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Asymptotic behavior of the solutions of the Dirichlet problem for the Laplace operator in a domain with a small hole. A functional analytic approach

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Summary: We consider a hypersurface in $\mathbb{R}^n$ parametrized by a diffeomorphism ϕ^o of the unit sphere in $\mathbb{R}^n$ into $\mathbb{R}^n$, and we take a point w in the domain $\mathbb{I}[\phi^o]$ enclosed by the image of ϕ^o, and we consider the 'hole' $\mathbb{I}[w + \epsilon\xi]$ enclosed by the image of the hypersurface $w + \epsilon\xi$, where ξ is a diffeomorphism as ϕ^o with $0 \in \mathbb{I}[\xi]$ and ϵ is a small positive real parameter. Then we consider the Dirichlet problem for the Laplace equation in the perforated domain $\mathbb{I}[\phi^o]$ with the hole $\mathbb{I}[w + \epsilon\xi]$ removed and show real analytic continuation properties of the solution u and of the corresponding energy integral as functionals of the sextuple of w, ϵ, ξ, ϕ^o, and of the Dirichlet data in the interior and exterior boundaries of the perforated domain, which we think of as a point in an appropriate Banach space, around a degenerate sextuple with $\epsilon = 0$.

1 Introduction

In this paper, we consider the Dirichlet problem for the Laplace equation in a perforated domain. We consider an open domain $\mathbb{I}[\phi^o]$ of $\mathbb{R}^n$ enclosed by the image $\phi^o(\partial\mathbb{B}_n)$ of a diffeomorphism ϕ^o of the boundary $\partial\mathbb{B}_n$ of the unit ball $\mathbb{B}_n$ in $\mathbb{R}^n$ into $\mathbb{R}^n$. Then we select a point w in $\mathbb{I}[\phi^o]$ and we consider the 'hole' $\mathbb{I}[w + \epsilon\xi]$ enclosed by the image of the hypersurface $w + \epsilon\xi$, where ξ is a diffeomorphism as ϕ^o with $0 \in \mathbb{I}[\xi]$ and ϵ is a small positive real parameter. Then we consider the annular domain $\mathbb{A}[w, \epsilon, \xi, \phi^o]$ bounded by the pair of 'sphere type' hypersurfaces $w + \epsilon\xi$ and ϕ^o, and a Dirichlet boundary value problem for the Laplace equation in $\mathbb{A}[w, \epsilon, \xi, \phi^o]$. We assign the Dirichlet data $g^i \circ (w + \epsilon\xi)^{(-1)}$ on the 'inner boundary' $w + \epsilon\xi(\partial\mathbb{B}_n)$ and the Dirichlet data $g^o \circ \phi^{o(-1)}$ on the 'outer boundary' $\phi^o(\partial\mathbb{B}_n)$. We note that we have chosen to write the boundary data

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in this form only in order to have both the functions g^i and g^o defined on the common domain $\partial\mathbb{B}_n$. Then we consider the problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{A}[w,\epsilon,\xi,\phi^o], \\ u = g^i \circ (w+\epsilon\xi)^{(-1)} & \text{on } w+\epsilon\xi(\partial\mathbb{B}_n), \\ u = g^o \circ \phi^{o(-1)} & \text{on } \phi^o(\partial\mathbb{B}_n). \end{cases} \tag{1.1}$$

and we ask what happens when the parameter ϵ shrinks to 0.

Such problem is by no means new and has been long investigated by the techniques of asymptotic analysis, which aim at giving complete asymptotic expansions of the solution of (1.1) in terms of the parameter ϵ. It is perhaps difficult to provide a complete list of the contributions. Here, we mention the work of Kozlov, Maz'ya and Movchan [8], Maz'ya, Nazarov and Plamenewskii [14], Movchan [16], Ozawa [17], Ward and Keller [19]. In particular, we mention that a complete asymptotic expansion of the solution of (1.1) in terms of ϵ for fixed values of w, ξ, ϕ^o can be found in Maz'ya, Nazarov and Plamenewskii [14, Thms. 2.1.1, 2.4.1, Vol. I] (see also Movchan [16, p. 206]) by the so-called compound asymptotic expansion method.

This paper aims at characterizing the behavior of the solution of (1.1) around $\epsilon = 0$ by a different approach, which in ideas stems from that of [9, 10].

We shall consider the solution $u[w,\epsilon,\xi,\phi^o,g^i,g^o](\cdot)$ of Problem (1.1) as a function of the complex of variables $(w,\epsilon,\xi,\phi^o,g^i,g^o)$, which we denote $\mathbf{p}$, around a 'degenerate' sextuple $\mathbf{p}_0 \equiv (w_0,0,\xi_0,\phi^o_0,g^i_0,g^o_0)$ with $\epsilon = 0$ and ask how such solution $u[\mathbf{p}](\cdot)$ or the corresponding energy integral depend on $\mathbf{p}$, which we think of as a point of an appropriate Banach space. Our main results are Theorem 5.3 and Theorem 6.1 and describe the behavior of $u[\mathbf{p}]$ in a fixed region of its domain and of the corresponding energy integral around $\mathbf{p}_0$.

Namely, we prove that the solution $u[\mathbf{p}]$ restricted to a compact subset of its domain can be written in the form

$$u[\mathbf{p}](\cdot) = U_1[\mathbf{p}](\cdot) + \frac{U_2[\mathbf{p}](\cdot)}{V_1[w,\epsilon,\xi,\phi^o] + V_2[w,\epsilon,\xi,\phi^o]\Upsilon_n(\epsilon)},$$

where $U_1[\mathbf{p}](\cdot)$ and $U_2[\mathbf{p}](\cdot)$ are functions which depend real analytically upon $\mathbf{p}$ and $V_1[w,\epsilon,\xi,\phi^o]$, $V_2[w,\epsilon,\xi,\phi^o]$ are reals which depend real analytically on $\mathbf{p}$ (actually only on (w,ϵ,ξ,ϕ^o)) around a degenerate sextuple, $\Upsilon_n(\epsilon)$ is the function of $]0,+\infty[$ to $\mathbb{R}$ such that $\Upsilon_n(|\cdot|)$ is the fundamental solution of the Laplace operator (cf. (2.2), Theorem 5.3.) In particular, if $n \geq 3$, one sees that $u[\cdot]$ admits a real analytic continuation, while for $n = 2$, $u[\cdot]$ has a logarithmic behavior around a degenerate sextuple.

Then we turn to consider the energy integral $\int_{\mathbb{A}[w,\epsilon,\xi,\phi^o]} |\nabla u[\mathbf{p}](t)|^2\, dt$ of $u[\mathbf{p}]$ and we show that it can be written in the form

$$F_1[\mathbf{p}] + \frac{F_2[\mathbf{p}]}{V_1[w,\epsilon,\xi,\phi^o] + V_2[w,\epsilon,\xi,\phi^o]\Upsilon_n(\epsilon)},$$

where $F_1[\mathbf{p}]$, $F_2[\mathbf{p}]$ are reals which depend real analytically on $\mathbf{p}$ around a degenerate sextuple (cf. Theorem 6.1.) As a Corollary, we prove that the electrostatic capacity of $\mathrm{cl}\mathbb{I}[w+\epsilon\xi]$ with respect to $\mathbb{I}[\phi^o]$ can be written as

$$\mathrm{Cap}[w,\epsilon,\xi,\phi^o] = \frac{-1}{V_1[w,\epsilon,\xi,\phi^o] + V_2[w,\epsilon,\xi,\phi^o]\Upsilon_n(\epsilon)}.$$

We now briefly outline our strategy. We first decompose the boundary data g^i, g^o into a term for which the Fredholm integral equations corresponding to problem (1.1) can be solved, and into a term for which such equations are not solvable. Then we write $u[\mathbf{p}]$ as a sum $u_r[\mathbf{p}] + u_s[\mathbf{p}]$, where $u_r[\mathbf{p}]$ is a double layer potential whose density solves the Fredholm equations (cf. (2.4)) and $u_s[\mathbf{p}]$ is a suitable multiple of a single layer potential whose density solves the equation transpose to the Fredholm equation of (1.1) (cf. (2.3).) As is well known, the presence of $u_s[\mathbf{p}]$ is due to the hole in the domain $\mathbb{I}[\phi^o]$ (cf. *e.g.*, Folland [5, Ch. 3].) Next we analyze the real analytic continuation properties of the solutions of the transpose equation (cf. Theorem 3.10) and then of the Fredholm equation (cf. Theorem 4.4) around $\mathbf{p}_0$. Since the corresponding problems are degenerate, we construct equivalent nondegenerate problems to which we apply a corollary of the Implicit Function Theorem (cf. Theorem A.1 of the Appendix.) Then by exploiting the description of the real analytic continuation properties of the density functions upon variation of $\mathbf{p}$ around $\mathbf{p}_0$ and the equality $u[\mathbf{p}] = u_r[\mathbf{p}]+u_s[\mathbf{p}]$, we deduce our main results.

Our results are in accordance with the behavior one would expect by looking at the above mentioned asymptotic expansions. Actually, the above mentioned expansions of $u[\mathbf{p}]$ (see Maz'ya, Nazarov and Plamenewskii [14, Thms. 2.1.1, 2.4.1, Vol. I], Movchan [16, p. 206]) and of the corresponding energy integral (cf. Maz'ya, Nazarov and Plamenewskii [14, Thms. 8.1.4, 8.1.5, Vol. I]) may *suggest* the validity of the above results, at least for fixed values of w, ξ, ϕ^o, g^i, g^o.

Perhaps, one could also try to prove such results for fixed values of w, ξ, ϕ^o, g^i, g^o by showing that the series of the above mentioned asymptotic expansions converge to the corresponding functions. Such approach however may be nontrivial, and for its possible feasibility the author takes no credit and refers to some expert in asymptotic analysis. We also mention that one could think of proving our results by considering a real analytic curve of sextuples $\mathbf{p}$ through a degenerate sextuple $\mathbf{p}_0$ with $\epsilon = 0$ depending on a real parameter and then by showing the appropriate continuation properties of the solution of (1.1) or of the corresponding energy integral as a function of the parameter of the curve. Such method, also known as 'parameter method' at least since the thirties would anyway yield a weaker form of our results. Indeed, it is well known that for an operator in a Banach space, even in the finite dimensional case, real analyticity on all real analytic curves does not imply real analyticity (cf. Boman [2]).

The paper is organized as follows. Section 2 is a section of preliminaries. In Sections 3 and 4, we construct nondegenerate systems of integral equations for the densities of the potentials we employ to represent solutions, and we analyze them by a consequence of the Implicit Function Theorem. In Sections 5 and 6 we apply the results of Sections 3 and 4 to prove our main results. At the end of the paper we have enclosed an Appendix, where we present a corollary of the Implicit Function Theorem which we exploit in Sections 3 and 4.

2 Preliminaries and notation

We denote the norm on a (real) normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the product space $\mathcal{X}\times\mathcal{Y}$ with the norm defined by $\|(x,y)\|_{\mathcal{X}\times\mathcal{Y}}\equiv\|x\|_{\mathcal{X}}+\|y\|_{\mathcal{Y}}$ $\forall(x,y)\in\mathcal{X}\times\mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. For standard definitions of Calculus in normed spaces, we refer to Prodi and Ambrosetti [18]. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. Throughout the paper, n is an element of $\mathbb{N}\setminus\{0,1\}$. The inverse function of an invertible function f is denoted $f^{(-1)}$, as opposed to the reciprocal of a complex-valued function g, or the inverse of a matrix A, which are denoted g^{-1} and A^{-1}, respectively. A dot '$\cdot$' denotes the inner product in $\mathbb{R}^n$, or the matrix product between matrices with real entries. Let $\mathbb{D}\subseteq\mathbb{R}^n$. Then $\mathrm{cl}\,\mathbb{D}$ denotes the closure of $\mathbb{D}$ and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. For all $R>0$, $x\in\mathbb{R}^n$, x_j denotes the j-th coordinate of x, $|x|$ denotes the Euclidean modulus of x in $\mathbb{R}^n$, and $\mathbb{B}_n(x,R)$ denotes the ball $\{y\in\mathbb{R}^n : |x-y|<R\}$. For short, we set $\mathbb{B}_n\equiv\mathbb{B}_n(0,1)$. Let Ω be an open subset of $\mathbb{R}^n$. The space of m times continuously differentiable real-valued functions on Ω is denoted by $C^m(\Omega,\mathbb{R})$, or more simply by $C^m(\Omega)$. Let $r\in\mathbb{N}\setminus\{0\}$, $f\in(C^m(\Omega))^r$. The s-th component of f is denoted f_s, and Df denotes the gradient matrix of f. Let $\eta\equiv(\eta_1,\dots,\eta_n)\in\mathbb{N}^n$, $|\eta|\equiv\eta_1+\dots+\eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|}f}{\partial x_1^{\eta_1}\dots\partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions f such that f and its derivatives $D^\eta f$ of order $|\eta|\leq m$ can be extended with continuity to $\mathrm{cl}\,\Omega$ is denoted $C^m(\mathrm{cl}\,\Omega)$. The subspace of $C^m(\mathrm{cl}\,\Omega)$ whose functions have m-th order derivatives that are Hölder continuous with exponent $\alpha\in]0,1]$ is denoted $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, (cf. *e.g.* Gilbarg and Trudinger [6]). Let $\mathbb{D}\subseteq\mathbb{R}^n$. Then $C^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{D})$ denotes $\left\{f\in(C^{m,\alpha}(\mathrm{cl}\,\Omega))^n : f(\mathrm{cl}\,\Omega)\subseteq\mathbb{D}\right\}$. Now let Ω be a bounded open subset of $\mathbb{R}^n$. Then $C^m(\mathrm{cl}\,\Omega)$ endowed with the norm $\|f\|_m\equiv\sum_{|\eta|\leq m}\sup_{\mathrm{cl}\,\Omega}|D^\eta f|$ is a Banach space. If $f\in C^{0,\alpha}(\mathrm{cl}\,\Omega)$, then its Hölder quotient $|f:\Omega|_\alpha$ is defined as $\sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} : x,y\in\mathrm{cl}\,\Omega, x\neq y\right\}$. The space $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, equipped with its usual norm $\|f\|_{m,\alpha}=\|f\|_m+\sum_{|\eta|=m}|D^\eta f|_\alpha$, is well-known to be a Banach space. We say that a bounded open subset of $\mathbb{R}^n$ is of class C^m or of class $C^{m,\alpha}$, if it is a manifold with boundary imbedded in $\mathbb{R}^n$ of class C^m or $C^{m,\alpha}$, respectively (cf. *e.g.*, Gilbarg and Trudinger [6, §6.2]). For standard properties of the functions of class $C^{m,\alpha}$ both on a domain of $\mathbb{R}^n$ or on a manifold imbedded in $\mathbb{R}^n$ we refer to Gilbarg and Trudinger [6] (see also [12, §2, Lem. 3.1, 4.26, Thm. 4.28], Lanza and Rossi [13, §2]). We note that throughout the paper 'analytic' means 'real analytic'. For the definition and properties of analytic operators, we refer to Prodi and Ambrosetti [18, p. 89].

The set

$$\mathcal{A}_{\partial\mathbb{B}_n}\equiv\Big\{\phi\in C^1\left(\partial\mathbb{B}_n,\mathbb{R}^n\right) : \\ \phi \text{ is injective, } d\phi(y) \text{ is injective for all } y\in\partial\mathbb{B}_n\Big\}$$

is open in $C^1(\partial\mathbb{B}_n,\mathbb{R}^n)$ (see [12, Prop. 4.29], Lanza and Rossi [13, Lem. 2.5]). We now note that if $\phi\in\mathcal{A}_{\partial\mathbb{B}_n}$, then by the Jordan–Leray separation Theorem (cf. *e.g.*, Deimling [3, Thm. 5.2]), the set $\mathbb{R}^n\setminus\phi(\partial\mathbb{B}_n)$ has exactly two connected components. We denote by $\mathbb{I}[\phi]$ the bounded connected component, and by $\mathbb{E}[\phi]$ the unbounded connected component.

If we assume that $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1]$ and $\phi \in C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}$, then $\mathbb{I}[\phi]$ is a bounded open connected set of class $C^{m,\alpha}$, and

$$\partial\mathbb{I}[\phi] = \phi(\partial\mathbb{B}_n) = \partial\mathbb{E}[\phi] \tag{2.1}$$

(cf. *e.g.*, Lanza and Rossi [13, Lem. 2.6]).

