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GLOBAL WORLD FUNCTIONS

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Abstract. Starting from the Amann-Conley-Zehnder finite reduction framework in the non-compact Viterbo's version, we discuss the existence of global generating function with a finite number of auxiliary parameters describing the two-points Characteristic Relation related to the geodesic problem in the Hamiltonian formalism. This applies both to Analytical Mechanics and to General Relativity - we construct a global object generalizing the World Function introduced by Synge, which is well-defined only locally. Whenever the auxiliary parameters can be fully removed, Synge's World Function is restored.

1. Introduction

In the textbook by Synge [19] one can find the following definition:

Let $P'(x')$ and $P(x)$ be two points of space-time, joined by a geodesic Γ with equations $x^i = \xi^i(u)$ where u is a special[1] *parameter. Then the integral*

$$\Omega(P'P) = \Omega(x', x) = \frac{1}{2}(u_1 - u_0)\int_{u_0}^{u_1} g_{ij}U^iU^j\, du$$

taken along Γ with $U^i = d\xi^i/du$, has a value independent of the particular special parameter chosen. If, as we shall suppose, the points P', P determine a unique geodesic passing through them, then Ω is a function of these two points; it is a function of the eight variables x', x and we shall call it the world-function *of space-time.*

The World Function had a rather troubled history. The main criticism is that it has a local meaning only, and even in simple cases we cannot use it for global analysis. Really, Synge recognized this limitation few lines after the definition.

[1] special parameters are the representative elements of a class of parameters invariant by affine transformations.

He used this object for approximations and to solve geodesic triangles in spaces of small curvature. In more recent times, in the textbook by De Felice-Clarke [13], it is used in measure theory to define spatial length between an observer and an event, to measure the effect of curvature in the measure of angles, and in the study of Doppler effect.

From a strictly analytical mechanical point of view, the World Function appears as the generating function of the canonical transformation induced by the differential equations of the geodesic system, thought of in the Hamiltonian format, and it there exists whenever the phenomenon of the entanglement of geodesics in the base manifold (the space-time) does not occur. Such a generating function is solution of a certain Hamilton-Jacobi equation, and it has been a rather intriguing matter along past years to establish a useful notion of 'global solution' in Hamilton-Jacobi theory in a rigorous way. In order to give a comprehensible explanation of our proposal about globalization of the World Function, we briefly review a few elements of the geometric setting for the global theory of the Hamilton-Jacobi equation.

Given a Hamilton-Jacobi problem $H \circ \mathrm{d}S = e,\ H : T^*Q \to \mathbb{R}$, the symplectic geometry environment provides a new concept of solutions, called geometric solutions, i.e. Lagrangian submanifolds Λ in the coisotropic fiber $H^{-1}(e)$ of T^*Q. These geometric solutions have meaning only locally no longer. The description of the Lagrangian submanifolds is a crucial problem in the symplectic arena: a geometric solution of a Hamilton-Jacobi problem $H = e$ is meaningful —and then comparable to the classical (weak) solutions [4]— if we are able to write for it some generating functions, see e.g. [5]-[10], [20]. A fundamental tool in this area is the Maslov-Hörmander theorem: it shows that we can locally describe Lagrangian submanifolds $\Lambda \subset T^*Q$ as image of the differential —with respect to $q \in Q$— of a smooth function $S = S(q, u)$, valued at the stationary points u of a set of auxiliary parameters $u \in \mathbb{R}^k$, for a suitable integer $k \in \mathbb{N}$. In other words, *locally* at each point $\lambda \in \Lambda \subset T^*Q$,

$$\Lambda = \left\{ (q^i, p_j) : \ q^i \in Q, \ p_j = \frac{\partial S}{\partial q^j}(q^i, u^A), \ \frac{\partial S}{\partial u^B}(q^i, u^A) = 0 \right\}$$

with the transversality condition:

$$\mathrm{rk} \left(\frac{\partial^2 S}{\partial u^A \partial q^i} \quad \frac{\partial^2 S}{\partial u^A \partial u^B} \right) \Bigg|_{\Lambda} = k = \max .$$

The importance of having *global* generating function S is evident when we try to construct classical solutions for H-J from geometric ones. In fact, if the relation

$$0 = \frac{\partial S}{\partial u^B}(q^i, u^A)$$

can be solved with respect to u^A, i.e. if for every q there exists one and only one $u^A = \tilde{u}^A(q)$ solving it, then we may define a classical (smooth) solution $\tilde{S}(q) := S(q, \tilde{u}(q))$. It is well known that in general such a classical solution does not exist globally (i.e. one cannot remove all auxiliary parameters u): this is due mainly to the nonlinearity of the problem (Hamiltonian function and initial data) and the related lack of transversality of Λ with respect to the fibers of $\pi_Q : T^*Q \to Q$. Nevertheless, when the above stationarization procedure to make *classical* solutions fails, by the global generating function for Λ one can try to build up *viscosity* solutions —in the sense of Crandall-Evans-Lions— by removing the auxiliary parameters through suitable $\inf$-$\sup$ procedures, e.g.:

$$s = s(q) := \inf_{u^A \in \mathbb{R}^{k_1}} \sup_{u^\alpha \in \mathbb{R}^{k_2}} S(q, u^A, u^\alpha), \qquad k_1 + k_2 = k.$$

This program actually works in some interesting cases, see e.g. [4], [9].

