq -C\mathrm{e}^{\rho(T-t)}(1+|x|^2) + C\mathrm{e}^{\rho(T-t)}(1+|\alpha|^2),$$
$$-2C(1+|x|^2) \leq \bar{\psi}(x) \leq 0,$$

where $\widetilde{C}$ depends only on $C$ and $\nu$. For all real $R > 0$ and all integer $n > 0$, we set $A_n := \{\alpha \in A : |\alpha| \leq n\}$, $\bar{\ell}_R(x,t,\alpha) := \max\{\bar{\ell}(x,t,\alpha), -R\}$ and $\bar{\psi}_R(x,t) := \max\{\bar{\psi}(x,t), -R\}$. We observe that $\bar{\ell}_R \colon \mathbb{R}^N \times [\tau,T] \times A_n \to \mathbb{R}$ is bounded and uniformly continuous in $x \in \mathbb{R}^N$ uniformly with respect to $(t,\alpha) \in [\tau,T] \times A_n$ and $\psi_R \colon \mathbb{R}^N \to \mathbb{R}$ is bounded and uniformly continuous in $\mathbb{R}^N$. Set

$$\bar{H}(x,t,p,X) := \sup_{\alpha\in A}\left\{-\frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,t,\alpha)X) - \langle b(x,t,\alpha), p\rangle - \bar{\ell}(x,t,\alpha)\right\},$$
$$H_n^R(x,t,p,X) := \sup_{\alpha\in A_n}\left\{-\frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x,t,\alpha)X) - \langle b(x,t,\alpha), p\rangle - \bar{\ell}_R(x,t,\alpha)\right\}$$

and define

$$V_n^R(x,t) = \inf_{\alpha_s\in\mathcal{A}_t^n} \mathrm{E}_{tx}\left\{\int_t^T \bar{\ell}_R(X_s^{x,t},s,\alpha_s)\,ds + \bar{\psi}_R(X_T^{x,t})\right\},$$

where $\mathcal{A}_t^n$ is the set of $A_n$-valued $\mathcal{F}_t$-progressively measurable controls such that (3.2) holds. The function $V_n^R$ is now the value function of a stochastic control problem with bounded controls and uniformly continuous datas $b$, $\sigma$, $\bar{\ell}_R$, and $\bar{\psi}_R$. These assumptions enter the framework of Yong and Zhou [41]. We deduce that $V_n^R$ is the unique continuous viscosity solution of

$$-\frac{\partial V_n^R}{\partial t} + H_n^R(x,t,DV_n^R,D^2V_n^R) = 0 \quad \text{in } \mathbb{R}^N \times (\tau,T), \tag{3.19}$$

with terminal condition $V_n^R(x,T) = \bar{\psi}_R(x)$ in $\mathbb{R}^N$. Moreover, for all compact subsets $K \subset \mathbb{R}^N$ there exists $M > 0$ independent on $R$ and $n$ such that $||V_n^R||_\infty \leq M$ in $K \times [\tau,T]$. Indeed, take any constant control $\alpha_s = \bar{\alpha} \in A_1$, by definition of $V_n^R$ for all

$R, n > 0$ and for every $(x,t) \in \mathbb{R}^N \times [\tau, T]$ we have

$$V_n^R(x,t) \leq \mathrm{E}_{tx}\left\{\int_t^T \bar{\ell}_R(X_s^{x,t}, s, \bar{\alpha})\, ds + \bar{\psi}_R(X_T^{x,t})\right\}$$
$$\leq \int_t^T C\mathrm{e}^{\rho(T-s)}(1+\bar{\alpha}^2)\, ds \leq \frac{C}{\rho}\mathrm{e}^{\rho(T-t)}(1+\bar{\alpha}^2),$$

and on the other hand we have $V_n^R(x,t) \geq \widetilde{V}(x,t) \geq -\widetilde{C}\mathrm{e}^{2\rho(T-t)}(1+|x|^2)$.