For all pairs $\boldsymbol{\phi} \equiv (\phi^i, \phi^o) \in \mathcal{A}^2_{\partial\mathbb{B}_n}$ such that $\phi^i(\partial\mathbb{B}_n) \subseteq \mathbb{I}[\phi^o]$, we set

$$\mathbb{A}[\boldsymbol{\phi}] \equiv \mathbb{I}[\phi^o] \cap \mathbb{E}[\phi^i].$$

Then by applying the Jordan–Leray separation Theorem to ϕ^i, ϕ^o, it is easy to see that

$$\partial\mathbb{A}[\boldsymbol{\phi}] = \phi^i(\partial\mathbb{B}_n) \cup \phi^o(\partial\mathbb{B}_n).$$

If we further assume that $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1]$ and that $\boldsymbol{\phi} \equiv (\phi^i, \phi^o)$ belongs to $(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n})^2$, then as we have said above both $\phi^i(\partial\mathbb{B}_n)$ and $\phi^o(\partial\mathbb{B}_n)$ are manifolds of class $C^{m,\alpha}$. Then again by exploiting (2.1), one can easily show that $\partial\mathbb{A}[\boldsymbol{\phi}]$ is an open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. We shall consider the Dirichlet problem in $\mathbb{A}[\boldsymbol{\phi}]$ with $\phi^i = w + \epsilon\xi$, where $w \in \mathbb{I}[\phi^o]$, ϵ is a real nonzero parameter and $0 \in \mathbb{I}[\xi]$, $w + \epsilon\xi(\partial\mathbb{B}_n) \subseteq \mathbb{I}[\phi^o]$. Thus now the corresponding annular domain $\mathbb{A}[\boldsymbol{\phi}]$ can be identified by the variables w, ϵ, ξ, ϕ^o. Thus we denote by $\mathbb{A}[w, \epsilon, \xi, \phi^o]$ the set $\mathbb{A}[\boldsymbol{\phi}]$ with $\boldsymbol{\phi} \equiv (w + \epsilon\xi, \phi^o)$. We find also convenient to set $\mathbb{A}[w, 0, \xi, \phi^o] \equiv \mathbb{I}[\phi^o] \setminus \{w\}$. As we have just seen, w, ϵ, ξ, ϕ^o are subject to certain conditions. We now introduce the set of quadruples $(w, \epsilon, \xi, \phi^o)$ which we retain as admissible. For all fixed $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$, we set

$$\mathcal{E}^{m,\alpha} \equiv \Big\{\mathbf{a} \equiv (w, \epsilon, \xi, \phi^o) \in \mathbb{R}^n \times \mathbb{R} \times \left(C^{m,\alpha}\left(\partial\mathbb{B}_n, \mathbb{R}^n\right) \cap \mathcal{A}_{\partial\mathbb{B}_n}\right)^2 : \\ 0 \in \mathbb{I}[\xi],\, w + \epsilon\xi(\partial\mathbb{B}_n) \subseteq \mathbb{I}[\phi^o]\Big\},$$

and

$$\mathcal{E}^{m,\alpha}_+ \equiv \left\{\mathbf{a} \equiv (w, \epsilon, \xi, \phi^o) \in \mathcal{E}^{m,\alpha} : \epsilon > 0\right\}.$$

The sets $\mathcal{E}^{m,\alpha}$ and $\mathcal{E}^{m,\alpha}_+$ are easily seen to be open in the Banach space $\mathbb{R}^n \times \mathbb{R} \times (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n))^2$. A simple topological argument shows that if $(w, \epsilon, \xi, \phi^o) \in \mathcal{E}^{m,\alpha}$ with $\epsilon \neq 0$, then $\mathrm{cl}\mathbb{I}[w + \epsilon\xi] \subseteq \mathbb{I}[\phi^o]$. To simplify our notation we shall sometimes write $\mathbf{a}$ instead of $(w, \epsilon, \xi, \phi^o)$.

For each given pair of Dirichlet data $(g^i, g^o) \in C^{m,\alpha}(\partial\mathbb{B}_n)^2$ and for a quadruple $(w, \epsilon, \xi, \phi^o) \in \mathcal{E}^{m,\alpha}$ with $\epsilon > 0$, we consider the Dirichlet problem (1.1), which is well known to have a unique solution $u[w, \epsilon, \xi, \phi^o, g^i, g^o]$ of class $C^{m,\alpha}(\mathrm{cl}\mathbb{A}[w, \epsilon, \xi, \phi^o])$, and we investigate the behavior of $u[w, \epsilon, \xi, \phi^o, g^i, g^o]$ upon perturbation of the sextuple $(w, \epsilon, \xi, \phi^o, g^i, g^o)$ around a given degenerate sextuple $(w_0, 0, \xi_0, \phi^o_0, g^i_0, g^o_0)$ of $\mathcal{E}^{m,\alpha} \times (C^{m,\alpha}(\partial\mathbb{B}_n))^2$. To simplify our notation, we shall write $\mathbf{p}$ instead of $(w, \epsilon, \xi, \phi^o, g^i, g^o)$ and $\mathbf{p}_0$ instead of $(w_0, 0, \xi_0, \phi^o_0, g^i_0, g^o_0)$.

Since we shall exploit classical Potential Theory, we now introduce the normal field on the boundary of an annular domain and then some preliminaries.

For each $\phi \in \mathcal{A}_{\partial\mathbb{B}_n}$, we denote by $\nu_\phi(\cdot)$ the outward unit normal to $\mathbb{I}[\phi]$ on $\partial\mathbb{I}[\phi] = \phi(\partial\mathbb{B}_n)$. For all pairs $\boldsymbol{\phi} \equiv (\phi^i, \phi^o) \in \mathcal{A}^2_{\partial\mathbb{B}_n}$ such that $\phi^i(\partial\mathbb{B}_n) \subseteq \mathbb{I}[\phi^o]$, we denote by $\nu_{\boldsymbol{\phi}}(\cdot)$ the outward unit normal to $\mathbb{A}[\boldsymbol{\phi}]$ on $\partial\mathbb{A}[\boldsymbol{\phi}]$. Clearly,

$$\nu_{\boldsymbol{\phi}}(t) = -\nu_{\phi^i}(t) \quad \forall t \in \phi^i(\partial\mathbb{B}_n), \qquad \nu_{\boldsymbol{\phi}}(t) = \nu_{\phi^o}(t) \quad \forall t \in \phi^o(\partial\mathbb{B}_n).$$

In the specific case $\mathbf{a} \equiv (w, \epsilon, \xi, \phi^o) \in \mathcal{E}^{m,\alpha}_+$, $\phi^i = w + \epsilon\xi$, we write $\nu_{\mathbf{a}}$ to denote the exterior unit normal field to $\mathbb{A}[\mathbf{a}] \equiv \mathbb{A}[w, \epsilon, \xi, \phi^o]$.

We denote by $d\sigma$ the standard surface measure on a manifold of codimension 1 of $\mathbb{R}^n$. We will sometimes attach to $d\sigma$ a subscript to indicate the integration variable.

We denote by Υ_n the function of $]0, +\infty[$ to $\mathbb{R}$ defined by

$$\Upsilon_n(r) \equiv \begin{cases} \frac{1}{s_n} \log r & \forall r \in]0, +\infty[, \quad \text{if } n = 2, \\ \frac{1}{(2-n)s_n} r^{2-n} & \forall r \in]0, +\infty[, \quad \text{if } n > 2, \end{cases} \tag{2.2}$$

where s_n denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n$. We denote by S_n the function of $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_n(\xi) \equiv \Upsilon_n(|\xi|) \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

S_n is well-known to be the fundamental solution of the Laplace operator. Clearly, we have $DS_n(\xi) = \frac{1}{s_n} \frac{\xi}{|\xi|^n}$ for $n \geq 2$. We collect in the following statement some known facts in classical Potential Theory.

Theorem 2.1 *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1[$. *Let* $\mathbf{a} \equiv (w, \epsilon, \xi, \phi^o) \in \mathcal{E}^{m,\alpha}_+$, $\phi^i \equiv w + \epsilon\xi$.

(i) If $T \in C^{m-1,\alpha}(\partial\mathbb{A}[\mathbf{a}])$, $\int_{\phi^i(\partial\mathbb{B}_n)} T d\sigma = 0$, *then the problem*

$$\begin{cases} T(t) = \frac{1}{2}\tau(t) + \int_{\partial\mathbb{A}[\mathbf{a}]} \tau(s) \frac{\partial}{\partial\nu_{\mathbf{a}}(t)} \left(S_n(t-s)\right) d\sigma_s & \forall t \in \partial\mathbb{A}[\mathbf{a}], \\ \int_{\phi^i(\partial\mathbb{B}_n)} \tau d\sigma = 1, \end{cases} \tag{2.3}$$

has one and only one solution $\tau \in C^{m-1,\alpha}(\partial\mathbb{A}[\mathbf{a}])$.

(ii) Let τ_0 *be the only solution of problem (2.3) with* $T = 0$. *Let* $\tilde{\Gamma} \in C^{m,\alpha}(\partial\mathbb{A}[\mathbf{a}])$ *be such that* $\int_{\partial\mathbb{A}[\mathbf{a}]} \tilde{\Gamma}\tau_0 \, d\sigma = 0$. *Then the problem*

$$\begin{cases} \tilde{\Gamma}(t) = \frac{1}{2}\mu(t) + \int_{\partial\mathbb{A}[\mathbf{a}]} \mu(s) \frac{\partial}{\partial\nu_{\mathbf{a}}(s)} \left(S_n(t-s)\right) d\sigma_s & \forall t \in \partial\mathbb{A}[\mathbf{a}], \\ \int_{\phi^i(\partial\mathbb{B}_n)} \mu d\sigma = 0, \end{cases} \tag{2.4}$$

has one and only one solution $\mu \in C^{m,\alpha}(\partial\mathbb{A}[\mathbf{a}])$.

(iii) Let τ_0 *be the only solution of problem (2.3) with* $T = 0$. *Let* $\Gamma \in C^{m,\alpha}(\partial\mathbb{A}[\mathbf{a}])$. *Then the boundary value problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{A}[\mathbf{a}], \\ u = \Gamma & \text{on } \partial\mathbb{A}[\mathbf{a}], \end{cases}$$

has one and only one solution $u \in C^{m,\alpha}(\mathrm{cl}\mathbb{A}[\mathbf{a}])$ *and*

$$u(t) = \int_{\partial\mathbb{A}[\mathbf{a}]} \mu(s) \frac{\partial}{\partial \nu_{\mathbf{a}}(s)} \left(S_n(t-s)\right) \, d\sigma_s \qquad (2.5)$$
$$+ \frac{\int_{\partial\mathbb{A}[\mathbf{a}]} \Gamma\tau_0 \, d\sigma}{\left\{\int_{\phi^i(\partial\mathbb{B}_n)} d\sigma\right\}^{-1}} \frac{\int_{\partial\mathbb{A}[\mathbf{a}]} \tau_0(s) S_n(t-s) \, d\,\sigma_s}{\int_{\phi^i(\partial\mathbb{B}_n)} \int_{\partial\mathbb{A}[\mathbf{a}]} \tau_0(s) S_n(t-s) \, d\sigma_s d\sigma_t}$$

for all $t \in \mathbb{A}[\mathbf{a}]$*, where* μ *is the only solution of (2.4) with*

$$\tilde{\Gamma}(t) \equiv \Gamma(t) - \left(\int_{\partial\mathbb{A}[\mathbf{a}]} \Gamma\tau_0 \, d\sigma\right) \chi_{\phi^i(\partial\mathbb{B}_n)}(t) \qquad \forall t \in \partial\mathbb{A}[\mathbf{a}]. \qquad (2.6)$$

(iv) If $T \in C^{m-1,\alpha}(\xi(\partial\mathbb{B}_n))$*,* $\int_{\xi(\partial\mathbb{B}_n)} T \, d\sigma = 0$*,* $\beta \in \mathbb{R}$*, then the problem*

$$\begin{cases} T(t) = \frac{1}{2}\tau(t) - \int_{\xi(\partial\mathbb{B}_n)} \tau(s) \frac{\partial}{\partial\nu_\xi(t)} \left(S_n(t-s)\right) \, d\sigma_s & \forall t \in \xi(\partial\mathbb{B}_n), \\ \int_{\xi(\partial\mathbb{B}_n)} \tau d\sigma = \beta, \end{cases} \qquad (2.7)$$

has one and only one solution $\tau \in C^{m-1,\alpha}(\xi(\partial\mathbb{B}_n))$*.*

(v) Let τ_0 *be the only solution of problem (2.7) with* $T = 0$*,* $\beta = 1$*. Let* $\tilde{\Gamma} \in C^{m,\alpha}(\xi(\partial\mathbb{B}_n))$ *be such that* $\int_{\xi(\partial\mathbb{B}_n)} \tilde{\Gamma}\tau_0 \, d\sigma = 0$*,* $\omega \in \mathbb{R}$*. Then the problem*

$$\begin{cases} \tilde{\Gamma}(t) = \frac{1}{2}\mu(t) - \int_{\xi(\partial\mathbb{B}_n)} \mu(s) \frac{\partial}{\partial\nu_\xi(s)} \left(S_n(t-s)\right) \, d\sigma_s & \forall t \in \xi(\partial\mathbb{B}_n), \\ \int_{\xi(\partial\mathbb{B}_n)} \mu \, d\sigma = \omega, \end{cases} \qquad (2.8)$$

has one and only one solution $\mu \in C^{m,\alpha}(\xi(\partial\mathbb{B}_n))$*.*

(vi) Let τ_0 *be the only solution of problem (2.7) with* $T = 0$*,* $\beta = 1$*. Let* $\Gamma \in C^{m,\alpha}(\xi(\partial\mathbb{B}_n))$*. Then the boundary value problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{E}[\xi], \\ u = \Gamma & \text{on } \xi(\partial\mathbb{B}_n), \\ \sup_{t\in\mathbb{E}[\xi]} |u(t)||t|^{n-2} < +\infty, \end{cases} \qquad (2.9)$$

has one and only one solution $u^i \in C^{m,\alpha}(\mathrm{cl}\mathbb{E}[\xi])$*.*

If $n \geq 3$*, the unique solution* u *of (2.9) is delivered by the formula*

$$u(t) = -\int_{\xi(\partial\mathbb{B}_n)} \mu(s) \frac{\partial}{\partial\nu_\xi(s)} \left(S_n(t-s)\right) \, d\sigma_s \qquad (2.10)$$
$$+ \frac{\int_{\xi(\partial\mathbb{B}_n)} \Gamma\tau_0 \, d\sigma}{\left\{\int_{\xi(\partial\mathbb{B}_n)} d\sigma\right\}^{-1}} \frac{\int_{\xi(\partial\mathbb{B}_n)} \tau_0(s) S_n(t-s) \, d\sigma_s}{\int_{\xi(\partial\mathbb{B}_n)} \int_{\xi(\partial\mathbb{B}_n)} \tau_0(s) S_n(t-s) \, d\sigma_s d\sigma_t}$$

for all $t \in \mathbb{E}[\xi]$*, where* μ *is the only solution of (2.8) with* $\omega = 0$ *and*

$$\tilde{\Gamma}(t) \equiv \Gamma(t) - \left(\int_{\xi(\partial\mathbb{B}_n)} \Gamma\tau_0 \, d\sigma \right) \qquad \forall t \in \xi(\partial\mathbb{B}_n). \tag{2.11}$$

If $n = 2$*, the unique solution* u *of (2.9) is delivered by the formula*

$$u(t) = -\int_{\xi(\partial\mathbb{B}_n)} \mu(s) \frac{\partial}{\partial\nu_\xi(s)} \left(S_n(t-s)\right) \, d\sigma_s + \int_{\xi(\partial\mathbb{B}_n)} \Gamma\tau_0 \, d\sigma, \tag{2.12}$$

for all $t \in \mathbb{E}[\xi]$*, where* μ *is the only solution of (2.8) with* $\tilde{\Gamma}$ *as in (2.11) and* $\omega = 0$.

(vii) *If* $T \in C^{m-1,\alpha}(\phi^o(\partial\mathbb{B}_n))$*, then the problem*

$$T(t) = \frac{1}{2}\tau(t) + \int_{\phi^o(\partial\mathbb{B}_n)} \tau(s) \frac{\partial}{\partial\nu_{\phi^o}(t)} \left(S_n(t-s)\right) \, d\sigma_s \qquad \forall t \in \phi^o(\partial\mathbb{B}_n), \tag{2.13}$$

has one and only one solution $\tau \in C^{m-1,\alpha}(\phi^o(\partial\mathbb{B}_n))$.

