Randomly perturbed dynamical systems and Aubry-Mather theory

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Abstract: We give a new PDE proof of a Freidlin-Wentzell theorem about the exit points from a domain of a random process, obtained by perturbing a dynamical system through the addition of a small noise. The relevant part of the analysis concerns an Hamilton-Jacobi equation, coupled with a Neumann boundary condition, which does not possess any strict subsolution. A metric method based on the introduction of an intrinsic length is adopted, and a notion of Aubry set, adjusted to the setting, is given.

Keywords: dynamical systems; random perturbations; large deviations; viscosity solutions; PDE; Aubry-Mather theory.

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1 Introduction

In this paper we study the problem of the exit from a bounded domain $\Omega \subset \mathbb{R}^N$ of a random process, arising as a result of small random perturbations of a dynamical system. We aim to give a new proof of a classical Freidlin-Wentzell theorem, based on the large deviation principle, by solely using PDE techniques and, in particular, by performing a qualitative analysis of a suitable Hamilton-Jacobi equation coupled with a Neumann boundary condition, in the framework of viscosity solutions theory.

Consider the differential equation

$$\dot{\xi}(t) = b(\xi(t)) \quad \text{in } \overline{\Omega}, \tag{1}$$

with b a smooth vector field in $\overline{\Omega}$, and assume the boundary condition

$$b(x) \cdot n(x) < 0 \qquad \text{for any } x \in \partial\Omega, \tag{2}$$

where $n(\cdot)$ is the exterior normal to $\partial\Omega$. The perturbed stochastic equation is given by

$$dX_\varepsilon(t) = b(X_\varepsilon(t))dt + \sqrt{\varepsilon}\, dW(t), \tag{3}$$

where W is a standard N-dimensional Wiener process, and $\varepsilon > 0$ a small parameter.

Despite the forward invariance of Ω for the deterministic dynamics, it is well known that the random trajectories X_ε, issued from points of Ω, leave the domain almost surely for any $\varepsilon > 0$. The asymptotic behaviour of the exit points of X_ε, for ε going to 0, has been widely investigated by means of probabilistic techniques and the setup is at present quite well delineated, see the monograph (Freidlin and Wentzell, 1998) by Freidlin and Wentzell, chapters 4 and 6, the expository paper by Day (1999), and the references therein.

The main output of this analysis is that the exit points minimise the so-called quasipotential from some distinguished subset of Ω_b, the set of the ω-limits of b. The quasipotential, see Section 2, is defined by minimising the action functional

$$\frac{1}{2}\int_0^T |\dot{\xi} - b(\xi)|^2 dt,$$

where ξ is a curve contained in $\overline{\Omega}$. Loosely speaking the value of this energy indicates how difficult is for X_ε to pass near ξ, when ε is small.

Since diffusion processes are involved, an alternative PDE approach for this problem and for similar ones, based on the asymptotic analysis of certain elliptic PDEs with a small parameter, is possible. Even though in this way the results obtained in the probabilistic literature are not yet fully recovered, this kind of investigation is justified for it allows to look at the topic from a new angle and makes mostly use of quite simple techniques, in comparison with the sophisticated probabilistic tools usually employed.

Relevant contributions in this line of research can, for instance, be found in Eizenberg (1990), Fleming (1977/1978), Kamin (1978, 1982). Viscosity solutions theory has proved to be particularly effective in this context because of its nice stability results, and the fact that representation formulae are, in many cases, available for such type of weak solutions. Starting from Evans and Ishii (1985), the viscosity solutions approach has been pursued in Barles and Perthame (1990), Bardi (1987), Fleming and Souganidis (1986), Ishii and Koike (1991), and Perthame (1990).

Our results are actually a generalisation of those obtained in Perthame (1990), where the above outlined exit problem is treated assuming the limit set Ω_b to consist of a single equilibrium of attractive type. We instead require that

(i) Ω_b is the finite union of equivalence classes $\mathcal{K}_i$, under $\sim$

(ii) all the $\mathcal{K}_i$ but one, say $\mathcal{K}_1$, are repulsors for b, and $\operatorname{div} b > 0$ on $\bigcup_{i>1} \mathcal{K}_i$,

where the relation $\sim$ is defined by

$$x \sim y \Leftrightarrow V(x,y) = V(y,x) = 0,$$

and $V(\cdot,\cdot)$ is the quasipotential. Some comments on these assumptions are provided in Section 2.

After a procedure that we describe in Section 3, our analysis focuses on the study of the convergence in $\overline{\Omega}$, for $\varepsilon \to 0$, of the solutions V_ε to the family of viscous Hamilton-Jacobi equations

$$-\frac{\varepsilon}{2}\,\Delta V_\varepsilon + H(x, DV_\varepsilon) = \varepsilon \operatorname{div} b, \qquad \text{in } \Omega$$

coupled with the Neumann condition

$$\frac{\partial V_\varepsilon}{\partial n} + 2b \cdot n = 0 \qquad \text{in } \partial\Omega.$$

The Hamiltonian H is given by the Fenchel transform of the Lagrangian appearing as integrand in the formula defining the action functional, namely

$$H(x,p) = \frac{|p|^2}{2} + b(x) \cdot p. \tag{4}$$

The functions V_ε are obtained, up to an additive constant, through a suitably rescaled *logarithmic transform* of the densities of the invariant measures associated to the reflected diffusions with drift b. This shows the relationship of this asymptotics with large deviations. The limit problem is given by

$$\begin{cases} H(x, Du) = 0 \quad \text{in } \Omega \\ \dfrac{\partial u}{\partial n} + 2b \cdot n = 0 \quad \text{on } \partial\Omega \end{cases} \tag{5}$$

The previous problem admits a viscosity solution. But the solution, even if the value at a point of Ω is fixed, is in general not unique because the non-emptiness of the ω-limits set Ω_b implies, and is actually equivalent to, the non-existence of a strict subsolution in Ω. It follows that Comparison Principles, which are based on the existence of strict subsolutions and are mandatory for applying the Barles-Perthame semi-relaxed limits technique to the analysis of the asymptotic behaviour of V_ε, are difficult to establish.

This difficulty has been overcome in Perthame (1990) through a clever construction by hands of a strict subsolution outside small balls centred at the unique equilibrium of b.

Aiming to include the problem in a general theory, we find parallels with the study of periodic Hamilton-Jacobi equations in the critical case, i.e., when periodic viscosity solutions, but no strict subsolutions, do exist (see Fathi (2009) for a general account on the subject, and its relevance in Hamiltonian dynamics as well as in weak KAM theory).

We adapt to our setting the so-called metric approach, proposed in Fathi and Siconolfi (2005) to study critical periodic equations (see Camilli and Siconolfi (2007) and also Siconolfi (2006) for an expository presentation), which is based on the introduction of an intrinsic length for curves of $\overline{\Omega}$. The key point in our analysis is to show that the quasi-potential V coincides with the geodetic distance induced in $\overline{\Omega}$ by minimising the intrinsic length in the class of curves joining two given points.

Following Fathi and Siconolfi (2005), we single out some special points of Ω by requiring that cycles with infinitesimal intrinsic length and natural length greater than some positive constant pass through them. They make up a set, called Aubry set and denoted by $\mathcal{A}$, enjoying the crucial property that there are subsolutions to the Hamilton-Jacobi equation, which are strict outside it. From this we derive a comparison principle for sub/supersolution of (5) with the same trace on $\mathcal{A}$, and we subsequently prove that the same holds when $\mathcal{A}$ is replaced by Ω_b. Recently in Ishii and Mitake (2007), Mitake (2008) analogous results have been obtained, giving representation formulas for viscosity solutions of Hamilton-Jacobi equations in bounded domains under state constraints.

If Ω_b consists of a unique equivalence class under $\sim$, i.e. $\Omega_b = \mathcal{K}_1$, we can deduce from these results that there is an unique solution w to (5) vanishing at some point $x_0 \in \mathcal{K}_1$, and such solution actually vanish on the whole $\mathcal{K}_1 = \Omega_b$. An application of the semi-relaxed limits technique then gives that the sequence V_ε of solutions to the viscous Hamilton-Jacobi problem with $V_\varepsilon(x_0) = 0$ uniformly converge to w in $\overline{\Omega}$, see Theorem 7.2.

The situation is quite different when, on the contrary, Ω_b is made up by several equivalence classes. The solution of (5) vanishing on $\mathcal{K}_1$ is, in fact, not any more unique. Nevertheless we are able to get also in this case the convergence result, proving in Theorem 7.3 that the vanishing viscosity method is capable to select at the limit, among such solutions, the maximal one. It seems to be a new fact in the viscosity literature on this subject. An important step for establishing it is to show that any subsolution to the Hamilton-Jacobi equation, which is strict outside $\mathcal{A}$, can be smoothed up around a repulsor of b, see Proposition 7.4. Results on analogous selection principles in the context of viscosity solutions and weak KAM theory have been obtained for a different problem in the periodic settig in Anantharaman et al. (2005) (see also Bessi, 2003).

The paper is organised as follows: Section 2 is devoted to fix some notations and to precisely state the assumptions, in Section 3 it is explained how the exit problem from Ω of the random trajectories is related to the previously described asymptotics. In Section 4 it is proved that the quasipotential coincides with the intrinsic distance. The Aubry set $\mathcal{A}$ is introduced in Section 5, and the relation between and Ω_b is investigated, as well. The comparison principles for the problem (5) are given in Section 6, and Section 7 contains the main results. Two final appendices concern the proof of results pertaining to the intrinsic length and to a smoothing procedure, respectively.

2 Preliminaries and assumptions

We assume the previously introduced open bounded domain $\Omega \subset \mathbb{R}^N$ and vector field b to possess C^2 boundary $\partial\Omega$, and to be of class C^1 in $\overline{\Omega} := \Omega \cup \partial\Omega$, respectively. The signed distance function $d^{\#}(\cdot, \Omega) := 2d(\cdot, \Omega) - d(\cdot, \partial\Omega)$ is consequently of class C^1 in a suitable neighbourhood $\partial\Omega$, and $Dd^{\#}(\cdot, \Omega) = n(\cdot)$ on $\partial\Omega$. The notation $d(\cdot, \Omega)$ stands for the Euclidean distance from Ω. We write d_Ω for the Euclidean geodetic distance in Ω, the inequality

$$d_\Omega(x, y) \leq l_\Omega |x - y| \tag{6}$$

holds true for any x, y in Ω and some constant l_Ω, thanks to the regularity assumption on $\partial\Omega$. We denote by $\operatorname{div} b$ the divergence of b. By ω-limit (resp. α-limit) of an integral trajectory, defined in a right-unbounded (resp. left-unbounded) interval, we mean a limit point for $t \to +\infty$ (resp. $t \to -\infty$). The condition (2) implies that the set Ω_b of ω-limit points of all such trajectories is nonempty and contained in Ω. The (possibly empty) set of equilibria of b, denoted by $\mathcal{E}$, is contained in Ω_b.

