

## Critical-Point Description of the Transition from Vibrational to Rotational Regimes in the Pairing Phase

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An approximate solution at the critical point of the pairing transition from harmonic vibration to deformed rotation in gauge space is found by analytic solution of the collective pairing Hamiltonian. The eigenvalues are expressed in terms of the zeros of Bessel functions of integer order. The results are compared to the pairing bands based on the Pb isotopes.

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Pair correlations in electron motion are directly related to macroscopic phenomena such as superconductivity [1]. The concepts that were developed to describe such correlations found immediate application in nuclear physics and provided a key to understanding the excitation spectra of even  $A$  nuclei, odd-even mass differences, rotational moments of inertia, and a variety of other phenomena [2,3]. Pair correlations are also of great importance in describing the behavior of other finite fermion systems such as  $^3\text{He}$  clusters, Fermi-gas condensates, fullerenes, quantum dots, and metal clusters [4]. Ideas to describe phenomena associated with pair correlations in any one system are likely to find application in the others.

An early approach to describing pair correlations in nuclei was the development of a collective model by Bès and co-workers [5]. The variables in the model are a pair deformation  $\alpha$  (which can be related to the gap parameter) and a gauge angle  $\phi$  (which is the canonical conjugate to the particle-number operator  $N$ ). The collective pairing Hamiltonian was derived in direct analogy to the Bohr collective Hamiltonian which describes the quadrupole degree of freedom for the nuclear shape [6].

Notable benchmarks of nuclear behavior such as the harmonic vibrator [7], the symmetrically deformed rotor [8], and the soft triaxial rotor [9] correspond to analytic solutions of the Bohr Hamiltonian. They also correspond to limits of the interacting boson model [10]. An algebraic description of the nature of the transition between these limits has been developed in direct analogy with classical phase transitions [11]. The Bohr Hamiltonian has recently received renewed attention due to the suggestion that simple analytic approximations can be made to describe the critical point of the transitions between nuclear shapes [12–14]. These can then serve as new benchmarks against which nuclear properties can be compared.

In this Letter, we apply similar approximations to obtain an analytic solution of the collective pairing Hamiltonian corresponding to the critical point of the transition from a “normal” to a “superconducting” nucleus. Nuclei with

two identical particles added or removed from a closed-shell configuration should be close to the normal limit, where there is no static deformation of the pair field and the fluctuations of the field give rise to a pairing vibrational spectrum [15]. Pairing vibrational structures have been observed around  $^{208}\text{Pb}$  [16], although large anharmonicities must be included in this interpretation. In nuclei with many particles outside of the closed-shell configuration, a static deformation of the pair field arises and rotational behavior results. This corresponds to the superconducting limit. The angular variable in the rotational motion is the gauge angle  $\phi$ , which describes the orientation in gauge space. This broken symmetry in gauge space results in a pair-rotational band [17] comprising the sequence of ground states of even-even nuclei, differing by pairs of identical nucleons, and with many nucleons outside a closed shell.

Here we would like to investigate the transition from pair-vibrational to pair-rotational regimes. To do this, we will find solutions to the collective pairing Hamiltonian [5]:

$$-\frac{\hbar^2}{2B} \frac{\partial^2 \psi}{\partial \alpha^2} - \frac{\hbar^2}{4} \left( \frac{1}{\mathfrak{I}B} \frac{\partial \mathfrak{I}}{\partial \alpha} - \frac{1}{B^2} \frac{\partial B}{\partial \alpha} \right) \frac{\partial \psi}{\partial \alpha} + \left( V(\alpha) + \frac{\hbar^2 M^2}{2\mathfrak{I}} - E \right) \psi = 0, \quad (1)$$

where  $\alpha$  is the deformation of the pair field,  $\mathfrak{I}$  is an inertia parameter,  $B$  is a mass parameter,  $M = A - A_0$  (number of particles  $A$  relative to a reference  $A_0$ ), and  $V(\alpha)$  is the potential. In general,  $\mathfrak{I}$  and  $B$  are functions of the pair deformation  $\alpha$ . By choosing suitable potentials, we can find analytical solutions of Eq. (1) in the different limits. Consider the potential energy surface as a function of the pair-field deformation parameter  $\alpha$ , as schematically illustrated in Fig. 1. In the vibrational limit, the potential is parabolic with a minimum at  $\alpha = 0$ . The transition to the rotational pairing regime gives rise to a deformed minimum in the potential. At the critical point, these two minima cross and the deformation of the pair field changes