(viii) *Let* $\tilde{\Gamma} \in C^{m,\alpha}(\phi^o(\partial\mathbb{B}_n))$*. Then the integral equation*

$$\tilde{\Gamma}(t) = \frac{1}{2}\mu(t) + \int_{\phi^o(\partial\mathbb{B}_n)} \mu(s) \frac{\partial}{\partial\nu_{\phi^o}(s)} \left(S_n(t-s)\right) \, d\sigma_s \qquad \forall t \in \phi^o(\partial\mathbb{B}_n), \tag{2.14}$$

has one and only one solution $\mu \in C^{m,\alpha}(\phi^o(\partial\mathbb{B}_n))$*. Moreover, the boundary value problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{I}[\phi^o], \\ u = \tilde{\Gamma} & \text{on } \phi^o(\partial\mathbb{B}_n), \end{cases} \tag{2.15}$$

has one and only one solution $u^o \in C^{m,\alpha}(\mathrm{cl}\mathbb{I}[\phi^o])$*, delivered by the formula*

$$u^o(t) = \int_{\phi^o(\partial\mathbb{B}_n)} \mu(s) \frac{\partial}{\partial\nu_{\phi^o}(s)} \left(S_n(t-s)\right) \, d\sigma_s \qquad \forall t \in \mathbb{I}[\phi^o]. \tag{2.16}$$

For the existence and uniqueness of continuous solutions of the integral equations in (i), (ii), (iv), (v), (vii), (viii), we refer to Folland [5, Ch. 3], and for the $C^{m,\alpha}$ regularity of the corresponding solutions, we refer for example to [11, App. C]. Then the corresponding $C^{m,\alpha}$ regularity of the solution of the boundary value problems of (iii), (vi), (viii) follows by standard properties of the layer potentials (cf. Miranda [15], see also [13, Thm. 3.1]).

By Theorem 2.1, we can introduce the following notation.

If $(\xi, g^i) \in \left(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}\right) \times C^{m,\alpha}(\partial\mathbb{B}_n)$, then we denote by $u^i[\xi, g^i]$ the unique solution of (2.9) with $\Gamma = g^i \circ \xi^{(-1)}$, and we denote by $u^i_r[\xi, g^i]$ the function $-\int_{\xi(\partial\mathbb{B}_n)} \mu(s) \frac{\partial}{\partial\nu_\xi(s)} (S_n(t-s)) \, d\sigma_s$ of the variable $t \in \mathbb{E}[\xi]$ with μ as in (vi), and by $u^i_s[\xi, g^i]$ the difference $u^i[\xi, g^i] - u^i_r[\xi, g^i]$.

If $(\phi^o, g^o) \in \left(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}\right) \times C^{m,\alpha}(\partial\mathbb{B}_n)$, then we denote by $u^o[\phi^o, g^o]$ the unique solution of (2.15) with $\Gamma = g^o \circ (\phi^o)^{(-1)}$.

Remark 2.2 If τ_0 is as in (iii) of Theorem 2.1, then $\int_{\partial\mathbb{A}[\mathbf{a}]} \tau_0(s)\, d\sigma_s = 0$. Then by the integral equation (2.3), the single layer potential in the second fraction in the right-hand side of (2.5) must be constant in $\mathbb{I}[w+\epsilon\xi]$ and zero in $\mathbb{E}[\phi^o]$. Since τ_0 is not identically zero, such a constant value is nonzero and accordingly the denominator of the second fraction in the right-hand side of (2.5) can never vanish. Similarly, one can show that the denominator of the second fraction in the right-hand side of (2.10) can never vanish.

3 A real analyticity theorem for the solutions of the transpose equation (2.3)

We start our analysis of the transpose equation (2.3), which is defined on the **a** dependent domain $\partial\mathbb{A}[\mathbf{a}]$, by transforming it into an equation on the fixed domain $\partial\mathbb{B}_n$. To do so, we need the following technical Lemma, which can be verified by standard calculus (see also [13, p. 166]).

Lemma 3.1 *Let* $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$, $\phi, \xi \in C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}$.

(i) There exists a positive function $\tilde{\sigma}[\phi] \in C^{m-1,\alpha}(\partial\mathbb{B}_n)$ *such that*

$$\int_{\phi(\partial\mathbb{B}_n)} \omega(s)\, d\sigma_s = \int_{\partial\mathbb{B}_n} \omega\circ\phi(y)\tilde{\sigma}[\phi](y)\, d\sigma_y \qquad \forall \omega \in L^1\left(\phi\left(\partial\mathbb{B}_n\right)\right).$$

(ii) If $w \in \mathbb{R}^n$, $\epsilon \in \mathbb{R}\setminus\{0\}$, $0 \in \mathbb{I}[\xi]$, *then* $w+\epsilon\xi \in \mathcal{A}_{\partial\mathbb{B}_n}$ *and*

$$\nu_{w+\epsilon\xi}\circ(w+\epsilon\xi) = \mathrm{sgn}(\epsilon)\,\nu_\xi\circ\xi \qquad \text{on } \partial\mathbb{B}_n,$$

where $\mathrm{sgn}(\epsilon) \equiv 1$ *if* $\epsilon > 0$, $\mathrm{sgn}\,(\epsilon) \equiv -1$ *if* $\epsilon < 0$.

Then we have the following.

Proposition 3.2 *Let* $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. *Let* $M \equiv \left(M_j\right)_{j=1,2,3}$ *be the map of* $\mathcal{E}^{m,\alpha} \times \left(C^{m-1,\alpha}(\partial\mathbb{B}_n)\right)^2$ *to* $\left(C^{m-1,\alpha}(\partial\mathbb{B}_n)\right)^2 \times \mathbb{R}$ *defined by*

$$\begin{aligned}
&M_1[w,\epsilon,\xi,\phi^o,\eta^i,\rho^o](x) \qquad\qquad (3.1)\\
&\equiv \frac{1}{2}\eta^i(x) - \int_{\partial\mathbb{B}_n} \eta^i(y)(\nu_\xi\circ\xi(x))\cdot DS_n(\xi(x)-\xi(y))\tilde{\sigma}[\xi](y)\, d\sigma_y \\
&\quad - \epsilon^{n-1}\int_{\partial\mathbb{B}_n} \rho^o(y)(\nu_\xi\circ\xi(x))\cdot DS_n(w+\epsilon\xi(x)-\phi^o(y))\tilde{\sigma}[\phi^o](y)d\sigma_y \quad \forall x \in \partial\mathbb{B}_n,\\
&M_2[w,\epsilon,\xi,\phi^o,\eta^i,\rho^o](x)\\
&\equiv \frac{1}{2}\rho^o(x) + \int_{\partial\mathbb{B}_n} \eta^i(y)(\nu_{\phi^o}\circ\phi^o(x))\cdot DS_n(\phi^o(x)-w-\epsilon\xi(y))\tilde{\sigma}[\xi](y)\, d\sigma_y\\
&\quad + \int_{\partial\mathbb{B}_n} \rho^o(y)(\nu_{\phi^o}\circ\phi^o(x))\cdot DS_n(\phi^o(x)-\phi^o(y))\tilde{\sigma}[\phi^o](y)\, d\sigma_y \quad \forall x \in \partial\mathbb{B}_n,\\
&M_3[w,\epsilon,\xi,\phi^o,\eta^i,\rho^o] \equiv \int_{\partial\mathbb{B}_n} \eta^i(y)\tilde{\sigma}[\xi](y)\, d\sigma_y - 1
\end{aligned}$$

for all $(w,\epsilon,\xi,\phi^o,\eta^i,\rho^o)\in\mathcal{E}^{m,\alpha}\times\left(C^{m-1,\alpha}(\partial\mathbb{B}_n)\right)^2$. *If* $(w,\epsilon,\xi,\phi^o)\in\mathcal{E}^{m,\alpha}_+$, *then the pair of functions* $(\eta^i,\rho^o)\in\left(C^{m-1,\alpha}(\partial\mathbb{B}_n)\right)^2$ *satisfies equation*

$$M[w,\epsilon,\xi,\phi^o,\eta^i,\rho^o]=0 \tag{3.2}$$

if and only if the function $\tau\in C^{m-1,\alpha}(\partial\mathbb{A}[w,\epsilon,\xi,\phi^o])$ *defined by*

$$\begin{aligned}\tau(s) &\equiv \epsilon^{1-n}\eta^i\circ(w+\epsilon\xi)^{(-1)}(s) && \forall s\in w+\epsilon\xi(\partial\mathbb{B}_n),\\ \tau(s) &\equiv \rho^o\circ(\phi^o)^{(-1)}(s) && \forall s\in\phi^o(\partial\mathbb{B}_n),\end{aligned}$$

satisfies (2.3) with $T=0$. *In particular, for each fixed* $(w,\epsilon,\xi,\phi^o)\in\mathcal{E}^{m,\alpha}_+$, *equation (3.2) has exactly one solution* $(\eta^i,\rho^o)\in\left(C^{m-1,\alpha}(\partial\mathbb{B}_n)\right)^2$.

If $(w,0,\xi,\phi^o)\in\mathcal{E}^{m,\alpha}$, *then the pair of functions* $(\eta^i,\rho^o)\in\left(C^{m-1,\alpha}(\partial\mathbb{B}_n)\right)^2$ *satisfies equation*

$$M[w,0,\xi,\phi^o,\eta^i,\rho^o]=0, \tag{3.3}$$

if and only if both the following two conditions are fulfilled.

(j) The function τ *defined by*

$$\tau(s)\equiv\rho^o\circ(\phi^o)^{(-1)}(s)\qquad\forall s\in\phi^o(\partial\mathbb{B}_n),$$

belongs to $C^{m-1,\alpha}(\phi^o(\partial\mathbb{B}_n))$ *and satisfies (2.13) with*

$$T(t)=-\nu_{\phi^o}(t)\cdot DS_n(t-w)\qquad\forall t\in\phi^o(\partial\mathbb{B}_n).$$

(jj) The function τ *defined by*

$$\tau(s)\equiv\eta^i\circ\xi^{(-1)}(s)\qquad\forall s\in\xi(\partial\mathbb{B}_n),$$

belongs to $C^{m-1,\alpha}(\xi(\partial\mathbb{B}_n))$ *and satisfies (2.7) with* $T=0$, $\beta=1$.

In particular, for each fixed $(w,0,\xi,\phi^o)\in\mathcal{E}^{m,\alpha}$, *equation (3.3) has exactly one solution* $(\eta^i,\rho^o)\in\left(C^{m-1,\alpha}(\partial\mathbb{B}_n)\right)^2$.

Proof: The statement follows by a straightforward verification based on the theorem of change of variables in integrals. We only observe that if $(w,0,\xi,\phi^o)\in\mathcal{E}^{m,\alpha}$ is fixed, then by Theorem 2.1 (iv) the first and third component of (3.3) admit a unique solution $\eta^i\in C^{m-1,\alpha}(\partial\mathbb{B}_n)$. Then by Theorem 2.1 (vii), the second component of equation (3.3) has a unique solution $\rho^o\in C^{m-1,\alpha}(\partial\mathbb{B}_n)$. □

By Proposition 3.2, it makes sense to introduce the following.

Definition 3.3 *Let* $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. *For each* $\mathbf{a}\equiv(w,\epsilon,\xi,\phi^o)\in\mathcal{E}^{m,\alpha}$ *with* $\epsilon>0$ *or* $\epsilon=0$, *we denote by* $(\hat\eta^i[\mathbf{a}],\hat\rho^o[\mathbf{a}])$ *the unique solution* $(\eta^i,\rho^o)\in\left(C^{m-1,\alpha}(\partial\mathbb{B}_n)\right)^2$ *of equation (3.2) or (3.3), respectively. Finally, if* $\epsilon>0$, *we set*

$$\hat\rho^i[\mathbf{a}]\equiv\epsilon^{1-n}\hat\eta^i[\mathbf{a}].$$

Our goal is now to show that $\hat{\eta}^i[\cdot]$, $\hat{\rho}^o[\cdot]$ admit a real analytic continuation around a 'degenerate' point $\mathbf{a}_0 \equiv (w_0, 0, \xi_0, \phi_0^o) \in \mathcal{E}^{m,\alpha}$. By Proposition 3.2, it suffices to show that locally around $(\mathbf{a}_0, \hat{\eta}^i[\mathbf{a}_0], \hat{\rho}^o[\mathbf{a}_0])$ the set of zeros of M is the graph of a real analytic operator. We plan to do so by applying the Implicit Function Theorem in Banach space around $(\mathbf{a}_0, \hat{\eta}^i[\mathbf{a}_0], \hat{\rho}^o[\mathbf{a}_0])$ As a first step we must understand the properties of M. Now in the defintion of M we have both operators which display no singularity and integral operators with a singularity. To analyze their regularity, we shall need the following Proposition.

Proposition 3.4 *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1[$.

(i) The map $\tilde{\sigma}[\cdot]$ *of* $C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}$ *to* $C^{m-1,\alpha}(\partial\mathbb{B}_n)$ *which takes* ϕ *to* $\tilde{\sigma}[\phi]$ *is real analytic.*

(ii) The map of $C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}$ *to* $C^{m-1,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n)$ *which takes* ϕ *to* $\nu_\phi \circ \phi$ *is real analytic.*

(iii) Let F *be a real analytic map of* $\mathbb{R}^n \setminus \{0\}$ *to* $\mathbb{R}$. *Then the map* H_1 *of* $\{(\phi^i, \phi^o, f) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n))^2 \times L^1(\partial\mathbb{B}_n) : \phi^i(x) \neq \phi^o(y)\ \forall x, y \in \partial\mathbb{B}_n\}$ *to* $C^{m,\alpha}(\partial\mathbb{B}_n)$ *which takes* (ϕ^i, ϕ^o, f) *to the function* $H_1[\phi^i, \phi^o, f]$ *defined by*

$$H_1[\phi^i, \phi^o, f](x) \equiv \int_{\partial\mathbb{B}_n} F\left(\phi^i(x) - \phi^o(y)\right) f(y)\, d\sigma_y, \qquad \forall x \in \partial\mathbb{B}_n,$$

is real analytic.

(iv) Let F *be a real analytic map of* $\mathbb{R}^n \setminus \{0\}$ *to* $\mathbb{R}$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$. *Then the map* H_2 *of* $\{(\phi, f) \in C^0(\partial\mathbb{B}_n, \mathbb{R}^n) \times L^1(\partial\mathbb{B}_n) : \phi(\partial\mathbb{B}_n) \cap \mathrm{cl}\Omega = \emptyset\}$ *to* $C^0(\mathrm{cl}\Omega)$ *which takes* (ϕ, f) *to the function* $H_2[\phi, f]$ *of* $\mathrm{cl}\Omega$ *to* $\mathbb{R}$ *defined by*

$$H_2[\phi, f](t) \equiv \int_{\partial\mathbb{B}_n} F\left(t - \phi(y)\right) f(y)\, d\sigma_y, \qquad \forall t \in \mathrm{cl}\Omega,$$

is real analytic.

(v) Let F *be a real analytic map of* $\mathbb{R}^n \setminus \{0\}$ *to* $\mathbb{R}$. *Let* Ω *be a bounded connected open subset of* $\mathbb{R}^n$ *of class* C^1. *Then the map* H_3 *of* $\{(\phi, \Phi, f) \in C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \times C^{m,\alpha}(\mathrm{cl}\Omega, \mathbb{R}^n) \times L^1(\partial\mathbb{B}_n) : \phi(\partial\mathbb{B}_n) \cap \Phi(\mathrm{cl}\Omega) = \emptyset\}$ *to* $C^{m,\alpha}(\mathrm{cl}\Omega)$ *which takes* (ϕ, Φ, f) *to the function* $H_3[\phi, \Phi, f]$ *of* $\mathrm{cl}\Omega$ *to* $\mathbb{R}$ *defined by*

$$H_3[\phi, \Phi, f](t) \equiv \int_{\partial\mathbb{B}_n} F\left(\Phi(t) - \phi(y)\right) f(y)\, d\sigma_y, \qquad \forall t \in \mathrm{cl}\Omega,$$

is real analytic.

Proof: For statements (i), (ii), see Lanza and Rossi [13, Prop. 3.13]. Statement (iii) is a corollary of a known result for composition operators (cf. Böhme and Tomi [1, p. 10], Henry [7, p. 29], Valent [20, Thm. 5.2, p. 44]), and its proof is a straightforward modification of the corresponding elementary argument of Lanza and Rossi [13, Lem. 3.9]). We just observe that the map which takes (ϕ^i, ϕ^o) to $\phi^i(x) - \phi^o(y) \in C^{m,\alpha}(\partial\mathbb{B}_n \times \partial\mathbb{B}_n, \mathbb{R}^n \setminus \{0\})$, and the map which takes a function of $C^{m,\alpha}(\partial\mathbb{B}_n \times \partial\mathbb{B}_n, \mathbb{R}^n \setminus \{0\})$ to its composite function with F in $C^{m,\alpha}(\partial\mathbb{B}_n \times \partial\mathbb{B}_n)$ are real analytic and that the map which takes a pair of functions (g, f) of $C^{m,\alpha}(\partial\mathbb{B}_n \times \partial\mathbb{B}_n) \times L^1(\partial\mathbb{B}_n)$ to $\int_{\partial\mathbb{B}_n} g(\cdot, y) f(y)\, d\sigma_y$ in $C^{m,\alpha}(\partial\mathbb{B}_n)$ is real analytic. The proof of statements (iv), (v) is similar (see also [11, §1]). □

We now turn our attention to operators which appear in the definition of M, and which present a singularity in their definition. The integral operators which appear in the definition of M and other operators which we shall consider later are single layers, double layers and corresponding derivatives 'pulled back' by means of diffeomorphisms of $\mathcal{A}_{\partial\mathbb{B}_n}$. To handle such operators, we introduce some notation and results of Lanza and Rossi [13]. For each bounded open connected subset Ω of $\mathbb{R}^n$ of class C^1, the set

$$\mathcal{A}_{\mathrm{cl}\,\Omega} \equiv \left\{ \Phi \in C^1(\mathrm{cl}\,\Omega, \mathbb{R}^n) : \ \Phi \text{ is injective, } \det D\Phi(x) \neq 0, \ \forall x \in \mathrm{cl}\,\Omega \right\}$$

is open in $C^1(\mathrm{cl}\,\Omega, \mathbb{R}^n)$ (cf. [12, Cor. 4.24, Prop. 4.29]). If $\delta > 0$, we set

$$\mathbb{A}_\delta \equiv \{x \in \mathbb{R}^n : 1 - \delta < |x| < 1 + \delta\}$$

$$\mathbb{A}_\delta^+ \equiv \{x \in \mathbb{R}^n : 1 - \delta < |x| < 1\}, \qquad \mathbb{A}_\delta^- \equiv \{x \in \mathbb{R}^n : 1 < |x| < 1 + \delta\},$$

Then we have the following (cf. Lanza and Rossi [13, Prop. 2.8]).