Finally one can readily see that $H_n^R$ converges locally uniformly to $\bar{H}$ as $n, R \to \infty$. Thus by applying the half-relaxed limits method (see Barles and Perthame [9]), the functions

$$\overline{V}(x,t) = \limsup{}^* V_n^R(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ n,R\to+\infty}} V_n^R(y,s)$$
$$\text{and}\quad \underline{V}(x,t) = \liminf{}^* V_n^R(x,t) = \liminf_{\substack{(y,s)\to(x,t)\\ n,R\to+\infty}} V_n^R(y,s)$$

are, respectively, viscosity sub- and supersolution of (3.17). Theorem 2.1 yields $\overline{V}(x,t) \leq \widetilde{U}(x,t) \leq \underline{V}(x,t)$. On the other hand by construction we have also $\widetilde{V}^*(x,t) \leq \limsup^* V_n^R(x,t)$. It follows $\widetilde{V}^*(x,t) \leq \widetilde{U}(x,t)$ and we can conclude. ◻

**3.2. Some examples.**

*Example* 3.1. A model case we have in mind is the so-called stochastic linear regulator problem which is a stochastic perturbation of the deterministic linear quadratic problem. In this case, the stochastic differential (3.1) is linear and reads

$$dX_s^{t,x} = [B(s)X_s^{t,x} + C(s)\alpha_s]ds + \sum_j [C_j(s)X_s^{t,x} + D_j(s)]dW_s^j$$

and the expected total cost to minimized is

$$J(x,t,\alpha_s) = \mathrm{E}_{tx}\left\{\int_t^T [\langle X_s^{t,x}, Q(s)X_s^{t,x}\rangle + \langle \alpha_s, R(s)\alpha_s\rangle]\, ds + \langle X_T^{t,x}, GX_T^{t,x}\rangle\right\}.$$

The previous results apply if the functions $B(\cdot), C(\cdot), C_j(\cdot), D_j(\cdot), Q(\cdot), R(\cdot)$, and $G$ are deterministic continuous matrix-valued functions of suitable size and if $R(s)$ is a positive definite symmetric matrix. Deterministic and stochastic linear quadratic problems were extensively studied. For a survey we refer for instance to the books of Bensoussan [10], Fleming and Rishel [18], Fleming and Soner [19], Øksendal [38], and Yong and Zhou [41] and references therein.

*Example* 3.2. Equations of the type (3.4) are largely considered in mathematical finance. See the introductory books quoted in the introduction or Pham [39]. In particular, recently Benth and Karlsen [11] studied the following semilinear elliptic partial equation

$$-w_t - \frac{1}{2}\beta^2 w_{xx} + F(x, w_x) = 0 \qquad \text{in } \mathbb{R}\times(0,T) \tag{3.20}$$

with the final condition $w(x,T) = 0$. The nonlinear function $F$ is given by

$$F(x,p) = \frac{1}{2}\delta^2 p^2 - \left\{\alpha(x) - \frac{\mu(x)\beta\rho}{\sigma(x)}\right\}p - \frac{\mu^2(x)}{\sigma^2(x)},$$

where $\delta, \beta, \rho$ are real constant and $\alpha(x)$ and $\frac{\mu}{\sigma}(x)$ are $C^1$ functions satisfying

$$|\alpha(x)|, \left|\frac{\mu}{\sigma}(x)\right| \leq C|x| \quad \text{for all } x \in \mathbb{R}.$$

Their main motivation is to determine via the solution of (3.20) the minimal entropy martingale measure in stochastic market. The conditions they assume on data fall within the assumptions of section 2.

*Example* 3.3. Another application of the results obtained in section 2 is given by the finite time-horizon risk-sensitive limit problem for nonlinear systems. In the stochastic risk-sensitive problem, (3.1) reads

$$dX_s^{t,x,\varepsilon} = g(X_s^{t,x,\varepsilon}, \beta_s)\, dt + \sqrt{\frac{\varepsilon}{\gamma^2}} c(X_s^{t,x,\varepsilon})\, dW_s, \tag{3.21}$$

where $X_s^{t,x,\varepsilon} \in \mathbb{R}^N$ depends on the parameter $\varepsilon > 0$, $g$ represents the nominal dynamics with control $\beta_s \in B$, a compact normed space, and $c$ is an $N \times k$-valued diffusion coefficient, $\varepsilon$ is a measure of the risk-sensitivity and $\gamma$ is the disturbance attenuation level. The cost criterion is of the form

$$J^\varepsilon(x,t,\beta_s) = \mathrm{E} \exp\left\{\frac{1}{\varepsilon}\left[\int_t^T f(X_s^{t,x,\varepsilon}, \beta_s)\, dt + \psi(X_T^{t,x,\varepsilon})\right]\right\} \tag{3.22}$$

and the value function is

$$V^\varepsilon(x,t) = \inf_{\beta_s \in \mathcal{B}_t} \varepsilon \log J^\varepsilon(x,t,\beta_s) = \varepsilon \log \inf_{\beta_s \in \mathcal{B}_t} J^\varepsilon(x,t,\beta_s), \tag{3.23}$$