A first rigorous attempt to globalize the Maslov-Hörmander theorem was made by Laudenbach [16], Sikorav [18] and Chaperon [11] for the compact case. Later, Viterbo [21] built up a global version in the non-compact case $\mathbb{R}^n$ using the reduction techniques of Amann, Conley and Zehnder [2, 12]. Viterbo's theorem states the existence of global generating function for Lagrangian submanifolds, geometric solutions of an evolution problem of Hamilton-Jacobi, starting from zero-section of $T^*\mathbb{R}^n$ with Hamiltonian function $H : T^*\mathbb{R}^n \to \mathbb{R}$ with second partial derivatives uniformly bounded. It is not yet available in literature a version for non-compact and non-parallelizable manifolds. The aim of Chaperon and Viterbo was mainly to construct a new global theory of weak solutions for H-J problems: the so-called *min-max* solutions based on the Lusternik-Schnirelman theory.

Here is a scheme of Viterbo's statement proof:

1. Viterbo's theorem describes Lagrangian submanifolds connected by a Hamiltonian isotopy to the zero-section of $T^*\mathbb{R}^n$. The curves in the Sobolev space $H^1([0,T]; T^*\mathbb{R}^n)$, starting from zero section, are described by the couple (q_T, ϕ) where q_T is the q-projection of the final point (at the time $t = T$) and $\phi \in L^2$ is the velocity of the curve.

2. The Action Functional is considered as a formal global generating function with infinite parameters (in L^2). In fact, the variation of the Action Functional (obtained with a variation of velocity) is zero exactly in correspondence to the solutions of the related Hamilton system.

3. Developing velocities in Fourier series and keeping the finite kernel (terms from $-N$ to N) and the infinite tail, one can observe that, for N large enough, the finite kernel of velocity of a solution determines the full solution (by a fixed-point lemma). So, one may consider only a finite number of parameters, obtaining in this way a global generating function.

We notice that Viterbo considers curves starting from zero-section, i.e. with vanishing initial conjugate momentum, and constructs an object $\Lambda \subset T^*\mathbb{R}^n$. This fact simplifies the proof a lot; by minor changes, one could carry out the same construction by starting from any exact Lagrangian submanifold.

Now, in order to construct a generating function of a symplectic relation, that is, an object inside $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$, we have to reformulate the framework drastically. Moreover, in the original theory all works for fixed initial vanishing momentum $p_0 = 0$ and fixed final configuration q_T: on the other side, since our aim here is to construct a two-points generating function with auxiliary parameters describing the above symplectic relation, we have to fix both initial and final configurations q_0 and q_T.

Curves will be identified by the the straight segment between q_0 and q_T plus loops. The variations of a curve will be still given by loops. This approach restoring loops can be also found in a recent paper [14].
It is worth noticing that, after the seminal paper [20] by Tulczyjew on the H-J theory, Hamilton Principal Functions with auxiliary parameters were considered for the first time in [5] and [7].

Under some suitable hypothesis, when the Lagrangian function L is the (half of the) quadratic form valued on the 4-velocity related to a Riemannian or a Semi-Riemannian metric, our construction leads to a generating function with auxiliary parameters which is exactly the announced globalization of the World Function of Synge, see Theorem 1 and Corollary 3. For the related Hamilton Principal Function, see Corollary 5.

2. A two-point version of Viterbo's construction

In this section $Q = \mathbb{R}^n$ and H will denote a Hamiltonian function $H : T^*\mathbb{R}^n \to \mathbb{R}$. Adopting standard notations, as in [1, 3, 6], we denote by $\omega = \mathrm{d}p \wedge \mathrm{d}q = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q^i$ the standard symplectic 2-form on $T^*\mathbb{R}^n$. The Hamiltonian vector field X_H is defined by $i_{X_H}\omega = -\mathrm{d}H$. Denoting curves by $\gamma = (q,p)$ and setting $J = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ -\mathbb{I} & \mathbb{O} \end{pmatrix}$, the related Hamilton equations $\dot{\gamma} = X_H(\gamma) = J\nabla H(\gamma)$ read

$$\dot{q} = \frac{\partial H}{\partial p}(q,p), \qquad \dot{p} = -\frac{\partial H}{\partial q}(q,p).$$