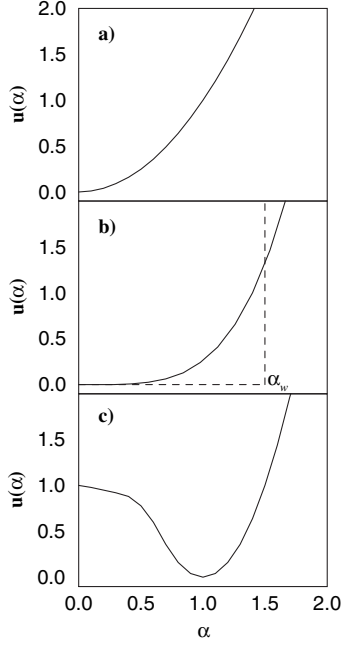


FIG. 1. Schematic of the potential energy surfaces  $u(\alpha)$  as functions of the deformation of the pair field  $\alpha$  for the transition from (a) spherical vibrations, through (b) the critical point (the infinite-square-well approximation with an outer wall at  $\alpha = \alpha_w$  is shown with a dashed line), and to (c) deformed rotation.

from spherical to deformed. This picture is supported by boson calculations of potential surfaces [18].

In the pair-rotational limit, we can approximate the potential by assuming a static deformation of the pair field,  $\alpha = \alpha_0$ . Under this assumption, the derivatives in Eq. (1) tend to zero, implying that

$$E \propto (A - A_0)^2, \quad (2)$$

and we find the expected parabolic dependence between energy and particle number for pair rotations.

In the case in which the equilibrium deformation is zero and fluctuations of the pair field are small, then  $B$  is a constant and  $\mathfrak{S} = 4B\alpha^2$  [5]. Equation (1) then becomes

$$-\frac{\hbar^2}{2B} \frac{\partial^2 \psi}{\partial \alpha^2} - \frac{\hbar^2}{2B\alpha} \frac{\partial \psi}{\partial \alpha} + \left( \frac{\hbar^2 M^2}{8B\alpha^2} + V(\alpha) - E \right) \psi = 0. \quad (3)$$

Introducing the reduced energy  $\epsilon = (2B/\hbar^2)E$  and reduced potential  $u(\alpha) = (2B/\hbar^2) V(\alpha)$ , Eq. (3) can be rewritten as:

$$\frac{\partial^2 \psi}{\partial \alpha^2} + \frac{1}{\alpha} \frac{\partial \psi}{\partial \alpha} + \left( \epsilon - u(\alpha) - \frac{M^2}{4\alpha^2} \right) \psi = 0. \quad (4)$$

For pair vibrations, the potential can be taken to be a parabola with a minimum at zero pair deformation [see Fig. 1(a)]. With  $u(\alpha) = \alpha^2$  and  $m = M/2$ , Eq. (4) can be expressed as:

$$\frac{\partial^2 \psi}{\partial \alpha^2} + \frac{1}{\alpha} \frac{\partial \psi}{\partial \alpha} + \left( \epsilon - \alpha^2 - \frac{m^2}{\alpha^2} \right) \psi = 0. \quad (5)$$

Equation (5) has the same form as the radial equation of an isotropic oscillator (see, for example, Ref. [19]) and can be solved by using a trial wave function of the form:

$$\psi = \alpha^m e^{-\alpha^2/2} W(\alpha). \quad (6)$$

Solving Eq. (5), we find that

$$E \propto (A - A_0), \quad (7)$$

which is the expected linear dependence between energy and particle number in the vibrational limit.

We now turn to finding an analytical solution for the critical point of the transition from vibrational to rotational pairing regimes. As pointed out by Iachello [12], the situation in which a potential has a flat behavior as a function of some coordinate appears typically when the system undergoes a phase transition at a critical point. A simple approximation to the critical-point potential [see Fig. 1(b)] is an infinite-square well:

$$\begin{aligned} u(\alpha) &= 0, & \alpha &\leq \alpha_w, \\ u(\alpha) &= \infty, & \alpha &> \alpha_w. \end{aligned} \quad (8)$$

This approximation to the potential at the critical point of the pairing-phase transition is identical to the assumption of infinite-square well potentials used in the critical-point descriptions of nuclear shape transitions [12–14]. Using this potential in Eq. (4), we obtain a Bessel equation:

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{1}{z} \frac{\partial \psi}{\partial z} + \left( 1 - \frac{M^2}{4z^2} \right) \psi = 0, \quad (9)$$

where  $z = \alpha k$  with  $k = \epsilon^{1/2}$ . The boundary condition  $\psi(\alpha_w) = 0$  determines the eigenfunctions to be related to Bessel functions of integer order such that:

$$\psi_{\xi,M}(z) = c_{\xi,M} J_{M/2}(z), \quad (10)$$

where  $c_{\xi,M}$  are constants of normalization. The associated eigenvalues are given by:

$$\epsilon_{\xi,M} = k_{\xi,M}^2, \quad k_{\xi,M} = \frac{x_{\xi,M}}{\alpha_w}, \quad (11)$$

where  $x_{\xi,M}$  is the  $\xi$ th zero of the Bessel function  $J_{M/2}(z)$ .

In this Letter, we will use the eigenvalues given by Eq. (11) to find the spectrum of states for comparison to experimental data. Transition matrix elements, related to two-nucleon transfer probabilities [5], could also be determined, since

$$\langle \psi_{\xi',M'} | \hat{O} | \psi_{\xi,M} \rangle \propto \int_0^{\alpha_w} \psi_{\xi',M'}^* \alpha^2 \psi_{\xi,M} d\alpha, \quad (12)$$

where  $\hat{O}$  is the pair transfer operator. An extensive comparison of all data, including pair transfer probabilities, will be the subject of a longer article.

We now turn to finding the energy spectrum of the states from the zeros of the related Bessel functions using Eq. (11). Normalizing the energies of excited states to that of the first excited state, we can form a reduced spectrum of states defined as:

$$E_N = \frac{x_{\xi,M}^2 - x_{1,0}^2}{x_{1,2}^2 - x_{1,0}^2}. \quad (13)$$

The normalized excitation energies found in this way for some states are given in Table I. In Fig. 2, it can be seen that the energies for the  $\xi = 1$  sequence of states follow a behavior which is between the linear dependence for a pure harmonic vibrator [Eq. (7)] and the parabolic dependence for a deformed rotor [Eq. (2)] as expected for this description of the transition between the two limits. The sequence of states with  $\xi = 1$  corresponds empirically to the sequence formed by the  $0^+$  ground states of neighboring even-even nuclei along an isotopic or isotonic chain. States with  $\xi > 1$  correspond to excited  $0^+$  states formed from pair excitations.

Before proceeding, it is worth commenting on the algebraic structure associated with the solutions of the pairing Hamiltonian. Following Iachello [13], the square-well approximation of the critical point in a generalized phase transition of the form  $U(n) \leftrightarrow SO(n+1)$ , with  $n \geq 2$ , has the  $E(n)$  dynamic symmetry, where  $E(n)$  is the  $n$ -dimensional Euclidean group. For the pairing-phase transition,  $n = 2$  and the corresponding symmetry at the critical point is  $E(2)$ . The eigenfunctions of the critical-point solution are Bessel functions of integer order and form a basis for the representations of this group.

We will now compare our calculations with experiment by using the known data on the mass excesses [20]  $\varepsilon(A)$  along an isotopic sequence. Such a comparison for the Pb isotopes is shown in Fig. 3. We define the empirical neutron pairing energy  $E_{\text{pair}}(A)$  as:

$$E_{\text{pair}} = [\varepsilon(A) - \varepsilon(A_0)] - C(A - A_0), \quad (14)$$

where  $\varepsilon(A) - \varepsilon(A_0)$  is the difference between the mass excess for a given isotope with mass number  $A$  and the mass excess of the chosen reference nucleus with mass number  $A_0$ . A linear term is subtracted and the constant  $C$  is chosen to make  $E_{\text{pair}}(A_0 - 2) = E_{\text{pair}}(A_0 + 2)$  [21]. The values of  $E_{\text{pair}}$  in Fig. 3 are again normalized to the first excited state.

For  $A_0 = 202$  or 204, we find that the sequence follows the parabolic dependence of the rotational pairing regime.

TABLE I. Excitation energies of the critical-point description.