Definition 2.1: We say that $\mathcal{K} \subset \Omega$ is a local stable *attractor* for b if

(i) there is a neighbourhood $\widetilde{\Omega}$ of $\mathcal{K}$ such that all the ω-limit points of integral curves of b issued from $\widetilde{\Omega}$, lie in $\mathcal{K}$

(ii) for any $\rho > 0$ there is a $\delta = \delta(\rho)$ such that $0 < \delta \leq \rho$ and the integral curves of b issued from points at distance less than δ from $\mathcal{K}$ remains at distance less than ρ, for any $t > 0$.

The set made up by initial points for which item (i) is satisfied is called *basin of attraction* of $\mathcal{K}$. A local instable *repulsor* is an attractor for the inverse dynamics $-b$. The notion of *basin of repulsion* is given accordingly.

The Freidlin-Wentzell quasipotential, related to b, is defined by

$$V(y,x) = \inf\left\{ \int_0^T \frac{1}{2}|\dot{\xi}(s) - b(\xi(s))|^2\, ds : \xi(0) = y, \right.$$
$$\left. \xi(T) = x, \xi \in W^{1,\infty}([0,T], \overline{\Omega}),\ T > 0 \right\},$$

for any pair of points y, x in $\overline{\Omega}$ (see Friedlin and Wentzell, 1998). Note that $L(x,v) = \frac{1}{2}|v - b(x)|^2$ is the Lagragian function associated to the Hamiltonian defined in (4). V is nonnegative, and satisfies the triangle inequality $V(x,y) \leq V(x,z) + V(z,y)$ for every $x,y,z \in \overline{\Omega}$. We deduce from (6) that

$$V(x,y) \leq C|x-y| \quad \text{for any } x,y, \text{ some } C > 0, \tag{7}$$

we set

$$V(x, \partial\Omega) = \min_{y \in \partial\Omega} V(x,y) \quad \text{for any } x \in \overline{\Omega}.$$

It is associated with V an equivalence relation $\sim$ given, for any y, x in $\overline{\Omega}$, by

$$x \sim y \Leftrightarrow V(x,y) = V(y,x) = 0.$$

An equivalence class $\mathcal{K}$ under $\sim$ is closed, and $V(\cdot, y)$, $V(y,\cdot)$ are constant in $\mathcal{K}$, for any fixed $y \in \overline{\Omega}$, thanks to the triangle inequality. We denote these common values by $V(\mathcal{K}, y)$, $V(y, \mathcal{K})$, respectively. We similarly define $V(\mathcal{K}, \partial\Omega)$ and $V(\mathcal{K}, \mathcal{K}')$, when $\mathcal{K}'$ is another equivalence class.

Besides the boundary condition (2), we will require, to get our main results, the following conditions on Ω_b:

$$\Omega_b \text{ is the finite union of } \sim -\text{equivalence classes } \mathcal{K}_i,$$
$$i = 1, \ldots, M, \quad \text{for some } M, \tag{8}$$
$$\text{all the } \mathcal{K}_i \text{ but one, say } \mathcal{K}_1, \text{ are repulsors for } b, \quad \text{and} \quad \operatorname{div} b > 0 \text{ in } \bigcup\nolimits_{i>1} \mathcal{K}_i. \tag{9}$$

We need the positiveness of $\operatorname{div} b$ in the proof of Theorem 7.3, see in particular (59). This is a quite strong assumption. Observe, however, that if the $\mathcal{K}_i$ reduce to repulsive equilibria, for $i > 1$, it holds up to a local perturbation of the vector field b.

It comes from (8)–(9) that $\mathcal{K}_1$ must satisfy an attractiveness property. In fact the instability condition on the $\mathcal{K}_i$, for $i > 1$, directly implies that the ω-limits of integral trajectories of b issued from points of $\overline{\Omega} \setminus \bigcup_{i>1} \mathcal{K}_i$ belong to $\mathcal{K}_1$. This, in turn, entails

$$V(\mathcal{K}_i, \mathcal{K}_1) = 0 \quad \text{for } i > 1. \tag{10}$$

As a consequence of (2), we also have

$$V(\mathcal{K}_i, \partial\Omega) > 0 \quad \text{for } i = 1, \ldots, M. \tag{11}$$

3 The exit problem

The goal of this section is to relate the exit from Ω of the random trajectories, solutions to (3), to a vanishing viscosity asymptotic problem, as $\varepsilon \to 0$. We basically follow the approach of Perthame (1990), see also Matkowsky and Schuss (1977). We start by considering the singular perturbation problem

$$\begin{cases} -\dfrac{\varepsilon}{2}\Delta u_\varepsilon - b(x)Du_\varepsilon = 0 & x \in \Omega \\ u_\varepsilon(x) = \varphi(x) & x \in \partial\Omega \end{cases} \tag{12}$$

with φ continuous datum. The solution can be represented, for any ε, by the formula $u_\varepsilon(x) = \mathbb{E}_x\varphi(X_\varepsilon(\tau_\varepsilon))$, where $\mathbb{E}$ indicates the expectation, X_ε is the trajectory of (3) with initial data x, and τ_ε is its exit time from Ω.

To study the asymptotic behaviour of the u_ε, it has been introduced in Eizenberg (1990) the condition

$$\max_{j=1,\dots,M} \min_{i=1,\dots,M} V(\mathcal{K}_i, \partial\Omega) - V(\mathcal{K}_i, \mathcal{K}_j) > 0. \tag{13}$$

Note that (9) implies (13). In fact, we can derive from (10)–(11)

$$\min_{i=1,\dots,M} V(\mathcal{K}_i, \partial\Omega) - V(\mathcal{K}_i, \mathcal{K}_1) > 0.$$

Theorem 3.1 (Eizenberg, 1990): *Under the assumptions* (8), (13) *the solutions* u_ε *of* (12) *converge locally uniformly in* Ω *to a constant, with an exponential decay in compact subsets of* Ω, *as* ε *goes to* 0.

We will denote by ν_φ this constant value in the following. We proceed by recalling that, as output of the boundary layer analysis performed by Kamin (1978) and by Perthame (1990), we know that, when $\varphi \in C^{1,\beta}(\partial\Omega)$ for some $\beta > 0$, such solutions u_ε, besides attaining the datum φ on $\partial\Omega$, satisfies a sort of approximated Neumann condition. Namely, for $x_0 \in \Omega$,

$$\frac{\varepsilon}{2}\frac{\partial u_\varepsilon}{\partial n} + (\varphi - u_\varepsilon(x_0))b \cdot n \to 0 \quad \text{uniformly in } \partial\Omega \text{ as } \varepsilon \to 0. \tag{14}$$

Such a formula is obtained, see Perthame (1990, pp.740–743), by exploiting the boundary condition (2) and the strong maximum principle for uniformly elliptic operators. It is then natural to consider the adjoint problem associated to (12):

$$\begin{cases} -\dfrac{\varepsilon}{2}\Delta v_\varepsilon + \operatorname{div}(b\, v_\varepsilon) = 0 & x \in \Omega, \\ \dfrac{\varepsilon}{2}\dfrac{\partial v_\varepsilon}{\partial n} - b \cdot n v_\varepsilon = 0 & x \in \partial\Omega. \end{cases} \tag{15}$$

It is well established, see Bensoussan (1989, Theorem II,4.4), that (15) admits a nontrivial variational solution $v_\varepsilon \in W^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$, $1 \le p < \infty$, which is unique up to a multiplicative constant. When normalised by setting $\int_\Omega v_\varepsilon \, dx = 1$, it remains positive on the whole $\overline{\Omega}$, and can be interpreted as the density of the invariant measure associated to the diffusion (3) with reflection on the boundary. By applying Green formula, we get

$$\begin{aligned} 0 &= \int_\Omega v_\varepsilon \left(-\frac{\varepsilon}{2}\Delta u_\varepsilon - b\cdot Du_\varepsilon\right) - u_\varepsilon\left(-\frac{\varepsilon}{2}\Delta v_\varepsilon + \operatorname{div}(b\, v_\varepsilon)\right) dx \\ &= \int_{\partial\Omega} -\frac{\varepsilon}{2} v_\varepsilon \frac{\partial u_\varepsilon}{\partial n} + u_\varepsilon\left(\frac{\varepsilon}{2}\frac{\partial v_\varepsilon}{\partial n} - b\cdot n v_\varepsilon\right) d\varsigma = \int_{\partial\Omega} -\frac{\varepsilon}{2} v_\varepsilon \frac{\partial u_\varepsilon}{\partial n} d\varsigma, \end{aligned} \tag{16}$$

where ς indicates the surface measure on $\partial\Omega$. We combine (14) and (16) to find

$$\int_{\partial\Omega} (\varphi - u_\varepsilon(x_0)) b\cdot n\, v_\varepsilon \, d\varsigma \to 0,$$

and, taking into account Theorem 3.1

$$\nu_\varphi = \lim_{\varepsilon\to 0} \frac{\int_{\partial\Omega} \varphi b\cdot n v_\varepsilon \, d\varsigma}{\int_{\partial\Omega} b\cdot n v_\varepsilon \, d\varsigma}. \tag{17}$$

By letting φ vary in $C^{1,\beta}(\partial\Omega)$, we see from this last relation that the probability measures on $\partial\Omega$ appearing in the right-hand side weakly converge to a limit measure P, so that

$$\nu_\varphi = \int_{\partial\Omega} \varphi(x) dP(x). \tag{18}$$

The support of P then represents the most probable place of exit of X_ε to the boundary of Ω, for small ε, in view of the stochastic representation formula for the solutions to (12) previously recalled.

Since the v_ε are positive, we can define $V_\varepsilon := -\varepsilon \log v_\varepsilon$, for any ε. If such functions converge uniformly in $\partial\Omega$, as $\varepsilon \to 0$, to some w, we replace in (17) v_ε by $e^{-\frac{V_\varepsilon}{\varepsilon}}$, to get through standard arguments (see Friedlin and Wentzell, 1998; Perthame, 1990) that the measure P in (18) is supported by the minimisers of w in $\partial\Omega$.

On the other side, the V_ε are viscosity solutions (unique, up to an additive constant) of the viscous Hamilton-Jacobi problem

$$\begin{cases} -\frac{\varepsilon}{2}\Delta V_\varepsilon + H(x, DV_\varepsilon) = \varepsilon \operatorname{div} b & x \in \Omega \\ \frac{\partial V_\varepsilon}{\partial n} + 2b\cdot n = 0 & x \in \partial\Omega \end{cases} \tag{19}$$

where H is defined as in (4). Hence their uniform limit in $\overline{\Omega}$, provided it exists, is solution of the limit problem (5). The issue of the convergence of the sequence V_ε will be treated in Section 7. We can, for the moment, deduce its local equicontinuity in Ω. This is done by first applying the Harnack inequality to v_ε, which gives

$$\inf_{B(x,r)} v_\varepsilon \geq \exp(-C - Cr\|b\|_\infty/\varepsilon) \sup_{B(x,r)} v_\varepsilon,$$

($B(x,r)$ is the Euclidean ball centred at x, with radius r) for any ε, any $x \in \Omega$, r such that $B(x,4r) \subset \Omega$, and some constant C independent from ε, and then by deriving the estimate

$$\sup_{y \in B(x,r)} V_\varepsilon(y) \leq \inf_{y \in B(x,r)} V_\varepsilon(y) + C\varepsilon + C\|b\|_\infty r. \tag{20}$$

4 Intrinsic length and quasipotential

Here we present a metric interpretation of the quasi-potential V. We essentially use the regularity of $\partial\Omega$, note however that in this section we do not need conditions (2), (8), (9).