We denote by ϕ_H^t the flow of X_H, $\frac{d}{dt}\phi_H^t = J\nabla H(\phi_H^t)$. We consider the standard projections

$$T^*\mathbb{R}^n \overset{pr_1}{\longleftarrow} T^*\mathbb{R}^n \times T^*\mathbb{R}^n \overset{pr_2}{\longrightarrow} T^*\mathbb{R}^n$$

and we carry out the symplectic structure $\bar{\omega}$ on $T^*\mathbb{R}^n \times T^*\mathbb{R}^n \cong T^*\left(\mathbb{R}^n \times \mathbb{R}^n\right)$ by the following twofold pull-back of the standard symplectic 2-form on $\mathbb{R}^n$,

$$\bar{\omega} := pr_2^*\omega - pr_1^*\omega = \mathrm{d}p_2 \wedge \mathrm{d}q_2 - \mathrm{d}p_1 \wedge \mathrm{d}q_1.$$

We recall that the *Characteristic Relation* of the Hamiltonian system H is the set $\mathcal{C}$ of points $(q_1,p_1;q_2,p_2) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ such that, for some $T \in \mathbb{R}$, one has $(q_2,p_2) = \phi_H^T(q_1,p_1)$; it comes out that $\mathcal{C}$ is a Lagrangian submanifold of $(T^*\mathbb{R}^n \times T^*\mathbb{R}^n, \bar{\omega})$, that is: $\bar{\omega}\big|_{\mathcal{C}} = 0$ and $\dim\mathcal{C} = 2n \ = \frac{1}{2}\dim(T^*\mathbb{R}^n \times T^*\mathbb{R}^n)$.

Theorem 1. *Let $H : T^*\mathbb{R}^n \to \mathbb{R}$ be a C^2 Hamiltonian function. Suppose the following condition is satisfied:*

$$\sup_{(q,p)\in T^*\mathbb{R}^n} \left|\nabla^2 H\right| = C < +\infty.$$

Then the set $\Lambda \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ defined by:

$$\Lambda := \left\{(q_0,p_0,q_T,p_T) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n : \phi_H^T(q_0,p_0) = (q_T,p_T)\right\} \qquad (*)$$

admits a global generating function with finite auxiliary parameters

$$S : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}, \quad (q_0,q_T,u) \mapsto S(q_0,q_T,u),$$

such that:

$$p_0 = -\frac{\partial S}{\partial q_0}, \qquad p_T = \frac{\partial S}{\partial q_T}, \qquad 0 = \frac{\partial S}{\partial u}.$$

Remark 2. *For general Hamiltonian systems, the set Λ would not be the characteristic relation $\mathcal{C}$, since the final time $t = T$ is fixed; anyway, this set Λ is actually Lagrangian (maximal isotropic) with respect to the symplectic 2-form $\bar{\omega}$; in particular, for geodesic-like Hamiltonian $H = \frac{1}{2}g_{ij}^{-1}(q)p_ip_j$, Λ coincides (by choosing $T = 1$) precisely with the Characteristic Relation $\mathcal{C}$, see Corollary 3.*

The above requested boundness of the second derivatives of H is a hard task to verify whenever the Hamiltonian function is not compactly supported, or definitively quadratic, in $\mathbb{R}^{2n}$; this difficulty arises especially by concerning with geodesic problems (in the Riemannian or Semi-Riemannian case), for which

$$H = \frac{1}{2} g^{-1}(q)(p,p), \qquad \nabla^2 H = \begin{pmatrix} \frac{1}{2}\frac{\partial^2 g^{ij}}{\partial q^l \partial q^m}(q) p_i p_j & \frac{1}{2}\frac{\partial g^{ij}}{\partial q^l}(q) p_i \\ \\ \frac{1}{2}\frac{\partial g^{ij}}{\partial q^l}(q) p_i & g^{ij}(q) \end{pmatrix},$$

so that we cannot assume any 'a priori' boundness on p's, especially in the Semi-Riemannian case (in the Riemannian one, which is a p-convex case, energy conservation does help us to it.)

In order to remove this difficulty and to avoid extreme sorts of pathology, we will restrict inside a fixed (large) compact sub-set $\mathcal{K} \subset \mathbb{R}^n$ together with the request of *finite* cardinality of the sets of geodesics linking the pairs $(q_0, q_1) \in \mathcal{K} \times \mathcal{K}$; this property —see $(**)$ below— seems to be reasonably enjoyed by Riemannian [Semi-Riemannian] metric tensors which are suitably asymptotic to Euclidean [Minkowskian] metrics. We do not delve further into this matter.

