Infinitesimally stable and unstable singularities of 2 degrees of freedom completely integrable systems

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Abstract

In this article we give a list of 10 rank zero and 6 rank one singularities of 2 degrees of freedom completely integrable systems. Among such singularities, 14 are the singularities that satisfy a non-vanishing condition on the quadratic part, the remaining 2 are rank 1 singularities that play a role in the geometry of completely integrable systems with fractional monodromy. We describe which of them are stable and which are unstable under infinitesimal completely integrable deformations of the system.

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1 Introduction

A. Setting. A n degrees of freedom completely integrable system is a map $f = (f_1, ..., f_n) : M \to \mathbb{R}^n$, where M is a $2n$-dimensional symplectic manifold, the functions f_i Poisson commute, and the level-sets of f are compact.

Given a n degrees of freedom completely integrable system f, a theorem of Liouville-Arnol'd states in particular that, chosen a regular value r of f there is a neighbourhood U of r such that $f^{-1}(U)$ is isomorphic to $U \times (T^n \cup \cdots \cup T^n)$ and f conjugates to the projection on the first component. A consequence of the Liouville-Arnol'd theorem is that to each completely integrable system is associated a torus bundle with singularities.

Typically, singular values of a completely integrable system are responsible for the topology of the torus bundle [7], and the topology of the torus bundle has remarkable consequences on the distribution of quantum energy levels of the associated quantum system [22, 12]. It follows that the global analysis of, and the quantum mechanics associated to a completely integrable system rely on its singularities.

Singularities admit a rough description by their rank. A singularity of f is a point of M whose rank is less than n, the *critical values* of f are the images in $\mathbb{R}^n$ of all critical points. A more detailed description of singularities is called *bifurcation diagram.* A bifurcation diagram is a stratification of the set of values attained by f in subsets labelled with an indication of their rank and of the topology of the f-level sets above them. This is usually summarized in a few curves and labels as in Figure 1 below. The singularities of 2 degrees of freedom completely integrable systems are an ample representation of possible singularities, we hence concentrate on completely integrable systems on a 4-dimensional manifold M, that is maps $f : M \to \mathbb{R}^2$ such that $\{f_1, f_2\} = 0$, and with compact level-sets.

For a systematic investigation of the singularities of completely integrable systems, it is natural to resort to singularity theory (see [1] for a comprehensive treatment). Singularity theory deals with critical points of maps $f : M \to N$, where M is a m-dimensional manifold and N is a n-dimensional manifold. Its main goal is to provide a classification of 'reasonable' singularities. To achieve this classification one gives the following definitions

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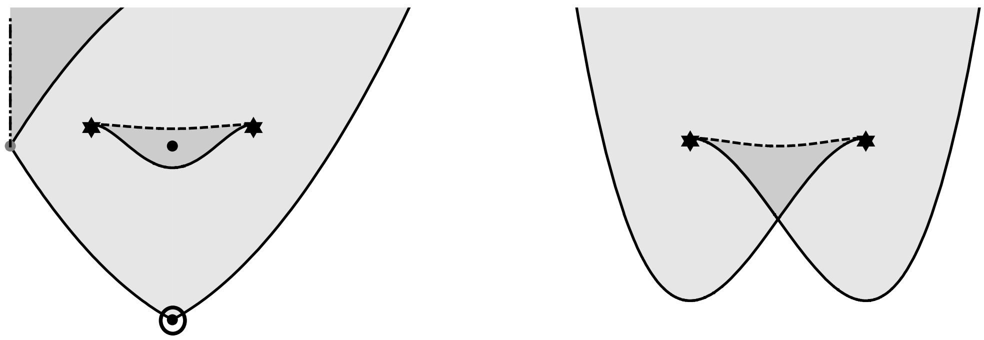


Figure 1: Two possible bifurcation diagrams of 2 degrees of freedom completely integrable systems. The light-gray regions correspond to regular values with connected preimage, the darker-gray regions correspond to regular values with two components in the preimage, the continuous lines correspond to elliptic (E) singularities, the dashed lines to hyperbolic (H) singularities, the dashed-dotted line corresponds to fold (P) singularities, the stars to cuspidal (C) singularities, the black point to a focus-focus (FF) singularity, the circled black point to an elliptic-elliptic (EE) singularity, the gray point to an elliptic-fold (EP) singularity.

Definition 1 *A* **germ** *of a function from M to N at a point p of M is the equivalence class of functions defined in a neighbourhood of p under the equivalence relation $f \sim g$ if $f = g$ in a neighbourhood of p. The space of germs at p is denoted $\mathrm{Germs}_p(M, N)$, its elements are couples (f, p).*

Definition 2 *Two germs (f, p) and $(\widetilde{f}, \widetilde{p})$ are* **equivalent***, and we write $(f, p) \equiv (\widetilde{f}, \widetilde{p})$, if there exists a germ of diffeomorphism φ in a neighbourhood of p with $\varphi(p) = \widetilde{p}$ and a germ of diffeomorphism ψ in a neighbourhood of $f(p)$ with $\psi(f(p)) = \widetilde{f}(\widetilde{p})$ such that $(\widetilde{f}, \widetilde{p}) = (\psi \circ f \circ \varphi^{-1}, \widetilde{p})$ in $\mathrm{Germs}_{\widetilde{p}}(M, N)$.*

Definition 3 *A germ (f, p) is* **stable** *if there exists a neighbourhood U of p and a $\varepsilon > 0$ such that for every germ $(\widetilde{f}, p)$ with $\sup_{q \in U} ||f(q) - \widetilde{f}(q)|| < \varepsilon$ there is a $\widetilde{p}$ in U for which $(f, p) \equiv (\widetilde{f}, \widetilde{p})$.*

Definition 4 *A* **singularity** *is the class of germs up to the above equivalence, a singularity will be denoted as $[f, p]$. A singularity $[f, p]$ is* **stable** *if the germ (f, p) is stable.*

One of the major achievements in singularity theory is the proof of Mather [17] that a germ (f, p) is stable if and only if it is **infinitesimally stable**: consider a system of coordinates $x_1, ..., x_m$ near p and $y_1, ..., y_n$ near $f(p)$, then the germ $x \mapsto f(x)$ is a function from a neighbourhood of 0 in $\mathbb{R}^m$ to a neighbourhood of 0 in $\mathbb{R}^n$. Let g be a function defined in a neighbourhood of 0 in $\mathbb{R}^m$ with values in $\mathbb{R}^n$, and let $f_\lambda = f + \lambda g$ with $\lambda \in \mathbb{R}$. The family of functions f_λ are called **parametric deformation** of f, and g is called **infinitesimal deformation** of f. If $(f(x), 0)$ is stable then for λ small there exist ψ_λ local automorphism of $\mathbb{R}^n$ near 0 and φ_λ local automorphism of $\mathbb{R}^m$ near 0 such that $\psi_\lambda \circ f \circ \varphi_\lambda^{-1} = f_\lambda$. Differentiating with respect to λ in $\lambda = 0$ one obtains that: *if the germ $(f(x), 0)$ is stable, then for every infinitesimal deformation g there exist a function h defined in a neighbourhood of 0 in $\mathbb{R}^m$ with values in $\mathbb{R}^m$ and a function k defined in a neighbourhood of 0 in $\mathbb{R}^n$ with values in $\mathbb{R}^n$ such that $g = \nabla f \cdot h + k \circ f$.* The other implication is the theorem of Mather.

The application of singularity theory to completely integrable systems appears very natural: it seems one needs only to change the notion of infinitesimal deformation into that of **infinitesimal completely integrable deformation**. The only reasonable requirement for a function g to be an infinitesimal completely integrable deformation of f is that $\{f_i, g_j\} = \{f_j, g_i\}$ for every $i, j = 1, ..., n$. In fact, a singularity of an integrable system is represented by a germ f on a symplectic manifold M to $\mathbb{R}^n$ satisfying the differential equations $\{f_i, f_j\} = 0$ for every $i, j = 1, ..., n$, hence the components of a deformation $f + \lambda g$ must satisfy $\{(f + \lambda g)_i, (f + \lambda g)_j\} = 0$. Differentiating with respect to λ one obtains the above condition on the components of g. There are two main complications with the attempt of mimicking Mather's construction:

R1 germs of completely integrable systems $\widetilde{f}$ neighbouring a given germ of completely integrable system f form a dense and not pathwise-connected sub-manifold in the manifold $\text{Germs}_p(M, \mathbb{R}^n)$ [25, 16].

R2 There exist completely integrable systems f and infinitesimal completely integrable deformations g of f such that no parametric deformation of f, $\lambda \mapsto f_\lambda$, consisting of completely integrable systems satisfies $\partial_\lambda f_\lambda|_{\lambda=0} = g$.

B. Goals and layout of the paper. The main goal of the theory of singularities of completely integrable systems is to provide a classification of 'relevant' singularities together with their normal form expression. After defining infinitesimal stability, we analize the infinitesimal stability of the list of all representatives of singularities of 2 degrees of freedom completely integrable systems whose quadratic differential has maximal dimension, and of other two singularities described in [4]. Our investigation is motivated by the study of the singularities appearing in fractional monodromy [19, 20] and in bidromy [23], and is an advance in answering natural questions regarding their local expression and their codimension (i.e. the number of parameters needed to expose them).