Proposition 3.5 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1]$. If $\phi_0 \in C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}$, then there exist $\delta \in]0, 1[$, an open neighborhood $\mathcal{W}_0$ of ϕ_0 contained in $C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}$, and a real analytic extension operator $\mathbf{E}_0$ of $\mathcal{W}_0$ to $C^{m,\alpha}(\mathrm{cl}\mathbb{A}_\delta, \mathbb{R}^n) \cap \mathcal{A}_{\mathrm{cl}\mathbb{A}_\delta}$ such that*

(i) $\mathbf{E}_0[\phi]_{|\partial\mathbb{B}_n} = \phi$, for all $\phi \in \mathcal{W}_0$.

(ii) $\mathbf{E}_0[\phi]\left(\mathbb{A}_\delta^+\right)$ is a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ contained in $\mathbb{I}[\phi]$, $\mathbf{E}_0[\phi]\left(\mathbb{A}_\delta^-\right)$ is a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ contained in $\mathbb{E}[\phi]$, for all $\phi \in \mathcal{W}_0$.

The set

$$\mathcal{A}'_{\mathrm{cl}\mathbb{A}_\delta} \equiv \left\{ \Phi \in \mathcal{A}_{\mathrm{cl}\mathbb{A}_\delta} : \ \Phi(\mathbb{A}_\delta^+) \subseteq \mathbb{I}[\Phi_{|\partial\mathbb{B}_n}] \right\}$$

is open in $\mathcal{A}_{\mathrm{cl}\mathbb{A}_\delta}$ and $\mathbf{E}_0$ maps $\mathcal{W}_0$ to $\mathcal{A}'_{\mathrm{cl}\mathbb{A}_\delta}$ (cf. Lanza and Rossi [13, Lem. 2.9]). We now introduce some notation on single and double layer potentials. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1[$. We set

$$v^+[\phi, f](t) \equiv \int_{\phi(\partial\mathbb{B}_n)} S_n(t - s) f \circ \phi^{(-1)}(s)\, d\sigma_s \quad \forall t \in \mathbb{I}[\phi], \tag{3.4}$$

$$v^-[\phi, f](t) \equiv \int_{\phi(\partial\mathbb{B}_n)} S_n(t - s) f \circ \phi^{(-1)}(s)\, d\sigma_s \quad \forall t \in \mathbb{E}[\phi],$$

for each $\phi \in C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}$, and $f \in C^{m-1,\alpha}(\partial\mathbb{B}_n)$,

$$w^+[\phi, f](t) \equiv \int_{\phi(\partial\mathbb{B}_n)} \frac{\partial}{\partial\nu_\phi(s)} (S_n(t-s))\ f \circ \phi^{(-1)}(s)\, d\sigma_s \quad \forall t \in \mathbb{I}[\phi], \tag{3.5}$$
$$w^-[\phi, f](t) \equiv \int_{\phi(\partial\mathbb{B}_n)} \frac{\partial}{\partial\nu_\phi(s)} (S_n(t-s))\ f \circ \phi^{(-1)}(s)\, d\sigma_s \quad \forall t \in \mathbb{E}[\phi],$$

for each $\phi \in C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}$, and $f \in C^{m,\alpha}(\partial\mathbb{B}_n)$. Then by classical Potential Theory, we know that $v^+[\phi, f]$, $w^+[\phi, f]$, and $v^-[\phi, f]$, $w^-[\phi, f]$ can be extended with continuity to $\mathrm{cl}\mathbb{I}[\phi]$ and to $\mathrm{cl}\mathbb{E}[\phi]$, respectively. We denote the corresponding extensions by the same symbol. Then we have the following two statements, which have been proved in Lanza and Rossi [13, Prop. 3.11, Thm. 3.12].

Proposition 3.6 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$, $\delta \in]0,1[$. Then the following statements hold.*

(i) Let $V^+[\Phi, f]$ and $V^-[\Phi, f]$ denote the continuous extensions to $\mathrm{cl}\mathbb{A}_\delta^+$ and to $\mathrm{cl}\mathbb{A}_\delta^-$ of $v^+[\Phi_{|\partial\mathbb{B}_n}, f] \circ \Phi_{|\mathbb{A}_\delta^+}$ and of $v^-[\Phi_{|\partial\mathbb{B}_n}, f] \circ \Phi_{|\mathbb{A}_\delta^-}$, respectively, for all (Φ, f) in $\left(C^{m,\alpha}(\mathrm{cl}\mathbb{A}_\delta, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\mathbb{A}_\delta}\right) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$.

Then the maps of $\left(C^{m,\alpha}(\mathrm{cl}\mathbb{A}_\delta, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\mathbb{A}_\delta}\right) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$ to $C^{m,\alpha}\left(\mathrm{cl}\mathbb{A}_\delta^+\right)$ and to $C^{m,\alpha}\left(\mathrm{cl}\mathbb{A}_\delta^-\right)$ which take (Φ, f) to $V^+[\Phi, f]$ and to $V^-[\Phi, f]$ are real analytic, respectively.

(ii) Let $W^+[\Phi, f]$ and $W^-[\Phi, f]$ denote the continuous extensions to $\mathrm{cl}\mathbb{A}_\delta^+$ and to $\mathrm{cl}\mathbb{A}_\delta^-$ of $w^+[\Phi_{|\partial\mathbb{B}_n}, f] \circ \Phi_{|\mathbb{A}_\delta^+}$ and of $w^-[\Phi_{|\partial\mathbb{B}_n}, f] \circ \Phi_{|\mathbb{A}_\delta^-}$, respectively, for all (Φ, f) in $\left(C^{m,\alpha}(\mathrm{cl}\mathbb{A}_\delta, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\mathbb{A}_\delta}\right) \times C^{m,\alpha}(\partial\mathbb{B}_n)$.

Then the maps of $\left(C^{m,\alpha}(\mathrm{cl}\mathbb{A}_\delta, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\mathbb{A}_\delta}\right) \times C^{m,\alpha}(\partial\mathbb{B}_n)$ to $C^{m,\alpha}\left(\mathrm{cl}\mathbb{A}_\delta^+\right)$ and to $C^{m,\alpha}\left(\mathrm{cl}\mathbb{A}_\delta^-\right)$ which take (Φ, f) to $W^+[\Phi, f]$ and to $W^-[\Phi, f]$ are real analytic, respectively.

Theorem 3.7 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. Then the following statements hold.*

(i) The map $V[\cdot,\cdot]$ of $\left(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}\right) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$ to $C^{m,\alpha}(\partial\mathbb{B}_n)$ defined by

$$V[\phi, f](x) \equiv \int_{\phi(\partial\mathbb{B}_n)} S_n(\phi(x) - s)\ f \circ \phi^{(-1)}(s)\, d\sigma_s \quad \forall x \in \partial\mathbb{B}_n,$$

for all $(\phi, f) \in \left(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}\right) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$, is real analytic.

(ii) The map $W[\cdot,\cdot]$ of $\left(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}\right) \times C^{m,\alpha}(\partial\mathbb{B}_n)$ to $C^{m,\alpha}(\partial\mathbb{B}_n)$ defined by

$$W[\phi, f](x) \equiv \int_{\phi(\partial\mathbb{B}_n)} \frac{\partial}{\partial\nu_\phi(s)} (S_n(\phi(x) - s))\ f \circ \phi^{(-1)}(s)\, d\sigma_s \quad \forall x \in \partial\mathbb{B}_n,$$

for all $(\phi, f) \in \left(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}\right) \times C^{m,\alpha}(\partial\mathbb{B}_n)$, is real analytic.

Then we can deduce the validity of the following (cf. [11, Prop. 7 §2]).

Proposition 3.8 *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. *Then the map* V_* *of* $(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$ *to* $C^{m-1,\alpha}(\partial\mathbb{B}_n)$ *defined by*

$$V_*[\phi, f] \equiv \int_{\partial\mathbb{B}_n} f(y)(\nu_\phi \circ \phi(x)) \cdot DS_n(\phi(x) - \phi(y))\tilde{\sigma}[\phi](y)\, d\sigma_y,$$

for all $(\phi, f) \in \left(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}\right) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$, *is real analytic.*

We now prove the following Theorem.

Theorem 3.9 *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. *The set* $\mathcal{E}^{m,\alpha} \times \left(C^{m-1,\alpha}(\partial\mathbb{B}_n)\right)^2$ *is open in* $\mathbb{R}^n \times \mathbb{R} \times (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n))^2 \times \left(C^{m-1,\alpha}(\partial\mathbb{B}_n)\right)^2$ *and the operator* M *defined in (3.1) is real analytic. If* $\mathbf{b}_0 \equiv (w_0, 0, \xi_0, \phi_0^o, \eta_0^i, \rho_0^o) \in \mathcal{E}^{m,\alpha} \times \left(C^{m-1,\alpha}(\partial\mathbb{B}_n)\right)^2$ *is a zero of* M*, then the differential* $\partial_{(\eta^i,\rho^o)} M[\mathbf{b}_0]$ *of* M *with respect to the variable* (η^i, ρ^o) *at* $\mathbf{b}_0$ *is delivered by the formula*

$$\partial_{(\eta^i,\rho^o)} M_1[\mathbf{b}_0](\overline{\eta}^i, \overline{\rho}^o)(x) = \frac{1}{2}\overline{\eta}^i(x) - V_*[\xi_0, \overline{\eta}^i](x) \qquad \forall x \in \partial\mathbb{B}_n, \tag{3.6}$$

$$\begin{aligned}
&\partial_{(\eta^i,\rho^o)} M_2[\mathbf{b}_0](\overline{\eta}^i, \overline{\rho}^o)(x) \\
&\qquad = \frac{1}{2}\overline{\rho}^o(x) + \int_{\partial\mathbb{B}_n} \overline{\eta}^i(y)(\nu_{\phi_0^o} \circ \phi_0^o(x)) \cdot DS_n(\phi_0^o(x) - w_0)\tilde{\sigma}[\xi_0](y)\, d\sigma_y \\
&\qquad\quad + V_*[\phi_0^o, \overline{\rho}^o](x) \qquad \forall x \in \partial\mathbb{B}_n,
\end{aligned}$$

$$\partial_{(\eta^i,\rho^o)} M_3[\mathbf{b}_0](\overline{\eta}^i, \overline{\rho}^o) = \int_{\partial\mathbb{B}_n} \overline{\eta}^i(y)\tilde{\sigma}[\xi_0](y)\, d\sigma_y,$$

for all $(\overline{\eta}^i, \overline{\rho}^o) \in (C^{m-1,\alpha}(\partial\mathbb{B}_n))^2$, *and is a linear homeomorphism of* $(C^{m-1,\alpha}(\partial\mathbb{B}_n))^2$ *onto the space*

$$V_{\xi_0}^{m,\alpha} \equiv \left\{(f^i, f^o, \beta) \in (C^{m-1,\alpha}(\partial\mathbb{B}_n))^2 \times \mathbb{R} : \int_{\partial\mathbb{B}_n} f^i(y)\tilde{\sigma}[\xi_0](y)\, d\sigma_y = 0\right\}. \tag{3.7}$$

Moreover,

$$\int_{\partial\mathbb{B}_n} M_1[\mathbf{b}]\tilde{\sigma}[\xi](y)\, d\sigma_y = 0 \qquad \forall \mathbf{b} \in \mathcal{E}^{m,\alpha} \times (C^{m-1,\alpha}(\partial\mathbb{B}_n))^2. \tag{3.8}$$

Proof: By continuity of the pointwise product and by standard properties of composition of functions in Schauder spaces, by Propositions 3.4, 3.8 and the definition of M, and by standard calculus in Banach space, we immediately deduce that M is real analytic and that (3.6) holds. We now prove that $\partial_{(\eta^i,\rho^o)} M[\mathbf{b}_0]$ is a linear homeomorphism. By the Open Mapping Theorem it suffices to show that $\partial_{(\eta^i,\rho^o)}M[\mathbf{b}_0]$ is an isomorphism. Let $(f^i, f^o, \beta) \in V^{m,\alpha}_{\xi_0}$. We must show that there exists a unique $(\overline{\eta}^i, \overline{\rho}^o) \in (C^{m-1,\alpha}(\partial\mathbb{B}_n))^2$ such that

$$(f^i, f^o, \beta) = \partial_{(\eta^i,\rho^o)}M[\mathbf{b}_0](\overline{\eta}^i, \overline{\rho}^o), \tag{3.9}$$

which is a system in three equations, one for each component of M. We first observe that by setting

$$T(t) \equiv f^i \circ \xi_0^{(-1)}(t) \qquad \forall t \in \xi_0(\partial\mathbb{B}_n),$$

by changing the variable with the function ξ_0 in the first and third equation of (3.6), and by exploiting the unique solvability of (2.7), we deduce that the system of the first and third components of (3.9) has one and only one solution $\overline{\eta}^i \in C^{m-1,\alpha}(\partial\mathbb{B}_n)$.

Then we observe that

$$f^o \circ \phi_0^{o(-1)}(t) - \int_{\partial\mathbb{B}_n} \overline{\eta}^i(y)\nu_{\phi_0^o}(t) \cdot DS_n(t-w_0)\tilde{\sigma}[\xi_0](y)\,d\sigma_y$$

is an element of $C^{m-1,\alpha}(\phi_0^o(\partial\mathbb{B}_n))$ (see Proposition 3.4 (iv)). Then the unique solvability of the second equation of (3.9) follows by Theorem 2.1 (vii).

We now turn to the proof of (3.8). By Fubini's Theorem and by the well known identity $\int_{\xi(\partial\mathbb{B}_n)} \frac{\partial}{\partial\nu_{\xi}(t)}(S_n(s-t))\,d\sigma_t = \frac{1}{2}$ for all $s \in \xi(\partial\mathbb{B}_n)$, we have

$$\int_{\partial\mathbb{B}_n} M_1[\mathbf{b}]\tilde{\sigma}[\xi](y)\,d\sigma_y = -\epsilon^{n-1}\int_{\partial\mathbb{B}_n}\int_{\partial\mathbb{B}_n} \rho^o(y)(\nu_\xi \circ \xi(x)) \tag{3.10}$$
$$\cdot DS_n(w+\epsilon\xi(x)-\phi^o(y))\tilde{\sigma}[\phi^o](y)\,d\sigma_y\tilde{\sigma}[\xi](x)\,d\sigma_x,$$

for all $\mathbf{b} \in \mathcal{E}^{m,\alpha} \times \left(C^{m-1,\alpha}(\partial\mathbb{B}_n)\right)^2$. If $\epsilon = 0$ equality (3.8) follows. If $\epsilon \neq 0$, we note that the right-hanide of (3.10) equals

$$-\mathrm{sgn}^n(\epsilon)\int_{w+\epsilon\xi(\partial\mathbb{B}_n)} \frac{\partial u}{\partial\nu_{w+\epsilon\xi}}\,d\sigma,$$

where u is the function of $\mathbb{I}[\phi^o]$ to $\mathbb{R}$ defined by

$$u(t) \equiv \int_{\phi^o(\partial\mathbb{B}_n)} \rho^o \circ \phi^{o(-1)}(s)S_n(t-s)\,d\sigma_s \qquad \forall t \in \mathbb{I}[\phi^o].$$

Since u is harmonic in $\mathbb{I}[\phi^o]$, it is also harmonic in a neighborhood of $\mathrm{cl}\mathbb{I}[w+\epsilon\xi]$, and accordingly the last integral in (3.10) vanishes. Hence, (3.8) follows. □

We are now ready to prove the real analyticity of $\hat{\eta}^i[\cdot]$, $\hat{\rho}^o[\cdot]$.