where $\mathcal{B}_t$ is the set of $B$-valued, $\mathcal{F}_t$-progressively measurable controls such that there exists a strong solution to (3.21). The dynamic programming equation associated to this problem is

$$\begin{cases} -\dfrac{\partial w}{\partial t} - \dfrac{1}{2\gamma^2}\langle Dw, a(x)Dw\rangle + \tilde{G}(x, Dw) - \dfrac{\varepsilon}{2\gamma^2}\mathrm{Tr}(a(x)D^2 w) = 0 & \text{in } \mathbb{R}^N \times [\tau, T], \\ w(x,T) = \psi(x), \end{cases} \tag{3.24}$$

where $a(x) = c(x)c(x)^T$ and

$$\tilde{G}(x,p) = \max_{\beta \in B}\{\langle -g(x,\beta), p\rangle - f(x,\beta)\}.$$

We note that

$$-\frac{1}{2\gamma^2}\langle p, a(x)p\rangle = \min_{\alpha \in \mathbb{R}^k}\left\{\frac{\gamma^2}{2}|\alpha|^2 - \langle c(x)\alpha, p\rangle\right\}. \tag{3.25}$$

In [16] it is shown that as $\varepsilon \downarrow 0$ (i.e., as the problem becomes infinitely risk averse), the value function of the risk-sensitive problem converges to that of an $H_\infty$ robust control problem. This problem can be considered as a differential game with the cost functional

$$J(x,t,\alpha,\beta) = \int_t^T \Big(f(y_x(s), \beta_s) - \frac{\gamma^2}{2}|\alpha_s|^2\Big) ds + \psi(y_x(T)), \tag{3.26}$$

where $\alpha_s \in \mathcal{A}_t := L^2([t,T],\mathbb{R}^k)$ is the control of the maximizing player, $\beta_s \in \mathcal{B}_t :=$ {measurable functions $[t,T] \to B$} is the control of the minimizing player, and $y_x(\cdot)$ is the unique solution of the following dynamical system

$$\begin{cases} y'(s) = g(y(s),\beta_s) + c(y(s))\alpha_s, \\ y(t) = x. \end{cases}$$

Note that we switch notation from $X_t$ to $y(t)$ to emphasize that the paths are now (deterministic) solutions of ordinary differential equations rather than (stochastic) solutions of stochastic differential equations. The dynamic programing equation associated to the robust control problem is a first-order equation given by

$$-\frac{\partial w}{\partial t} + \min_{\alpha\in\mathbb{R}^k}\left\{\frac{\gamma^2}{2}|\alpha|^2 - \langle c(x)\alpha, Dw\rangle\right\} + \tilde{G}(x,Dw) = 0 \quad \text{in } \mathbb{R}^N \times (\tau,T) \tag{3.27}$$

with the terminal condition $w(T,x) = \psi(x)$. One of the key tools to get this convergence result is the uniqueness property for (3.27). In Da Lio and McEneaney [16], the authors characterized the value function of the $H_\infty$ control as the unique solution to (3.27) in the set of locally Lipschitz continuous functions growing at most quadratically with respect to the state variable. Moreover, the uniqueness result in [16] is obtained by using representation formulas of locally Lipschitz solutions of (3.27). We remark that the comparison theorem, Theorem 2.1, not only improves the uniqueness result for (3.27) obtained in [16] (in the sense it holds in a larger class of functions) but it also should allow us to prove the convergence result in [16] under weaker assumptions, only the equiboundedness estimates of the solutions of (3.24) being enough by means of the half-relaxed limit method.

## 4. Study of related equations.

In this section we focus our attention on Hamilton–Jacobi equations of the form

$$\begin{cases} \dfrac{\partial w}{\partial t} + \langle \Sigma(x,t)Dw, Dw\rangle + G(x,t,Dw,D^2w) = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ w(x,0) = \psi(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{4.1}$$

where $\Sigma(x,t) \in \mathcal{M}_N(\mathbb{R})$ and $G$ is given by (1.3) and

$$\begin{cases} \dfrac{\partial w}{\partial t} + h(x)|Dw|^2 = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ w(x,0) = \psi(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{4.2}$$

where $h : \mathbb{R}^N \to \mathbb{R}$. Our aim is to investigate comparison (and existence) results for (4.1) and (4.2) under assumptions which include Hamiltonians like (2.6) in Remark 2.2(iii) (see Remark 4.1 for further comments). More precisely, we introduce two new assumptions:

**(A4)** $\Sigma \in C(\mathbb{R}^N \times [0,T]; \mathcal{S}_N^+(\mathbb{R}))$ and for all $x \in \mathbb{R}^N, t \in [0,T]$,

$$0 < \Sigma(x,t) \quad \text{and} \quad |\Sigma(x,t)| \le \bar{C}.$$

**(A5)** $h \in C(\mathbb{R}^N;\mathbb{R})$, $h \in W^{2,\infty}(\tilde{\Gamma})$, where $\tilde{\Gamma}$ is an open neighborhood of $\Gamma := \{x \in \mathbb{R}^N : h(x) = 0\}$ and for all $x \in \Gamma$,

$$Dh(x) = 0.$$

THEOREM 4.1. *Assume* (A2), (A3) *and* (A4) (*respectively,* (A3) *and* (A5)). *Let* $U \in USC(\mathbb{R}^N \times [0,T])$ *be a viscosity subsolution of* (4.1) (*respectively,* (4.2)) *and* $V \in LSC(\mathbb{R}^N \times [0,T])$ *be a viscosity supersolution of* (4.1) (*respectively,* (4.2)) *satisfying the quadratic growth condition* (2.4). *Then* $U \leq V$ *in* $\mathbb{R}^N \times [0,T]$.

The question of existence faces the same problems as in section 2. We have existence and uniqueness of a continuous viscosity solution for (4.1) and (4.2) in the class of functions with quadratic growth at least for short time (as in Corollary 2.1). But solutions can blow up in finite time.

Before giving the proof of the theorem, we give some comments on the equations and the assumptions.

*Remark* 4.1. (i) In the same way as in Remark 2.2, the above results hold replacing $\Sigma$ by $-\Sigma$ in (4.1) and $h$ by $-h$ in (4.2) and when dealing with terminal data in both equations.

(ii) Coming back to Remark 2.2(iii), we note $(a_1+a_2)(a_2-a_1)^T$ is not necessarily symmetric in (2.6), whereas we assume $\Sigma$ to be symmetric in (A4) (and $h$ is real-valued and therefore symmetric in (A5)). This is not a restriction of generality since comparison for (4.1) with $\Sigma$ symmetric implies obviously a comparison for (4.1) for any matrix $\Sigma$, using that, for any $\Sigma \in \mathcal{M}_N(\mathbb{R})$, $(\Sigma+\Sigma^T)/2 \in \mathcal{S}_N(\mathbb{R})$.

(iii) Coming back to Remark 2.2(iv) again, we see that (A4) corresponds to the case where the convex Hamiltonian is predominant with respect to the concave one in (1.1) with (2.6). Assumption (A5) corresponds to (2.6) when $a_1$ and $a_2$ are real valued but $h$ is allowed to change its sign. For example, we have comparison for

$$w_t + \phi(x)^3|Dw|^2 = 0,$$

where $\phi \in C^2(\mathbb{R}^N;\mathbb{R})$ is any bounded function. Let us compare (1.12) with first-order Hamilton–Jacobi equations whose Hamilonians are Lipschitz continuous both in the state and gradient variables, or, to make it simple, with the Eikonal equation

$$\frac{\partial w}{\partial t} + a(x)|Dw| = 0 \quad \text{in } \mathbb{R}^N \times (0,T),$$

where $a$ is Lipschitz continuous function. We know [13], [30] we have existence and uniqueness of a continuous viscosity solution for any continuous initial data without any restriction on the growth. In this case, the sign of $a$ does not play any role, whereas it seems to be the case for the sign of $h$ in (1.12). We would like to know if comparison for (1.12) is true under weaker assumptions than (A5) and in dimension $N > 1$ (i.e., when $\Sigma$ is neither positive nor negative definite in (4.1)).

(iv) There are some links between Theorem 2.1 and Theorem 4.1. Nevertheless we point out that if (1.1) under assumptions of section 2 is naturally associated with a control problem, this is not necessarily the case under the assumptions of the current section. To be more precise, let us compare (1.1) with $H$ given by (2.3) and (4.1) under (A4). The matrix $a$ can be singular in (2.3), whereas we impose the nondegeneracy condition $\Sigma > 0$ in (4.1). The counterpart is that the regularity assumption with respect to $x$ on $\Sigma$ is weaker ($\Sigma$ is supposed to be merely continuous) than the locally Lipschitz regularity we assume for $a$. Therefore (4.1) does not enter in the framework of section 2 in general. A natural consequence is that proofs differ: in both proofs of Theorem 2.1 and 4.1, the main argument is a kind of linearization procedure, but while in the proof of Theorem 2.1 we use essentially the convexity (or concavity) of the operator corresponding to the unbounded control set, here we use the locally Lipschitz continuity of the Hamiltonian with respect to the gradient uniformly in the state variable.