All the curves considered here, and throughout the paper, will be Lipschitz-continuous, and their Euclidean (natural) length will be indicated by $\ell(\cdot)$. Given such a curve ξ in $[0,T]$, for some positive T, contained in $\overline{\Omega}$, an *intrinsic length*, associated to the 0-sublevel sets $Z(\cdot) := \{p : H(\cdot,p) \leq 0\}$, where H is defined as in (4), can be defined through

$$\ell_b(\xi) = \int_0^T \sigma(\xi, \dot{\xi})\, dt, \tag{21}$$

where $\sigma(x,q) := \max_{p \in Z(x)} p \cdot q$ for any x, q, is the support function of $Z(x)$ at q. Since $Z(x)$ is the ball of radius $|b(x)|$ centred at $-b(x)$, for any x, we have

$$\sigma(x,q) = |b(x)|\,|q| - b(x) \cdot q \tag{22}$$

and

$$\ell_b(\xi) \leq 2|b|_\infty \ell(\xi). \tag{23}$$

Note that ℓ_b is not affected by change of parameter, being σ is positively homogeneous in q, moreover the intrinsic length is lower semicontinuous with respect to the uniform convergence of curves.

We consider the Hamilton-Jacobi equation

$$H(x, Du) = 0 \quad \text{in } \Omega. \tag{24}$$

In view of studying it in the framework of viscosity solutions theory, we recall some definitions.

Definition 4.1: Given an u.s.c. function w defined in $\overline{\Omega}$, $x \in \Omega$, a C^1 function ϕ is said to be (strict) *supertangent* to w at x if such point is a (strict) local maximiser for $w - \phi$ in $\overline{\Omega}$ (Note that if $x \in \partial\Omega$, then it is a constrained local maximiser).

We indicate by $D^+w(x)$ the (possibly empty) set made up by the differentials of all such C^1-supertangent.

The notion of *subtangent* for a l.s.c. function v is given by replacing maximiser by minimiser, and $D^-v(x)$ is defined as the set of the differentials of all subtangents at a point x.

See Barles (1994), Bardi and Capuzzo-Dolcetta (1997), Koike (2004) for other related definitions, as well as for a general treatment of viscosity solutions theory.

Since the Hamiltonian is convex and coercive, the notions of viscosity subsolution and a.e. locally Lipschitz-continuous subsolution for (24) coincide, see Bardi and Capuzzo-Dolcetta (1997); such a function, say u, is also characterised by the property that $H(x, p) \leq 0$ for any $x \in \Omega$, $p \in \partial u(x)$, where ∂u indicates the (Clarke) generalised gradient of u. It is then readily seen that

$$u(x) - u(y) \leq \ell_b(\xi) \quad \text{for any } x, y \text{ in } \Omega, \text{ any } \xi \text{ contained in } \Omega \text{ and joining } y \text{ to } x.$$

This implies, in view of (6), (23), that any subsolution is Lipschitz-continuous in Ω and can be consequently extended by continuity in $\overline{\Omega}$. Unless otherwise specified, we always think a subsolution to (24) as defined in $\overline{\Omega}$.

The *geodetic distance* associated to ℓ_b is defined in $\Omega \times \Omega$ as the infimum of the intrinsic length of the curves, contained in Ω, joining a given pair of points. It is, more precisely, a nonnegative nonsymmetric semidistance.

Due to the regularity condition on $\partial\Omega$, such a notion can be equivalently given by taking the connecting curves lying in $\overline{\Omega}$, instead of Ω. Notice that for a general domain, on the contrary, one obtains, in this way, a new distance which can be, for some pairs of points, strictly less than the previous one. This property will be used for proving that quasipotential and intrinsic distance coincide. Since we did not find a precise reference in the literature, we supply a proof of it in Appendix A.

Proposition 4.2: *The quasipotential and the intrinsic distance coincide. More precisely:*

$$V(y, x) = \inf\{\ell_b(\xi) : \xi \textit{ joining } y \textit{ to } x \textit{ and contained in } \overline{\Omega}\},$$

For any y, x in Ω.

Proof: Taking into account Lemma A.1, we see that the quasi-potential V is greater than or equal to the intrinsic distance, since $\sigma(x, q) \leq L(x, q)$, for any x q.

The argument to get the converse inequality is somehow complicated by the fact that b can vanish in Ω, this is the reason why we introduce an arbitrarily small ε.

We consider a curve $\eta(t)$, parametrised by the arc-length and contained in $\overline{\Omega}$, joining two given points y, x of $\overline{\Omega}$, and $\varepsilon > 0$. We define a new parameter $s = \Lambda(t)$, where Λ is the antiderivative of

$$\frac{1}{|b(\eta)| + \varepsilon + \sqrt{\varepsilon^2 + 2\varepsilon|b(\eta)|}},$$

vanishing at $t = 0$. The curve $\xi(s) = \eta(\Lambda^{-1}(s))$, defined in an interval $[0, T]$, for some positive T, satisfies

$$|\dot{\xi}(s)| = \left|\dot{\eta}(\Lambda^{-1}(s))\frac{d}{ds}\Lambda^{-1}(s)\right| = |b(\xi(s))| + \varepsilon + \sqrt{\varepsilon^2 + 2\varepsilon|b(\xi(s))|},$$

for a.e. s. We consequently have, taking into account (22)

$$\begin{aligned}\sigma(\xi(s), \dot{\xi}(s)) + \varepsilon|\dot{\xi}(s)| = (|b(\xi(s))| + \varepsilon)(|b(\xi(s))| + \varepsilon + \sqrt{\varepsilon^2 + 2\varepsilon|b(\xi(s))|}) \\ - b(\xi(s)) \cdot \dot{\xi}(s) = L(\xi(s), \dot{\xi}(s)),\end{aligned}$$

for a.e. s, and find by integrating

$$\ell_b(\eta) + \varepsilon\ell(\eta) = \ell_b(\xi) + \varepsilon\ell(\xi) = \int_0^T L(\xi, \dot{\xi})dt \geq V(y, x),$$

which concludes the proof in view of Lemma A.1. □

Thanks to the previous result we know, see Fathi and Siconolfi (2005), Siconolfi (2006), that the functions $x \mapsto V(y, x)$, for $y \in \overline{\Omega}$, constitute a class of fundamental solution to (24), they more precisely are viscosity solutions in $\Omega\backslash\{y\}$ and subsolutions in the whole Ω. They will appear in representation formulae for solutions of (24) satisfying suitable boundary conditions, that will be in use in Sections 6 and 7. The following characterisation holds:

Proposition 4.3: *A function u is a subsolution to* (24) *if and only if*

$$-V(x, y) \leq u(x) - u(y) \leq V(y, x) \quad \text{for any } x, y \text{ in } \overline{\Omega}. \tag{25}$$

In particular any subsolution to (24) *is Lipschitz continuous in* $\overline{\Omega}$.

Proof: See Fathi and Siconolfi (2005). □

5 Aubry set and ω-limits

In this section we assume (2).

Definition 5.1: A subsolution u of (24) is called *strict* in a open subset Ω' of Ω if $H(x, Du(x)) \leq -\delta$ for some $\delta > 0$ and a.e. $x \in \Omega'$, or equivalently $H(x, Du) \leq -\delta$ in the viscosity sense in Ω'.

We have already pointed out in Section 1 the importance of strict subsolutions in Ω to get comparison principle for boundary problems involving (24). Unfortunately we have:

Lemma 5.2: *The equation* (24) *does not admit any strict subsolution in* Ω.

Proof: Given $y \in \partial\Omega$, we see by (2) that there is $x \in \Omega$ with $V(y, x) = 0$. Proposition 4.3 then implies that $u(x) \leq u(y)$ for any subsolution u to (24). We deduce that a minimiser x_0 of u must lie in Ω. The relations $0 \in D^- u(x_0)$, $H(x_0, 0) = 0$ thus tell us that u cannot be a strict subsolution of (24). □

Hamilton-Jacobi equations which admit subsolutions, but not strict ones, are called critical in Fathi and Siconolfi (2005), where they are studied in the periodic setting. Following the approach of Fathi and Siconolfi (2005), we show that a crucial role for the analysis of (24) is played by a subset of Ω named after Aubry and denoted by $\mathcal{A}$. It acts, in a sense, as an hidden boundary on which a datum must be fixed in order to obtain a unique solution of (24) coupled with suitable Neumann boundary conditions. We postpone to the end of the section the statement of the crucial proposition from which such a property stems. We, instead, start by defining $\mathcal{A}$ and putting it in relation with the dynamics (1).

Definition 5.3: The Aubry set is the set of points $y \in \overline{\Omega}$ such that there is a sequence of cycles, ξ_n, passing through y and contained in $\overline{\Omega}$, which satisfy

$$\lim_n \ell_b(\xi_n) = 0, \quad \inf_n \ell(\xi_n) > 0.$$

It directly comes from the definition and (23) that $\mathcal{A}$ is a closed subset of $\overline{\Omega}$. Indeed let $y^k \in \mathcal{A}$ be a sequence converging to y. Then for every y^k there is a sequence of cycles ξ_n^k satisfying the previous condition. Then we can define a sequence of cycles ξ_n passing through y by juxtaposition of ξ_n^n and two segments between y to y^n. It is immediate to check that $\lim_n \ell_b(\xi_n^n) = 0$ and $\inf_n \ell(\xi_n^n) > 0$, and then by definition $y \in \mathcal{A}$. We deduce from condition (2):

Proposition 5.4: *$\mathcal{A}$ is contained in Ω.*

We need a preliminary result, which is a partial converse of the trivial fact that any integral curve of b has zero intrinsic length.