Warning: We are going to deal with

i) Riemannian $g = g_R$, sgn$(g_R) = (1, ..., 1)$, and

ii) Semi-Riemannian $g = g_{SR}$, sgn$(g_{SR}) = (-1, 1, ..., 1)$, manifolds topologically equivalent to $\mathbb{R}^n$ with trivial atlas (one chart is sufficient).

Anyway, the Euclidean metric (and norm) many times used below could be replaced by minor changes

i) just by the Riemannian metric, and

ii) in the Semi-Riemannian case, by the following Riemannian metric [15, 17]

$$\hat{g}_R := g_{SR} + 2\, u \otimes u,$$

where u is a time-like vector field, $\langle g_{SR}\, u, u\rangle \equiv -1$, giving the time-orientation to $(\mathbb{R}^n, g_{SR})$.

We are ready to state the following

Corollary 3. (World Function) *Let $H : T^*\mathbb{R}^n \to \mathbb{R},\ H = \frac{1}{2} g^{-1}(q)(p,p)$, be the C^2 Hamiltonian function of a Riemannian or Semi-Riemannian system. Consider the set of geodesics (for $T = 1$):*

$$\begin{aligned} G(q_0, q_1) \quad &= \Big\{ \gamma(\cdot) = (q(\cdot), p(\cdot)) \in H^1([0,1]; \mathbb{R}^{2n}) : q(0) = q_0, \\ &\qquad q(1) = q_1, \ \ \dot{\gamma} = J \nabla H(\gamma) \Big\}. \end{aligned}$$

For a fixed (arbitrarily large) compact $\mathcal{K}$ of $\mathbb{R}^n$, we suppose that

$$\#G(q_0, q_1) < +\infty \qquad \forall (q_0, q_1) \in \mathcal{K} \times \mathcal{K}. \tag{$**$}$$

Then the above set Λ in $(*)$, restricted to $T^*\mathcal{K}\times T^*\mathcal{K}$, is precisely *the expected Lagrangian submanifold denoting the Characteristic Relation $\mathcal{C}$ of the Hamiltonian system H and the related generating function S,*

$$\mathcal{K} \times \mathcal{K} \times \mathbb{R}^k \ni (q_0, q_1, u) \mapsto S(q_0, q_1, u) \in \mathbb{R},$$

represents the looked for generalized global World Function.

Remark 4. *Alternatively, the hypothesis $(**)$ may be replaced by the following one:*
There exists a compact set $\mathcal{G} \subset H^1$ such that:

$$G(q_0, q_1) \subset \mathcal{G} \qquad \forall (q_0, q_1) \in \mathcal{K} \times \mathcal{K} \tag{$**$}'$$

Indeed $H^1 \hookrightarrow C^0$ (by Sobolev embedding) so $\mathcal{G}$ is bounded and we have an a priori *bound on every point of each geodesic joining pairs of points of $\mathcal{K}$.*

Finally, we have

Corollary 5. (Hamilton Principal Function) *Let $S(q_0, q_1, u)$ be a generalized World Function related to the Hamiltonian $H : T^*\mathbb{R}^n \to \mathbb{R},\ H = \frac{1}{2}g^{-1}(q)(p, p)$, for a Riemannian or Semi-Riemannian metric g and for $(q_0, q_1) \in \mathcal{K} \times \mathcal{K}$, where $\mathcal{K}$ is a compact set of $\mathbb{R}^n$. Then the function*

$$\mathcal{S}(q_0, q_1, u) = 2\sqrt{|e\, S(q_0, q_1, u)|}$$

is a (generalized) global Hamilton Principal Function for the Hamilton-Jacobi equation $H = e$.

Remark 6. *Given the above generalized World Function $S = S(q_0, q_1, u)$, if we are able to remove the auxiliary parameters by means of the stationarization*

$$\frac{\partial S}{\partial u}(q_0, q_1, u) = 0,$$

we find the classical World Function again; but, in general, we cannot remove them and this fact is due to the lack of transversality of Λ with respect the fibers of $\pi : T^*(\mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R}^n \times \mathbb{R}^n$, which tells us that there are more geodesics connecting $q_0,\ q_1$.

More precisely, it can happen that for the above relation $\partial S/\partial u = 0$,

1. there is no solution: we have no geodesics connecting $q_0,\ q_1$,
2. we find h functionally independent solutions $u_\alpha = \tilde{u}_\alpha(q_0,\ q_1)$, $\alpha \in \{1, ..., h\}$, i.e. we have h geodesics connecting $q_0,\ q_1$.