(v) Note that we are able to deal with a second-order term in (4.1) but not in (4.2).

*Proof of Theorem* 4.1. We divide the proof into two parts, corresponding, respectively, to the case of (4.1) and (4.2). The main argument in both proofs is a kind of linearization procedure as the one of Lemma 2.2. The rest of the proof is close to the one of Theorem 2.1.

*Part* 1. We assume (A2), (A3), and (A4).

The estimates of $G$ are exactly the same than in Lemma 2.2 so, for sake of simplicity, we choose to take $G \equiv 0$. The linearization procedure is the subject of the following lemma, whose proof is postponed to the end of the section.

LEMMA 4.1. *Let* $0 < \lambda < 1$ *and set* $\Psi = \lambda U - V$. *Then* $\Psi$ *is a USC viscosity subsolution of*

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{2\bar{C}}{1-\lambda}|Dw|^2 \leq 0 & \textit{in } \mathbb{R}^N \times (0,T), \\ w(x,0) \leq (\lambda - 1)\psi(x) & \textit{in } \mathbb{R}^N. \end{cases} \tag{4.3}$$

If $\psi \geq 0$, then it follows $\Psi(x,0) \leq 0$ in $\mathbb{R}^N$ and using the above lemma with the same supersolution as in the proof of Theorem 2.1, we obtain $\Psi \leq 0$ in $\mathbb{R}^N \times [0,T]$. Note that the boundedness of $\Sigma$ (see (A4)) is crucial to build the supersolution. In the particular case when $G = 0$, we can also choose a simpler supersolution such as, for instance,

$$(x,t) \mapsto K\frac{[(|x|-R)^+]^2}{1-Lt} + \eta t, \quad K, L, R, \eta > 0. \tag{4.4}$$

Letting $\lambda$ go to 1, we conclude that $U \leq V$ in $\mathbb{R}^N \times [0,T]$.

It remains to prove that we can assume $\psi \geq 0$ without loss of generality. To this end we use an argument similar to the one of Step 2 in the proof of Theorem 2.1: let $\bar{U} = U + M(1+|x|^2)\mathrm{e}^{\rho t}$, $\bar{V} = V + M(1+|x|^2)\mathrm{e}^{\rho t}$, for some positive constants $M$ and $\rho$ and set $\bar{\Psi} = \lambda\bar{U} - \bar{V}$. We can easily prove that for $\rho > 16M^2\bar{C}\mathrm{e}$, $\bar{\Psi}$ is a subsolution of (4.3) (with a larger constant $4\bar{C}$ instead of $2\bar{C}$) in $\mathbb{R}^N \times (0,\tau]$ for $\tau = 1/\rho$. Moreover, $\bar{\Psi}(x,0) \leq (\lambda-1)(\psi(x) + M(1+|x|^2)\mathrm{e}^{\rho t}) \leq 0$ for $M$ sufficiently large (since $\psi$ has quadratic growth). Letting $\lambda$ go to 1, it follows that $U \leq V$ in $\mathbb{R}^N \times [0,\tau]$ and we conclude by a step-by-step argument.

*Part* 2. We assume (A3) and (A5).

Set $\Omega^+ = \{x \in \mathbb{R}^N : h(x) > 0\}$ and $\Omega^- = \{x \in \mathbb{R}^N : h(x) < 0\}$ which are open subsets of $\mathbb{R}^N$. Define $\bar{\Psi} = \lambda\bar{U} - \bar{V}$ for $0 < \lambda < 1$, $\bar{U} = U + M(1+|x|^2)\mathrm{e}^{\rho t}$ and $\bar{V} = V + M(1+|x|^2)\mathrm{e}^{\rho t}$ with $M$ larger than the constants which appear in (A3) and in the growth condition (2.4). Arguing as in Lemma 4.1 and noticing that all arguments are local, we obtain that $\bar{\Psi}$ is a subsolution in $\Omega^+$ of the same kind of equation as (4.3), namely,

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{C}{1-\lambda}|Dw|^2 \leq 0 & \text{in } \Omega^+ \times (0,\tau), \\ w(x,0) \leq 0 & \text{in } \Omega^+, \end{cases} \tag{4.5}$$

for some constant $C = 16\,||h||_\infty$, $\rho > 4MC\mathrm{e}$, and $\tau = 1/\rho > 0$ which are independent of $\lambda$. Note that $\bar{\Psi}(\cdot,0) \leq 0$ in $\mathbb{R}^N$ because of the choice of $M$.

The sign of $\bar{\Psi}$ on $\Gamma$ is the subject of the following lemma the proof of which uses (A5) and is postponed.