Theorem 3.10 *Let* $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. *Let* $\mathbf{a}_0 \equiv (w_0, 0, \xi_0, \phi_0^o) \in \mathcal{E}^{m,\alpha}$. *Then there exist an open neighborhood* $\mathcal{U}_0$ *of* $\mathbf{a}_0$ *in* $\mathcal{E}^{m,\alpha}$, *and an open neighborhood* $\mathcal{V}_0$ *of*

$(\hat{\eta}^i[\mathbf{a}_0], \hat{\rho}^o[\mathbf{a}_0])$ in $\left(C^{m-1,\alpha}(\partial\mathbb{B}_n)\right)^2$ and a real analytic operator (E^i, R^o) of $\mathcal{U}_0$ to $\mathcal{V}_0$ such that

$$(E^i[\mathbf{a}], R^o[\mathbf{a}]) = (\hat{\eta}^i[\mathbf{a}], \hat{\rho}^o[\mathbf{a}]) \tag{3.11}$$

for all $\mathbf{a} \equiv (w, \epsilon, \xi, \phi^o) \in \mathcal{U}_0$ such that $\epsilon \geq 0$. Moreover,

$$E^i[\mathbf{a}] = \epsilon^{n-1}\hat{\rho}^i[\mathbf{a}]$$

for all $\mathbf{a} \in \mathcal{U}_0$ such that $\epsilon > 0$. Finally, the graph of (E^i, R^o) coincides with the set of zeros of M in $\mathcal{U}_0 \times \mathcal{V}_0$.

Proof: We plan to apply Theorem A.1 of the Appendix, a corollary of the Implicit Function Theorem, by taking F equal to M and G equal to the function of $\mathcal{H} \equiv \mathcal{E}^{m,\alpha} \times \left(C^{m-1,\alpha}(\partial\mathbb{B}_n)\right)^2 \times \left(C^{m-1,\alpha}(\partial\mathbb{B}_n)\right)^2 \times \mathbb{R}$ to $\mathbb{R}$ defined by

$$G[\mathbf{b}, f^i, f^o, \beta] = \int_{\partial\mathbb{B}_n} f^i(y)\tilde{\sigma}[\xi](y)\, d\sigma_y,$$

for all $(\mathbf{b}, f^i, f^o, \beta) \in \mathcal{H}$, with $\mathbf{b} \equiv (w, \epsilon, \xi, \phi^o, \eta^i, \rho^o)$. Obviously, G is real analytic, and $G[\mathbf{b}, 0, 0, 0] = 0$ and $G[\mathbf{b}, M[\mathbf{b}]] = 0$ for $\mathbf{b}$ in $\mathcal{E}^{m,\alpha} \times (C^{m-1,\alpha}(\partial\mathbb{B}_n))^2$, and the partial differential $\partial_{(f^i, f^o, \beta)}G[\mathbf{b}_0, 0, 0, 0]$, with $\mathbf{b}_0 \equiv (w_0, 0, \xi_0, \phi_0^o, \hat{\eta}^i[\mathbf{a}_0], \hat{\rho}^o[\mathbf{a}_0])$ coincides with the linear map which takes $(\overline{f}^i, \overline{f}^o, \beta)$ to $\int_{\partial\mathbb{B}_n} \overline{f}^i(y)\tilde{\sigma}[\xi_0](y)\, d\sigma_y$ which is surjective onto $\mathbb{R}$, and has kernel equal to $V^{m,\alpha}_{\xi_0}$, which is a closed subspace of $(C^{m-1,\alpha}(\partial\mathbb{B}_n))^2 \times\mathbb{R}$ of codimension 1, which admits a closed topological supplement of dimension 1. Then the Theorem follows by Theorem 3.9 and by Theorem A.1 of the Appendix. □

4 A real analyticity theorem for the solutions of the Fredholm equation (2.4)

We start our analysis of the Fredholm equation (2.4), which is defined on the $\mathbf{a}$ dependent domain $\partial\mathbb{A}[\mathbf{a}]$, by transforming it into an equation on the fixed domain $\partial\mathbb{B}_n$. We do so by means of the following.

Theorem 4.1 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1[$. Let $\mathbf{a}_0 \equiv (w_0, 0, \xi_0, \phi_0^o) \in \mathcal{E}^{m,\alpha}$. Let $\mathcal{U}_0$ be an open neighborhood of $\mathbf{a}_0$ as in Theorem 3.10. Let $\Lambda \equiv \left(\Lambda_j\right)_{j=1,2,3}$ be the map of*

$$\mathcal{O}_0 \equiv \mathcal{U}_0 \times \left(C^{m,\alpha}(\partial\mathbb{B}_n)\right)^4 \tag{4.1}$$

to $\left(C^{m,\alpha}(\partial\mathbb{B}_n)\right)^2 \times \mathbb{R}$ *defined by*

$$\begin{aligned}
\Lambda_1[\mathbf{q}](x) \equiv\ & -\left\{ g^i(x) - \left[\int_{\partial\mathbb{B}_n} g^i E^i[\mathbf{a}]\tilde{\sigma}[\xi]\, d\sigma + \int_{\partial\mathbb{B}_n} g^o R^o[\mathbf{a}]\tilde{\sigma}[\phi^o]\, d\sigma \right] \right\} \\
& + \frac{1}{2}\theta^i(x) + \int_{\partial\mathbb{B}_n} \theta^i(y)(\nu_\xi \circ \xi(y)) \cdot DS_n(\xi(x) - \xi(y))\tilde{\sigma}[\xi](y)\, d\sigma_y \\
& - \int_{\partial\mathbb{B}_n} \theta^o(y)(\nu_{\phi^o} \circ \phi^o(y)) \cdot DS_n(w + \epsilon\xi(x) - \phi^o(y))\tilde{\sigma}[\phi^o](y)\, d\sigma_y \\
& \qquad \forall x \in \partial\mathbb{B}_n, \qquad (4.2) \\
\Lambda_2[\mathbf{q}](x) \equiv\ & -g^o(x) + \frac{1}{2}\theta^o(x) \\
& + \epsilon^{n-1} \int_{\partial\mathbb{B}_n} \theta^i(y)(\nu_\xi \circ \xi(y)) \cdot DS_n(\phi^o(x) - w - \epsilon\xi(y))\tilde{\sigma}[\xi](y)\, d\sigma_y \\
& - \int_{\partial\mathbb{B}_n} \theta^o(y)(\nu_{\phi^o} \circ \phi^o(y)) \cdot DS_n(\phi^o(x) - \phi^o(y))\tilde{\sigma}[\phi^o](y)\, d\sigma_y \\
& \qquad \forall x \in \partial\mathbb{B}_n, \\
\Lambda_3[\mathbf{q}] \equiv\ & \int_{\partial\mathbb{B}_n} \theta^i \tilde{\sigma}[\xi]\, d\sigma,
\end{aligned}$$

for all $\mathbf{q} \equiv (w, \epsilon, \xi, \phi^o, g^i, g^o, \theta^i, \theta^o) \in \mathcal{O}_0$*, where as usual we have abbreviated the quadruple* $(w, \epsilon, \xi, \phi^o)$ *as* $\mathbf{a}$.

If $\mathbf{p} \equiv (w, \epsilon, \xi, \phi^o, g^i, g^o) \in \left(\mathcal{E}_+^{m,\alpha} \cap \mathcal{U}_0\right) \times \left(C^{m,\alpha}(\partial\mathbb{B}_n)\right)^2$*, then the pair of functions* $(\theta^i, \theta^o) \in \left(C^{m,\alpha}(\partial\mathbb{B}_n)\right)^2$ *satisfies equation*

$$\Lambda[\mathbf{p}, \theta^i, \theta^o] = 0 \qquad (4.3)$$

if and only if the function $\mu \in C^{m,\alpha}(\partial\mathbb{A}[\mathbf{a}])$ *defined by*

$$\begin{aligned}
\mu(s) &\equiv \theta^i \circ (w + \epsilon\xi)^{(-1)}(s) && \forall s \in w + \epsilon\xi(\partial\mathbb{B}_n), \\
\mu(s) &\equiv \theta^o \circ (\phi^o)^{(-1)}(s) && \forall s \in \phi^o(\partial\mathbb{B}_n),
\end{aligned}$$

satisfies (2.4) with $\tilde{\Gamma} \in C^{m,\alpha}(\partial\mathbb{A}[\mathbf{a}])$ *defined as in (2.6) with* $\phi^i \equiv w + \epsilon\xi$ *and* τ_0 *as in Theorem 2.1 (ii), and with* $\Gamma \in C^{m,\alpha}(\partial\mathbb{A}[\mathbf{a}])$ *defined by*

$$\begin{aligned}
\Gamma(s) &\equiv g^i \circ (w + \epsilon\xi)^{(-1)}(s) && \forall s \in w + \epsilon\xi(\partial\mathbb{B}_n), \\
\Gamma(s) &\equiv g^o \circ (\phi^o)^{(-1)}(s) && \forall s \in \phi^o(\partial\mathbb{B}_n).
\end{aligned}$$

In particular, for each fixed $\mathbf{p} \equiv (w, \epsilon, \xi, \phi^o, g^i, g^o)$ *in* $\left(\mathcal{E}_+^{m,\alpha} \cap \mathcal{U}_0\right) \times \left(C^{m,\alpha}(\partial\mathbb{B}_n)\right)^2$*, equation (4.3) has exactly one solution* $(\theta^i, \theta^o) \in \left(C^{m,\alpha}(\partial\mathbb{B}_n)\right)^2$.

If $(w, 0, \xi, \phi^o, g^i, g^o) \in \mathcal{U}_0 \times \left(C^{m,\alpha}(\partial\mathbb{B}_n)\right)^2$*, then the pair of functions* (θ^i, θ^o) *of* $(C^{m,\alpha}(\partial\mathbb{B}_n))^2$ *satisfies equation*

$$\Lambda[w, 0, \xi, \phi^o, g^i, g^o, \theta^i, \theta^o] = 0 \tag{4.4}$$

if and only if both the following two conditions are fulfilled

(j) The function μ *of* $\phi^o(\partial\mathbb{B}_n)$ *to* $\mathbb{R}$ *defined by*

$$\mu(s) \equiv \theta^o \circ (\phi^o)^{(-1)}(s) \qquad \forall s \in \phi^o(\partial\mathbb{B}_n),$$

belongs to $C^{m,\alpha}(\phi^o(\partial\mathbb{B}_n))$ *and satisfies (2.14) with* $\tilde{\Gamma} = g^o \circ (\phi^o)^{(-1)}$.

(jj) The function μ *of* $\xi(\partial\mathbb{B}_n)$ *to* $\mathbb{R}$ *defined by*

$$\mu(s) \equiv \theta^i \circ \xi^{(-1)}(s) \qquad \forall s \in \xi(\partial\mathbb{B}_n),$$

belongs to $C^{m,\alpha}(\xi(\partial\mathbb{B}_n))$ *and satisfies (2.8) with* $\omega = 0$ *and*

$$\Gamma = g^i \circ \xi^{(-1)} - \int_{\partial\mathbb{B}_n} g^i E^i[w, 0, \xi, \phi^o]\tilde{\sigma}[\xi]\, d\sigma.$$

In particular, for each $(w, 0, \xi, \phi^o, g^i, g^o) \in \mathcal{U}_0 \times (C^{m,\alpha}(\partial\mathbb{B}_n))^2$*, equation (4.4) has exactly one solution* $(\theta^i, \theta^o) \in (C^{m,\alpha}(\partial\mathbb{B}_n))^2$.

Proof: The first part of the statement follows by a straightforward verification based upon the theorem of change of variables in integrals. We only consider the last statement relative to case $\epsilon = 0$. Then we now assume that $(w, 0, \xi, \phi^o, g^i, g^o) \in \mathcal{U}_0 \times \left(C^{m,\alpha}(\partial\mathbb{B}_n)\right)^2$. If $(\theta^i, \theta^o) \in \left(C^{m,\alpha}(\partial\mathbb{B}_n)\right)^2$ satisfies equation (4.4), then we first observe that a simple computation based on the second component of (4.4), and on the second component of equation (3.2) with $\epsilon = 0$, which holds by Theorem 3.10, and on Fubini's Theorem, and on the equality

$$\int_{\partial\mathbb{B}_n} E^i[w, 0, \xi, \phi^o]\tilde{\sigma}[\xi]\, d\sigma = 1, \tag{4.5}$$

which follows by Theorem 3.10, shows that

$$\begin{aligned} \int_{\partial\mathbb{B}_n} g^o R^o[w, 0, \xi, \phi^o]\tilde{\sigma}[\phi^o]\, d\sigma &= \int_{\partial\mathbb{B}_n} E^i[w, 0, \xi, \phi^o](y) \\ &\quad \cdot \left\{ \int_{\partial\mathbb{B}_n} \theta^o(x)(\nu_{\phi^o} \circ \phi^o(x)) \cdot DS_n(w - \phi^o(x))\tilde{\sigma}[\phi^o](x)\, d\sigma_x \right\} \tilde{\sigma}[\xi](y)\, d\sigma_y \\ &= \int_{\partial\mathbb{B}_n} \theta^o(x)(\nu_{\phi^o} \circ \phi^o(x)) \cdot DS_n(w - \phi^o(x))\tilde{\sigma}[\phi^o](x)\, d\sigma_x. \end{aligned} \tag{4.6}$$

Then by changing the variables in equation (4.4) by means of the functions ξ, ϕ^o, we obtain that (j), (jj) must hold. Similarly, we can show that if (j), (jj) hold, then θ^i, θ^o

belong to $C^{m,\alpha}(\partial\mathbb{B}_n)$ and satisfy (4.4). The existence of a unique solution θ^o as in (j) follows by Theorem 2.1 (viii). Then we note that if we set $\tau_0 \equiv E^i[w, 0, \xi, \phi^o] \circ \xi^{(-1)}$, then the first and third component of (3.2) with $\epsilon = 0$ imply that equation (2.7) holds with $T = 0$, $\beta = 1$. By equality (4.5), we have $\int_{\xi(\partial\mathbb{B}_n)} \Gamma\tau_0 \, d\sigma = 0$. Then Theorem 2.1 (v) with $\omega = 0$ implies the existence of θ^i as in (jj). □

By Theorem 4.1, it makes sense to introduce the following.

Definition 4.2 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1[$. Let $\mathbf{a}_0 \equiv (w_0, 0, \xi_0, \phi_0^o) \in \mathcal{E}^{m,\alpha}$. Let $\mathcal{U}_0$ be an open neighborhood of $\mathbf{a}_0$ as in Theorem 3.10. For each $\mathbf{p} \equiv (w, \epsilon, \xi, \phi^o, g^i, g^o)$ of $\mathcal{U}_0 \times (C^{m,\alpha}(\partial\mathbb{B}_n))^2$ with $\epsilon > 0$ or $\epsilon = 0$, we denote by $(\hat{\theta}^i[\mathbf{p}], \hat{\theta}^o[\mathbf{p}])$ the unique solution $(\theta^i, \theta^o) \in (C^{m,\alpha}(\partial\mathbb{B}_n))^2$ of equation (4.3) or (4.4), respectively.*

Our goal is now to show that $\hat{\theta}^i, \hat{\theta}^o$ have a real analytic continuation around a 'degenerate' sextuple $\mathbf{p}_0 \equiv (w_0, 0, \xi_0, \phi_0^o, g_0^i, g_0^o)$.

By Theorem 4.1, it suffices to show that locally around $(\mathbf{p}_0, \hat{\theta}^i[\mathbf{p}_0], \hat{\theta}^o[\mathbf{p}_0])$ the set of zeros of Λ is the graph of a real analytic function. We plan to do so by applying Theorem A.1 of the Appendix, a corollary of the Implicit Function Theorem, around $(\mathbf{p}_0, \hat{\theta}^i[\mathbf{p}_0], \hat{\theta}^o[\mathbf{p}_0])$. As a first step, we show the following.