The investigation of Lagrangian bundles with singularities has been considered in many other works. In [24, 10] the concept of deformation of singular Lagrangian foliations is posed in terms of algebraic-topology. In [6], among other things, are discussed cuspidal singularities, while in [15] it is proven in the smooth cathegory that rank 0, non-degenerate singularities (in our terminology the singularities labelled EE, EH, HH, FF) are infinitesimally stable. In our work we systematize the analysis and give a real analytic approach to investigate rank 0 and rank 1 singularities of 2 degrees of freedom completely integrable systems.

The layout of the paper is the following: in Section 2 we define infinitesimal stability for singularities of completely integrable systems. In Section 3 we perform a systematic analysis of the rank 0 singularities whose quadratic differential (see [5] for a definition) spans a 2-dimensional subspace of the space of quadratic forms. This analysis produces a list that includes the 7 classical rank zero singularities: the non-degenerate *elliptic-elliptic (EE)*, *elliptic-hyperbolic (EH)*, *hyperbolic-hyperbolic (HH)*, *focus-focus (FF)* (see [4], page 299), the degenerate *elliptic-fold (EP)*, *hyperbolic-fold (HP)* and *fold-fold (PP)* and other 3 singularities that do not appear to play a role in physical nor geometric theory. We then prove that the singularities (EE), (EH), (HH), (FF) are infinitesimally stable, that the others are not, and we describe some deformations that make these latter unstable. In Section 4 we do the same for rank 1 singularities whose quadratic differential spans a 1-dimensional subspace of the quadratic forms. We analyze the non-degenerate *hyperbolic (H)* and *elliptic (E)*, together with the degenerate *fold (P)* and *cuspidal (C)* singularities (for this last singularity see [4] Section 10.6 and [6]), and we show that (E), (H) and (C) are infinitesimally stable, while (P) is not. At the end of Section 4 we apply the same technique to other two singularites described in [4] Section 10.6, and we show they are infinitesimally unstable.

2 Infinitesimal stability

A. Infinitesimal stability in singularity theory. We work in the analytic category. A singularity $[f, p]$ can always be represented by an element $\left(\sum_{J\in\mathbb{N}^m} a_{1,J} x^J, ..., \sum_{J\in\mathbb{N}^m} a_{n,J} x^J\right) = \sum_{J\in\mathbb{N}^m} \vec{a}_J x^J$ of the space of n copies of convergent power series in m variables. (From now on we indicate with x the collection of variables $x_1, ..., x_m$, we do the same later with the variables y and z.)

Definition 5 *A singularity $f = \sum_{J\in\mathbb{N}^m} \vec{a}_J x^J$ is* **stable** *if there exist $\delta > 0$ and $\varepsilon > 0$ such that every power series $\widetilde{f} = \sum_{J\in\mathbb{N}^m} \vec{b}_J x^J$ with radius of convergence bigger than δ and*

$$\sup_{||x||\leq\delta} ||f(x) - \widetilde{f}(x)|| < \varepsilon,$$

admits a local analytic automorphism φ of $\mathbb{R}^m$ near 0 and a local analytic automorphism ψ of $\mathbb{R}^n$ near 0 such that $\widetilde{f} = \psi \circ f \circ \varphi^{-1}$.

The theorem of Mather can be formulated in an explicit condition.

Theorem 6 *A singularity $f = \sum_{J\in\mathbb{N}^m} \vec{a}_J x^J$ is stable if and only if there exists $\delta > 0$ such that every element $g = \sum_{J\in\mathbb{N}^m} \vec{b}_J x^J$ with radius of convergence bigger than δ satisfies*

$$(IA) \qquad g(x) = \frac{\partial f}{\partial x_1}(x)H^1(x) + \cdots + \frac{\partial f}{\partial x_m}(x)H^m(x) + K \circ f(x)$$

with $H^i(x) = \sum_{J\in\mathbb{N}^m} H^i_J x^J$ and $K(z) = \sum_{J\in\mathbb{N}^n} \vec{K}_J z^J$ convergent power series (the coefficients $H^i_J \in \mathbb{R}$, while $\vec{a}_J, \vec{b}_J, \vec{K}_J \in \mathbb{R}^n$).

Definition 7 *We call* **infinitesimal action space** *the subvector space of convergent power series that can be written as in equation (IA).*

B. Infinitesimal stability for completely integrable systems. Let us now specialize to the case in which M is a $2n$-dimensional symplectic manifold, $N = \mathbb{R}^n$, and $(f, p) \in \mathrm{Germs}_p(M, N)$ represents a singularity of a completely integrable system. In view of remark R1, a statement as that of Mather's theorem is impossible: the space of integrable systems do not form a closed submanifold in the space of all functions. On the other hand, completely integrable systems typically depend on parameters. It is hence as fundamental to investigate the stability of singularities under parametric deformations of a given completely integrable system. Without loss of generality, from now on we shall always make a choice of local canonical coordinates x, y so that the singularity is placed in the origin. A singularity of a completely integrable system is hence a convergent power series $f(x, y) = \sum_{I,J\in\mathbb{N}^m} \vec{a}_{I,J} x^I y^J$ with $\vec{a}_{I,J} \in \mathbb{R}^n$ and with the appropriate commutation relations.

Definition 8 *A* **parametric deformation of a completely integrable system** *f is a parametric family of functions $f_\lambda(x, y) = \sum_{I,J\in\mathbb{N}^m} \vec{a}^{\lambda}_{I,J} x^I y^J$, such that $f_0 = f$ and, for every fixed λ the map f_λ is a completely integrable system.*

Which singularities are parametrically stable? How does a singularity of a completely integrable system change under parametric deformations? We already observed that, given a parametric deformation f_λ of a completely integrable system f, the identity $\{(f_\lambda)_i, (f_\lambda)_j\} = 0$ implies that the function $g = \partial_\lambda f_\lambda|_{\lambda=0}$ satisfies $\{f_i, g_j\} = \{f_j, g_i\}$.

Definition 9 *An* **infinitesimal completely integrable deformation** *of a completely integrable system $f(x, y) = \sum_{I,J\in\mathbb{N}^m} \vec{a}_{I,J} x^I y^J$ is a convergent power series $g(x, y) = \sum_{I,J\in\mathbb{N}^m} \vec{b}_{I,J} x^I y^J$ that satifies the homological equation*

$$(HOM) \qquad \{f_i, g_j\} = \{f_j, g_i\}.$$

We plan to use this space as the tangent space to the manifold of parametric deformations of a given completely integrable system f. The goal would be to prove that a germ at p of a completely integrable system f is **parametrically stable** if and only if equation (IA) can be solved when g is any infinitesimal completely integrable deformation. Remark R2 indicates that also this idea is not viable. In fact the space tangent to parametric deformations of a completely integrable system is a subspace of the space of infinitesimal completely integrable deformations. This inclusion is strict, as is shown in the example below.

Example: In $\mathbb{R}^4$ with canonical coordinates x_1, y_1, x_2, y_2, consider the completely integrable system $f = (x_1^2 + y_1^2, x_2^2)$. A good choice of infinitesimal completely integrable deformation of f is the function $g = (x_2, y_2)$. We show that there is no 1-parametric analytic family of completely integrable systems whose derivative with respect to λ at $\lambda = 0$ is g. Every such 1-parametric family would be of the form

$$f_\lambda(x_1, y_1, x_2, y_2) = (x_1^2 + y_1^2 + \lambda x_2 + \lambda^2 h_1(x, y) + o(2), x_2^2 + \lambda y_2 + \lambda^2 h_2(x, y) + o(2)),$$

where $o(d)$ is infinitesimal with respect to λ^d. It follows that the Poisson brackets of the two components of f_λ are $\lambda^2(1 + \{x_1^2 + y_1^2, h_2\} + \{h_1, x_2^2\}) + o(2) = \lambda^2(1 + 2x_1\{x_1, h_2\} + 2y_1\{y_1, h_2\} - 2x_2\{x_2, h_1\}) + o(2)$. This function cannot be zero in a neighbourhood of zero for every $\lambda \neq 0$.

To cope with this difficulty we first give the following definition

Definition 10 *A singularity of completely integrable system f is* **infinitesimally stable** *if every infinitesimal completely integrable deformation g of f belongs to the infinitesimal action space, that is*

$$(SIA) \qquad g(x,y) = \frac{\partial f}{\partial x_1}(x,y)H^1(x,y) + \cdots + \frac{\partial f}{\partial y_n}(x,y)H^{2n}(x,y) + K \circ f(x,y).$$

for appropriate germs of functions $H^i(x) = \sum_{J \in \mathbb{N}^m} H^i_J x^J$ and $K(z) = \sum_{J \in \mathbb{N}^n} \vec{K}_J z^J$.

We keep calling **infinitesimal action space** *the subvector space of convergent power series that can be written as in equation (SIA).*

And then we observe that, if a singularity is parametrically unstable, then there exists an infinitesimal completely integrable deformation g such that equation (SIA) has no solutions. Vice-versa, if one determines an infinitesimal completely integrable deformation g such that (SIA) has no solutions and such that $\{g_1, g_2\} = 0$ (this will always happen in the examples we discuss), then the singularity is unstable.