Proposition 5.5: *Let ξ a curve defined in some open interval I, contained in $\overline{\Omega}$ with positive natural length and zero intrinsic length, which satisfies $\xi \cap \mathcal{E} = \emptyset$. Then ξ is an integral curve of b, up to change of parameter.*

Proof: The curve ξ can be assumed, without loss of generality, to be parametrised by the arc-length. From (22), $\xi \cap \mathcal{E} = \emptyset$ and $\ell_b(\xi) = 0$ we derive

$$\dot{\xi}(t) = \frac{b(\xi(t))}{|b(\xi(t))|} \quad \text{for a.e. } t \in I, \tag{26}$$

hence ξ is of class C^1 and the previous equality holds for every t. The required parametrisation is then $\sigma = \Lambda(t)$, where Λ is an antiderivative of $\frac{1}{|b(\xi)|}$ in I. Taking into account (26), we in fact find that the curve $\zeta(s) = \xi(\Lambda^{-1}(s))$ satisfies

$$\dot{\zeta}(s) = \dot{\xi}(\Lambda^{-1}(s)) \frac{d}{ds} \Lambda^{-1}(s) = b(\zeta(s)) \quad \text{for any } s.$$

□

Proof of Proposition 5.4: Assume, for purposes of contradiction, that there is an $x_0 \in \mathcal{A} \cap \partial\Omega$, then $x_0 \notin \mathcal{E}$, thanks to (2). We can find a sequence of cycles ξ_n contained in $\overline{\Omega}$, parametrised by the arc length, with

$$\ell(\xi_n) \geq 2\delta, \quad \xi_n(\delta) = x_0 \quad \text{for any } n \text{ and some } \delta > 0,$$
$$\lim_n \ell_b(\xi_n) = 0.$$

Given $\varepsilon \in (0, \min\{\delta/2, d(x_0, \mathcal{E})/2\})$, we apply Ascoli theorem to see that the ξ_n uniformly converge in $I := (\delta - \varepsilon, \delta + \varepsilon)$, up to a subsequence, to some curve ξ, contained in $\overline{\Omega}$. We, in addition, have that $\inf_I d(\xi(t), \mathcal{E}) > 0$ and $\ell_b(\xi) = 0$, because of the lower semicontinuity of ℓ_b. Hence ξ is an integral curve of the dynamics, up to change of parameter, in the light of Proposition 5.5. Since $\xi(\delta) = x_0 \in \partial\Omega$, δ is a maximiser of $d^{\#}(\xi(\cdot), \Omega)$ in I, and so

$$\frac{d}{dt} d^{\#}(\xi(t), \Omega)\big|_{t=\delta} = 0. \tag{27}$$

We thus deduce that $n(x_0) \cdot b(x_0) = 0$, in contrast with (2). □

The next proposition shows, among other things, that the Aubry set is nonempty.

Proposition 5.6: *$\mathcal{A}$ contains Ω_b.*

Proof: Assume first x_0 to be an equilibrium of b, so that $Z(x_0)$ reduces to $\{0\}$. We therefore find, by the continuity of H, for any given $\varepsilon > 0$

$$Z(\cdot) \subseteq \{p \in \mathbb{R}^N : |p| \le \varepsilon\} \quad \text{in a suitable neighbourhood } U_\varepsilon \text{ of } x_0.$$

This yields $\sigma(x, q) \le \varepsilon |q|$ for any $x \in U_\varepsilon$, $q \in \mathbb{R}^N$, which, in turn, implies that any cycle ξ passing through x_0, contained in U_ε and of natural length in between 1 and 2, satisfies

$$\ell_b(\xi) \le \varepsilon \ell(\xi) \le 2\varepsilon.$$

Since ε has been arbitrarily fixed, we conclude that $x_0 \in \mathcal{A}$.

Let now x_0 be an ω-limit point not in $\mathcal{E}$, then $x_0 = \lim_n \gamma(t_n)$ for some nonconstant integral trajectory γ of b, and a positively diverging sequence t_n. When t varies in any right-unbounded interval, the curve γ has infinite natural length, since otherwise it should have an equilibrium as unique ω-limit point, and vanishing intrinsic length, by the very definition of ℓ_b.

Taking into account these properties and (22), we can select, for any $\varepsilon > 0$, some indices m, k, with $m > k$, such that the cycle ξ_ε obtained by juxtaposition of the segment joining x_0 to $\gamma(t_k)$, the portion of γ between t_k and t_m, and the segment joining $\gamma(t_m)$ to x_0, satisfies $\ell(\xi_\varepsilon) \ge 1$, $\ell_b(\xi_\varepsilon) < \varepsilon$. This again shows that x_0 is in $\mathcal{A}$. □

Remark 5.7: The limit set Ω_b is in general strictly contained in $\mathcal{A}$. Take for instance a vector field $\tilde{b}$ possessing a closed integral curve γ with the property that no other trajectory of $\tilde{b}$ has ω-limit set intersecting γ.

Set $b = \psi \tilde{b}$, with ψ smooth nonnegative function with two zeroes on γ. Accordingly, γ comprises two equilibria of b and two heteroclinic trajectories connecting them.

It is easy to check that all the points of γ belong to $\mathcal{A}$ but only the two equilibria are in Ω_b.

Proposition 5.8: *Let $\mathcal{K}$ be a repulsor for b, then there exists a neighbourhood $\widetilde{\Omega}$ of $\mathcal{K}$ with $\mathcal{A} \cap \widetilde{\Omega} \subset \mathcal{K}$.*

Proof: Let $\widetilde{\Omega}$ be a neighbourhood of $\mathcal{K}$ contained in its repulsion basin and assume by contradiction that there is an $x_0 \in \mathcal{A} \cap \widetilde{\Omega}$ such that $d(x_0, \mathcal{K}) > 0$.

Using the fact that $x_0 \in \mathcal{A}$, we choose ξ_n a sequence of cycles contained in $\overline{\Omega}$, parametrised by the arc-length with periods T_n, such that $\inf_n T_n > 0$, $\xi_n(0) = x_0$, for any n, and $\lim_n \ell_b\big(\xi_n\big|_{[0,T_n]}\big) = 0$. By extending it by periodicity, we can assume the ξ_n to be defined in $(-\infty, +\infty)$. The sequence ξ_n converges locally uniformly in $(-\infty, +\infty)$, up to a subsequence, to a curve $\xi \subset \overline{\Omega}$, by Ascoli Theorem. Since the periods T_n are not infinitesimal, we see that the number of loops performed by the ξ_n in a given finite time is equibounded, from which we derive, taking into account the lower semicontinuity property of the intrinsic length

$$0 = \lim \ell_b(\xi_n\big|_I) = \ell_b(\xi\big|_I) \quad \text{for any compact interval } I,$$

and so

$$\ell_b(\xi) = 0.$$

We claim that

$$\inf_{(-\infty,0]} d(\xi(t), \mathcal{K}) = 0. \tag{28}$$

To show this, we set $-T = \sup\{t \in (-\infty, 0] : \xi(t) \in \mathcal{E}\}$. Note that such quantity is strictly negative, and possibly infinite if the set appearing in the previous formula were empty. We have that $\xi\big|_{(-T,0)}$ is an integral curve of b, up to change of parameter, by Proposition 5.5, and the limit points, for $t \longrightarrow T^+$, of ξ are α-limits of the reparametrised curve. This shows the claim (28), since x_0 belongs to the repulsion basin of $\mathcal{K}$.

We fix now $\rho = d(x_0, \mathcal{K})/2 > 0$. Then, by the instability condition (say, the stability condition in Definition 2.1 for the backward dynamics), that there is a $0 < \delta \le d(x_0, \mathcal{K})/2$ such that

$$d(\zeta(0), \mathcal{K}) \le \delta \quad \text{implies } \zeta(t) \in \widetilde{\Omega}, \quad d(\zeta(t), \mathcal{K}) \le d(x_0, \mathcal{K})/2 \tag{29}$$

for any solution ζ of (1) and any $t < 0$. By (28) the sets

$$I_n := \{t \in [-T_n, 0] : d(\xi_n(t), \mathcal{K}) \le \delta\}$$

are nonempty, for n sufficiently large. We define for such n

$$-T'_n = \min I_n, \quad \gamma_n = \xi_n(\cdot - T'_n).$$

We claim that, even if the periods T_n can be in general unbounded, the sequence $T_n - T'_n$ is, on the contrary, always bounded and is convergent, up to a subsequence, to some $T > 0$. We invoke Ascoli Theorem to see that γ_n converges locally uniformly in $(-\infty, +\infty)$, up to a subsequence, to some curve γ. We moreover have, arguing as in the first part of the proof, that $\ell_b(\gamma) = 0$ and $\inf_{(-\infty,0]} d(\gamma(t), \mathcal{K}) = 0$. We can find, by the last relation, a $t_1 > 0$ with $d(\gamma(-t_1), \mathcal{K}) < \delta/2$, and consequently $d(\gamma_n(-t_1), \mathcal{K}) < \delta$, for n sufficiently large. We infer from the very definition of T'_n that $-t_1 - T'_n < -T_n$, for n large enough. This implies the announced boundedness of $(T_n - T'_n)$, which is then convergent, up to a subsequence, to some $T \geq 0$. Now observe that $T > 0$. Indeed if $T_n - T'_n \to 0$ along a subsequence, we would get

$$\delta \geq d(\xi_n(T'_n), \mathcal{K}) = d(x_0, \mathcal{K}) + d(\xi_n(T'_n), \mathcal{K}) - d(\xi_n(T_n), \mathcal{K}) \to d(x_0, \mathcal{K})$$

in contrast with the fact that $\delta \leq d(x_0, \mathcal{K})/2$.

We have

$$\gamma(-T) = \lim_n \gamma_n(-T_n + T'_n) = \lim_n \xi_n(-T_n) = x_0,$$

and, in addition

$$d(\gamma(t), \mathcal{K}) = \lim_n d(\gamma_n(t), \mathcal{K}) = \lim_n d(\xi_n(t - T'_n), \mathcal{K}) \geq \delta \quad \text{for } t \in (-T, 0),$$

thanks to the definition of T'_n and the fact that $t - T'_n \geq -T_n$, when n is sufficiently large.

Since $\gamma(t) \in \widetilde{\Omega} \backslash \mathcal{K}$, for $t \in [-T, 0]$ by the previous relation, and so $\gamma\big|_{[-T,0]} \cap \mathcal{E} = \emptyset$, we deduce from Proposition 5.5 that $\gamma|_{[-T,0]}$ is an integral curve of the dynamics, up to change of parameter. The relations $d(\gamma(0), \mathcal{K}) = \delta$, $\gamma(T) = x_0$ are then in contrast with (29).

Proposition 5.9:

(i) *The Aubry set is forward invariant for the dynamical system* (1)*,*

(ii) *Given $x_0 \in \mathcal{A}$, the integral curve η of b, starting at x_0 and defined in $[0, +\infty)$, satisfies $V(\eta(t), \eta(s)) = 0$, for any t, s in $[0, +\infty)$.*

The interesting point in item (ii) is that $V(\eta(t), \eta(s)) = 0$ holds true for $t > s$, when $t < s$, this identity immediately comes from the definition of quasi-potential.

Proof: If x_0 is an equilibrium, there is nothing to prove, assume then x_0 to be in $\mathcal{A} \backslash \mathcal{E}$. We can find a sequence of cycles ξ_n parametrised by the arc-length, with $T_n := \ell(\xi_n)$ bounded by below by a positive constant, such that

$$\xi_n(0) = \xi_n(T_n) = x_0 \quad \text{for any } n, \quad \lim_n \ell_b(\xi_n) = 0. \tag{30}$$

For $0 < T < \inf_n T_n$, the ξ_n converge uniformly in $[0, T]$, up to a subsequence, to a curve ξ thanks to Ascoli Theorem, and, by the lower semicontinuity property of the intrinsic length

$$0 = \lim \ell_b(\xi_n\big|_{[0,T]}) = \ell_b(\xi). \tag{31}$$

By taking T sufficiently small, we also have

$$\min_{[0,T]} d(\xi(t), \mathcal{E}) > 0,$$

and deduce from Proposition 5.5 that ξ is the integral curve of b starting at x_0, restricted to $[0, T]$, up to change of parameter.