If the points $q_0,\ q_1$ are close enough, we always have only *one* geodesic connecting them, $u = \tilde{u}(q_0\, q_1)$, so we restore the classical local World Function $\Omega(q_0, q_1) := S(q_0, q_1, \tilde{u}(q_0, q_1))$; but the first *conjugate point* to q_0 along the prolungation of this geodesic (towards the future) signals the end of this condition of transversality.

Remark 7. *Let us consider the Geometric Cauchy Problem in $\mathcal{K}$, for the Hamilton-Jacobi equation*

$$H \circ \mathrm{d}S = e,$$

for initial data $\sigma : \Sigma^{n-1} \to \mathbb{R}$ on the hypersurface $j : \Sigma^{n-1} \hookrightarrow \mathcal{K} \subset \mathbb{R}^n$ which can be represented by the following initial Lagrangian submanifold

$$\Lambda_0 = \{(q,p) : \langle p, Tj(\chi)\, v\rangle = \langle \mathrm{d}\sigma, v\rangle,\ q = j(\chi),\ \forall v \in T_\chi\Sigma^{n-1},\quad \forall \chi \in \Sigma^{n-1}\}.$$

It is solved —see [8] for details— by the Lagrangian manifold Λ globally described by the following generating function $\hat{\mathcal{S}}$ with auxiliary parameters (u, χ):

$$\hat{\mathcal{S}}(q; u, \chi) = \mathcal{S}(\tilde{q}(\chi), q, u) + \sigma(\chi).$$

The global Hamilton Principal Function $\mathcal{S}(q_0, q_1, u)$ can be interpreted as a sort of Geometric Green Kernel for Hamilton-Jacobi problem.

3. Proofs

Proof of Theorem 1. Let us consider $q_0, q_T \in \mathbb{R}^n$. We define the following set of curves ($H^s \equiv W^{s,2}$):

$$\Gamma = \left\{\gamma(\cdot) = (q(\cdot), p(\cdot)) \in H^1([0,T], \mathbb{R}^{2n}) : \quad q(0) = q_0, \quad q(T) = q_T\right\}$$

An element of Γ is a curve in $T^*\mathbb{R}^n$ whose canonical projection on $\mathbb{R}^n$ is a curve connecting q_0 and q_T. A curve $\Gamma \ni \gamma : [0,T] \to \mathbb{R}^{2n}$ solves the Hamiltonian system if the equation $\dot{\gamma} = J\nabla H(\gamma)$ is satisfied for almost every $t \in [0,T]$. Actually, by Sobolev embedding theorem, these curves are C^0. Given $\phi \in L^2([0,T], \mathbb{R}^{2n})$, for every fixed $N \in \mathbb{N}$, we consider the two projection maps (which are Lipschitz of unitary norm):

$$\mathbb{P}_N\phi(s) := \sum_{|k|\leq N} \phi_k \mathrm{e}^{i\frac{2\pi k}{T}s}, \qquad \mathbb{Q}_N\phi(s) := \phi(s) - \mathbb{P}_N\phi(s),$$

$$L^2 = \mathbb{P}_N L^2 \oplus \mathbb{Q}_N L^2, \qquad u \in \mathbb{P}_N L^2, \qquad v \in \mathbb{Q}_N L^2.$$

The subspaces $\mathbb{P}_N L^2$ and $\mathbb{Q}_N L^2$ are orthogonal with respect to the scalar product of L^2. Furthermore, we define the map

$$h : \mathbb{R}^n \times \mathbb{R}^n \times L^2 \longrightarrow \Gamma$$

$$(q_0, q_T, \phi(\cdot)) \longmapsto (q(t), p(t)) = h(q_0, q_T, \phi(\cdot))(t),$$

where

$$\begin{cases} q(t) & := & q_0 + \frac{t}{T}(q_T - q_0) + \displaystyle\int_0^t \Big(\phi_q(s) - \frac{1}{T}\int_0^T \phi_q(\tau)d\tau\Big)ds, \\ p(t) & := & \displaystyle\frac{1}{T}\int_0^T \phi_q(\tau)d\tau + \int_0^t \phi_p(s)\,ds. \end{cases}$$

Essentially, $q(\cdot)$ is a variation of the straight line connecting q_0 to q_T. This is a bijective map; indeed, by computing its inverse, we find:

$$\begin{cases} q_0 & = & q(0), \\ q_T & = & q(T), \\ \phi_q(t) - \frac{1}{T}\int_0^T \phi_q(\tau)d\tau & = & \dot{q}(t) - \frac{1}{T}(q_T - q_0), \\ \phi_p(t) & = & \dot{p}(t), \end{cases}$$

and the indeterminacy on the mean value of ϕ_q, i.e. $\phi_q^{k=0} := \frac{1}{T}\int_0^T \phi_q(\tau)d\tau$, is fully removed

$$p(0) = \phi_q^{k=0}.$$

Now we consider the Hamilton-Helmholtz Action Functional:

$$A : \mathbb{R}^n \times \mathbb{R}^n \times L^2 \to \mathbb{R}$$

$$A[q_0, q_T, \phi] = \int_0^T (p \cdot \dot{q} - H)|_{\gamma = h(q_0, q_T, \phi)}\, dt$$