3 Rank zero singularities

In this section, we analyze the infinitesimal stability of some rank 0 singularities of 2 degrees of freedom completely integrable systems. To achieve this goal we compute a list of singularities based on conditions on the quadratic part of their Taylor expansion. For each of the elements of such list we characterize its infinitesimal completely integrable deformations (the g that satisfy the homological equation HOM), and then determine wether all such deformations do belong to the infinitesimal action space or not.

A. Local classification. Assume to be given a rank 0 singularity of an analytic, 2 degrees of freedom completely integrable system, that is 2 power series $f = (f_1, f_2)$ in the canonical variables x_1, y_1, x_2, y_2 such that $f(0) = 0$ and $d_0 f = 0$. The quadratic differential ([5] page 85 and [1] page 60) of f defines a linear map from $\mathbb{R}^2$ to the quadratic functions on $\mathbb{R}^4$. Let us make the assumption that the image $\mathfrak{a}$ of this map is 2-dimensional. The set of quadratic functions on $\mathbb{R}^4$ with Poisson brackets is isomorphic to the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ with Lie brackets, and to $\mathfrak{a}$ corresponds a 2-dimensional commutative subalgebra of $\mathfrak{sp}(4,\mathbb{R})$.

The classification of real commutative subalgebras of semi-simple Lie algebras due to Kostant [13] and Sugiura [26], and later applied to 2,4 and 6-dimensional real symplectic Lie algebras in [21], allows us to prove that, up to a sympletic change of coordinates in the domain and a change of coordinates in the range, there are 10 canonical couples of Poisson commuting quadratic polynomials, that we list in the lemma below. The proof of the lemma is a translation in the space of quadratic functions of the statements in [21]. The only complication appears in dealing with the fold-fold (PP) singularity, since the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ contains a 3-dimensional commutative subalgebra isomorphic to that spanned by the polynomials x_1^2, x_2^2, x_1x_2, and one should prove that every 2-dimensional subspace of this 3-dimensional space is conjugate to that spanned by x_1^2, x_2^2.

Lemma 11 *Let f be a 2 degrees of freedom completely integrable system as above. Then there exist a symplectic change of coordinates in the domain and a change of coordinates in the range such that, in the new coordinates, one of the following happens*

(EE) $f(x_1, y_1, x_2, y_2) = (x_1^2 + y_1^2, x_2^2 + y_2^2) + o(2)$, *the elliptic-elliptic form,*

(EH) $f(x_1, y_1, x_2, y_2) = (x_1^2 + y_1^2, x_2^2 - y_2^2) + o(2)$, *the elliptic-hyperbolic form,*

(HH) $f(x_1, y_1, x_2, y_2) = (x_1^2 - y_1^2, x_2^2 - y_2^2) + o(2)$, *the hyperbolic-hyperbolic form,*

(FF) $f(x_1, y_1, x_2, y_2) = (x_1y_1 + x_2y_2, x_1y_2 - x_2y_1) + o(2)$, *the focus-focus form,*

(EP) $f(x_1, y_1, x_2, y_2) = (x_1^2 + y_1^2, x_2^2) + o(2)$, *the elliptic-fold form,*

(HP) $f(x_1, y_1, x_2, y_2) = (x_1^2 - y_1^2, x_2^2) + o(2)$, *the hyperbolic-fold form,*

(PP) $f(x_1, y_1, x_2, y_2) = (x_1^2, x_2^2) + o(2)$, *the fold-fold form, and*

(X1) $f(x_1,y_1,x_2,y_2) = (2x_2y_1 + y_2^2, y_1^2) + o(2)$,

(X2) $f(x_1,y_1,x_2,y_2) = (x_1^2 + y_1^2 - x_2^2 - y_2^2, x_1^2 + y_1^2 - x_2y_1 - x_1y_2) + o(2)$,

(X3) $f(x_1,y_1,x_2,y_2) = (x_2y_1, x_1y_1 + x_2y_2) + o(2)$.

B. Infinitesimally stable and unstable rank 0 singularities. Let us consider the singularities associated to the quadratic part of the germs above. To prove infinitesimal stability of a singularity one has to characterize the infinitesimal deformations g that satisfy the homological equation (HOM) and then prove that all such infinitesimal completely integrable deformations belong to the infinitesimal action space (SIA). On the other hand, a singularity can be proven to be infinitesimally unstable by providing an infinitesimal completely integrable deformation that does not belong to the infinitesimal action space, in other words, by determining low-order polynomials g_1, g_2 that satisfy equation (HOM) but that are not of the form (SIA).

Theorem 12 *The singularities (EE), (EH), (HH) and (FF) are infinitesimally stable, the other singularities are not.*

Proof. To characterize the germs of infinitesimal deformation $g = (g_1, g_2)$ that satisfy the homological equation we use a technique that can be found in [2].

The (EE) case: Let $z_j = x_j + iy_j$ and $w_j = x_j - iy_j$, $j = 1, 2$. Each analytic function g has Taylor expansion $\sum a(\nu_1,\nu_2,\nu_3,\nu_4) z_1^{\nu_1} z_2^{\nu_2} w_1^{\nu_3} w_2^{\nu_4}$, such expansion can be rearranged as follows

$$g = \sum_{\mu\in\mathbb{Z}^2} \left(\sum_{m\in\mathbb{N}^2} b(m_1,m_2,\mu_1,\mu_2)(z_1w_1)^{m_1}(z_2w_2)^{m_2} \right) E_{\mu_1}(z_1,w_1)E_{\mu_2}(z_2,w_2),$$

where

$$b(m_1,m_2,\mu_1,\mu_2) = \begin{cases} a(m_1+\mu_1, m_1, m_2+\mu_2, m_2) & \text{if } 0 \le \mu_1,\mu_2 \\ a(m_1, m_1-\mu_1, m_2+\mu_2, m_2) & \text{if } \mu_1 \le 0 \le \mu_2 \\ a(m_1+\mu_1, m, m_2, m_2-\mu_2) & \text{if } \mu_2 \le 0 \le \mu_1 \\ a(m_1, m_1-\mu_1, m_2, m_2-\mu_2) & \text{if } \mu_1,\mu_2 \le 0 \end{cases} \qquad E_\mu(x,y) = \begin{cases} x^\mu & \text{if} \quad \mu \ge 0 \\ y^{-\mu} & \text{if} \quad \mu \le 0. \end{cases}$$

We will write $\sum_m b(m_1,m_2,\mu_1,\mu_2)(z_1w_1)^{m_1}(z_2w_2)^{m_2} = \widehat{g}^\mu$ and $\widehat{g}^\mu E_{\mu_1}(z_1,w_1)E_{\mu_2}(z_2,w_2) = g^\mu$. From

$$\{f_1,z_1\} = 2iz_1, \qquad \{f_1,w_1\} = -2iw_1, \qquad \{f_1,z_2\} = 0, \qquad \{f_1,w_2\} = 0,$$

$$\{f_2,z_1\} = 0, \qquad \{f_2,w_1\} = 0, \qquad \{f_2,z_2\} = 2iz_2, \qquad \{f_2,w_2\} = -2iw_2,$$

it follows that the functions z_1w_1 and z_2w_2 are in the kernel of both differential operators $\{f_1,\cdot\}, \{f_2,\cdot\}$, and

$$\{f_1, E_\mu(z_1,w_1)\} = i\mu E_\mu(z_1,w_1), \qquad \{f_1, E_\mu(z_2,w_2)\} = 0,$$

$$\{f_2, E_\mu(z_1,w_1)\} = 0, \qquad \{f_2, E_\mu(z_2,w_1)\} = i\mu E_\mu(z_2,w_2).$$

The homological equation reads

$$\{f_1,g_2\} = \sum \widehat{g}_2^\mu i\mu_1 E_{\mu_1}(z_1,w_2)E_{\mu_2}(z_2,w_1) = \{f_2,g_1\} = \sum \widehat{g}_1^\mu i\mu_2 E_{\mu_1}(z_1,w_2)E_{\mu_2}(z_2,w_1),$$

hence, the Taylor coefficients of the generic infinitesimal completely integrable deformation must satisfy $\mu_2\,\widehat{g}_1^\mu = \mu_1\,\widehat{g}_2^\mu$. In other words, the generic infinitesimal completely integrable deformation has the form

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} g_1^{(0,0)} \\ g_2^{(0,0)} \end{pmatrix} + \sum_{\mu_1\neq 0} \begin{pmatrix} g_1^{(\mu_1,0)} \\ 0 \end{pmatrix} + \sum_{\mu_2\neq 0} \begin{pmatrix} 0 \\ g_2^{(0,\mu_2)} \end{pmatrix} + \sum_{\mu_1,\mu_2\neq 0} \begin{pmatrix} g_1^{(\mu_1,\mu_2)} \\ \frac{\mu_2}{\mu_1} g_1^{(\mu_1,\mu_2)} \end{pmatrix}.$$