Theorem 4.3 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1[$. Let $\mathbf{a}_0 \equiv (w_0, 0, \xi_0, \phi_0^o) \in \mathcal{E}^{m,\alpha}$. Let $\mathcal{U}_0$ an the open neighborhood of $\mathbf{a}_0$ as in Theorem 3.10. The set $\mathcal{O}_0$ introduced in (4.1) is open in $\mathbb{R}^{n+1} \times (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n))^2 \times (C^{m,\alpha}(\partial\mathbb{B}_n))^4$ and the operator Λ defined in (4.2) is real analytic. If $(g_0^i, g_0^o, \theta_0^i, \theta_0^o) \in (C^{m,\alpha}(\partial\mathbb{B}_n))^4$ and $\mathbf{q}_0 \equiv (w_0, 0, \xi_0, \phi_0^o, g_0^i, g_0^o, \theta_0^i, \theta_0^o) \in \mathcal{O}_0$ is a zero of Λ, then the differential of Λ with respect to the variable (θ^i, θ^o) at $\mathbf{q}_0$ is delivered by the formula*

$$
\begin{aligned}
\partial_{(\theta^i,\theta^o)}\Lambda_1[\mathbf{q}_0](\overline{\theta}^i, \overline{\theta}^o)(x) &= \frac{1}{2}\overline{\theta}^i(x) - W[\xi_0, \overline{\theta}^i](x) \\
&\quad - \int_{\partial\mathbb{B}_n} \overline{\theta}^o(y)(\nu_{\phi_0^o} \circ \phi_0^o(y)) \cdot DS_n(w_0 - \phi_0^o(y))\tilde{\sigma}[\phi_0^o](y)\, d\sigma_y \qquad \forall x \in \partial\mathbb{B}_n, \\
\partial_{(\theta^i,\theta^o)}\Lambda_2[\mathbf{q}_0](\overline{\theta}^i, \overline{\theta}^o)(x) &= \frac{1}{2}\overline{\theta}^o(x) + W[\phi_0^o, \overline{\theta}^o](x) \qquad \forall x \in \partial\mathbb{B}_n, \\
\partial_{(\theta^i,\theta^o)}\Lambda_3[\mathbf{q}_0](\overline{\theta}^i, \overline{\theta}^o) &= \int_{\partial\mathbb{B}_n} \overline{\theta}^i(y)\tilde{\sigma}[\xi_0](y)\, d\sigma_y
\end{aligned}
\tag{4.7}
$$

for all $(\overline{\theta}^i, \overline{\theta}^o) \in (C^{m,\alpha}(\partial\mathbb{B}_n))^2$, and is a linear homeomorphism of $(C^{m,\alpha}(\partial\mathbb{B}_n))^2$ onto the space

$$
\begin{aligned}
W^{m,\alpha}_{\mathbf{a}_0} \equiv \Big\{ &(\gamma^i, \gamma^o, \omega) \in \big(C^{m,\alpha}(\partial\mathbb{B}_n)\big)^2 \times \mathbb{R} : \\
&\int_{\partial\mathbb{B}_n} \gamma^i E^i[\mathbf{a}_0]\tilde{\sigma}[\xi_0]\, d\sigma + \int_{\partial\mathbb{B}_n} \gamma^o R^o[\mathbf{a}_0]\tilde{\sigma}[\phi_0^o]\, d\sigma = 0 \Big\}.
\end{aligned}
\tag{4.8}
$$

Moreover,

$$\int_{\partial\mathbb{B}_n} \Lambda_1[w,\epsilon,\xi,\phi^o,g^i,g^o,\theta^i,\theta^o] E^i[w,\epsilon,\xi,\phi^o]\tilde{\sigma}[\xi]\,d\sigma \tag{4.9}$$
$$+\int_{\partial\mathbb{B}_n} \Lambda_2[w,\epsilon,\xi,\phi^o,g^i,g^o,\theta^i,\theta^o] R^o[w,\epsilon,\xi,\phi^o]\tilde{\sigma}[\phi^o]\,d\sigma = 0$$

for all $(w,\epsilon,\xi,\phi^o,g^i,g^o,\theta^i,\theta^o) \in \mathcal{O}_0$.

Proof: Since $\mathcal{E}^{m,\alpha}$ and $\mathcal{U}_0$ are open, then $\mathcal{O}_0$ is also open. By continuity of the pointwise product and by standard properties of composition of functions in Schauder spaces, by Proposition 3.4, Theorems 3.7, and 3.10, and the definition of Λ, and by standard calculus in Banach space, we immediately deduce that Λ is real analytic and that formula (4.7) holds. We now prove that $\partial_{(\theta^i,\theta^o)}\Lambda[\mathbf{q}_0]$ is a linear homeomorphism. By the Open Mapping Theorem, it suffices to show that $\partial_{(\theta^i,\theta^o)}\Lambda[\mathbf{q}_0]$ is an isomorphism. Let $(\gamma^i,\gamma^o,\omega) \in W^{m,\alpha}_{\mathbf{a}_0}$. We must show that there exists a unique $(\overline{\theta}^i,\overline{\theta}^o) \in (C^{m,\alpha}(\partial\mathbb{B}_n))^2$ such that

$$(\gamma^i,\gamma^o,\omega) = \partial_{(\theta^i,\theta^o)}\Lambda[\mathbf{q}_0](\overline{\theta}^i,\overline{\theta}^o), \tag{4.10}$$

which is a system in three equations, one for each component of Λ. We first observe that by setting $\tilde{\Gamma} \equiv \gamma^o \circ (\phi^o_0)^{(-1)}$ and by changing the variables with the function ϕ^o_0 in the second component of (4.10), and by exploiting the unique solvability of (2.14), we deduce that the second equation of (4.10) has a unique solution $\overline{\theta}^o \in C^{m,\alpha}(\partial\mathbb{B}_n)$. Next we consider the system of the first and third equation of (4.10), and we define the function T_1 of $\xi_0(\partial\mathbb{B}_n)$ to $\mathbb{R}$ by setting

$$T_1(t) = \gamma^i \circ \xi_0^{(-1)}(t) + \int_{\partial\mathbb{B}_n} \overline{\theta}^o(y)(\nu_{\phi^o_0}\circ\phi^o_0(y))\cdot DS_n(w_0-\phi^o_0(y))\tilde{\sigma}[\phi^o_0](y)\,d\sigma_y$$

for all $t \in \xi_0(\partial\mathbb{B}_n)$. By Theorem 3.10, $E^i[\mathbf{a}_0]$, $R^o[\mathbf{a}_0]$ must satisfy (3.2). By a simple computation based on the Fubini Theorem, on the second and third component of (3.2) for $\epsilon = 0$, and on the second component of (4.10) shows that

$$\int_{\partial\mathbb{B}_n}\left\{\int_{\partial\mathbb{B}_n} \overline{\theta}^o(y)(\nu_{\phi^o_0}\circ\phi^o_0(y))\cdot DS_n(w_0-\phi^o_0(y))\tilde{\sigma}[\phi^o_0](y)\,d\sigma_y\right\}$$
$$\cdot E^i[\mathbf{a}_0](x)\tilde{\sigma}[\xi_0](x)\,d\sigma_x = \int_{\partial\mathbb{B}_n}\gamma^o R^o[\mathbf{a}_0]\tilde{\sigma}[\phi^o_0]\,d\sigma,$$

where $\mathbf{a}_0 \equiv (w_0,0,\xi_0,\phi^o_0)$. Accordingly, $\int_{\xi_0(\partial\mathbb{B}_n)} T_1 E^i[\mathbf{a}_0]\circ\xi_0^{(-1)}\,d\sigma = 0$. Hence, we can change the variables in the first and third equation of (4.10) and exploit the unique solvability of (2.8) to deduce that the first and third equation of (4.10) has a unique solution $\overline{\theta}^i \in C^{m,\alpha}(\partial\mathbb{B}_n)$.

To prove (4.9), we first replace $\Lambda_1[\mathbf{p}]$, $\Lambda_2[\mathbf{p}]$ in the left-hand side of (4.9) by the expressions delivered by (4.2). Then we exploit the Fubini Theorem and (3.2), which is satisfied by $(\mathbf{a}, E^i[\mathbf{a}], R^o[\mathbf{a}])$ by virtue of Theorem 3.10. □

We are now ready to prove the main result of this section.

Theorem 4.4 *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. *Let* $\mathbf{p}_0 \equiv (w_0, 0, \xi_0, \phi^o_0, g^i_0, g^o_0) \in \mathcal{E}^{m,\alpha} \times \left(C^{m,\alpha}(\partial\mathbb{B}_n)\right)^2$. *Let* $\mathcal{U}_0$, $\mathcal{O}_0$ *be as in Theorem 4.1. Then there exists an open neighborhood* $\mathcal{U}_1$ *of* $\mathbf{p}_0$ *in* $\mathcal{U}_0 \times \left(C^{m,\alpha}(\partial\mathbb{B}_n)\right)^2$ *and an open neighborhood* $\mathcal{V}_1$ *of* $(\hat{\theta}^i[\mathbf{p}_0], \hat{\theta}^o[\mathbf{p}_0])$ *in* $\left(C^{m,\alpha}(\partial\mathbb{B}_n)\right)^2$ *with* $\mathcal{U}_1 \times \mathcal{V}_1 \subseteq \mathcal{O}_0$, *and a real analytic operator* (Θ^i, Θ^o) *of* $\mathcal{U}_1$ *to* $\mathcal{V}_1$ *such that*

$$(\Theta^i[\mathbf{p}], \Theta^o[\mathbf{p}]) = (\hat{\theta}^i[\mathbf{p}], \hat{\theta}^o[\mathbf{p}]) \tag{4.11}$$

for all $\mathbf{p} \equiv (w, \epsilon, \xi, \phi^o, g^i, g^o) \in \mathcal{U}_1$ *such that* $\epsilon \geq 0$. *Moreover, the graph of* (Θ^i, Θ^o) *coincides with the set of zeros of* Λ *in* $\mathcal{U}_1 \times \mathcal{V}_1$.

Proof: We first observe that the functional G of $\mathcal{H}_1 \equiv \mathcal{O}_0 \times (C^{m,\alpha}(\partial\mathbb{B}_n))^2 \times \mathbb{R}$ to $\mathbb{R}$ defined by

$$G[\mathbf{q}, \gamma^i, \gamma^o, \omega] \equiv \int_{\partial\mathbb{B}_n} \gamma^i E^i[w, \epsilon, \xi, \phi^o]\tilde{\sigma}[\xi]\, d\sigma + \int_{\partial\mathbb{B}_n} \gamma^o R^o[w, \epsilon, \xi, \phi^o]\tilde{\sigma}[\phi^o]\, d\sigma$$

for all $(\mathbf{q}, \gamma^i, \gamma^o, \omega) \in \mathcal{H}_1$, is real analytic and satisfies equality $G[\mathbf{q}, 0, 0, 0] = 0$ and $G[\mathbf{q}, \Lambda[\mathbf{q}]] = 0$ for all $\mathbf{q} \in \mathcal{O}_0$, and that its differential at $(\mathbf{q}_0, 0, 0, 0)$ with respect to the variables $(\gamma^i, \gamma^o, \omega)$ has image equal to $\mathbb{R}$ and kernel equal to $W^{m,\alpha}_{\mathbf{a}_0}$, which is a closed subspace of $(C^{m,\alpha}(\partial\mathbb{B}_n))^2 \times \mathbb{R}$ of codimension 1. Then the Theorem follows by Theorem 4.3 and by Theorem A.1 of the Appendix. □

5 A functional analytic representation Theorem for the solution of the singularly perturbed Dirichlet problem

We now show some applications of Theorem 4.4. To do so, we first introduce the following technical representation Lemma, which can be verified by a change of variables in formula (2.5), and by Theorems 3.10, 4.4.

Lemma 5.1 *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. *Let* $\mathbf{p}_0 \equiv (w_0, 0, \xi_0, \phi^o_0, g^i_0, g^o_0) \in \mathcal{E}^{m,\alpha} \times \left(C^{m,\alpha}(\partial\mathbb{B}_n)\right)^2$. *Let* $\mathcal{U}_1$ *be as in Theorem 4.4. For each* $\mathbf{p} \equiv (w, \epsilon, \xi, \phi^o, g^i, g^o) \in \mathcal{U}_1$ *with* $\epsilon > 0$, *we have*

$$u[\mathbf{p}] = u_r[\mathbf{p}] + u_s[\mathbf{p}], \tag{5.1}$$

where

$$\begin{aligned} u_r[\mathbf{p}](t) = &-\int_{\partial\mathbb{B}_n} \Theta^o[\mathbf{p}](y)(\nu_{\phi^o} \circ \phi^o(y)) \cdot DS_n(t - \phi^o(y))\tilde{\sigma}[\phi^o](y)\, d\sigma_y \\ &+ \epsilon^{n-1} \int_{\partial\mathbb{B}_n} \Theta^i[\mathbf{p}](y)(\nu_\xi \circ \xi(y)) \cdot DS_n(t - w - \epsilon\xi(y))\tilde{\sigma}[\xi](y)\, d\sigma_y, \end{aligned} \tag{5.2}$$

for all $t \in \mathbb{A}[\mathbf{a}]$ *and*

$$\begin{aligned} u_s[\mathbf{p}](t) = {} & \left\{ \int_{\partial\mathbb{B}_n} g^i E^i[\mathbf{a}] \tilde{\sigma}[\xi]\, d\sigma + \int_{\partial\mathbb{B}_n} g^o R^o[\mathbf{a}] \tilde{\sigma}[\phi^o]\, d\sigma \right\} \\ & \cdot \left\{ \int_{\partial\mathbb{B}_n} E^i[\mathbf{a}](y) S_n(t - w - \epsilon\xi(y)) \tilde{\sigma}[\xi](y)\, d\sigma_y \right. \\ & \qquad \left. + \int_{\partial\mathbb{B}_n} R^o[\mathbf{a}](y) S_n(t - \phi^o(y)) \tilde{\sigma}[\phi^o](y)\, d\sigma_y \right\} \\ & \qquad \cdot \left\{ \int_{\partial\mathbb{B}_n} \tilde{\sigma}[\xi]\, d\sigma \right\} \\ & \cdot \left\{ \int_{\partial\mathbb{B}_n} \left\{ \int_{\partial\mathbb{B}_n} E^i[\mathbf{a}](y) S_n(\epsilon(\xi(x) - \xi(y))) \tilde{\sigma}[\xi](y)\, d\sigma_y \right. \right. \\ & \qquad \left. \left. + \int_{\partial\mathbb{B}_n} R^o[\mathbf{a}](y) S_n(w + \epsilon\xi(x) - \phi^o(y)) \tilde{\sigma}[\phi^o](y)\, d\sigma_y \right\} \tilde{\sigma}[\xi](x)\, d\sigma_x \right\}^{-1} \end{aligned} \tag{5.3}$$

for all $t \in \mathbb{A}[\mathbf{a}]$.

Then we also have the following elementary technical Lemma.

Lemma 5.2 *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. *Let* $\mathbf{a}_0 \in \mathcal{E}^{m,\alpha}$. *Then there exist an open neighborhood* $\mathcal{U}_0$ *of* $\mathbf{a}_0$ *in* $\mathcal{E}^{m,\alpha}$ *and real analytic operators* V_1 *and* V_2 *of* $\mathcal{U}_0$ *to* $\mathbb{R}$ *such that*

$$\begin{aligned} & V_1[\mathbf{a}] + V_2[\mathbf{a}]\Upsilon_n(\epsilon) \\ & = \left\{ \int_{\partial\mathbb{B}_n} \tilde{\sigma}[\xi]\, d\sigma \right\}^{-1} \\ & \quad \cdot \left\{ \int_{\partial\mathbb{B}_n} \left\{ \int_{\partial\mathbb{B}_n} E^i[\mathbf{a}](y) S_n(\epsilon(\xi(x) - \xi(y))) \tilde{\sigma}[\xi](y)\, d\sigma_y \right. \right. \\ & \qquad \left. \left. + \int_{\partial\mathbb{B}_n} R^o[\mathbf{a}](y) S_n(w + \epsilon\xi(x) - \phi^o(y)) \tilde{\sigma}[\phi^o](y)\, d\sigma_y \right\} \tilde{\sigma}[\xi](x)\, d\sigma_x \right\} \end{aligned} \tag{5.4}$$

for all $\mathbf{a} \in \mathcal{U}_0$ *with* $\epsilon > 0$, *and such that* $V_2[\mathbf{a}] \neq 0$ *if* $\mathbf{a} \in \mathcal{U}_0$, *and such that* $V_1[\mathbf{a}] + V_2[\mathbf{a}]\Upsilon_n(\epsilon) \neq 0$ *if* $\mathbf{a} \in \mathcal{U}_0$ *with* $\epsilon > 0$. *If* $n = 2$, *we can take* $V_2[\cdot]$ *identically equal to* 1.

Proof: Let $\mathcal{U}_0$ be as in Theorem 3.10. If $n = 2$, we note that

$$\frac{\int_{\partial\mathbb{B}_n} \left\{ \int_{\partial\mathbb{B}_n} E^i[\mathbf{a}](y) \tilde{\sigma}[\xi](y)\, d\sigma_y \right\} \tilde{\sigma}[\xi](x)\, d\sigma_x}{\int_{\partial\mathbb{B}_n} \tilde{\sigma}[\xi]\, d\sigma} = 1$$

for all $\mathbf{a} \in \mathcal{U}_0$. Then we take $V_2[\cdot]$ identically equal to 1 and define $V_1[\mathbf{a}]$ as the difference between the right-hand side of (5.4) and $\Upsilon_n(\epsilon)$.