	$\xi = 1$	$\xi = 2$	$\xi = 3$	$\xi = 4$
$ M  = 0$	0.00	2.77	7.77	14.97
$ M  = 2$	1.00	4.88	10.98	19.30
$ M  = 4$	2.31	7.31	14.52	23.95
$ M  = 6$	3.92	10.06	18.39	28.93

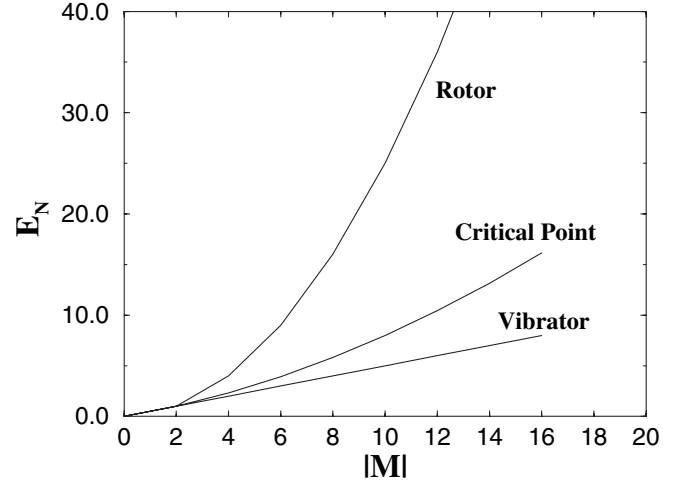


FIG. 2. Plots of normalized energies for the lowest sequence of states of the rotor, vibrator, and critical-point descriptions.

This indicates a large static deformation of the pair field (associated with a superconducting phase). With  $A_0 = 206$ , deviations from the rotational parabolic dependence are seen. With  $A_0 = 208$ , we find that the sequence is closer to the vibrational (normal) phase. Empirically, the transition to the rotational regime requires only a few pairs outside of the closed-shell configuration. This result is closely related to the fact that only a few nucleon pairs contribute to the pairing gap [22].

The isotopes around  $^{208}\text{Pb}$  have been used as the textbook example of pair vibrations in nuclei [21]. Deviations from the pure vibrational spectrum were described in terms of large anharmonicities. These deviations in the energies are clearly seen in Fig. 3, and the sequence around

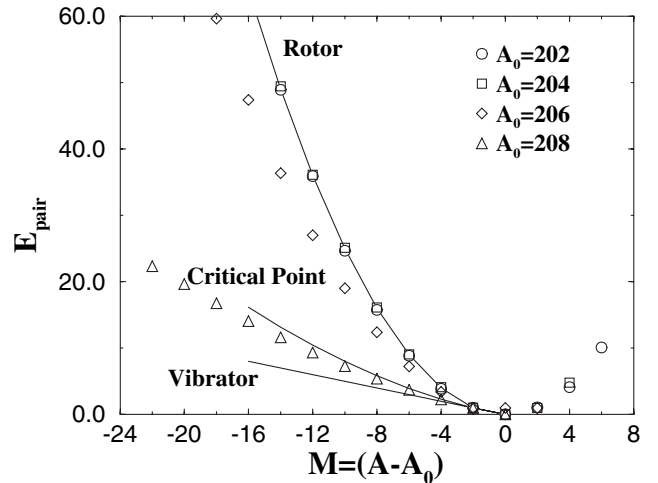


FIG. 3. Plots of the empirical neutron pair energies for the sequence formed by the  $0^+$  ground states of the Pb isotopes using as a reference  $^{202}\text{Pb}$  (open circles),  $^{204}\text{Pb}$  (open squares),  $^{206}\text{Pb}$  (open diamonds), and  $^{208}\text{Pb}$  (open triangles). For comparison are shown the expectations of the pure vibrator, pure deformed rotor, and the critical-point description (solid lines).

$A_0 = 208$  lies much closer to the transitional description. We have also examined the sequences of Ni and Sn isotopes using the doubly magic nuclei  $^{56}\text{Ni}$  and  $^{132}\text{Sn}$  as references. Again, the spectra of neutron pair energies lie closer to the transitional description than to the vibrational description. These observations suggest a general phenomenon. In using the collective pairing Hamiltonian, a square-well potential provides a simple analytic approach that can naturally account for the observed anharmonicities associated with the harmonic oscillator solution. The measurement of properties of new doubly magic nuclei such as  $^{100}\text{Sn}$  [23] and  $^{78}\text{Ni}$  [24], and their even-even neighbors, will be of great interest in testing this idea.