Thanks to (30), (31) and the uniform convergence of ξ_n to ξ, we can moreover choose, for any positive ε, the index n in such a way that the cycle γ obtained by juxtaposition of ξ, the segment joining $\xi(T)$ to $\xi_n(T)$ and $\xi_n\big|_{[T,T_n]}$, satisfies $\ell_b(\gamma) < \varepsilon$ and, clearly, $\ell(\gamma) \geq \ell(\xi)$.

This implies that the whole ξ is contained in $\mathcal{A}$, and, in addition, $V(\xi(t), \xi(s)) \leq \ell_b(\gamma) < \varepsilon$, we consequently have

$$V(\xi(t), \xi(s)) = 0 \quad \text{for any } t, s \text{ in } [0, T]. \tag{32}$$

The property that any integral curve of b starting in $\mathcal{A}$ remains in $\mathcal{A}$ for some open interval with left endpoint 0, previously found out, directly gives the asserted forward invariance of the Aubry set, taking into account that it is compact.

We denote by I the maximal closed interval with left endpoint 0 for which item (ii) holds, note that by (32) it does not reduce to a point. We assume by contradiction that the right endpoint of I, say S_0, is finite. Arguing as above, we see that

$$V(\eta(t), \eta(s)) = 0 \quad \text{for any } t, s \text{ in } [S_0, S_1],$$

for some $S_1 > S_0$. For $t \in [S_0, S_1]$, $s \in [0, S_0]$, we find

$$0 \leq V(\eta(t), \eta(s)) \leq V(\eta(t), \eta(S_0)) + V(\eta(S_0), \eta(s)) = 0,$$

which shows that (ii) holds in $[0, S_1]$, in contrast with the maximality of I. □

The item (ii) of the previous result, combined with Proposition 4.3, immediately gives:

Corollary 5.1: *Any subsolution of* (24) *is constant on every integral curve of* b *contained in* $\mathcal{A}$.

As already announced, we proceed to state the main property of $\mathcal{A}$. The intuitive meaning of the assertion is that the obstruction to the existence of strict subsolutions to (24), see Lemma 5.2, is concentrated in the Aubry set.

Proposition 5.10: *There exists a subsolution to* (24) *which is strict in any open subset of* Ω *whose closure is disjoint from* $\mathcal{A}$.

See Fathi and Siconolfi (2005), Proposition 6.1, we sketch the proof for reader's convenience.

Proof: We consider $y \in \Omega \backslash \mathcal{A}$, then $y \notin \mathcal{E}$ by Proposition 5.6. We can therefore take p_0 in the interior of $Z(y)$. We have

$$p_0 \in \operatorname{int} Z(x) \quad \text{for } x \text{ in a suitable neighbourhood } U \text{ of } y, \tag{33}$$

thanks to the continuous character of the set-valued map Z, equivalently $H(x, p_0) < 0$ for $x \in U$. We know, see Fathi and Siconolfi (2005), Lemma 5.5, that there is a neighbourhood $\overline{U} \subset U$ of y such that, for any $x \in \overline{U}$, $V(y, x)$ can be obtained as the infimum of the intrinsic length of the curves joining y to x and contained in U.

The function $x \mapsto p_0(x-y)$ is therefore a strict subtangent to $V(y,\cdot)$ at y, in view of (33), so that the definition

$$u_y = \begin{cases} \max\{p_0(\cdot - y) + \varepsilon_y, V(y,\cdot)\} & \text{for } x \in B_y \\ V(y,\cdot) & \text{otherwise} \end{cases} \tag{34}$$

provides a subsolution to (24), satisfying $H(x, Du_y) \le -\delta_y$ in B_y, for a suitable choice of the positive constants ε_y, δ_y, and of the Euclidean ball B_y centred at y.

When y varies in $\Omega \setminus \mathcal{A}$, the family B_y make up an open cover of this set, from which a countable locally finite subcover B_{y_i}, $i \in \mathbb{N}$, can be extracted.

The functions $u_i := u_{y_i}$, $i \in \mathbb{N}$, are equibounded, up to the addition of suitable constants, and equiLipschitz-continuous, being subsolutions to (24). As a consequence, the series $\sum_i \lambda_i u_i$, $\sum_i \lambda_i Du_i$, where λ_i is any sequence of positive constants satisfying $\sum_i \lambda_i = 1$, are convergent in $L^\infty(\Omega)$, and the limit $u := \sum_i \lambda_i u_i$ is a subsolution to (24) with $Du = \sum_i \lambda_i Du_i$.

If Ω' is an open subset of Ω with closure disjoint from $\mathcal{A}$, we can select a finite family $B_{y_{i_k}}$, $k = 1, \dots, m$, for some $m \in \mathbb{N}$, covering $\overline{\Omega'}$. We exploit the convex character of H to get

$$H(x, Du(x)) \le \sum_{k=1}^{m} \lambda_{i_k} H(x, Du_{i_k}(x)) \le -\delta,$$

where $\delta = \min_k \lambda_{i_k} \delta_{y_{i_k}}$. This proves the assertion. □

It has also been proved in Fathi and Siconolfi (2005) that regular subsolution to (24) can be obtained by smoothing up outside $\mathcal{A}$ the strict subsolutions, whose existence has been proved in the previous Proposition. We will, in particular, use in Section 7 the following result:

Proposition 5.11: *There exists a subsolution to* (24) *which is strict in any open subset of* Ω *whose closure is disjoint from* $\mathcal{A}$, *and of class* C^∞ *in* $\Omega \setminus \mathcal{A}$.

6 Comparison principles

In this section we solely assume the condition (2). We study the problem (5), made up by the equation (24) coupled with the Neumann boundary condition

$$(Du(x) + 2b(x)) \cdot n(x) = 0 \qquad x \in \partial\Omega. \tag{35}$$

We record for later use that the left-hand side of the previous formula can be equivalently rewritten as

$$H_p(x, Du(x)) \cdot n(x) + b(x) \cdot n(x).$$

The relation in (35) must be understood in the viscosity weak sense. Namely an u.s.c. (resp. l.s.c.) function u is said to be a subsolution (resp. supersolution) of (24), (35), if it is a standard viscosity subsolution (resp. supersolution) to (24) in Ω, and, in addition

$$\min\{H(x,p),(p+2b(x))\cdot n(x)\}\le 0 \quad \text{for } x\in\partial\Omega, p\in D^+u(x)$$
$$\max\{H(x,p),(p+2b(x))\cdot n(x)\}\ge 0 \quad \text{for } x\in\partial\Omega, p\in D^-u(x).$$

We also consider the state constraint boundary condition

$$H(x,Du(x))\ge 0 \qquad x\in\partial\Omega. \tag{36}$$

We recall that a solution v to (24), (36) is required to be a subsolution in Ω, and to satisfy the supersolution condition up to the boundary, i.e.,

$$H(x,p)\ge 0 \quad \text{for } x\in\overline{\Omega}, p\in D^-v(x).$$

As already pointed out in the Introduction, the problem (24), (35) does not admit, in general, unique solution, even if we fix the value at some point of Ω. We exhibit a one-dimensional example of multiplicity of solutions.

Example 6.1: Let $\overline{\Omega}$ be the interval $[-2,2]$, $U(x)=(x^2-1)^2$, and define $b(x)=-U'(x)$. The Aubry set in this case is $\{-1,0,1\}$, and coincides with the set of equilibria of b. It is easy to check that $U(x)$ and

$$w(x)=\begin{cases}0 & x\in[-1,1]\\ U(x) & x\in\overline{\Omega}\backslash[-1,1]\end{cases}$$

are two C^1 solutions of (24), (35) with $U=w$ at -1, 1. Note, however, that they are different at $0\in\mathcal{A}$.

We plan to show that a solution of the Neumann problem is uniquely selected if an admissible trace is prescribed on $\mathcal{A}$ or Ω_b. We start by investigating the comparison properties of sub/supersolutions, when they are assumed to satisfy some inequality on the Aubry set, and then replace $\mathcal{A}$ by Ω_b in the result obtained, by means of Proposition 5.9 and Corollary 5.1.

We more precisely aim to prove the following two main facts:

Theorem 6.2: *Let v, u be a l.s.c. super and an u.s.c. subsolution to* (24), (35), *respectively. If $u\le v$ on $\mathcal{A}$ then $u\le v$ in $\overline{\Omega}$.*

Theorem 6.3:

(*i*) *Let u and v be an u.s.c. subsolution and a solution to* (24), (35), *respectively, with $u\le v$ on Ω_b, then $u\le v$ in $\overline{\Omega}$,*

(*ii*) *Given a function g on Ω_b with $g(x)-g(y)\le V(y,x)$, there exists an unique solution w to* (24), (35) *equaling g on Ω_b, and it is given by*

$$w(x)=\min\{g(y)+V(y,x): y\in\Omega_b\}, \quad x\in\overline{\Omega}. \tag{37}$$

We first discuss the equivalence of problems (24), (35) and (24), (36). For this we quote from Lions (1985) the inequality

$$(y-x)\cdot n(y) \geq -C_0|y-x|^2 \quad \text{for any } y \in \partial\Omega, x \in \overline{\Omega}, \text{ and some } C_0 > 0. \quad (38)$$

and the following result:

Proposition 6.4: *Let* u *be a subsolution (resp. a l.s.c. supersolution) of* $H(x, Du) = a$, *for some* $a \in \mathbb{R}$, *in a neighbourhood of* $\partial\Omega$ *intersected with* Ω. *Take* $x_0 \in \partial\Omega$, *and* $p_0 \in D^+u(x_0)$ *(resp.* $p_0 \in D^-u(x_0)$*). If*

$$\lambda_0 := \sup\{\lambda \,|\, p_0 + \lambda n(x_0) \in D^+u(x_0)\} \quad (39)$$

$$(\textit{resp. } \lambda_0 := \inf\{\lambda \,|\, p_0 + \lambda n(x_0) \in D^-u(x_0)\}) \quad (40)$$

is finite then $H(x_0, p_0 + \lambda_0 n(x_0)) \leq a$ *(resp.* $\geq a$*).*

Proposition 6.5: *Any subsolution* u *to* (24) *is also subsolution of Neumann problem.*

Proof: Recall that by Proposition 4.3, every subsolution u to (24) is Lipschitz-continuous up to the boundary. Assume now by contradiction that u does not satisfy the Neumann boundary condition in viscosity sense. This means that, for some $x_0 \in \partial\Omega$ and $p_0 \in D^+u(x_0)$, both $H(x_0, p_0)$ and $(H_p(x_0, p_0) + b(x_0)) \cdot n(x_0)$ are strictly positive. Then, by (2), $H_p(x_0, p_0) \cdot n(x_0) > 0$ and so

$$H(x_0, p_0 + \lambda n(x_0)) > 0 \quad \text{for any } \lambda \geq 0, \quad (41)$$

because the function $\lambda \mapsto H(x_0, p_0 + \lambda n(x_0))$ is convex. Given any $p \in D^+u(x_0)$, we take into account that u is Lipschitz-continuous and $x_0 - \lambda n(x_0) \in \Omega$ for $\lambda > 0$ suitably small, to find

$$p \cdot n(x_0) \leq \liminf_{\lambda \to 0^+} \frac{u(x_0) - u(x_0 - \lambda n(x_0))}{\lambda} \leq |Du|_\infty.$$

We deduce that the quantity $\lambda_0 > 0$, defined as in (39), is finite, and so

$$H(x_0, p_0 + \lambda_0 n(x_0)) \leq 0,$$

by Proposition 6.4. This contradicts (41). □

Proposition 6.6: *A l.s.c. function* v *is a supersolution to* (24), (36) *if and only if it is a supersolution to* (24), (35).