If instead $n \geq 3$, we set

$$V_2[\mathbf{a}] \equiv (2-n)s_n \frac{\int_{\partial\mathbb{B}_n} \left\{ \int_{\partial\mathbb{B}_n} E^i[\mathbf{a}](y) S_n(\xi(x)-\xi(y)) \tilde{\sigma}[\xi](y)\, d\sigma_y \right\} \tilde{\sigma}[\xi](x)\, d\sigma_x}{\int_{\partial\mathbb{B}_n} \tilde{\sigma}[\xi]\, d\sigma}$$

for all $\mathbf{a} \in \mathcal{U}_0$. Then we take define $V_1[\mathbf{a}]$ as the difference between the right-hand side of (5.4) and $V_2[\mathbf{a}]\Upsilon_n(\epsilon)$. We now show that $V_2[\mathbf{a}] \neq 0$ if $\mathbf{a} \in \mathcal{U}_0$ with $\epsilon = 0$. By Theorem 3.10 (see also the first component of M in (3.1) with $\epsilon = 0$), the single layer potential $\int_{\xi(\partial\mathbb{B}_n)} E^i[\mathbf{a}] \circ \xi^{(-1)}(s) S_n(t-s)\, d\sigma_s$ must be constant in $t \in \mathbb{I}[\xi]$ and zero at infinity. Since $E^i[\mathbf{a}] \circ \xi^{(-1)}$ cannot be identically 0, then the integral average on $\xi(\partial\mathbb{B}_n)$ of such single layer cannot vanish. Hence, $V_2[\mathbf{a}] \neq 0$ if $\mathbf{a} \in \mathcal{U}_0$ with $\epsilon = 0$.

The real analyticity of the operators V_j follows by Proposition 3.4, by Theorems 3.7 (i) and 3.10. By continuity of $V_2[\cdot]$, possibly shrinking $\mathcal{U}_0$, we can assume that $V_2[\cdot]$ does not vanish in $\mathcal{U}_0$.

By Remark 2.2, Proposition 3.2, Theorem 3.10, the right-hand side of (5.4) cannot vanish for $\mathbf{a} \in \mathcal{U}_0$ with $\epsilon > 0$. □

We are now ready to draw our conclusions from Theorems 3.10 and 4.4, and Lemmas 5.1 and 5.2. Namely, we prove the following.

Theorem 5.3 *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. *Let* $\mathbf{p}_0 \equiv (w_0, 0, \xi_0, \phi_0^o, g_0^i, g_0^o) \in \mathcal{E}^{m,\alpha} \times \left(C^{m,\alpha}(\partial\mathbb{B}_n)\right)^2$. *Let* $\mathcal{U}_0, V_1, V_2$ *be as in Lemma 5.2. Let* Ω *be an open subset of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\Omega \subseteq \mathbb{I}[\phi_0^o] \setminus \{w_0\}$. *Then there exist an open neighborhood* $\mathcal{U}$ *of* $\mathbf{p}_0$ *in* $\mathcal{E}^{m,\alpha} \times \left(C^{m,\alpha}(\partial\mathbb{B}_n)\right)^2$ *and two real analytic operators* U_1, U_2 *of* $\mathcal{U}$ *to the space*

$$C_h^0(\mathrm{cl}\Omega) \equiv \left\{ u \in C^0(\mathrm{cl}\Omega) \cap C^2(\Omega) : \Delta u(t) = 0 \ \forall t \in \Omega \right\}$$

endowed with the norm of the uniform convergence such that the following conditions hold.

(i) $\mathrm{cl}\Omega \subseteq \mathbb{A}[w, \epsilon, \xi, \phi^o]$ *for all* $(w, \epsilon, \xi, \phi^o, g^i, g^o) \in \mathcal{U}$.

(ii) $(w, \epsilon, \xi, \phi^o) \in \mathcal{U}_0$ *for all* $\mathbf{p} \in \mathcal{U}$.

(iii)

$$u[\mathbf{p}](t) = U_1[\mathbf{p}](t) + \frac{U_2[\mathbf{p}](t)}{V_1[w, \epsilon, \xi, \phi^o] + V_2[w, \epsilon, \xi, \phi^o]\Upsilon_n(\epsilon)} \qquad \forall t \in \mathrm{cl}\Omega,$$

for all $\mathbf{p} \equiv (w, \epsilon, \xi, \phi^o, g^i, g^o) \in \mathcal{U}$ *such that* $\epsilon > 0$.

(iv)

$$U_1[w, 0, \xi, \phi^o, g^i, g^o](t) = u^o[\phi^o, g^o](t) \qquad \forall t \in \mathrm{cl}\Omega,$$

for all $(w, 0, \xi, \phi^o, g^i, g^o) \in \mathcal{U}$.

Proof: Possibly shrinking the neighborhood $\mathcal{U}_1$ of Theorem 4.4, we can assume that condition (i) holds and that $(w, \epsilon, \xi, \phi^o)$ belongs to the domain $\mathcal{U}_0$ of V_1, V_2 for $\mathbf{p} \in \mathcal{U}$ (cf. Lemma 5.2). Then we take as U_1 the operator defined by the right-hand side of (5.2). By formula (5.2), by definition of $\Theta^o[\cdot]$, and by Theorem 4.1, statement (iv) holds. Then we take as U_2 the operator defined by the the product of first two factors in braces appearing in right-hand side of (5.3). Then by (5.3) also (iii) holds.

The real analyticity of the operators U_j follows by Proposition 3.4 and by Theorems 3.10 and 4.4. □

Remark 5.4 We note that if $n \geq 3$, then the right-hand side of the formula in (iii) of Theorem 5.3 can be continued real analytically in the whole of $\mathcal{U}$.

6 A real analytic continuation Theorem for the energy integral

Theorem 6.1 *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1[$. *Let* $\mathbf{p}_0 \equiv (w_0, 0, \xi_0, \phi_0^o, g_0^i, g_0^o) \in \mathcal{E}^{m,\alpha} \times (C^{m,\alpha}(\partial\mathbb{B}_n))^2$. *Let* $\mathcal{U}_0$, V_1, V_2 *be as in Lemma 5.2. Then there exist an open neighborhood* $\mathcal{U}$ *of* $\mathbf{p}_0$ *in* $\mathcal{E}^{m,\alpha} \times (C^{m,\alpha}(\partial\mathbb{B}_n))^2$ *and two real analytc operators* F_1, F_2 *of* $\mathcal{U}$ *to* $\mathbb{R}$ *such that* $(w, \epsilon, \xi, \phi^o) \in \mathcal{U}_0$ *if* $\mathbf{p} \in \mathcal{U}$ *and*

$$\int_{\mathbb{A}[w,\epsilon,\xi,\phi^o]} |\nabla u[\mathbf{p}](t)|^2 \, dt = F_1[\mathbf{p}] + \frac{F_2[\mathbf{p}]}{V_1[w, \epsilon, \xi, \phi^o] + V_2[w, \epsilon, \xi, \phi^o]\Upsilon_n(\epsilon)}, \tag{6.1}$$

for all $\mathbf{p} \equiv (w, \epsilon, \xi, \phi^o, g^i, g^o) \in \mathcal{U}$ *such that* $\epsilon > 0$. *Moreover,*

$$F_1[w, 0, \xi, \phi^o, g^i, g^o] = \int_{\mathbb{I}[\phi^o]} |\nabla u^o[\phi^o, g^o](t)|^2 \, dt + \delta_{2,n} \int_{\mathbb{E}[\xi]} |\nabla u^i[\xi, g^i](t)|^2 \, dt \tag{6.2}$$

for all $\mathbf{p} \equiv (w, 0, \xi, \phi^o, g^i, g^o) \in \mathcal{U}$, *where* $\delta_{2,n} = 1$ *if* $n = 2$, $\delta_{2,n} = 0$ *if* $n \geq 3$.

Proof: Let $\mathbf{E}_0$, $\mathbf{E}_1$ be extension operators of $\mathcal{W}_0$, $\mathcal{W}_1$ to $C^{m,\alpha}(\mathrm{cl}\mathbb{A}_\delta, \mathbb{R}^n) \cap \mathcal{A}_{\mathrm{cl}\mathbb{A}_\delta}$ with $\delta > 0$ as in Proposition 3.5 for ξ_0, ϕ_0^o, respectively. Possibly choosing a smaller δ, we can assume that $w_0 \notin \mathbf{E}_1[\phi_0^o](\mathrm{cl}\mathbb{A}_\delta)$. Let $\mathcal{U}$ be an open neighborhood of $\mathbf{p}_0$ contained in the neighborhood $\mathcal{U}_1$ of Theorem 4.4 and such that $\xi \in \mathcal{W}_0$, $\phi^o \in \mathcal{W}_1$, and $(w, \epsilon, \xi, \phi^o) \in \mathcal{U}_0$, and $(w + \epsilon\mathbf{E}_0[\xi](\mathrm{cl}\mathbb{A}_\delta)) \cap \phi^o(\partial\mathbb{B}_n) = \emptyset$, and $(w + \epsilon\xi(\partial\mathbb{B}_n)) \cap \mathbf{E}_1[\phi^o](\mathrm{cl}\mathbb{A}_\delta) = \emptyset$, for all $\mathbf{p} \in \mathcal{U}$. By the Divergence Theorem, we have

$$\begin{aligned} &\int_{\mathbb{A}[w,\epsilon,\xi,\phi^o]} |\nabla u[\mathbf{p}](t)|^2 \, dt \qquad (6.3) \\ &= -\epsilon^{n-2} \int_{\partial\mathbb{B}_n} \epsilon \left(\frac{\partial}{\partial\nu_{w+\epsilon\xi}} (u[\mathbf{p}]) \right) \circ (w + \epsilon\xi) \, g^i \tilde{\sigma}[\xi] \, d\sigma \\ &\quad + \int_{\partial\mathbb{B}_n} \left(\frac{\partial}{\partial\nu_\phi} (u[\mathbf{p}]) \right) \circ \phi^o \, g^o \tilde{\sigma}[\phi^o] \, d\sigma \end{aligned}$$

$$
\begin{aligned}
&= -\epsilon^{n-2}\int_{\partial\mathbb{B}_n} D\big[u[\mathbf{p}]\circ(w+\epsilon\mathbf{E}_0[\xi])\big](x)(D(\mathbf{E}_0[\xi])(x))^{-1}\nu_\xi\circ\xi(x)g^i(x)\tilde{\sigma}[\xi](x)\,d\sigma_x\\
&\quad+\int_{\partial\mathbb{B}_n} D\left[u[\mathbf{p}]\circ(\mathbf{E}_1[\phi^o])\right](x)(D(\mathbf{E}_1[\phi^o](x))^{-1}\nu_{\phi^o}\circ\phi^o(x)g^o(x)\tilde{\sigma}[\phi^o](x)\,d\sigma_x
\end{aligned}
$$

for all $\mathbf{p}\in\mathcal{U}_1$ with $\epsilon>0$, where as usual we have abbreviated (w,ϵ,ξ,ϕ^o) as $\mathbf{a}$.

We now consider the first integral in the right-hand side of (6.3). By Lemma 5.1, we have

$$
\begin{aligned}
&u_r[\mathbf{p}]\circ(w+\epsilon\mathbf{E}_0[\xi])(x) \qquad (6.4)\\
&\quad= -\int_{\partial\mathbb{B}_n}\Theta^o[\mathbf{p}](y)(\nu_{\phi^o}\circ\phi^o(y))\cdot DS_n\big(w+\epsilon\mathbf{E}_0[\xi](x)-\phi^o(y)\big)\tilde{\sigma}[\phi^o](y)\,d\sigma_y\\
&\qquad -W^-\big[\mathbf{E}_0[\xi],\Theta^i[\mathbf{p}]\big](x)\quad\forall x\in\mathbb{A}_\delta^-,
\end{aligned}
$$

for all $\mathbf{p}\in\mathcal{U}$ with $\epsilon>0$, and

$$
\begin{aligned}
&u_s[\mathbf{p}]\circ(w+\epsilon\mathbf{E}_0[\xi])(x) \qquad (6.5)\\
&\quad=\left\{\int_{\partial\mathbb{B}_n} g^iE^i[\mathbf{a}]\tilde{\sigma}[\xi]\,d\sigma+\int_{\partial\mathbb{B}_n} g^oR^o[\mathbf{a}]\tilde{\sigma}[\phi^o]\,d\sigma\right\}\\
&\qquad\cdot\left\{\int_{\partial\mathbb{B}_n} E^i[\mathbf{a}](y)S_n(\epsilon\big(\mathbf{E}_0[\xi](x)-\xi(y)\big))\tilde{\sigma}[\xi](y)\,d\sigma_y\right.\\
&\qquad\qquad\left.+\int_{\partial\mathbb{B}_n} R^o[\mathbf{a}](y)S_n\big(w+\epsilon\mathbf{E}_0[\xi](x)-\phi^o(y)\big)\tilde{\sigma}[\phi^o](y)\,d\sigma_y\right\}\\
&\qquad\cdot\left\{V_1[\mathbf{a}]+V_2[\mathbf{a}]\Upsilon_n(\epsilon)\right\}^{-1}\quad\forall x\in\mathbb{A}_\delta^-,
\end{aligned}
$$

for all $\mathbf{p}\in\mathcal{U}$ such that $\epsilon>0$. By Propositions 3.4–3.6, by Theorems 3.7, 3.10, and 4.4, and by (6.4), (6.5) there exist real analytic operators G_1, G_2, G_3 of $\mathcal{U}$ to $C^{m,\alpha}(\mathrm{cl}\mathbb{A}_\delta^-)$ and G_4 of $\mathcal{U}$ to $\mathbb{R}$ such that

$$
u[\mathbf{p}]\circ\big(w+\epsilon\mathbf{E}_0[\xi]\big)=G_1[\mathbf{p}]+G_4[\mathbf{p}]\frac{G_2[\mathbf{p}]+G_3[\mathbf{p}]\Upsilon_n(\epsilon)}{V_1[\mathbf{a}]+V_2[\mathbf{a}]\Upsilon_n(\epsilon)}\qquad\text{in }\mathbb{A}_\delta^-, \tag{6.6}
$$

for all $\mathbf{p}\in\mathcal{U}$ such that $\epsilon>0$. Indeed, we can take as $G_1[\mathbf{p}](\cdot)$ the function of $\mathrm{cl}\mathbb{A}_\delta^-$ to $\mathbb{R}$ defined by the right-hand side of (6.4) and we can take as $G_4[\mathbf{p}]$ the first factor in braces in the right-hand side of (6.5). Instead, to define G_3 we consider two separate cases.
If $n=2$ we take $G_3[\mathbf{p}]$ identically equal to 1.
If $n\geq 3$ we take $G_3[\mathbf{p}](\cdot)$ equal to

$$
(2-n)s_n\int_{\partial\mathbb{B}_n} E^i[\mathbf{a}](y)S_n\big(\mathbf{E}_0[\xi](\cdot)-\xi(y)\big)\tilde{\sigma}[\xi](y)\,d\sigma_y.
$$

Then for all $n\geq 2$ we take as $G_2[\mathbf{p}]$ the difference between the second factor in braces in the right-hand side of (6.5) and $G_3[\mathbf{p}]\Upsilon_n(\epsilon)$.

Clearly,

$$D\big[u[\mathbf{p}]\circ(w+\epsilon\mathbf{E}_0[\xi])\big]=D\big[G_1[\mathbf{p}]\big]+G_4[\mathbf{p}]\frac{D\big[G_2[\mathbf{p}]+G_3[\mathbf{p}]\Upsilon_n(\epsilon)\big]}{V_1[\mathbf{a}]+V_2[\mathbf{a}]\Upsilon_n(\epsilon)}\qquad \text{in } \mathbb{A}_\delta^-, \tag{6.7}$$

for all $\mathbf{p}\in\mathcal{U}$ such that $\epsilon>0$. By Theorems 4.1 and 4.4, we have

$$\begin{aligned} G_1[w,0,\xi,\phi^o,g^i,g^o] &= u^o[\phi^o,g^o](w)+u_r^i[\xi,g^i]\circ\mathbf{E}_0[\xi] \qquad \text{in } \mathbb{A}_\delta^-,\\ D\big[G_1[w,0,\xi,\phi^o,g^i,g^o]\big] &= D\big[u_r^i[\xi,g^i]\circ\mathbf{E}_0[\xi]\big]\qquad \text{in } \mathbb{A}_\delta^- \end{aligned}\tag{6.8}$$

for all $(w,0,\xi,\phi^o,g^i,g^o)\in\mathcal{U}$. Then

$$\begin{aligned} &\epsilon\bigg(\frac{\partial}{\partial\nu_{w+\epsilon\xi}}\big(u[\mathbf{p}]\big)\bigg)\circ(w+\epsilon\xi) \\ &\quad= D\big[G_1[\mathbf{p}]\big]\big(D\mathbf{E}_0[\xi]\big)^{-1}(\nu_\xi\circ\xi)\\ &\qquad+\frac{G_4[\mathbf{p}]}{V_1[\mathbf{a}]+V_2[\mathbf{a}]\Upsilon_n(\epsilon)}D\big[G_2[\mathbf{p}]+G_3[\mathbf{p}]\Upsilon_n(\epsilon)\big]\big(D\mathbf{E}_0[\xi]\big)^{-1}(\nu_\xi\circ\xi) \end{aligned}\tag{6.9}$$

on $\partial\mathbb{B}_n$, and thus the first integral in the right-hand side of (6.3) equals

$$\begin{aligned} &\int_{\partial\mathbb{B}_n} D\big[G_1[\mathbf{p}]\big]\big(D\mathbf{E}_0[\xi]\big)^{-1}(\nu_\xi\circ\xi)g^i\tilde{\sigma}[\xi]\,d\sigma \\ &\quad+\frac{G_4[\mathbf{p}]}{V_1[\mathbf{a}]+V_2[\mathbf{a}]\Upsilon_n(\epsilon)}\int_{\partial\mathbb{B}_n} D\big[G_2[\mathbf{p}]+G_3[\mathbf{p}]\Upsilon_n(\epsilon)\big]\big(D\mathbf{E}_0[\xi]\big)^{-1}(\nu_\xi\circ\xi)g^i\tilde{\sigma}[\xi]\,d\sigma \end{aligned}\tag{6.10}$$

for all $\mathbf{p}\in\mathcal{U}$ such that $\epsilon>0$. Now we note that if $\epsilon=0$ the first integral in (6.10) equals

$$\int_{\xi(\partial\mathbb{B}_n)} g^i\circ\xi^{(-1)}\frac{\partial}{\partial\nu_\xi}\big(u_r^i[\xi,g^i]\big)\,d\sigma=-\int_{\mathbb{E}[\xi]}\big|\nabla u_r^i[\xi,g^i](t)\big|^2\,dt,$$

(cf. *e.g.*, Folland [5, p. 118]). Moreover, if $n=2$ then $D[G_3[\mathbf{p}]\Upsilon_n(\epsilon)]=0$ and $\nabla u_r^i[\xi,g^i]=\nabla u^i[\xi,g^i]$.