LEMMA 4.2. *For all* $x \in \Gamma$, $t \in [0,T]$, *we have* $U(x,t) \le \psi(x) \le V(x,t)$.

Noticing that $\{\Gamma, \Omega^+, \Omega^-\}$ is a partition of $\mathbb{R}^N$, we set

$$\hat{\Psi} := \begin{cases} \sup\{\bar{\Psi}, 0\} & \text{in } (\Gamma \cup \Omega^+) \times [0,T], \\ 0 & \text{in } \Omega^- \times [0,T]. \end{cases}$$

From the lemma, $\bar{\Psi} \le 0$ in $\Gamma \times [0,T]$. Therefore the function $\hat{\Psi}$ is continuous in $\mathbb{R}^N \times (0,T)$. Moreover, we claim that $\hat{\Psi}$ is a subsolution of (4.5) in $\mathbb{R}^N \times (0,\tau)$. This claim is clearly true in $\Omega^- \times (0,T)$ (since 0 is clearly a subsolution) and in $\Omega^+ \times (0,\tau)$ (since $\hat{\Psi}$ is the supremum of two subsolutions). It remains to prove the result on $\Gamma$. Let $\varphi \in C^1(\mathbb{R}^N \times (0,\tau))$ such that $\hat{\Psi} - \varphi$ achieves a local maximum at a point $(\bar{x},\bar{t}) \in \Gamma \times (0,\tau)$. Let $(x,t)$ be in a neighborhood of $(\bar{x},\bar{t})$. If $x \in \Gamma \cup \Omega^+$, then $0 \le \hat{\Psi}(x,t)$, and if $x \in \Omega^-$, then $0 = \hat{\Psi}(x,t)$. In any case, $(0-\varphi)(x,t) \le (\hat{\Psi}-\varphi)(x,t) \le (\hat{\Psi}-\varphi)(\bar{x},\bar{t})$. But $\hat{\Psi}(\bar{x},\bar{t}) \le 0$ since $(\bar{x},\bar{t}) \in \Gamma \times (0,\tau)$. Therefore $(\bar{x},\bar{t})$ is a local maximum of $0-\varphi$ which ends the proof of the claim. The initial condition $\hat{\Psi} \le 0$ in $\mathbb{R}^N \times \{0\}$ is trivially satisfied. From the comparison principle proved in Part 1 (for example, using (4.4) as a supersolution), we obtain $\hat{\Psi} \le 0$ in $\mathbb{R}^N \times [0,\tau]$. Since $\tau$ does not depend on $\lambda$, we can send $\lambda$ to 1. We obtain $U \le V$ in $(\Gamma \cup \Omega^+) \times [0,\tau]$. We conclude in $(\Gamma \cup \Omega^+) \times [0,T]$ by a step-by-step procedure.

Repeating the same kind of arguments replacing $\Omega^+$ by $\Omega^-$ and $\bar{\Psi}$ by $U - M(1+|x|^2)\mathrm{e}^{\rho t} - \mu(V - M(1+|x|^2)\mathrm{e}^{\rho t})$, $0<\mu<1$, we obtain that $U - V \le 0$ in $(\Gamma \cup \Omega^-) \times [0,T]$, which ends the proof. □

*Proof of Lemma* 4.1. We proceed as in the proof of Lemma 2.2. We can assume that $G \equiv 0$ since the computations with $G$ are exactly the same than in Lemma 2.2.

Note first that $\lambda U$ is an $USC$ subsolution of

$$\begin{cases} \dfrac{\partial w}{\partial t} + \dfrac{1}{\lambda}\langle \Sigma(x,t)Dw, Dw\rangle \le 0 & \text{in } \mathbb{R}^N \times (0,T), \\ w(x,0) \le \lambda\psi(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{4.6}$$

Let $\varphi \in C(\mathbb{R}^N \times [0,T])$ and suppose that $\Psi - \varphi$ reaches a strict local maximum at $(\bar{x},\bar{t}) \in \mathbb{R}^N \times (0,T]$ in some compact subset $K \subset \mathbb{R}^N \times (0,T]$. We have

$$\max_{(x,t)(y,t)\in K} \left\{ \lambda U(x,t) - V(y,t) - \varphi\left(\frac{x+y}{2}, t\right) - \frac{|x-y|^2}{\varepsilon^2} \right\} \underset{\varepsilon \downarrow 0}{\longrightarrow} \Psi(\bar{x},\bar{t}) - \varphi(\bar{x},\bar{t}).$$