Gateaux derivative $\frac{DA}{D\phi}[q_0, q_T, \phi]\delta\phi$ with $\delta\phi \in L^2 (= T_\phi L^2)$ is defined by

$$\frac{DA}{D\phi}[q_0, q_T, \phi]\delta\phi = \frac{dA}{d\lambda}(q_0, q_T, \phi + \lambda\,\delta\phi)\Big|_{\lambda=0}.$$

We have that

$$\frac{DA}{D\phi}[q_0, q_T, \phi]\delta\phi \quad = \quad \int_0^T \left[\left(\dot{q} - \frac{\partial H}{\partial p} \right) \delta p - \left(\dot{p} + \frac{\partial H}{\partial q} \right) \delta q \right] \, ds + p \cdot \delta q \Big|_0^T$$

where the variations δq and δp are deduced directly from the above definition of h, i.e.

$$\delta q \quad = \quad \frac{d}{d\lambda} q(\cdot, q_0, q_T, \phi + \lambda \delta\phi) \Big|_{\lambda=0} = \int_0^t \left(\delta\phi_q(s) - \frac{1}{T} \int_0^T \delta\phi_q(\tau) d\tau \right) \, ds,$$

$$\delta p \quad = \quad \frac{d}{d\lambda} p(\cdot, q_0, q_T, \phi + \lambda \delta\phi) \Big|_{\lambda=0} = \delta\phi_q^{k=0} + \int_0^t \delta\phi_p(s) \, ds.$$

Notice that $\delta q(0) = \delta q(T) = 0$. We obtain also

$$\frac{DA}{D\phi}[q_0, q_T, \phi]\delta\phi = \int_0^T \left[\left(\dot{q} - \frac{\partial H}{\partial p} \right) \left(\delta\phi_q^{k=0} + \int_0^t \delta\phi_p(s) \, ds \right) \right.$$
$$\left. - \left(\dot{p} + \frac{\partial H}{\partial q} \right) \left(\int_0^t \left(\delta\phi_q(s) - \frac{1}{T} \int_0^T \delta\phi_q(\tau) d\tau \right) \, ds \right) \right] dt.$$

So the curve $\gamma(\cdot) = (q(\cdot), p(\cdot))$ solves Hamiltonian equations if and only if $\frac{DA}{D\phi}[q_0, q_T, \phi]\delta\phi = 0$ for every $\delta\phi \in L^2$. Along solutions we have

$$\frac{\partial A}{\partial q_0} \quad = \quad \int_0^T \left[\left(\dot{q} - \frac{\partial H}{\partial p} \right) \frac{\partial p}{\partial q_0} - \left(\dot{p} + \frac{\partial H}{\partial q} \right) \frac{\partial q}{\partial q_0} \right] \, ds + p \cdot \frac{\partial q}{\partial q_0} \Big|_0^T = -p(0),$$

$$\frac{\partial A}{\partial q_T} \quad = \quad \int_0^T \left[\left(\dot{q} - \frac{\partial H}{\partial p} \right) \frac{\partial p}{\partial q_T} - \left(\dot{p} + \frac{\partial H}{\partial q} \right) \frac{\partial q}{\partial q_T} \right] \, ds + p \cdot \frac{\partial q}{\partial q_T} \Big|_0^T = p(T).$$

This shows that the Action Functional may be regarded as a *formal* global generating function, with infinite parameters (in L^2). Now we will prove the following

Lemma 8. *Under above hypothesis on H, for fixed $(q_0, q_T) \in \mathbb{R}^n \times \mathbb{R}^n$ and $u \in \mathbb{P}_N L^2$, we consider the following further map $\hat{h}_N$*

$$\hat{h}_N : \mathbb{Q}_N L^2 \longrightarrow \mathbb{Q}_N L^2 \qquad v \longmapsto \hat{h}_N(v) := \mathbb{Q}_N J\nabla H(h(q_0, q_T, u+v)).$$

It is a contraction map for N large enough.