We have to prove that all infinitesimal completely integrable deformations belong to the infinitesimal action space. The generic element of the infinitesimal action space has the form

$$\left(\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right) \widetilde{H}_1(x,y) + \left(\begin{smallmatrix} y_1 \\ 0 \end{smallmatrix}\right) \widetilde{H}_2(x,y) + \left(\begin{smallmatrix} 0 \\ y_2 \end{smallmatrix}\right) \widetilde{H}_3(x,y) + \left(\begin{smallmatrix} 0 \\ x_2 \end{smallmatrix}\right) \widetilde{H}_4(x,y) + \left(\begin{smallmatrix} \widetilde{K}_1(x_1^2+y_1^2, x_2^2+y_2^2) \\ \widetilde{K}_2(x_1^2+y_1^2, x_2^2+y_2^2) \end{smallmatrix}\right) =$$

$$= \left(\begin{smallmatrix} z_1 \\ 0 \end{smallmatrix}\right) H_1(z,w) + \left(\begin{smallmatrix} w_1 \\ 0 \end{smallmatrix}\right) H_2(z,w) + \left(\begin{smallmatrix} 0 \\ z_2 \end{smallmatrix}\right) H_3(z,w) + \left(\begin{smallmatrix} 0 \\ w_2 \end{smallmatrix}\right) H_4(z,w) + \left(\begin{smallmatrix} K_1(z_1w_1, z_2w_2) \\ K_2(z_1w_1, z_2w_2) \end{smallmatrix}\right).$$

For appropriate $H_1, H_2, H_3, H_4, K_1, K_2$, one has that

$$\begin{aligned}
\begin{pmatrix} g_1^{(0,0)} \\ g_2^{(0,0)} \end{pmatrix} &= \begin{pmatrix} K_1(z_1w_1, z_2) \\ K_2(z_1w_2, w_1z_2) \end{pmatrix}, \\
\begin{pmatrix} g_1^{(\mu_1,0)} \\ 0 \end{pmatrix} &= \left(\begin{smallmatrix} z_1 \\ 0 \end{smallmatrix}\right) H_1(z,w) + \left(\begin{smallmatrix} w_1 \\ 0 \end{smallmatrix}\right) H_2(z,w), \\
\begin{pmatrix} 0 \\ g_1^{(0,\mu_2)} \end{pmatrix} &= \left(\begin{smallmatrix} 0 \\ z_2 \end{smallmatrix}\right) H_3(z,w) + \left(\begin{smallmatrix} 0 \\ w_2 \end{smallmatrix}\right) H_4(z,w), \\
\begin{pmatrix} g_1^{(\mu_1,\mu_2)} \\ \frac{\mu_2}{\mu_1} g_1^{(\mu_1,\mu_2)} \end{pmatrix} &= \left(\begin{smallmatrix} z_1 \\ 0 \end{smallmatrix}\right) H_1(z,w) + \left(\begin{smallmatrix} w_1 \\ 0 \end{smallmatrix}\right) H_2(z,w) + \left(\begin{smallmatrix} 0 \\ z_2 \end{smallmatrix}\right) H_3(z,w) + \left(\begin{smallmatrix} 0 \\ w_2 \end{smallmatrix}\right) H_4(z,w).
\end{aligned}$$

Hence all infinitesimal completely integrable deformations belong to the infinitesimal action space.

The (EH) case: Let $z_1 = x_1 + iy_1$, $w_1 = x_1 - iy_1$, $z_2 = x_2 + y_2$ and $w_2 = x_2 - y_2$. Each analytic function g admits an expansion formally identical to that above. In this case

$$\{f_1, E_\mu(z_1,w_1)\} = i\mu E_\mu(z_1,w_1) \qquad \{f_1, E_\mu(z_1,w_1)\} = 0$$

$$\{f_2, E_\mu(z_2,w_2)\} = 0 \qquad \{f_2, E_\mu(z_2,w_2)\} = \mu E_\mu(z_2,w_2).$$

The homological equation reads,

$$\{f_1, g_2\} = \sum \widehat{g}_2^\mu i\mu_1 E_{\mu_1}(z_1,w_2)E_{\mu_2}(z_2,w_1) = \{f_2, g_1\} = \sum \widehat{g}_1^\mu \mu_2 E_{\mu_1}(z_1,w_2)E_{\mu_2}(z_2,w_1),$$

hence, the Taylor coefficients of the generic infinitesimal completely integrable deformation must satisfy $\mu_2\,\widehat{g}_1^\mu = i\mu_1\,\widehat{g}_2^\mu$. The generic infinitesimal completely integrable deformation has the form

$$\left(\begin{smallmatrix} g_1 \\ g_2 \end{smallmatrix}\right) = \begin{pmatrix} g_1^{(0,0)} \\ g_2^{(0,0)} \end{pmatrix} + \sum_{\mu_1 \neq 0} \begin{pmatrix} g_1^{(\mu_1,0)} \\ 0 \end{pmatrix} + \sum_{\mu_2 \neq 0} \begin{pmatrix} 0 \\ g_2^{(0,\mu_2)} \end{pmatrix} + \sum_{\mu_1,\mu_2 \neq 0} \begin{pmatrix} g_1^{(\mu_1,\mu_2)} \\ -i\frac{\mu_2}{\mu_1} g_1^{(\mu_1,\mu_2)} \end{pmatrix},$$

while the generic element of the infinitesimal action space has the form

$$\left(\begin{smallmatrix} z_1 \\ 0 \end{smallmatrix}\right) H_1(z,w) + \left(\begin{smallmatrix} w_1 \\ 0 \end{smallmatrix}\right) H_2(z,w) + \left(\begin{smallmatrix} 0 \\ z_2 \end{smallmatrix}\right) H_3(z,w) + \left(\begin{smallmatrix} 0 \\ w_2 \end{smallmatrix}\right) H_4(z,w) + \left(\begin{smallmatrix} K_1(z_1w_1, z_2w_2) \\ K_2(z_1w_1, z_2w_2) \end{smallmatrix}\right).$$

The conclusions are the same as in the elliptic-elliptic case.

The (HH) case: Let $z_1 = x_1 + y_1$, $w_1 = x_1 - y_1$, $z_2 = x_2 + y_2$ and $w_2 = x_2 - y_2$. Arguments as those above prove that the generic infinitesimal completely integrable deformation has the form

$$\left(\begin{smallmatrix} g_1 \\ g_2 \end{smallmatrix}\right) = \begin{pmatrix} g_1^{(0,0)} \\ g_2^{(0,0)} \end{pmatrix} + \sum_{\mu_1 \neq 0} \begin{pmatrix} g_1^{(\mu_1,0)} \\ 0 \end{pmatrix} + \sum_{\mu_2 \neq 0} \begin{pmatrix} 0 \\ g_2^{(0,\mu_2)} \end{pmatrix} + \sum_{\mu_1,\mu_2 \neq 0} \begin{pmatrix} g_1^{(\mu_1,\mu_2)} \\ \frac{\mu_2}{\mu_1} g_1^{(\mu_1,\mu_2)} \end{pmatrix},$$

while the generic element of the infinitesimal action space has the form

$$\left(\begin{smallmatrix} z_1 \\ 0 \end{smallmatrix}\right) H_1(z,w) + \left(\begin{smallmatrix} w_1 \\ 0 \end{smallmatrix}\right) H_2(z,w) + \left(\begin{smallmatrix} 0 \\ z_2 \end{smallmatrix}\right) H_3(z,w) + \left(\begin{smallmatrix} 0 \\ w_2 \end{smallmatrix}\right) H_4(z,w) + \left(\begin{smallmatrix} K_1(z_1w_1, z_2w_2) \\ K_2(z_1w_1, z_2w_2) \end{smallmatrix}\right).$$

The conclusions are the same as in the elliptic-elliptic case.

The (FF) case: Let $z_1 = x_1 + ix_2$, $w_1 = y_1 - iy_2$, $z_2 = y_1 + iy_2$ and $w_2 = x_1 - ix_2$. In this case

$$\{f_1, E_\mu(z_1, w_1)\} = -\mu E_\mu(z_1, w_1) \qquad \{f_1, E_\mu(z_2, w_2)\} = \mu E_\mu(z_2, w_2)$$

$$\{f_2, E_\mu(z_1, w_1)\} = -i\mu E_\mu(z_1, w_1) \qquad \{f_2, E_\mu(z_2, w_2)\} = -i\mu E_\mu(z_2, w_2).$$

The homological equation reads

$$\{f_1, g_2\} = \sum \widehat{g}_2^\mu(\mu_2 - \mu_1) E_{\mu_1}(z_1, w_1) E_{\mu_2}(z_2, w_2) = \{f_2, g_1\} = \sum \widehat{g}_1^\mu(-i)(\mu_1 + \mu_2) E_{\mu_1}(z_1, w_1) E_{\mu_2}(z_2, w_2),$$

hence, the Taylor coefficients of an infinitesimal completely integrable deformation must satisfy $\widehat{g}_1^\mu(-i)(\mu_1 + \mu_2) = \widehat{g}_2^\mu(\mu_2 - \mu_1)$. The generic infinitesimal completely integrable deformation has the form