The above maximum is achieved at some point $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$. Writing the viscosity inequalities and subtracting them, we obtain

$$\begin{aligned} &\varphi_t\left(\frac{x_\varepsilon+y_\varepsilon}{2}, t_\varepsilon\right) + \frac{1}{\lambda}\left\langle \Sigma(x_\varepsilon,t_\varepsilon)\left(\frac{1}{2}D\varphi\left(\frac{x_\varepsilon+y_\varepsilon}{2}, t_\varepsilon\right) + 2\frac{x_\varepsilon-y_\varepsilon}{\varepsilon^2}\right),\right. \\ &\quad \left.\frac{1}{2}D\varphi\left(\frac{x_\varepsilon+y_\varepsilon}{2}, t_\varepsilon\right) + 2\frac{x_\varepsilon-y_\varepsilon}{\varepsilon^2}\right\rangle \\ &\quad -\left\langle \Sigma(y_\varepsilon,t_\varepsilon)\left(-\frac{1}{2}D\varphi\left(\frac{x_\varepsilon+y_\varepsilon}{2}, t_\varepsilon\right) + 2\frac{x_\varepsilon-y_\varepsilon}{\varepsilon^2}\right),\right. \\ &\quad \left.-\frac{1}{2}D\varphi\left(\frac{x_\varepsilon+y_\varepsilon}{2}, t_\varepsilon\right) + 2\frac{x_\varepsilon-y_\varepsilon}{\varepsilon^2}\right\rangle \le 0. \end{aligned}$$

In what follows, we omit writing the dependence in $t_\varepsilon$ and the point $((x_\varepsilon+y_\varepsilon)/2, t_\varepsilon)$ in the derivatives of $\varphi$. Set $p_\varepsilon = 2(x_\varepsilon - y_\varepsilon)/\varepsilon^2$, $p_x = D\varphi/2 + p_\varepsilon$, and $p_y = -D\varphi/2 + p_\varepsilon$.

We have

$$
\begin{aligned}
0 &\geq \varphi_t + \frac{1}{\lambda}\langle \Sigma(x_\varepsilon)p_x, p_x\rangle - \langle \Sigma(y_\varepsilon)p_y, p_y\rangle \\
&\geq \varphi_t + \left(\frac{1}{\lambda} - 1\right)\langle \Sigma(x_\varepsilon)p_x, p_x\rangle + \langle (\Sigma(x_\varepsilon) - \Sigma(y_\varepsilon))p_x, p_x\rangle + \langle \Sigma(y_\varepsilon)p_x, p_x\rangle \\
&\quad - \langle \Sigma(y_\varepsilon)p_y, p_y\rangle \\
&\geq \varphi_t + \left(\frac{1}{\lambda} - 1\right)\langle \Sigma(x_\varepsilon)p_x, p_x\rangle - m_K(|x_\varepsilon - y_\varepsilon|)|p_x|^2 + \langle \Sigma(y_\varepsilon)p_x, p_x\rangle - \langle \Sigma(y_\varepsilon)p_y, p_y\rangle,
\end{aligned}
$$

where $m_K$ is a modulus of continuity for $\Sigma$ in the compact subset $K$. Since $\Sigma(y_\varepsilon)$ is a symmetric matrix, we have

$$
\begin{aligned}
\langle \Sigma(y_\varepsilon)p_x, p_x\rangle - \langle \Sigma(y_\varepsilon)p_y, p_y\rangle &= \langle \Sigma(y_\varepsilon)(p_x + p_y), p_x - p_y\rangle = 2\,\langle \Sigma(y_\varepsilon)p_\varepsilon, D\varphi\rangle \\
&= -\langle \Sigma(y_\varepsilon)D\varphi, D\varphi\rangle + 2\,\langle \Sigma(y_\varepsilon)p_x, D\varphi\rangle.
\end{aligned}
$$

For any $\alpha > 0$, denoting by $\sqrt{\Sigma(y_\varepsilon)}$ the positive symmetric squareroot of the positive symmetric matrix $\Sigma(y_\varepsilon)$, we get

$$
\begin{aligned}
2\langle \Sigma(y_\varepsilon)p_x, D\varphi\rangle = 2\langle \sqrt{\Sigma(y_\varepsilon)}p_x, \sqrt{\Sigma(y_\varepsilon)}D\varphi\rangle &\leq \alpha\langle \Sigma(y_\varepsilon)D\varphi, D\varphi\rangle + \frac{1}{\alpha}\langle \Sigma(y_\varepsilon)p_x, p_x\rangle \\
&\leq \alpha\langle \Sigma(y_\varepsilon)D\varphi, D\varphi\rangle + \frac{1}{\alpha}\langle \Sigma(x_\varepsilon)p_x, p_x\rangle + \frac{1}{\alpha}\langle m_K(|x_\varepsilon - y_\varepsilon|)p_x, p_x\rangle.
\end{aligned}
$$