We now consider the second integral in (6.3). By Lemma 5.1, we have

$$\begin{aligned} &u_r[\mathbf{p}]\circ\mathbf{E}_1[\phi^o](x)\\ &\quad= W^+\big[\mathbf{E}_1[\phi^o],\Theta^o[\mathbf{p}]\big](x)\\ &\qquad+\epsilon^{n-1}\int_{\partial\mathbb{B}_n}\Theta^i[\mathbf{p}](y)\big(\nu_\xi\circ\xi(y)\big)\cdot DS_n\big(\mathbf{E}_1[\phi^o](x)-w-\epsilon\xi(y)\big)\tilde{\sigma}[\xi](y)\,d\sigma_y, \end{aligned}\tag{6.11}$$

for all $x \in \mathbb{A}_\delta^+$ and for all $\mathbf{p} \in \mathcal{U}$ with $\epsilon > 0$, and

$$\begin{aligned}
&u_s[\mathbf{p}] \circ \mathbf{E}_0[\phi^o](x) \\
&\quad = \left\{ \int_{\partial\mathbb{B}_n} g^i E^i[\mathbf{a}]\tilde{\sigma}[\xi]\, d\sigma + \int_{\partial\mathbb{B}_n} g^o R^o[\mathbf{a}]\tilde{\sigma}[\phi^o]\, d\sigma \right\} \\
&\qquad \cdot \left\{ \int_{\partial\mathbb{B}_n} E^i[\mathbf{a}](y) S_n\big(\mathbf{E}_1[\phi^o](x) - w - \epsilon\xi(y)\big)\tilde{\sigma}[\xi](y)\, d\sigma_y \right. \\
&\qquad\qquad \left. + \int_{\partial\mathbb{B}_n} R^o[\mathbf{a}](y) S_n\big(\mathbf{E}_1[\phi^o](x) - \phi^o(y)\big)\tilde{\sigma}[\phi^o](y)\, d\sigma_y \right\} \\
&\qquad \cdot \left\{ V_1[\mathbf{a}] + V_2[\mathbf{a}]\Upsilon_n(\epsilon) \right\}^{-1} \qquad \forall x \in \mathbb{A}_\delta^+,
\end{aligned} \tag{6.12}$$

for all $\mathbf{p} \in \mathcal{U}$ with $\epsilon > 0$. By Propositions 3.4–3.6, by Theorems 3.10 and 4.4, and by (6.11) and (6.12) there exist real analytic operators G_5, G_6 of $\mathcal{U}$ to $C^{m,\alpha}(\mathrm{cl}\mathbb{A}_\delta^+)$ such that

$$u[\mathbf{p}] \circ \big(\mathbf{E}_1[\phi^o]\big) = G_5[\mathbf{p}] + \frac{G_6[\mathbf{p}]}{V_1[\mathbf{a}] + V_2[\mathbf{a}]\Upsilon_n(\epsilon)} \qquad \text{in } \mathbb{A}_\delta^+, \tag{6.13}$$

for all $\mathbf{p} \in \mathcal{U}$ with $\epsilon > 0$. Indeed, we can take as $G_5[\mathbf{p}](\cdot)$ the function of $\mathrm{cl}\mathbb{A}_\delta^+$ to $\mathbb{R}$ defined by the right-hand side of (6.11), and we take as G_6 the product of the first two factors in the right-hand side of (6.12). Clearly,

$$D\big[u[\mathbf{p}] \circ \big(\mathbf{E}_1[\phi^o]\big)\big] = D\big[G_5[\mathbf{p}]\big] + \frac{D\big[G_6[\mathbf{p}]\big]}{V_1[\mathbf{a}] + V_2[\mathbf{a}]\Upsilon_n(\epsilon)} \qquad \text{in } \mathbb{A}_\delta^+, \tag{6.14}$$

for all $\mathbf{p} \in \mathcal{U}$ with $\epsilon > 0$. In particular, we observe that

$$\begin{aligned}
G_5[w, 0, \xi, \phi^o, g^i, g^o] &= u^o[\phi^o, g^o] \circ \mathbf{E}_1[\phi^o] && \text{in } \mathbb{A}_\delta^+, \\
D\big[G_5[w, 0, \xi, \phi^o, g^i, g^o]\big] &= D\big[u^o[\phi^o, g^o] \circ \mathbf{E}_1[\phi^o]\big] && \text{in } \mathbb{A}_\delta^+.
\end{aligned} \tag{6.15}$$

for all $(w, 0, \xi, \phi^o, g^i, g^o) \in \mathcal{U}$. Then

$$\begin{aligned}
\left(\frac{\partial}{\partial\nu_\phi}(u[\mathbf{p}])\right) \circ \phi^o &= D\big[G_5[\mathbf{p}]\big]\big(D\mathbf{E}_1[\phi^o]\big)^{-1}(\nu_{\phi^o} \circ \phi^o) \\
&\quad + \frac{1}{V_1[\mathbf{a}] + V_2[\mathbf{a}]\Upsilon_n(\epsilon)} D\big[G_6[\mathbf{p}]\big]\big(D\mathbf{E}_1[\phi^o]\big)^{-1}(\nu_{\phi^o} \circ \phi^o) \quad \text{on } \partial\mathbb{B}_n,
\end{aligned} \tag{6.16}$$

and thus the second integral in the right-hand side of (6.3) equals

$$\begin{aligned}
&\int_{\partial\mathbb{B}_n} D\big[G_5[\mathbf{p}]\big]\big(D\mathbf{E}_1[\phi^o]\big)^{-1}(\nu_{\phi^o} \circ \phi^o) g^o \tilde{\sigma}[\phi^o]\, d\sigma \\
&\quad + \frac{1}{V_1[\mathbf{a}] + V_2[\mathbf{a}]\Upsilon_n(\epsilon)} \int_{\partial\mathbb{B}_n} D\big[G_6[\mathbf{p}]\big]\big(D\mathbf{E}_1[\phi^o]\big)^{-1}(\nu_{\phi^o} \circ \phi^o) g^o \tilde{\sigma}[\phi^o]\, d\sigma
\end{aligned} \tag{6.17}$$

for all $\mathbf{p} \in \mathcal{U}$ such that $\epsilon > 0$. Now we note that by the Divergence Theorem the first integral in (6.17) for $\epsilon = 0$ equals

$$\int_{\phi^o(\partial\mathbb{B}_n)} g^o \circ (\phi^o)^{(-1)} \frac{\partial}{\partial\nu_{\phi^o}}\big(u^o[\phi^o, g^o]\big)\, d\sigma = \int_{\mathbb{I}[\phi^o]} \big|\nabla u^o[\phi^o, g^o](t)\big|^2\, dt.$$

By (6.3), (6.10) and (6.17), we immediately deduce the existence of F_1, F_2 and the validity of (6.1). The validity of (6.2) follows by the above computation of the first integrals of (6.10) and (6.17) at $\epsilon = 0$. $\square$

As is well known, if $(w, \epsilon, \xi, \phi^o) \in \mathcal{E}^{m,\alpha}$, then the relative electrostatic capacity $\mathrm{Cap}[w, \epsilon, \xi, \phi^o]$ of $\mathrm{cl}\mathbb{I}[w + \epsilon\xi]$ with respect to the domain $\mathbb{I}[\phi^o]$ is defined as

$$\mathrm{Cap}[w, \epsilon, \xi, \phi^o] = \int_{\mathbb{A}[w,\epsilon,\xi,\phi^o]} |\nabla u[w, \epsilon, \xi, \phi^o, 1, 0](t)|^2 \, dt.$$

Then we have the following special case of Theorem 6.1.

Theorem 6.2 *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1[$. *Let* $\mathbf{a}_0 \equiv (w_0, 0, \xi_0, \phi_0^o) \in \mathcal{E}^{m,\alpha}$. *Let* $\mathcal{U}_0$, V_1, V_2 *be as in Lemma 5.2. Then*

$$\mathrm{Cap}[\mathbf{a}] = \frac{-1}{V_1[\mathbf{a}] + V_2[\mathbf{a}]\Upsilon_n(\epsilon)} \tag{6.18}$$

for all $\mathbf{a} \equiv (w, \epsilon, \xi, \phi^o) \in \mathcal{U}_0$ *such that* $\epsilon > 0$.

Proof: Since $g^o = 0$, the second integral in the right-hand side of formula (6.3) must vanish. By Theorem 3.10, we have $\int_{\partial\mathbb{B}_n} E^i[\mathbf{a}]\tilde{\sigma}[\xi] \, d\sigma = 1$ for all $\mathbf{a} \equiv (w, \epsilon, \xi, \phi^o) \in \mathcal{U}_0$. Then by equalities $g^i = 1$, $g^o = 0$ and by Theorem 4.1, we have $\Theta^i[w, \epsilon, \xi, \phi^o, 1, 0] = \Theta^o[w, \epsilon, \xi, \phi^o, 1, 0] = 0$ and thus we conclude that G_1 in the proof of Theorem 6.1 must be identically zero. By Theorem 3.10 we have $G_4[w, \epsilon, \xi, \phi^o, 1, 0] = 1$. Then by formulas (6.3) and (6.10), we have

$$\begin{aligned} \mathrm{Cap}[\mathbf{a}] = {} & \frac{-\epsilon^{n-2}}{V_1[\mathbf{a}] + V_2[\mathbf{a}]\Upsilon_n(\epsilon)} \\ & \cdot \int_{\partial\mathbb{B}_n} D\big[G_2[\mathbf{p}] + G_3[\mathbf{p}]\Upsilon_n(\epsilon)\big]\big(D\mathbf{E}_0[\xi]\big)^{-1}(\nu_\xi \circ \xi)\tilde{\sigma}[\xi] \, d\sigma, \end{aligned} \tag{6.19}$$

for all $\mathbf{a} \in \mathcal{U}_0$ such that $\epsilon > 0$, where $G_2[\mathbf{p}] + G_3[\mathbf{p}]\Upsilon_n(\epsilon)$ equals the continuous extension to $\mathrm{cl}\mathbb{A}_\delta^-$ of the second factor in braces of (6.5). Now we introduce the following two auxiliary functions

$$\begin{aligned} v_1(t) &\equiv \epsilon^{1-n} \textstyle\int_{w+\epsilon\xi(\partial\mathbb{B}_n)} S_n(t-s)E^i[\mathbf{a}] \circ (w + \epsilon\xi)^{(-1)}(s) \, d\sigma_s \quad && \forall t \in \mathrm{cl}\mathbb{E}[w + \epsilon\xi], \\ v_2(t) &\equiv \textstyle\int_{\phi^o(\partial\mathbb{B}_n)} S_n(t-s)R^o[\mathbf{a}] \circ (\phi^o)^{(-1)}(s) \, d\sigma_s && \forall t \in \mathrm{cl}\mathbb{I}[\phi^o]. \end{aligned}$$

Then we have

$$\begin{aligned} & \mathrm{Cap}[\mathbf{a}] \\ & = \frac{-1}{V_1[\mathbf{a}] + V_2[\mathbf{a}]\Upsilon_n(\epsilon)} \left\{ \int_{w+\epsilon\xi(\partial\mathbb{B}_n)} \frac{\partial v_1}{\partial \nu_{w+\epsilon\xi}} \, d\sigma + \int_{w+\epsilon\xi(\partial\mathbb{B}_n)} \frac{\partial v_2}{\partial \nu_{w+\epsilon\xi}} \, d\sigma \right\}. \end{aligned} \tag{6.20}$$

By the classical Potential Theory, we have

$$\begin{aligned}\epsilon^{n-1}\frac{\partial v_1}{\partial \nu_{w+\epsilon\xi}}(t) &= \frac{1}{2}E^i[\mathbf{a}]\circ(w+\epsilon\xi)^{(-1)}(t)\\ &\quad + \int_{w+\epsilon\xi(\partial\mathbb{B}_n)}\frac{\partial}{\partial \nu_{w+\epsilon\xi}(t)}\big(S_n(t-s)\big)E^i[\mathbf{a}]\circ(w+\epsilon\xi)^{(-1)}(s)\,d\sigma_s,\end{aligned}$$

for all $t \in w+\epsilon\xi(\partial\mathbb{B}_n)$. Then by the Fubini Theorem and by the identity

$$\int_{w+\epsilon\xi(\partial\mathbb{B}_n)}\frac{\partial}{\partial \nu_{w+\epsilon\xi}(t)}\big(S_n(t-s)\big)\,d\sigma_t = \frac{1}{2},$$

for all $s \in w+\epsilon\xi(\partial\mathbb{B}_n)$ and by equality $\int_{w+\epsilon\xi(\partial\mathbb{B}_n)} E^i[\mathbf{a}]\circ(w+\epsilon\xi)^{(-1)}\,d\sigma = \epsilon^{n-1}$ which follows by Theorem 3.10, we have

$$\int_{w+\epsilon\xi(\partial\mathbb{B}_n)}\frac{\partial v_1}{\partial \nu_{w+\epsilon\xi}}\,d\sigma = 1.$$

Since v_2 is harmonic in a neighborhood of $\mathrm{cl}\mathbb{I}[w+\epsilon\xi]$, the second integral in (6.20) vanishes and thus (6.18) holds. □

We note that case $n=2$ of (6.18) is known (cf. [9, Cor. 3.2]).

A Appendix

In this appendix, we prove the following perhaps known consequence of the Implicit Function Theorem. For a proof, we refer to [11, App. B].

Theorem A.1 *Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, $\mathcal{Z}_1$ be Banach spaces. Let $\mathcal{O}$ be an open subset of $\mathcal{X}\times\mathcal{Y}$, $(x_0,y_0)\in\mathcal{O}$, F a real analytic map of $\mathcal{O}$ to $\mathcal{Z}$, $F(x_0,y_0)=0$. Let the partial differential $\partial_y F(x_0,y_0)$ with respect to the variable y be a linear homeomorphism of $\mathcal{Y}$ onto its image $V\equiv \mathrm{Im}\,\partial_y F(x_0,y_0)$. Assume that there exists a closed subspace V_1 of $\mathcal{Z}$ such that $\mathcal{Z}=V\oplus V_1$ algebraically. Let $\mathcal{O}_1$ be an open subset of $\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}$ containing $(x_0,y_0,0)$ such that $\mathcal{O}_1\supseteq\big\{(x,y,F(x,y)) : (x,y)\in\mathcal{O}\big\}$, $\mathcal{O}_1\supseteq\big\{(x,y,0) : (x,y)\in\mathcal{O}\big\}$. Let G be a real analytic map of $\mathcal{O}_1$ to $\mathcal{Z}_1$ such that $G(x,y,F(x,y))=0$ for all $(x,y)\in\mathcal{O}$, $G(x,y,0)=0$ for all $(x,y)\in\mathcal{O}$, and such that the partial differential $\partial_z G(x_0,y_0,0)$ is surjective onto $\mathcal{Z}_1$ and has kernel equal to V. Then there exist an open neighborhood $\mathcal{U}$ of x_0 in $\mathcal{X}$ and an open neighborhood $\mathcal{V}$ of y_0 in $\mathcal{Y}$ with $\mathcal{U}\times\mathcal{V}\subseteq\mathcal{O}$ and such that the set of zeros of F in $\mathcal{U}\times\mathcal{V}$ coincides with the graph of a real analytic function of $\mathcal{U}$ to $\mathcal{V}$.*

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