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} g_1^{(0,0)} \\ g_2^{(0,0)} \end{pmatrix} + \sum_{\mu_1 \neq 0} \begin{pmatrix} g_1^{(\mu_1, -\mu_1)} \\ 0 \end{pmatrix} + \sum_{\mu_1 \neq 0} \begin{pmatrix} 0 \\ g_2^{(\mu_1, \mu_1)} \end{pmatrix} + \sum_{\mu_1 \neq \pm\mu_2} \begin{pmatrix} g_1^{(\mu_1, \mu_2)} \\ i\frac{\mu_1 + \mu_2}{\mu_1 - \mu_2} g_1^{(\mu_1, \mu_2)} \end{pmatrix},$$

while the generic element of the infinitesimal action space has the form

$$\begin{pmatrix} z_1 \\ iz_1 \end{pmatrix} H_1(z, w) + \begin{pmatrix} w_1 \\ iw_1 \end{pmatrix} H_2(z, w) + \begin{pmatrix} z_2 \\ -iz_2 \end{pmatrix} H_3(z, w) + \begin{pmatrix} w_2 \\ -iw_2 \end{pmatrix} H_4(z, w) + \begin{pmatrix} K_1(z_1w_1, z_2w_2) \\ K_2(z_1w_1, z_2w_2) \end{pmatrix}.$$

The stability of the singularity can be proven by observing that:

- the term $\begin{pmatrix} g_1^{(0,0)} \\ g_2^{(0,0)} \end{pmatrix}$ is of the form $\begin{pmatrix} K_1(z_1w_2, w_1z_2) \\ K_2(z_1w_2, w_1z_2) \end{pmatrix}$,
- the terms $\begin{pmatrix} g_1^{(\mu,-\mu)} \\ 0 \end{pmatrix}$ contain either the factor $\begin{pmatrix} z_1w_2 \\ 0 \end{pmatrix}$ which can be obtained by using $\begin{pmatrix} z_1 \\ iz_1 \end{pmatrix} \frac{w_1}{2} + \begin{pmatrix} w_2 \\ -iw_2 \end{pmatrix} \frac{z_1}{2}$ or the factor w_1z_2 which can be obtained by using $\begin{pmatrix} w_1 \\ iw_1 \end{pmatrix} \frac{z_2}{2} + \begin{pmatrix} z_2 \\ -iz_2 \end{pmatrix} \frac{w_1}{2}$,
- the terms $\begin{pmatrix} 0 \\ g_2^{(\mu,\mu)} \end{pmatrix}$ contain either the factor $\begin{pmatrix} 0 \\ z_1z_2 \end{pmatrix}$ which can be obtained by using $\begin{pmatrix} z_1 \\ iz_1 \end{pmatrix} \frac{z_2}{2i} - \begin{pmatrix} z_2 \\ -iz_2 \end{pmatrix} \frac{z_1}{2i}$ or the factor $\begin{pmatrix} 0 \\ w_1w_2 \end{pmatrix}$ which can be obtained by using $\begin{pmatrix} w_1 \\ iw_1 \end{pmatrix} \frac{w_2}{2i} - \begin{pmatrix} w_2 \\ -iw_2 \end{pmatrix} \frac{w_1}{2i}$,
- the terms $\begin{pmatrix} g_1^{(\mu_1,\mu_2)} \\ i\frac{\mu_1+\mu_2}{\mu_1-\mu_2} g_1^{(\mu_1,\mu_2)} \end{pmatrix}$ certainly have either μ_1 or μ_2 non zero (since $\mu_1 \neq \mu_2$).
  - If $\mu_1 = 0$ then $\begin{pmatrix} g_1^{(0,\mu_2)} \\ -ig_1^{(0,\mu_2)} \end{pmatrix} = \widehat{g}_1^{(0,\mu_2)} E_{\mu_2 \mp 1}(z_2, w_2) \begin{pmatrix} E_{\pm 1}(z_2, w_2) \\ -iE_{\pm 1}(z_2, w_2) \end{pmatrix}$,
  - if $\mu_2 = 0$ then $\begin{pmatrix} g_1^{(\mu_1,0)} \\ ig_1^{(\mu_1,0)} \end{pmatrix} = \widehat{g}_1^{(\mu_1,0)} E_{\mu_1 \mp 1}(z_1, w_1) \begin{pmatrix} E_{\pm 1}(z_1, w_1) \\ iE_{\pm 1}(z_1, w_1) \end{pmatrix}$,
  - if both $\mu_1, \mu_2 \neq 0$ then $\begin{pmatrix} g_1^{(\mu_1,\mu_2)} \\ i\frac{\mu_1+\mu_2}{\mu_1-\mu_2} g_1^{(\mu_1,\mu_2)} \end{pmatrix}$ is sum of terms which contain factors $\begin{pmatrix} z_1z_2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} z_1w_2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} w_1z_2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} w_1w_2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ z_1z_2 \end{pmatrix}$, $\begin{pmatrix} 0 \\ z_1w_2 \end{pmatrix}$, $\begin{pmatrix} 0 \\ w_1z_2 \end{pmatrix}$, $\begin{pmatrix} 0 \\ w_2w_1 \end{pmatrix}$. Each of these factors can be obtained by using appropriate combinations with the functions H_i.

The (EP) case: Let $z_1 = x_1 + iy_1$, $w_1 = x_1 - iy_1$. The generic function has Taylor expansion

$$g = \sum_{\substack{\mu_1 \in \mathbb{Z} \\ \mu_2, \mu_3 \in \mathbb{N}}} g^\mu = \sum_{\substack{\mu_1 \in \mathbb{Z} \\ \mu_2, \mu_3 \in \mathbb{N}}} \widehat{g}^\mu E_{\mu_1}(z_1, w_1) x_2^{\mu_2} y_2^{\mu_3},$$

with $\widehat{g}^\mu$ function of (z_1w_1). From

$$\begin{matrix} \{f_1, z_1\} = 2iz_1 & \{f_1, x_2\} = 0 & \{f_1, w_1\} = -2iw_1 & \{f_1, y_2\} = 0 \\ \{f_2, z_1\} = 0 & \{f_2, x_2\} = 0 & \{f_2, w_1\} = 0 & \{f_2, y_2\} = 2x_2, \end{matrix}$$

it follows that $\{f_1, E_\mu(z_1,w_1)\} = i\mu E_\mu(z_1,w_1)$. The homological equation reads

$$\{f_1,g_2\} = \sum_{\substack{\mu_1\in\mathbb{Z}\\ \mu_2,\mu_3\in\mathbb{N}}} \widehat{g}_2^{\mu} i\mu_1 E_{\mu_1}(z_1,w_1)x_2^{\mu_2}y_2^{\mu_3} = \{f_2,g_1\} = \sum_{\substack{\mu_1\in\mathbb{Z}\\ \mu_2\in\mathbb{N},\mu_3\geq 1}} \widehat{g}_1^{\mu}\mu_3 E_{\mu_1}(z_1,w_1)x_2^{\mu_2+1}y_2^{\mu_3-1},$$

hence, the Taylor coefficients of the generic infinitesimal completely integrable deformation must satisfy no conditions on the coefficients $\widehat{g}_1^{(\mu_1,\mu_2,0)}$, while for $\mu_3 \geq 1, \mu_1 = 0$, $\widehat{g}_1^{(0,\mu_2,\mu_3)} = 0$, and for $\mu_3 \geq 1, \mu_1 \geq 1$, $\widehat{g}_1^{(\mu_1,\mu_2,\mu_3)} = i\mu_1/\mu_3\, \widehat{g}_1^{(\mu_1,\mu_2+1,\mu_3-1)}$. It follows that the generic infinitesimal completely integrable deformation has the form

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \sum_{\substack{\mu_1\in\mathbb{Z}\\ \mu_2\in\mathbb{N}}} \begin{pmatrix} g_1^{(\mu_1,\mu_2,0)} \\ 0 \end{pmatrix} + \sum_{\substack{\mu_2\in N\\ \mu_3\geq 1}} \begin{pmatrix} 0 \\ g_2^{(0,\mu_2,\mu_3)} \end{pmatrix} + \sum_{\substack{\mu_1\neq 0\\ \mu_2\in N,\mu_3\geq 1}} \begin{pmatrix} \frac{i\mu_1}{\mu_3} g_2^{(\mu_1,\mu_2+1,\mu_3-1)} \\ g_2^{(\mu_1,\mu_2,\mu_3)} \end{pmatrix},$$

while the generic element of the infinitesimal action space has the form

$$\begin{pmatrix} z_1 \\ 0 \end{pmatrix} H_1 + \begin{pmatrix} w_1 \\ 0 \end{pmatrix} H_2 + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} H_3 + \begin{pmatrix} K_1(z_1w_1,x_2^2) \\ K_2(z_1w_1,x_2^2) \end{pmatrix}.$$

The functions $\begin{pmatrix} x_2^{2n-1} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ y_2^n \end{pmatrix}$, $n \geq 1$ are infinitesimal completely integrable deformations that do not belong to the infinitesimal action space.