Proof. We recall that $\mathbb{Q}_N \mathrm{id}_{[0,T]}(t) = \sum_{|k|>N} \frac{\mathrm{i}T}{2\pi k} \mathrm{e}^{\mathrm{i}2\pi kt/T}$, $\langle f(\cdot), g(\cdot)\rangle_{L^2([0,T])} := \int_0^T f(t)\bar{g(t)}\frac{dt}{T}$, and $||w(\cdot)||^2_{L^2([0,T])} := \langle w(\cdot), w(\cdot)\rangle_{L^2([0,T])}$. Writing briefly

$$g(v) := h(q_0, q_T, u+v)$$

and $v := v_2 - v_1$, for $v_1, v_2 \in \mathbb{Q}_N L^2$, we get by direct computations:

$$\|g(v_1) - g(v_2)\|_{L^2} \leq \left\| \left(\int_0^t (v_{q2} - v_{q_1})(s)\, ds, \int_0^t (v_{p2} - v_{p_1})(s)\, ds \right) \right\|_{L^2}$$

$$= \left\| \int_0^t v(s)\, ds \right\|_{L^2} \leq \left\| \sum_{|k|>N} \frac{T}{\mathrm{i}2\pi k} v_k \mathrm{e}^{\mathrm{i}2\pi kt/T} - \sum_{|k|>N} \frac{T}{\mathrm{i}2\pi k} v_k \right\|_{L^2}$$

$$\leq \frac{T}{2\pi N}\|v\|_{L^2} + \left\| \sum_{|k|>N} \frac{\mathrm{i}T}{2\pi k} v_k \right\|_{L^2} \leq \frac{T}{2\pi N}\|v\|_{L^2} + \|\mathbb{Q}_N \mathrm{id}_{[0,T]}\|_{L^2} \|v\|_{L^2}$$

$$\leq \frac{T}{2\pi N}\|v\|_{L^2} + \frac{T}{2\pi}\sqrt{\frac{2}{N}}\,\|v\|_{L^2} = \frac{T}{2\pi N}(1+\sqrt{2N})\|v\|_{L^2}.$$

$$\|h(q_0, q_T, u+v_2) - h(q_0, q_T, u+v_1)\|_{L^2} \leq \frac{T}{2\pi N}(1+\sqrt{2N})\|v_2 - v_1\|_{L^2}.$$

Finally, we can estimate:

$$\|\mathbb{Q}_N J\nabla H(h(q_0, q_T, u+v_2)) - \mathbb{Q}_N J\nabla H(h(q_0, q_T, u+v_1))\|_{L^2}$$

$$\leq \sup_{\mathbb{R}^{2n}} |\nabla^2 H| \|g(v_1) - g(v_2)\|_{L^2} \leq \frac{CT}{2\pi N}(1+\sqrt{2N})\|v_2 - v_1\|_{L^2}.$$

For N large enough, we have:

$$\frac{CT}{2\pi N}(1+\sqrt{2N}) =: \alpha < 1$$

So this map is a contraction. The Banach-Caccioppoli Lemma ensures the existence of one and only one fixed point for this contraction. We will denote by $f(q_0, q_T, u)$ this fixed point:

$$f(q_0, q_T, u) = \mathbb{Q}_N J\nabla H\big(h(q_0, q_T, u + f(q_0, q_T, u))\big). \qquad (\star)$$

It can be proved that it depends smoothly by q_0, q_T, u. In fact, the fixed point function f solves the equation for the unknown v

$$\mathbf{G}(q_0, q_T, u, v) := \mathbb{Q}_N J\nabla H(h(q_0, q_T, u+v)) - v = 0$$

The implicit function theorem works, since

$$\frac{\partial}{\partial v}\mathbf{G}(q_0, q_T, u, v) = \frac{\partial}{\partial v}\mathbb{Q}_N J\nabla H(h(q_0, q_T, u+v)) - \mathbb{I}.$$

A bound for the derivatives in the above r.h.s. is given by the contraction Lipschitz constant α,

$$\left|\frac{\partial}{\partial v}\mathbb{Q}_N J\nabla H(h(q_0, q_T, u+v))\right| \leq \alpha < 1,$$

so that

$$\left|\left[\frac{\partial}{\partial v}\mathbf{G}(q_0, q_T, u, v)\right]^{-1}\right| \leq \frac{1}{1-\alpha}.$$

We gain that f is differentiable with respect to u, q_0, q_T, e.g.

$$\begin{aligned}
&\left|\frac{\partial f}{\partial u}(q_0, q_T, u, v)\right| \\
&= \left|-\left[\frac{\partial}{\partial v}\mathbf{G}(q_0, q_T, u, f(q_0, q_T, u))\right]^{-1}\frac{\partial}{\partial u}\mathbf{G}(q_0, q_T, u, f(q_0, q_T, u))\right| \\
&\leq \frac{\alpha}{1-\alpha} < +\infty.
\end{aligned}$$