Proof: If v is a supersolution of the state constraint problem, then it is obviously also a supersolution of the Neumann problem. To show the converse, we assume by contradiction that there is $x \in \partial\Omega$ and $p \in D^-v(x)$ with $H(x, p) < 0$. We then have by (2)

$$H_p(x, p) \cdot n(x) > 0. \quad (42)$$

We define the convex function $h(\lambda) := H(x_0, p + \lambda n(x_0))$, and set

$$\lambda_0 = \inf\{\lambda \,|\, p + \lambda n(x) \in D^- v(x)\}$$
$$\lambda_1 = \min\{\lambda \,|\, \lambda \in \arg\min\{h\}\},$$

note that the latter is well defined thanks to the coercivity of H, and is negative because of (42). By the convex character of h, $h(\lambda) < 0$ for $\lambda \in [\lambda_1, 0]$, and so $\lambda_0 < \lambda_1$ by Proposition 6.4. We can consequently select a sequence $\mu_k < \lambda_1$, converging to λ_1, with

$$p + \mu_k n(x) \in D^- v(x) \quad \text{and} \quad H_p(x, p + \mu_k n(x)) \cdot n(x) + b(x) \cdot n(x) < h'(\mu_k) < 0.$$

Taking into account that v is a supersolution to (24), (35), we deduce $h(\mu_k) \geq 0$ and, passing to the limit, $h(\lambda_1) = H(x, p + \lambda_1 n(x)) \geq 0$, which is impossible by the very definition of λ_1 and the inequality $H(x, p) < 0$. □

Exploiting Propositions 6.5, 6.6, and Proposition 5.10, we establish the first comparison principle for sub/supersolution of problem (24), (35). The proof relies on traditional viscosity solutions theory techniques.

Proposition 6.7: *Let v, u be a l.s.c. supersolution* (24), (35)*, and a subsolution to* (24)*, respectively. If $u \leq v$ on $\mathcal{A}$ then $u \leq v$ in $\overline{\Omega}$.*

Proof: We assume by contradiction

$$\arg\max_{\overline{\Omega}}\{u - v\} \subset \Omega', \tag{43}$$

where Ω' is an open set in $\overline{\Omega}$ possibly containing $\partial\Omega$, and with closure disjoint from $\mathcal{A}$. Recall that $\mathcal{A}$ is contained in Ω by Proposition 5.4. Thanks to Proposition 5.10, there exists a subsolution $\overline{u}$ to (24), which is strict in $\Omega' \setminus \partial\Omega$. These properties are inherited by the function $w := \lambda\overline{u} + (1 - \lambda)u$, for $\lambda \in (0, 1)$, thanks to the convex character of H. Moreover, by Proposition 4.3, w is Lipschitz-continuous in $\overline{\Omega}$. We can furthermore take λ so small that (43) still holds with w in place of u. To fix our ideas, we assume

$$H(x, Dw) < -\delta \quad \text{in } \Omega' \backslash \partial\Omega, \quad \text{for a positive } \delta.$$

This implies, arguing as in Proposition 6.5

$$\min\{H(x, p) + \delta, (p + 2b(x)) \cdot n(x)\} \leq 0 \quad \text{for } x \in \partial\Omega, p \in D^+ w(x). \tag{44}$$

We define for any $\rho > 0$, $x \in \overline{\Omega}$

$$w_\rho(x) = \max_{\overline{\Omega}}\left\{w(y) - \frac{1}{2\rho}|x - y|^2 + \alpha(d(x) - d(y))\right\},$$

where d is a C^1 function defined in $\mathbb{R}^N$, coinciding with $d^{\#}(\cdot, \Omega)$ in some neighbourhood of $\partial\Omega$, and α is a positive constant. We call any y realising the max in the above formula ρ-conjugate to x. We first aim to show that the inequality

$$\frac{|x-y|}{\rho} \le C(\alpha) \tag{45}$$

holds for any $x \in \overline{\Omega}$, any y ρ-conjugate to x, and some constant $C(\alpha)$ independent of x, y, ρ, and linearly dependent on α.

This can be actually derived as a consequence of the relation

$$\frac{y-x}{\rho} + \alpha Dd(y) \in D^+ w(y) \tag{46}$$

when $y \in \Omega$, since in this case $D^+ w(y) \subset \partial w(y)$, where ∂w is the generalised gradient of w. If instead $y \in \partial\Omega$, then $Dd(y) = n(y)$, and

$$p := \frac{y-x}{\rho} + (\alpha + \lambda) n(y) \in \partial w(y) \tag{47}$$

for some $\lambda \ge 0$. We thus exploit the estimate

$$\frac{|x-y|^2}{2\rho} \le (|Dw|_\infty + \alpha |Dd|_{\infty,\Omega}) l_\Omega \operatorname{diam} \Omega, \tag{48}$$

which comes from $w_\rho \ge w$ (diam stands for diameter and the constant l_Ω is defined in (6)), and (38) to get

$$\begin{aligned} |Dw|_\infty \ge p \cdot n(y) &\ge -C_0/\rho |x-y|^2 + \alpha + \lambda \\ &\ge -2C_0(|Dw|_\infty + \alpha |Dd|_{\infty,\Omega}) l_\Omega \operatorname{diam} \Omega + \alpha + \lambda, \end{aligned}$$

where C_0 is the constant appearing in (38). This shows that λ can be bounded by a constant independent of x, y, ρ, and linearly dependent on α. Such a property, together with (47), proves (45).

We moreover have

$$\begin{aligned} \left(\frac{y-x}{\rho} + \alpha n(y) + 2b(y) \right) \cdot n(y) &\ge -\frac{C_0}{\rho} |y-x|^2 + \alpha + 2b(y) \cdot n(y) \\ &\ge -C_0 \rho C(\alpha)^2 + \alpha + 2b(y) \cdot n(y) \end{aligned} \tag{49}$$

so that for $\alpha > 2 \sup_{\partial\Omega}(-b(y) \cdot n(y))$ and $\rho < (\alpha + 2 \inf_{\partial\Omega}(b(y) \cdot n(y)))(C_0 C(\alpha)^2)$, the quantity in the right hand-side is positive for any $x \in \overline{\Omega}$, $y \in \partial\Omega$ ρ-conjugate to x. Notice that this cannot be proved through the estimate (48), because of the presence of α in its right-hand side. This is a crucial point in the proof, and explains why the term containing α and $d(\cdot)$ appears in the formula defining w_ρ.

Let now x_ρ, y_ρ be, for any ρ, a maximiser of $w_\rho - v$ in $\overline{\Omega}$ and a ρ-conjugate to x_ρ, respectively. Since w is the uniform limit of w_ρ in $\overline{\Omega}$, when ρ goes to 0, by (45), the x_ρ, y_ρ both converge, up to a subsequence, to the same maximiser of $w - v$, and so x_ρ, y_ρ are in Ω', for ρ small enough.

By the very definition of w_ρ, the function $x \mapsto -\frac{1}{2\rho}|y_\rho - x|^2 + \alpha d(x)$ is subtangent to v at x_ρ. Therefore

$$\frac{1}{\rho}(y_\rho - x_\rho) + \alpha Dd(x_\rho) \in D^- v(x_\rho)$$

and

$$H\left(x_\rho, \frac{1}{\rho}(y_\rho - x_\rho) + \alpha Dd(x_\rho)\right) \geq 0, \tag{50}$$

by Proposition 6.6. Thanks to (46), (49) and (44), we get for y_ρ on the boundary as well as in the interior of Ω

$$H\left(y_\rho, \frac{1}{\rho}(y_\rho - x_\rho) + \alpha Dd(y_\rho)\right) \leq -\delta, \tag{51}$$

and we consequently reach a contradiction with (50), as ρ goes to 0, taking into account (45) and the fact that the functions H, Dd are continuous. □

Remark 6.8: Given any convex coercive Hamiltonian F and $a \in \mathbb{R}$, we can adapt the previous argument to show that a strict subsolution of the Hamilton-Jacobi equation $F(x, Du) = a$ in Ω, and a solution to the Neumann problem, with F replacing H in (35), cannot exist at the same time.

In the previous proof, we have essentially exploited the Lipschitz-continuous character of the subsolution u to (24). Hence the argument cannot be directly used to prove Theorem 6.2, which deals with subsolutions of the Neumann problem, since such functions can, as far as we know, present discontinuities on the boundary of Ω. However it is possible (see next Proposition) to rule out this possibility. This allows to invert the statement of Proposition 6.5, as well.

Proposition 6.9: *Any subsolution u to* (24), (35) *is Lipschitz-continuous in $\overline{\Omega}$.*

Proof: Let u a subsolution to (24), (35). Then we know that, since it is a subsolution to (24), it is Lipschitz-continuous in Ω (see Proposition 4.3) and can be extended to a Lipschtiz-continuous function in $\overline{\Omega}$. We will prove that actually this extension coincides with u.

Assume by contradiction that there is $x_0 \in \partial\Omega$ such that $u(x_0) > \limsup_{\Omega \ni x \to x_0} u(x)$. Let ϕ a smooth test function such that $u - \phi$ has a strict maximum at x_0 in $\overline{\Omega}$ and let $\lambda > 0$ be such that both $H(x_0, D\phi(x_0) + \lambda n(x_0)) > 0$ and $(D\phi(x_0) + \lambda n(x_0) + 2b(x_0)) \cdot n(x_0) > 0$ (using the definition of H and the boundedness of b).

Using the fact that u is Lipschitz-continuous in Ω and that $u(x_0) > \limsup_{\Omega \ni x \to x_0} u(x)$, it is possible to prove that also $u - \phi - \lambda d$ has a local maximum at x_0 in a sufficiently small neighbourhood $U \subseteq \overline{\Omega}$ of x_0 (possibly depending on λ). This will give a contradiction to the fact that u is a subsolution to (35) in viscosity sense. □

We proceed to establish the main results of the section.

Proof of Theorem 6.2: The assertion directly comes from Propositions 6.5, 6.7, 6.9. □

Proof of Theorem 6.3: We start by proving item (i). If $u \le v$ in $\mathcal{A}$ then the assertion directly comes from Theorem 6.2. Assume then the strict inequality $u(x) > v(x)$ to hold at some $x \in \mathcal{A} \backslash \Omega_b$. By Proposition 5.9 and Corollary 5.1, u and v are constant on the trajectory ξ of b starting at x. Hence $u(y) > v(y)$ at any ω-limit point of ξ, which is in contrast with the assumptions.