The (HP) case: Let $z_1 = x_1 + y_1$, $w_1 = x_1 - y_1$. Expanding the generic function as above, one obtains that the homological equation reads

$$\{f_1,g_2\} = \sum_{\substack{\mu_1\in\mathbb{Z}\\ \mu_2,\mu_3\in\mathbb{N}}} \widehat{g}_2^{\mu} \mu_1 E_{\mu_1}(z_1,w_1)x_2^{\mu_2}y_2^{\mu_3} = \{f_2,g_1\} = \sum_{\substack{\mu_1\in\mathbb{Z}\\ \mu_2\in\mathbb{N},\mu_3\geq 1}} \widehat{g}_1^{\mu}\mu_3 E_{\mu_1}(z_1,w_1)x_2^{\mu_2+1}y_2^{\mu_3-1},$$

hence, the generic infinitesimal completely integrable deformation has the form

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \sum_{\substack{\mu_1\in\mathbb{Z}\\ \mu_2\in\mathbb{N}}} \begin{pmatrix} g_1^{(\mu_1,\mu_2,0)} \\ 0 \end{pmatrix} + \sum_{\substack{\mu_2\in N\\ \mu_3\geq 1}} \begin{pmatrix} 0 \\ g_2^{(0,\mu_2,\mu_3)} \end{pmatrix} + \sum_{\substack{\mu_1\neq 0\\ \mu_2\in N,\mu_3\geq 1}} \begin{pmatrix} \frac{\mu_1}{\mu_3} g_2^{(\mu_1,\mu_2+1,\mu_3-1)} \\ g_2^{(\mu_1,\mu_2,\mu_3)} \end{pmatrix},$$

while the generic element of the infinitesimal action space has the form

$$\begin{pmatrix} z_1 \\ 0 \end{pmatrix} H_1 + \begin{pmatrix} w_1 \\ 0 \end{pmatrix} H_2 + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} H_3 + \begin{pmatrix} K_1(z_1w_1,x_2^2) \\ K_2(z_1w_1,x_2^2) \end{pmatrix}.$$

The same functions of the (EP) case are infinitesimal completely integrable deformations that do not belong to the infinitesimal action space.

The (PP) case: Using the ordinary Taylor expansion of a function, one has that the homological equation reads

$$\{f_1,g_2\} = \sum_{\substack{\mu_1,\mu_3,\mu_4\in\mathbb{N}\\ \mu_2\geq 1}} \widehat{g}_2^{\mu}\mu_2 x_1^{\mu_1+1}y_1^{\mu_2-1}x_2^{\mu_3}y_2^{\mu_4} = \{f_2,g_1\} = \sum_{\substack{\mu_1,\mu_2,\mu_3\in\mathbb{N}\\ \mu_4\geq 1}} \widehat{g}_1^{\mu}\mu_4 x_1^{\mu_1}y_1^{\mu_2}x_2^{\mu_3+1}y_2^{\mu_4-1}.$$

Hence, the Taylor coefficients of the generic infinitesimal completely integrable deformation must satisfy no conditions on $\widehat{g}_1^{(\mu_1,\mu_2,\mu_3,0)}$ and $\widehat{g}_2^{(\mu_1,0,\mu_3,\mu_4)}$, while for $\mu_2,\mu_4 \geq 1$, the coefficients must satisfy $\widehat{g}_1^{(\mu_1,\mu_2,\mu_3,\mu_4)} = (\mu_2+1)/\mu_4\, \widehat{g}_2^{(\mu_1-1,\mu_2+1,\mu_3+1,\mu_4-1)}$. It follows that the generic infinitesimal completely integrable deformation has the form

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \sum_{\mu_1,\mu_2,\mu_3\in\mathbb{N}} \begin{pmatrix} g_1^{(\mu_1,\mu_2,\mu_3,0)} \\ 0 \end{pmatrix} + \sum_{\mu_1,\mu_3,\mu_4\in\mathbb{N}} \begin{pmatrix} 0 \\ g_2^{(\mu_1,0,\mu_3,\mu_4)} \end{pmatrix} + \sum_{\substack{\mu_2,\mu_4\geq 1\\ \mu_1,\mu_3\in\mathbb{N}}} \begin{pmatrix} \frac{\mu_2+1}{\mu_4} g_2^{(\mu_1-1,\mu_2+1,\mu_3+1,\mu_4-1)} \\ g_2^{(\mu_1,\mu_2,\mu_3,\mu_4)} \end{pmatrix}.$$

The generic element of the infinitesimal action space has the form

$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix} H_1 + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} H_2 + \begin{pmatrix} K_1(x_1^2,x_2^2) \\ K_2(x_1^2,x_2^2) \end{pmatrix}.$$

It follows that the functions $\begin{pmatrix} y_1^m x_2^n \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ x_1^m y_2^n \end{pmatrix}$, are infinitesimal completely integrable deformations that do not belong to the infinitesimal action space.

The (X1-2-3) case: For each singularity we give a list of infinitesimal completely integrable deformations of degree at most 2 that do not belong to the infinitesimal action space. The deformations of X1 are

$$\begin{pmatrix} -x_1 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} x_1^2 \\ x_2^2-2x_1y_2 \end{pmatrix}, \quad \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -2x_1y_2 \\ y_2^2 \end{pmatrix}, \quad \begin{pmatrix} x_1y_2 \\ x_2y_1 \end{pmatrix},$$

of X2 are

$$\begin{pmatrix} -2x_1x_2+5y_2x_2+x_1y_1-2y_1y_2 \\ x_2y_2 \end{pmatrix}, \quad \begin{pmatrix} y_1^2-5y_2^2+4x_1y_2 \\ -y_2^2 \end{pmatrix}, \quad \begin{pmatrix} x_1^2-5y_2x_1+5y_2^2-x_2y_1 \\ y_2^2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x_2^2+y_2^2 \end{pmatrix},$$
$$\begin{pmatrix} -2y_2^2+x_1y_2+x_2y_1 \\ x_2y_1 \end{pmatrix}, \quad \begin{pmatrix} 4y_2^2-4x_1y_2 \\ y_1^2+y_2^2 \end{pmatrix}, \quad \begin{pmatrix} 2y_2^2-x_1y_2-x_2y_1 \\ x_1y_2 \end{pmatrix},$$

of X3 are

$$\begin{pmatrix} x_1y_2 \\ 0 \end{pmatrix}.$$

□

Remark. In all unstable cases, the infinitesimal completely integrable deformation g not belonging to the infinitesimal action space can be chosen so that $f + \lambda g$ is a completely integrable system for all λ. It follows that in all those cases there exist neighbouring completely integrable systems with singularities different from that defined by f. See Figure 2 for an example.

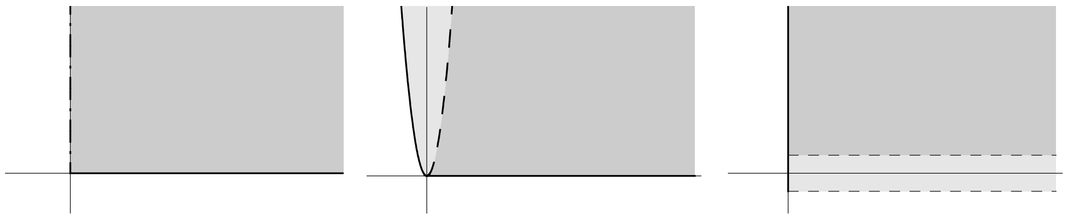
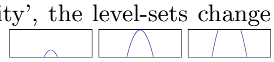


Figure 2: The bifurcation diagrams corresponding to deformations of the (EP) singularity. The first picture is the bifurcation diagram of the undeformed function $f_0 = (x_1^2 + y_1^2, x_2^2)$. The second bifurcation diagram is that of $f_\lambda = (x_1^2 + y_1^2 + \lambda x_2, x_2^2)$, all singular points have rank 1. The third bifurcation diagram is that of $f_\lambda = (x_1^2 + y_1^2, x_2^2 + \lambda y_2)$. In this last diagram, the two horizontal bifurcation lines correspond to 'singularities at infinity', the level-sets change number of components because part of them exits from the neighbourhood under consideration. 

4 Rank one singularities

In this section, we analyze the stability of some rank 1 singularities.

A. Local classification. Assume that f represents a rank 1 singularity of an analytic, 2 degrees of freedom, completely integrable system. By Caratheodory's theorem, it is possible to determine canonical coordinates x_1, y_1, x_2, y_2 and coordinates in the range of f such that $f(x_1, y_1, x_2, y_2) = (x_1, f_2(x_1, x_2, y_2))$, with $f_2(0, 0, 0) = 0$, $d_0 f_2 = 0$, and $\partial^2_{x_1} f_2(0, 0, 0) = 0$. We also make the generic assumption that the Hessian of f_2 with respect to x_1, x_2, y_2 is non-degenerate.

The quadratic differential $d_0^2 f$ is a quadratic map on the 3-dimensional vector space spanned by $\{\partial_{y_1}, \partial_{x_2}, \partial_{y_2}\}$, and has isotropic direction $\mathbb{R}\partial_{y_1}$. Let's assume $d_0^2 f$ to be non-zero, its signature can be either $\{0, +, +\}$, in which case we call the singularity **elliptic (e)**, or $\{0, +, -\}$, in which case we call the singularity **hyperbolic (h)**, or $\{0, 0, +\}$, in which cases we call the singularity **degenerate**. In the degenerate case, there are two possibilities:

(c) the third derivative of f_2 along one (hence any) degenerate direction of the quadratic map non-parallel to ∂_{y_1} is non-zero, in which case we call the singularity **cuspidal (c)**, or

(p) the third derivative of f_2 along the degenerate directions of the quadratic map is zero.