It follows

$$
\begin{aligned}
0 \geq \varphi_t &+ \left\langle \left[\left(\frac{1}{\lambda} - 1 - \frac{1}{\alpha}\right)\Sigma(x_\varepsilon) - \left(1 + \frac{1}{\alpha}\right)m_K(|x_\varepsilon - y_\varepsilon|)Id\right]p_x, p_x\right\rangle \\
&- (1+\alpha)\langle \Sigma(y_\varepsilon)D\varphi, D\varphi\rangle.
\end{aligned}
$$

Take $\alpha = \frac{2\lambda}{(1-\lambda)} > 0$ in order to have $\frac{1}{\lambda} - 1 - \frac{1}{\alpha} = \frac{1}{2}(\frac{1}{\lambda} - 1) > 0$. We recall that $\Sigma > 0$ by (A4); hence for $\varepsilon$ small enough the above scalar product is nonnegative. Thus

$$
0 \geq \varphi_t - (1+\alpha)\langle \Sigma(y_\varepsilon, t_\varepsilon)D\varphi, D\varphi\rangle = \varphi_t - \frac{1+\lambda}{1-\lambda}\langle \Sigma(y_\varepsilon, t_\varepsilon)D\varphi, D\varphi\rangle.
$$

Letting $\varepsilon$ go to 0, we get

$$
0 \geq \varphi_t(\bar{x}, \bar{t}) - \frac{2}{1-\lambda}\langle \Sigma(\bar{x}, \bar{t})D\varphi(\bar{x}, \bar{t}), D\varphi(\bar{x}, \bar{t})\rangle,
$$

which proves the result. □

*Proof of Lemma* 4.2. We make the proof for $U$, the second inequality being similar. Let $x_0 \in \Gamma$ and consider, for $\eta > 0$,

$$
\sup_{t \in [0,T]} \{U(x_0, t) - \eta t\}.
$$

This supremum is achieved at a point $t_0 \in [0, T]$ and we can assume, up to subtract $|t - t_0|^2$, that it is a strict local maximum. Consider, for $\varepsilon > 0$,

$$
\sup_{\bar{B}(x_0,1)\times[0,T]} \left\{U(x,t) - \frac{|x - x_0|^2}{\varepsilon^2} - \eta t\right\}.
$$

This supremum is achieved at a point $(x_\varepsilon, t_\varepsilon)$ and it is easy to see that $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$ when $\varepsilon \to 0$.

Suppose that $t_\varepsilon > 0$. It allows us to write the viscosity inequality for the subsolution $U$ at $(x_\varepsilon, t_\varepsilon)$ and we get

$$\eta + h(x_\varepsilon) \left| 2\frac{x_\varepsilon - x_0}{\varepsilon^2} \right|^2 \leq 0. \tag{4.7}$$

Since $h \in W^{2,\infty}$ in a neighborhood of $\Gamma$, we can write a Taylor expansion of $h$ at $x_0$ for $\varepsilon$ small enough

$$h(x_\varepsilon) = h(x_0) + \langle Dh(x_0), x_\varepsilon - x_0 \rangle + \frac{1}{2}\langle D^2 h(x_0)(x_\varepsilon - x_0), x_\varepsilon - x_0 \rangle + o(|x_\varepsilon - x_0|^2).$$

From (4.7) and (A5), it follows

$$\eta - (2C + m(|x_\varepsilon - x_0|^2)) \left( \frac{|x_\varepsilon - x_0|^2}{\varepsilon^2} \right)^2 \leq 0,$$

where $m$ is a modulus of continuity. Since $|x_\varepsilon - x_0|^2/\varepsilon^2 \to 0$ as $\varepsilon \to 0$, we obtain a contradiction for small $\varepsilon$.

Therefore $t_\varepsilon = 0$ for $\varepsilon$ small enough. It follows that for all $(x,t) \in \bar{B}(x_0, 1) \times [0, T]$, we have

$$U(x,t) - \frac{|x - x_0|^2}{\varepsilon^2} - \eta t \leq U(x_\varepsilon, 0) - \frac{|x_\varepsilon - x_0|^2}{\varepsilon^2} \leq \psi(x_\varepsilon).$$

Setting $x = x_0$ and sending $\varepsilon$ and then $\eta$ to 0, we obtain the conclusion. ∎

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