It is well known that the formula (37) provides a solution to (24), see Siconolfi (2006), Fathi and Siconolfi (2005). To show that the condition (36) is also satisfied, take, for purposes of contradiction, a strict subtangent ψ to w at some $x \in \partial\Omega$ not satisfying the required test inequality. Notice that it is also subtangent to $V(y, \cdot)$, for a suitable $y \in \Omega_b$. Then, arguing as in Proposition 5.10, we set

$$u = \begin{cases} \max\{\psi + \varepsilon, V(y, \cdot)\} & \text{for } x \in B \cap \overline{\Omega} \\ V(y, \cdot) & \text{otherwise.} \end{cases} \tag{52}$$

For a suitable choice of the constant ε and of the Euclidean ball B centred at x, u is a subsolution to $H = 0$ vanishing at y, and greater than $V(y, \cdot)$ in a neighbourhood of x. This is in contradiction with the maximality of $V(y, \cdot)$. □

7 Main results

In this section we exploit all the assumptions (2), (8), (9). We recall that, according to (9), $\mathcal{K}_1$ is the unique equivalence class making up Ω_b, which is not of repulsive type. We employ the semi-relaxed weak limits to deduce from the comparison results of the previous section and the estimate (20):

Proposition 7.1: *Let V_ε be the solution to* (19) *vanishing at some $x_0 \in \mathcal{K}_1$, for any ε. Then the V_ε uniformly converges in $\overline{\Omega}$ to a solution of* (24), (35) *vanishing on $\mathcal{K}_1$, up to a subsequence.*

Proof: By (20), the V_ε are locally equicontinuous in Ω. Hence they converge locally uniformly in Ω to some function w, up to a subsequence, in light of Ascoli theorem. We denote by V_{ε_k} such a subsequence. Note that w is a solution to (24), vanishing on $\mathcal{K}_1$ (recall that $\mathcal{K}_1$ is an equivalence class under $\sim$), because of straightforward stability results. Therefore it is Lipschitz-continuous in Ω, and so it can be extended by continuity in $\overline{\Omega}$. We set

$$\underline{w}(x) := \liminf_{\varepsilon_k \to 0, y \to x} V_{\varepsilon_k}(y), \quad \overline{w}(x) := \limsup_{\varepsilon_k \to 0, y \to x} V_{\varepsilon_k}(y),$$

for any $x \in \overline{\Omega}$. We know, by the Barles-Perthame argument, that $\underline{w}$, $\overline{w}$ are super and subsolution to (24), (35), respectively, and they equal w in Ω because of the local uniform convergence of V_{ε_k}. We consequently get $\underline{w} = \overline{w}$ on $\mathcal{A} \subset \Omega$, and deduce from Theorem 6.2 that $\underline{w} \geq \overline{w}$. This, in turn implies $\underline{w} = \overline{w}$ in the whole $\overline{\Omega}$, by the very definition of the lower/upper semi-relaxed weak limits.

Hence V_{ε_k} uniformly converge to w in $\overline{\Omega}$, and w satisfies the Neumann boundary condition (35). □

If Ω_b makes up an unique equivalence class under $\sim$, i.e. $\Omega_b = \mathcal{K}_1$, then there is an unique solution to (24), (35) vanishing on $\mathcal{K}_1$ by Theorem 6.3. We then deduce from Section 3 and the previous result:

Theorem 7.2: *If Ω_b makes up an unique equivalence class under $\sim$, then*

(i) *the solutions V_ε to* (19)*, vanishing at some $x_0 \in \Omega_b$, uniformly converge in $\overline{\Omega}$ to $V(\Omega_b, \cdot)$,*

(ii) *the probability measure P is supported by the set of minimisers of $V(\Omega_b, \cdot)$ in $\partial\Omega$.*

We proceed to discuss the case where Ω_b is divided in different equivalence classes. In this setting the vanishing viscosity method selects the maximal among the solutions to (24), (35) vanishing on $\mathcal{K}_1$. We, more precisely, get:

Theorem 7.3:

(i) *The solutions V_ε to* (19)*, vanishing at some $x_0 \in \mathcal{K}_1$, uniformly converge in $\overline{\Omega}$ to $V(\mathcal{K}_1, \cdot)$.*

(ii) *The probability measure P is supported by the set of minimisers of $V(\mathcal{K}_1, \cdot)$ in $\partial\Omega$.*

For proving it, we preliminarily need to strengthen the assertion of Proposition 5.11 showing that a C^∞ subsolution to (24) can be constructed around any repulsor which is, at the same time, an equivalence class under $\sim$. Such result is, in turn, based on a powerful lemma due to Fathi (1997, 2006), that we prove in the Appendix B.

Proposition 7.4: *Let $\mathcal{K}$ be a repulsor for b and an equivalence class under $\sim$, as well. There is an open neighbourhood $\widetilde{\Omega}$ of $\mathcal{K}$ and a C^∞ function ψ defined on $\widetilde{\Omega}$ such that*

(i) *the set of maximisers of ψ in $\widetilde{\Omega}$ coincides with $\mathcal{K}$,*

(ii) *the derivatives of all orders of ψ vanish on $\mathcal{K}$,*

(iii) $H(\cdot, D\psi(\cdot)) < 0$ *in $\widetilde{\Omega} \backslash \mathcal{K}$.*

Proof: Thanks to Proposition 5.8 and the fact that $\mathcal{K}$ is an equivalence class under $\sim$, there is an open neighbourhood $\widetilde{\Omega}$ of $\mathcal{K}$ with $\widetilde{\Omega} \cap \mathcal{A} = \mathcal{K}$. We can, in addition, assume that $\widetilde{\Omega}$ is contained in the repulsion basin of $\mathcal{K}$. There exists, by Proposition 5.11, a subsolution ϕ to (24), which is C^∞ in $\widetilde{\Omega} \backslash \mathcal{K}$, with

$$H(x, D\phi(x)) < 0 \quad \text{in } \widetilde{\Omega} \backslash \mathcal{K}. \tag{53}$$

We know, thanks to Proposition 4.3, that any subsolution must be constant on $\mathcal{K}$, and this constant value can be assumed to be 0 for ϕ, without loosing generality.

Since $D\phi(x) \in Z(x)$ for a.e. x, we have

$$0 = \sigma(x, b(x)) \geq D\phi(x) \cdot b(x) \quad \text{for a.e. } x \in \Omega, \tag{54}$$

and we can deduce from (53) that there is a positive constant $\delta = \delta(x_0)$, for any $x_0 \in \widetilde{\Omega} \backslash \mathcal{K}$, such that $B(x_0, \delta) \cap \mathcal{K} = \emptyset$ and

$$0 = \sigma(x, b(x)) > D\phi(x) \cdot b(x) + \delta |b(x)| \quad \text{for any } x \in B(x_0, \delta). \tag{55}$$

We denote by ξ the integral curve of (1) equaling x_0 at $t = 0$, and observe that the α-limit points of ξ belongs to $\mathcal{K}$, being x_0 in the repulsion basin of $\mathcal{K}$. We can accordingly find a $T > 0$ so large that $\xi(-T) \notin B(x_0, \delta)$ and $\phi(\xi(-T)) < \delta^2/2$. We invoke (54), (55) to get

$$\phi(x_0) - \delta^2/2 < \phi(x_0) - \phi(\xi(-T)) \leq \int_{-T}^{0} D\phi(\xi) \cdot b(\xi) dt < -\delta^2,$$

which shows that $\phi < 0$ in $\widetilde{\Omega} \backslash \mathcal{K}$. Following (Fathi, 2006), we now consider a sequence of C^∞ functions θ_n from $(-\infty, 0]$ to $(-\infty, 0]$ with

$$\begin{aligned} \theta_n(t) &= 0 \qquad t \in [-1/(n+2), 0] \\ \theta_n(t) &= t \qquad t \in (-\infty, -1] \\ 0 < \theta_n'(t) &< 2 \qquad t \in (-\infty, -1/(n+2)). \end{aligned}$$

Note that $\theta_n \circ \phi \in C^\infty(\widetilde{\Omega})$, for any n. Thanks to Lemma B.1, there thus are positive constants ε_n such that the series $\sum_n \varepsilon_n (\theta_n \circ \phi)$ is convergent to a function ψ in $C^\infty(\widetilde{\Omega})$. We can, in addition, assume $\sum_n \varepsilon_n < 1/2$. We claim that such ψ satisfies the statement. It is, in fact, readily seen that ψ vanishes on $\mathcal{K}$ and is strictly negative in $\widetilde{\Omega} \backslash \mathcal{K}$, which shows (i). The item (ii) comes from the fact that all the θ_n vanish in a left neighbourhood of 0. We finally have

$$D\psi(x) = \left(\sum_n \varepsilon_n \theta_n'(\phi(x)) \right) D\phi(x) \quad \text{for } x \in \widetilde{\Omega} \backslash \mathcal{K},$$

with $0 < \sum_n \varepsilon_n \theta_n'(\phi(x)) < 1$, which implies (iii) since H is convex in p, $H(x, D\phi(x)) < 0$, and $H(x, 0) = 0$. □

Proof of Theorem 7.3: The case $M = 1$ has been already treated in Theorem 7.2. If $M > 1$, we consider a solution u to (24), (35), vanishing on $\mathcal{K}_1$, and different from $V(\mathcal{K}_1, \cdot)$. Notice that

$$u(x) \leq V(\mathcal{K}_1, x) \quad \text{for } x \in \overline{\Omega}, \tag{56}$$

because of (25). We denote by a_i the value of u on $\mathcal{K}_i$, for any i, and take for granted that these sets have been labelled in such a way that $a_i \leq a_{i+1}$ for $i = 2, \ldots, M-1$.

We claim that there is an index $j > 1$ such that all the points of $\mathcal{K}_j$ are local minimisers of u. In fact, if such a property does not hold for $j = 2$, we can find a sequence x_n converging to some $x_0 \in \mathcal{K}_2$, and satisfying $u(x_n) < u(x_0) = a_2$, for any n. By the representation formula (37)

$$u(x_n) = u(y_n) + V(y_n, x_n) \quad \text{for some } y_n \in \Omega_b, \tag{57}$$

hence $u(y_n) < a_2$, and from the fact that $a_2 \leq a_i$ for $i > 1$, we deduce that $y_n \in \mathcal{K}_1$, and $u(y_n) = 0$ for any n.

Passing to the limit, up to subsequences, in (57) we therefore get

$$u(x_0) = V(y_0, x_0),$$

where $y_0 \in \mathcal{K}_1$ is a limit point of y_n. In conclusion:

$$u(\cdot) = V(\mathcal{K}_1, \cdot) \quad \text{on } \mathcal{K}_2 \tag{58}$$

In the case where $M = 2$, (58) amounts to say that $u(\cdot) = V(\mathcal{K}_1, \cdot)$ on the whole Ω_b, which is impossible in view of Theorem 6.3, and the fact that u has be taken different from $V(\mathcal{K}_1, \cdot)$.