Lemma 13 *In this setup, if the non-degenerate conditions (e) or (h) are fullfilled, there exist a symplectic change of coordinates in the domain and a change of coordinates in the range such that, in the new coordinates, the system is respectively*

(E) $f(x_1, y_1, x_2, y_2) = (x_1, x_1y_2 + x_2^2 + y_2^2 + o(2))$*, the elliptic case,*

(H) $f(x_1, y_1, x_2, y_2) = (x_1, x_1y_2 + x_2^2 - y_2^2 + o(2))$*, the hyperbolic case.*

If the degenerate conditions (c) or (p) are fullfilled, there exist a change of coordinates in the domain and a change of coordinates in the range such that, in the new coordinates, the system is respectively

(C) $f(x_1, y_1, x_2, y_2) = (x_1, x_1y_2 + x_2^2 + y_2^3 + o(3))$*, the cuspidal case, and*

(P) $f(x_1, y_1, x_2, y_2) = (x_1, x_1y_2 + x_2^2 + o(3))$*, the fold case.*

These last coordinates are non-canonical, but x_1, y_1 *are conjugate variables, which implies that* $\{x_1, \cdot\} = \partial_{y_1}(\cdot)$.

Proof. The Hessian of f_2 is the matrix $\begin{pmatrix} 0 & a & b \\ a & c & d \\ b & d & e \end{pmatrix}$. Up to a rotation in the space x_2, y_2 one can assume that $d = 0$ and $c > 0$. Condition (e) corresponds to $0 < c, e$, (h) corresponds to $e < 0 < c$, the degenerate condition corresponds to $0 = e < c$.

In the (e) case the symplectic change of coordinates $x_2 = \alpha X_2$, $y_2 = 1/\alpha\, Y_2$, $x_1 = X_1, y_1 = Y_1$ with $\alpha = \sqrt[4]{e/c}$ puts the Hessian of f_2 into the quadratic form whose associated matrix is $\begin{pmatrix} 0 & \alpha a & \alpha^{-1}b \\ \alpha a & \sqrt{ce} & 0 \\ \alpha^{-1}b & 0 & \sqrt{ce} \end{pmatrix}$, a rotation in the x_2, y_2-plane and a rescaling of f_2 yields (E).

In the (h) case the symplectic change of coordinates $x_2 = \alpha X_2$, $y_2 = 1/\alpha\, Y_2$, $x_1 = X_1, y_1 = Y_1$ with $\alpha = \sqrt[4]{-e/c}$ puts the Hessian of f_2 into the quadratic form whose associated matrix is $\begin{pmatrix} 0 & \alpha a & \alpha^{-1}b \\ \alpha a & \sqrt{-ce} & 0 \\ \alpha^{-1}b & 0 & -\sqrt{-ce} \end{pmatrix}$, a hyperbolic rotation in the x_2, y_2-plane and a rescaling of f_2 yields (H).

In the degenerate case the symplectic change of coordinates $X_1 = cb^2x_1, Y_1 = y_1/(cb^2), X_2 = x_2/b, Y_2 = ax_2 + by_2$ puts the Hessian of f_2 into the quadratic form whose associated matrix is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. In these coordinates

$$f_2(x_1, x_2, y_2) = x_1y_2 + x_2^2 + \varepsilon y_2^3 + y_2^2\left(a(1,0)x_1 + a(0,1)x_2\right) + \\ + y_2\left(a(2,0)x_1^2 + a(1,1)x_2x_1 + a(0,2)x_2^2\right) + \left(a(3,0)x_1^3 + a(2,1)x_1^2x_2 + a(1,2)x_1x_2^2 + a(0,3)x_2^3\right) + o(3),$$

with $\varepsilon = 1$ in the cuspidal (c) case and $\varepsilon = 0$ in the fold (p) case. The symplectic linear change of coordinates $X_1 = x_1, Y_1 = y_1 + d^2x_2 + dy_2, X_2 = x_2 - dx_1, Y_2 = y_2 - d^2x_1 + 2dx_2$ with $d = -a(0,1)/6$ cancels the monomial $y_2^2x_2$. The further quadratic change of coordinates

$$X_1 = x_1, \quad Y_1 = y_1, \quad Y_2 = y_2 - \widetilde{a}(1,0)x_2^2 - \widetilde{a}(1,0)y_2^2 - 2\widetilde{a}(1,1)x_2y_2 - \widetilde{a}(2,0)x_1y_2,$$

$$X_2 = x_2 - \tfrac{1}{2}\Big(\widetilde{a}(0,2)x_2\, y_2 - \widetilde{a}(0,3)x_2^2 + \widetilde{a}(1,1)x_1y_2 + \big(\widetilde{a}(1,0) - \widetilde{a}(1,2)\big)x_1x_2 - \widetilde{a}(2,1)x_1^2\Big),$$

yields the result (the $\widetilde{a}$ are the coefficients of the cubic part of f_2 after the first change of coordinates) . □

B. Infinitesimally stable and unstable rank 1 singularities. The infinitesimal deformations that satisfy the homological equation are much easier to determine in this case. In fact the condition $\{f_1, g_2\} = \{f_2, g_1\}$ implies that $g_2 = \int_{y_1}\{f_2, g_1\}$. Once chosen g_1, the second component of the infinitesimal completely integrable deformation is determined (up to a function constant with respect to y_1).

Theorem 14 *The singularities (E),(H) and (C) are infinitesimally stable. The singularity (P) is not.*

Proof. The (E) case: The generic infinitesimal completely integrable deformation has the form

$$\begin{pmatrix} g_1 \\ y_2 g_1(x,y)+2x_2\int \frac{\partial g_1}{\partial y_2}dy_1-(x_2+2y_2)\int \frac{\partial g_1}{\partial x_2}dy_1+c(x_1,x_2,y_2)\end{pmatrix},$$

while the generic element of the infinitesimal action space has the form

$$\begin{pmatrix} 1 \\ y_2\end{pmatrix} H_1(x,y)+\begin{pmatrix} 0 \\ 2x_2\end{pmatrix} H_2(x,y)+\begin{pmatrix} 0 \\ x_1+2y_2\end{pmatrix} H_3(x,y)+\begin{pmatrix} K_1(x_1,x_1y_2+x_2^2+y_2^2) \\ K_2(x_1,x_1y_2+x_2^2+y_2^2)\end{pmatrix}.$$

Whatever g_1 is, choosing $H_1=g_1-K_1$ one is reduced to prove that

$$c(x_1,x_2,y_2)=y_2\widetilde{K}_1(x_1,x_1y_2+x_2^2+y_2^2)+x_2\widetilde{H}_2(x,y)+(x_1+2y_2)\widetilde{H}_3(x,y)+\widetilde{K}_2(x_1,x_1y_2+x_2^2+y_2^2)$$

for appropriate $\widetilde{H}_2,\widetilde{H}_3,\widetilde{K}_2$. The function $c(x_1,x_2,y_2)$ can be decomposed into $x_2c_1(x_1,x_2,y_2)+(x_1+2y_2)c_2(x_1,y_2)+c_3(x_1)$, and hence there exists a solution. The hyperbolic (H) case is identical to the elliptic case.
The (C) case: The generic infinitesimal completely integrable deformation has the form

$$\begin{pmatrix} g_1 \\ y_2 g_1 2+x_2\int \{x_2,g_1\}dy_1+(x_1+3y_2^2)\int \{y_2,g_1\}dy_1+c(x_1,x_2,y_2)\end{pmatrix},$$

while the generic element of the infinitesimal action space has the form

$$\begin{pmatrix} 1 \\ y_2\end{pmatrix} H_1(x,y)+\begin{pmatrix} 0 \\ 2x_2\end{pmatrix} H_2(x,y)+\begin{pmatrix} 0 \\ x_1+3y_2^2\end{pmatrix} H_3(x,y)+\begin{pmatrix} K_1(x_1,x_1y_2+x_2^2+y_2^3) \\ K_2(x_1,x_1y_2+x_2^2+y_2^3)\end{pmatrix}.$$

Whatever g_1 is, choosing $H_1=g_1-K_1$ one is reduced to prove that

$$c(x_1,x_2,y_2)=y_2\widetilde{K}_1(x_1,x_1y_2+x_2^2+y_2^3)+x_2\widetilde{H}_2(x,y)+(x_1+3y_2^2)\widetilde{H}_3(x,y)+\widetilde{K}_2(x_1,x_1y_2+x_2^2+y_2^3)$$

for appropriate $\widetilde{H}_2,\widetilde{H}_3,\widetilde{K}_1,\widetilde{K}_2$. The function $c(x_1,x_2,y_2)$ can be decomposed into $x_2c_1(x_1,x_2,y_2)+c_2(x_1,y_2)$, so we are left to consider whether the equation

$$c_2(x_1,y_2)=y_2\widetilde{\widetilde{K}}_1(x_1,x_1y_2+x_2^2+y_2^3)+x_2\widetilde{\widetilde{H}}_2(x,y)+(x_1+3y_2^2)\widetilde{\widetilde{H}}_3(x,y)+\widetilde{\widetilde{K}}_2(x_1,x_1y_2+x_2^2+y_2^3)$$

admits a solution. The term $x_2\widetilde{\widetilde{H}}_2(x,y)$ reconduces the solution of the above equation to the solution of the equation

$$p(x_1,y_2)=y_2k_1(x_1,x_1y_2+y_2^3)+(x_1+3y_2^2)h(x_1,y_2)+k_2(x_1,x_1y_2+y_2^3). \tag{1}$$