In the future, we should be able to apply many of the modifications and ideas that have arisen as a consequence of the introduction of the critical-point descriptions of shape transitions to the description of the pairing-phase transition. For instance, modifying the infinite-square-well potential to a finite-square-well potential [25], varying the stiffness of the wall of the potential [26], and changing the softness of the deformation [27] are all important problems open to analytical solution. An advantage of the collective model is that the role of fluctuations of the pair gap are naturally accounted for by the choice of potential. For instance, by varying the softness, we can investigate the extent to which these fluctuations might alter the nature of the phase transition.

We could also expand the comparison with the experimental data to include energies of excited states (corresponding to  $\xi > 1$  in our description) and transfer strengths between the different states. In the case of excited  $0^+$  states in  $^{206}\text{Pb}$  and  $^{208}\text{Pb}$ , suggested as pair excitations, the energies lie close to the vibrational limit. It would be interesting to see if the first-excited pairing band continues this trend over a longer sequence of states.

To summarize, we have presented analytical solutions of the collective pairing Hamiltonian [5] by using simple approximations to the potential in the limits of harmonic vibrations (zero deformation of the pair field corresponding to normal behavior), deformed rotation (static deformation of the pair field corresponding to superconducting behavior), and at an intermediate transitional point. In the latter situation, the potential is approximated as an infinite-square well. The eigenvalues are expressed in terms of the zeros of Bessel functions of integer order. Comparison to the pairing bands based on the Pb isotopes suggests that this description may provide a simple approach to explaining the observed anharmonicities of the pairing vibrational structure around  $^{208}\text{Pb}$ .

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- [1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Phys. Rev.* **106**, 162 (1957).
  - [2] A. Bohr, B. R. Mottelson, and D. Pines, *Phys. Rev.* **110**, 936 (1958).
  - [3] S. T. Belyaev, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **31**, 1 (1959).
  - [4] *Pair Correlations in Many-Fermion Systems*, edited by V. Z. Kresin (Plenum, New York, 1988).
  - [5] D. R. Bès, R. A. Broglia, R. P. J. Perazzo, and K. Kumar, *Nucl. Phys.* **A143**, 1 (1970).
  - [6] A. Bohr, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **26**, 1 (1952).
  - [7] G. Scharff-Goldhaber and J. Weneser, *Phys. Rev.* **98**, 212 (1955).
  - [8] A. Bohr and B. R. Mottelson, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **27**, 1 (1953).
  - [9] L. Wilets and M. Jean, *Phys. Rev.* **102**, 788 (1956).
  - [10] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, England, 1987).
  - [11] A. E. L. Dieperink, O. Scholten, and F. Iachello, *Phys. Rev. Lett.* **44**, 1747 (1980).
  - [12] F. Iachello, *Phys. Rev. Lett.* **85**, 3580 (2000).
  - [13] F. Iachello, *Phys. Rev. Lett.* **87**, 052502 (2001).
  - [14] F. Iachello, *Phys. Rev. Lett.* **91**, 132502 (2003).
  - [15] D. R. Bès and R. A. Broglia, *Nucl. Phys.* **80**, 289 (1966).
  - [16] R. A. Broglia, O. Hansen, and C. Riedel, *Adv. Nucl. Phys.* **6**, 287 (1973).
  - [17] R. A. Broglia, J. Terasaki, and N. Giovanardi, *Phys. Rep.* **335**, 1 (2000).
  - [18] B. Sorensen, *Nucl. Phys.* **A134**, 1 (1969).
  - [19] B. H. Bransden and C. J. Joachain, *Introduction to Quantum Mechanics* (Longman Scientific and Technical, Harlow, England, 1989).
  - [20] G. Audi, A. H. Wapstra, and C. Thibault, *Nucl. Phys.* **A729**, 337 (2003).
  - [21] A. Bohr and B. R. Mottelson, *Nuclear Structure* (Benjamin, New York, 1974).
  - [22] D. M. Brink and R. A. Broglia, *Nuclear Superfluidity: Pairing in Finite Systems* (Cambridge University Press, Cambridge, England, 2005).
  - [23] M. Chartier *et al.*, *Phys. Rev. Lett.* **77**, 2400 (1996).
  - [24] P. T. Hosmer *et al.*, *Phys. Rev. Lett.* **94**, 112501 (2005).
  - [25] M. Caprio, *Phys. Rev. C* **65**, 031304 (2002).
  - [26] M. Caprio, *Phys. Rev. C* **69**, 044307 (2004).
  - [27] N. Pietralla and O. M. Gorbachenko, *Phys. Rev. C* **70**, 011304 (2004).