If $M > 2$, we pass to examine $\mathcal{K}_3$ under the condition that (58) is true. When $a_3 = a_2$, the same argument of above shows that either the claim holds for $j = 3$ or $u(\cdot) = V(\mathcal{K}_1, \cdot)$ on $\mathcal{K}_3$. If instead $a_3 > a_2$, and the claim does not hold for $j = 3$, we again find sequences x_n converging to some point $x_0 \in \mathcal{K}_3$, and y_n, contained in Ω_b, satisfying (57). This implies that $u(y_n) < a_3$, and so $y_n \in \mathcal{K}_1 \cup \mathcal{K}_2$, for any n. Passing to the limit, when n goes to infinity, we derive from this and (58)

$$u(x_0) = V(\mathcal{K}_1, y_0) + V(y_0, x_0) \quad \text{for some } y_0 \in \mathcal{K}_1 \cup \mathcal{K}_2.$$

We deduce, in light of (25), $u(\cdot) \geq V(\mathcal{K}_1, \cdot)$ on $\mathcal{K}_3$, and so $u(\cdot) = V(\mathcal{K}_1, \cdot)$, because of (56). An iteration of this argument, for every i, shows that if the claim were not true, then $u(\cdot) = V(\mathcal{K}_1, \cdot)$ in $\overline{\Omega}$, which is impossible. Note that every point of $\mathcal{K}_i$, $i > 1$, is a local maximiser of u since $u(x) \geq u(\mathcal{K}_i) - V(x, \mathcal{K}_i)$ (by (25)) and $V(\mathcal{K}_i, x) = 0$ for x in a sufficiently small neighbourhood of $\mathcal{K}_i$ (by the very definition of repulsor). Then we have actually proved that u is constant in a neighbourhood of some $\mathcal{K}_j$.

We denote by ψ a C^∞ function defined in some neighbourhood $\widetilde{\Omega}$ of such $\mathcal{K}_j$ and satisfying the statement of Proposition 7.4 with $\mathcal{K}_j$ in place of $\mathcal{K}$. The set of minimisers of $u - \psi$ in $\widetilde{\Omega}$ coincides with $\mathcal{K}_j$. Since $\operatorname{div} b > 0$ on $\mathcal{K}_j$, we have

$$\limsup_{\substack{\varepsilon \to 0 \\ x \to \mathcal{K}_j}} -\frac{1}{2}\Delta\psi(x) + \frac{1}{\varepsilon}H(x, D\psi(x)) - \operatorname{div} b(x) < 0. \tag{59}$$

This implies that no subsequence V_{ε_k} of solutions to (19) can have u as uniform limit in $\overline{\Omega}$, otherwise there should be a k large with ψ subtangent to V_{ε_k} at a point x_k so close to $\mathcal{K}_j$ that the supersolution test inequality for the equation appearing in problem (19), with ε_k in place of ε, should fail at x_k. The item (i) is then proved, in view of Proposition 7.1.

The item (ii) finally comes from Section 3. □

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Appendix A

Here we show that the intrinsic distance can be equivalently defined by minimising the intrinsic length of curves joining a given pair of points and lying in $\overline{\Omega}$, instead of Ω. This property is invoked in the proof of Proposition 4.2.

We first recall that, for a Lipschitz-continuous function u, the (Clarke) generalised gradient ∂u is defined via the formula

$$\partial u(x) = \mathrm{co}\{p = \lim_i D\,u(x_i) : x_i \in \mathrm{dom}\,Du,\, \lim_i x_i = x\},$$

where co stands for the closed convex hull, and dom Du denotes the full measure set of points where u is differentiable. If u is defined in $\overline{\Omega}$ then $u\big|_{\partial\Omega}$ is differentiable in $\partial\Omega$, except for a subset of vanishing surface measure. We denote by $D\big|_{\partial\Omega}u$ the differential of u restricted to $\partial\Omega$, and by $\mathrm{dom}\, D\big|_{\partial\Omega}$ the subset of $\partial\Omega$ where it does exist.

We will also use that the Euclidean projection on $\partial\Omega$, denoted by $\mathrm{proj}_{\partial\Omega}$, is uniquely defined for points belonging to a suitable neighbourhood of $\partial\Omega$, and the function $x \mapsto \mathrm{proj}_{\partial\Omega}(x)$ is of class C^1.

Lemma A.1: *Given any y, x in* Ω*, the intrinsic length between them is given by*

$$\inf\{\ell_b(\xi) : \xi \text{ joining } y \text{ to } x \text{ and contained in } \overline{\Omega}\}.$$

Proof: We fix $y \in \Omega$ and denote by u the function defined as the intrinsic length between y and x, when x varies in Ω. It is clear that

$$u(x) \geq \inf\{\ell_b(\xi) : \xi \text{ joining } y \text{ to } x \text{ and contained in } \overline{\Omega}\} \quad \text{for any } x \in \Omega. \quad \text{(A.1)}$$

We know, see Fathi and Siconolfi (2005), that u is a subsolution to (24), so that

$$p \cdot q \leq \sigma(x, q) \quad \text{for any } x \in \Omega, p \in \partial u(x), \;\; q \in \mathbb{R}^N. \quad \text{(A.2)}$$

The difficulty is to show that a similar inequality also holds on the boundary of Ω. Since u is Lipschitz-continuous, it can be extended by continuity in $\overline{\Omega}$. We furthermore extend it, without redenoting, in a suitable neighbourhood Θ of Ω by setting

$$u(x) = u(\mathrm{proj}_{\partial\Omega}(x)) \quad \text{for } x \in \Theta \backslash \overline{\Omega}.$$

With this choice the generalised gradient $\partial u(x)$ is given, at any $x \in \partial\Omega$, by the convex hull of the union of the sets

$$\Big\{p = \lim_i D\, u(x_i) : x_i \in \Omega \cap \mathrm{dom}\, Du, \lim_i x_i = x\Big\}$$

and

$$\Big\{p = \lim_i D|_{\partial\Omega} u(z_i) : z_i \in \partial\Omega \cap \mathrm{dom}\, D\big|_{\partial\Omega} u, \lim_i z_i = x\Big\}.$$

We claim that

$$p \cdot q \leq \sigma(x, q) \quad \text{for any } x \in \partial\Omega, p \in \partial u(x), \;\; q \in T_{\partial\Omega}(x), \quad \text{(A.3)}$$

where $T_{\partial\Omega}(x)$ stands for the tangent space to $\partial\Omega$ at x. To show it, let us consider $z \in \mathrm{dom}\, Du\big|_{\partial\Omega}$, then $D\big|_{\partial\Omega}u(z) - \lambda n(z) \in D^+u(z)$, for $\lambda > 0$ suitably large, being, in this case, the function $u - \lambda d^{\#}(\cdot, \Omega)$ supertangent to u at z. We deduce from Proposition 6.4, arguing as in Proposition 6.5, that $D|_{\partial\Omega}u(z) + \nu n(z) \in Z(z)$ for some $\nu \in \mathbb{R}$, and consequently

$$D|_{\partial\Omega}u(z) \cdot q \leq \sigma(z, q) \quad \text{for any } q \in T_{\partial\Omega}(z).$$

From this we finally get (A.3), taking into account the previously described form of ∂u on $\partial\Omega$ and (A.2). We now take a curve ξ, defined in $[0,T]$, for some $T>0$, and contained in $\overline{\Omega}$, joining y to some $x\in\Omega$. We have

$$u(x)=\int_0^T \frac{d}{dt}u(\xi)dt.$$

We know from Clarke (1983) that

$$\frac{d}{dt}u(t)=p\cdot\dot{\xi}(t)\quad\text{for some } p\in\partial u(\xi(t)),$$

at any point t where ξ is differentiable. If, in addition, for such a t, $\xi(t)\in\partial\Omega$, then $\dot{\xi}(t)$ must belong to $T_{\partial\Omega}(\xi(t))$, being ξ contained in $\overline{\Omega}$. This implies, in light of (A.2), (A.3), $\frac{d}{dt}u(\xi(t))\le\sigma(\xi(t),\dot{\xi}(t))$ for a.e. t, and so $u(x)\le\ell_b(\xi)$. We finally derive

$$u(x)\le\inf\{\ell_b(\xi):\xi\text{ joining } y \text{ to } x \text{ and contained in } \overline{\Omega}\}.$$

The assertion is then obtained in view of (A.1) and the very definition of u. □

Appendix B

We give here the proof of the lemma on which Proposition 7.4 is based. It simplifies to some extent several smoothing results appeared in the literature. See for instance Lin et al. (1996), Lemma 4.3.

Lemma B.1 (Fathi, 1997, 2006)**:** *Let Θ be an open subset of $\mathbb{R}^N$. Give $X:=C^\infty(\Theta)$ the usual topology induced by a countable family of seminorm μ_i expressing the locally uniform convergence in Θ of all derivatives.*

For any sequence ϕ_n of X there are positive numbers ε_n such that $\sum_n\varepsilon_n\phi_n$ is convergent.

Proof: Let λ_i be a positive decreasing sequence with $\sum_i\lambda_i=1$. We know that the topology of X is given by the complete distance, see Rudin (1991)

$$\delta(\phi,\psi):=\sum_i\lambda_i\frac{\mu_i(\phi-\psi)}{\mu_i(\phi-\psi)+1},$$

for ϕ, ψ in X. We choose ε_n in such a way that

$$\sum_{j=1}^n\mu_j(\varepsilon_n\phi_n)<\lambda_n\quad\text{for any } n,$$

which implies

$$\mu_i(\varepsilon_n\phi_n)\le\sum_{j=1}^n\mu_j(\varepsilon_n\phi_n)<\lambda_n\quad\text{if } i\le n. \tag{B.1}$$

Given $\varepsilon > 0$, we take n_0 such that $\sum_{i=n_0}^{\infty} \lambda_i < \varepsilon/2$, and $l > m > n_0$. We have, in view of (B.1)

$$\delta\Big(\sum_{n=m}^{l} \varepsilon_n \phi_n, 0\Big) \leq \sum_{i \leq n_0} \lambda_i \frac{\sum_{n=m}^{l} \mu_i(\varepsilon_n \phi_n)}{\mu_i(\sum_{n=m}^{l} \varepsilon_n \phi_n) + 1} + \sum_{i > n_0} \lambda_i \frac{\sum_{n=m}^{l} \mu_i(\varepsilon_n \phi_n)}{\mu_i(\sum_{n=m}^{l} \varepsilon_n \phi_n) + 1}$$

$$\leq \Big(\sum_{n=n_0}^{\infty} \lambda_n\Big)\Big(\sum_{i \leq n_0} \lambda_i\Big) + \sum_{i=n_0}^{\infty} \lambda_i < \varepsilon.$$

This shows the assertion. □

Remark B.2: It is worth noticing that the previous lemma holds in any Fréchet space, i.e. in any vector space endowed with a complete translation invariant metric such that the induced topology is locally convex. Recall that such topology can be equivalently defined by a countable separating family of seminorms.