By induction on the degree of p, observe that once a polynomial $p(x_1,y_2)$ can be written as in (1), then the polynomial $x_1p(x_1,y_2)$ can be written as in (1). Assume to be given a polynomial $p(x_1,y_2)=\sum a_{i,j}x_1^iy_2^j$ of degree n, then

- if $n=3m$, subtracting to p the function obtained by using $k_2=a_{0,3m}(x_1y_2+y_2^3)^m$, $h=k_1=0$,
- if $n=3m+1$, subtracting to p the function obtained by using $k_1=a_{0,3m+1}(x_1y_2+y_2^3)^m$, $h=k_2=0$,
- if $n=3m+2$, subtracting to p the function obtained by using $h=\frac{a_{0,3m+2}}{3}y_2^{3m}$, $k_1=k_2=0$,

one is reduced to prove that a polynomial of the form $x_1q_1(x_1,y_2)+q_2(x_1,y_2)$ can be written as in (1), where q_1,q_2 are polynomials of degree $n-1$. One can hence conclude by inductive hypothesis, together with the observation above and the ascertainment that all polynomials up to 3rd degree can be written as in (1).
The (P) case: The generic infinitesimal completely integrable deformation has the form

$$\begin{pmatrix} g_1 \\ y_2 g_1+2x_2\int \{x_2,g_1\}dy_1+x_1\int \{y_2,g_1\}dy_1+c(x_1,x_2,y_2)\end{pmatrix},$$

while the generic element of the infinitesimal action space has the form

$$\begin{pmatrix} 1 \\ y_2\end{pmatrix} H_1(x,y)+\begin{pmatrix} 0 \\ 2x_2\end{pmatrix} H_2(x,y)+\begin{pmatrix} 0 \\ x_1\end{pmatrix} H_3(x,y)+\begin{pmatrix} K_1(x_1,x_1y_2+x_2^2) \\ K_2(x_1,x_1y_2+x_2^2)\end{pmatrix}.$$

Whatever g_1 is, choosing $H_1 = g_1 - K_1$ one is reduced to prove that

$$c(x_1, x_2, y_2) = y_2 \widetilde{K}_1(x_1, x_1 y_2 + x_2^2) + x_2 \widetilde{H}_2(x, y) + x_1 \widetilde{H}_3(x, y) + \widetilde{K}_2(x_1, x_1 y_2 + x_2^2)$$

for appropriate $\widetilde{H}_2, \widetilde{H}_3, \widetilde{K}_1, \widetilde{K}_2$. The function $c(x_1, x_2, y_2)$ can be decomposed into $x_2 c_1(x_1, x_2, y_2) + x_1 c_2(x_1, y_2) + c_3(y_2)$, so we are left to consider whether the equation

$$c_3(y_2) = y_2 \widetilde{\widetilde{K}}_1(x_1, x_1 y_2 + x_2^2) + x_2 \widetilde{\widetilde{H}}_2(x, y) + x_1 \widetilde{\widetilde{H}}_3(x, y) + \widetilde{\widetilde{K}}_2(x_1, x_1 y_2 + x_2^2)$$

admits a solution. It is easy to see that $c_3(y_2) = y_2^2$ does not belong to the space $y_2 K_1(x_1, x_1 y_2 + x_2^2) + x_2 H_2(x, y) + x_1 H_3(x, y) + K_2(x_1, x_1 y_2 + x_2^2)$. In fact there is not term that can contain y_2^2 (the first term can produce y_2 or powers of y_2 multiplied with x_1, x_2, the second term can produce powers of y_2 multiplied with x_2. the third and last terms can produce powers of y_2 multiplied with x_1). □

C. Other relevant rank 1 singularities. In [4] page 389 the list of relevant rank 1 singularities includes the singularities

(A) $f(x_1, y_1, x_2, y_2) = (x_1, x_1 y_2^2 + x_2^2 + y_2^4)$,

(B) $f(x_1, y_1, x_2, y_2) = (x_1, x_1 y_2^2 + x_2^2 - y_2^4)$.

These singularities play an important role in fractional monodromy, and in explicit systems [19, 9, 20, 11, 18, 8] appear to be stable under parametric deformations.

Theorem 15 *The singularities (A) and (B) are infinitesimally unstable.*

Proof. The (A) case: The generic infinitesimal completely integrable deformation has the form

$$\begin{pmatrix} g_1 \\ y_2^2 g_1 + 2x_2 \int \{x_2, g_1\} dy_1 + (2x_1 y_2 + 4y_2^3) \int \{y_2, g_1\} dy_1 + c(x_1, x_2, y_2) \end{pmatrix},$$

while the generic element of the infinitesimal action space has the form

$$\begin{pmatrix} 1 \\ y_2^2 \end{pmatrix} H_1(x, y) + \begin{pmatrix} 0 \\ 2x_2 \end{pmatrix} H_2(x, y) + \begin{pmatrix} 0 \\ 2x_1 y_2 + 4y_2^3 \end{pmatrix} H_3(x, y) + \begin{pmatrix} K_1(x_1, x_1 y_2^2 + x_2^2 + y_2^4) \\ K_2(x_1, x_1 y_2^2 + x_2^2 + y_2^4) \end{pmatrix}.$$

Whatever g_1 is, choosing $H_1 = g_1 - K_1$ one is reduced to prove that

$$c(x_1, x_2, y_2) = y_2^2 \widetilde{K}_1(x_1, x_1 y_2^2 + x_2^2 + y_2^4) + x_2 \widetilde{H}_2(x, y) + (x_1 y_2 + 2y_2^3) \widetilde{H}_3(x, y) + \widetilde{K}_2(x_1, x_1 y_2^2 + x_2^2 + y_2^4)$$

for appropriate $\widetilde{H}_2, \widetilde{H}_3, \widetilde{K}_1, \widetilde{K}_2$. The function $c(x_1, x_2, y_2)$ can be decomposed into $x_2 c_1(x_1, x_2, y_2) + c_2(x_1, y_2)$ and, as in the cuspidal (C) case, the term $x_2 \widetilde{H}_2(x, y)$ reconduces the solution of the above equation to the solution of the equation

$$p(x_1, y_2) = y_2^2 k_1(x_1, x_1 y_2^2 + y_2^4) + (x_1 y_2 + 2y_2^3) h(x_1, y_2) + k_2(x_1, x_1 y_2^2 + y_2^4). \quad (2)$$

The polynomial $p(x_1, y_2) = y_2$ cannot be written as in (2), and as a matter of facts, the bifurcation diagram associated to the completely integrable system $f_\lambda(x_1, y_1, x_2, y_2) = (x_1, x_1 y_2^2 + x_2^2 + y_2^4 + \lambda y_2)$ for $\lambda \neq 0$ has two disjoint singular lines, one of elliptic (E) singularities, the other of hyperbolic (H) singularities (see Figure 3). □

Remark. Globally, hyperbolic singularities belong to two possible topological types of level-sets: the bi-tori (a figure eight times an interval whose two boundaries are glued with an orientation preserving diffeomorphism), and the curled-tori (same but with an orientation-reversing glueing diffeomorphism). The hyperbolic line attached to a cuspidal singularity necessarily has bi-tori as level-sets, while the hyperbolic line crucial in systems with fractional monodromy necessarily has curled-tori as level-sets.

Fractional monodromy is a topological invariant, and hence is a stable phenomenon. The singularities that play a role in fractional monodromy must hence be stable, but they locally are of type (A) and (B). It follows that their stability requires a non-local justification.

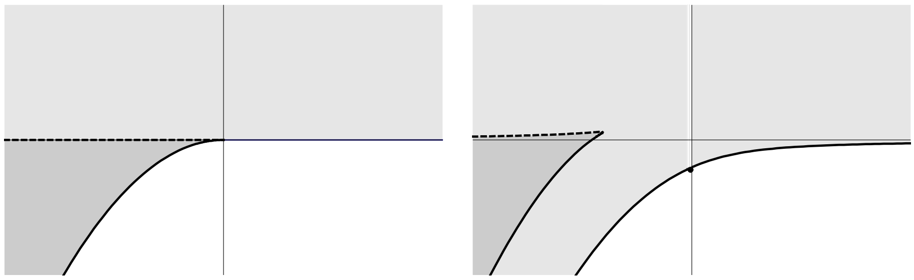


Figure 3: The two bifurcation diagrams corresponding to f_0 and to f_λ with λ small. The preimage of dark gray points has two connected components, the preimage of light gray points is connected. The singularities after deformation are elliptic, hyperbolic and cuspidal.

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