Finally, it is crucial to observe that if we are able to solve the *finite*[2] equation for $u = \left(\left(u_q^k\right)_{|k|\leq N}; \left(u_p^k\right)_{|k|\leq N}\right) \in \mathbb{P}_N L^2 \equiv \mathbb{R}^{2n(N+1)} = \mathbb{R}^{k(n,N)}$:

$$\bar{u} = \mathbb{P}_N J\nabla H\big(h(q_0, q_T, u + f(q_0, q_T, u))\big), \qquad (\star\star)$$

where

$$\bar{u} := \left(u_q^{-N}, \dots, u_q^{-1}, \frac{q_T - q_0}{T}, u_q^1, \dots, u_q^N; \left(u_p^k\right)_{|k|\leq N}\right),$$

then, by adding term by term the above two relations $(\star)$ and $(\star\star)$, the curve $\gamma = h(q_0, q_T, u + f(q_0, q_T, u))$ solves the Hamiltonian equations, its projection on $\mathbb{R}^n$ starts from q_0 and ends in q_T. □

[2] $\dim(\mathbb{P}_N L^2([0,T], \mathbb{R}^{2n})) = 2n(N+1) := k(n,N)$

Remark 9. *The above hypothesis of boundness of the second derivatives of H could be weakened by an analogous Lipschitz boundness for the Hamiltonian vector field X_H, but in this case the differentiability of the fixed point f, inherited from the implicit function theorem, does not work anymore.*

Lemma 10. *The map*

$$\begin{array}{rcl} S : \mathbb{R}^n \times \mathbb{R}^n \times R^{k(n,N)} & \to & \mathbb{R} \\ (q_0, q_T, u) & \mapsto & S(q_0, q_T, u) := A[q_0, q_T, u + f(q_0, q_T, u)] \end{array}$$

is a global generating function for Λ.

Proof. Put $\gamma = h(q_0, q_T, u + f(u, q_0, q_T)) = \big(q(\cdot, u, q_0, q_T), p(\cdot, u, q_0, q_T)\big)$, so we have that

$$\begin{aligned} \frac{\partial S}{\partial u} &= \int_0^T \left[\left(\dot{q} - \frac{\partial H}{\partial p} \right) \frac{\partial p}{\partial u} - \left(\dot{p} + \frac{\partial H}{\partial q} \right) \frac{\partial q}{\partial u} \right] ds + p \cdot \frac{\partial q}{\partial u} \bigg|_0^T, \\ &= \int_0^T (\dot{\gamma} - J \nabla H) \cdot J \nabla_u \gamma = \int_0^T (u - \mathbb{P}_N J \nabla H) \cdot J \nabla_u \gamma, \end{aligned}$$

because $(\star)$ holds. The boundary term $\frac{\partial q}{\partial u}\Big|_0^T$ vanishes, because

$$\begin{aligned} q(0, u, q_0, q_T) &\equiv q_0, \\ q(T, u, q_0, q_T) &\equiv q_T. \end{aligned}$$

Now it is easy to see that if $(q_0, q_T, \phi) \in \mathbb{R}^n \times \mathbb{R}^n \times L^2$ satisfies

$$p(0) = -\frac{\partial A}{\partial q_0}, \qquad p(T) = \frac{\partial A}{\partial q_T}, \qquad 0 = \frac{DA}{D\phi},$$

then $(q_0, q_T, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{k(n,N)}$ with $u = \mathbb{P}_N \phi$ satisfies:

$$p(0) = -\frac{\partial S}{\partial q_0}, \qquad p(T) = \frac{\partial S}{\partial q_T}, \qquad 0 = \frac{\partial S}{\partial u}.$$

On the other hand, if $(q_0, q_T, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{k(n,N)}$ satisfies the previous statement, then, setting $\phi = u + f(q_0, q_T, u)$, we have that (q_0, q_T, ϕ) satisfies

$$p(0) = -\frac{\partial A}{\partial q_0}, \qquad p(T) = \frac{\partial A}{\partial q_T}, \qquad 0 = \frac{DA}{D\phi}.$$

Note that the extremal momenta $p(0)$ and $p(T)$ are determined by the unique uniform continuous extension of γ. $\square$

Proof of Corollary 3.
Meaning $\sup_{\gamma\in G(q_0,q_1)} \sup_{t\in[0,1]} |\gamma(t)| = 0$ if $G(q_0, q_1) = \emptyset$, we easily see that

$$\sup_{(q_0,q_1)\in\mathcal{K}\times\mathcal{K}} \ \sup_{\gamma\in G(q_0,q_1)} \ \sup_{t\in[0,1]} |\gamma(t)| = c < +\infty,$$

that is $q(\cdot)$ and $p(\cdot)$ are bounded and by the constant c we can determine an upper bound for $|\nabla^2 H|$ which is essential in the proof of the above Theorem 1. Indeed, arguing by contradiction, if $c = +\infty$, we could define a sequence $\{(q_0^{(m)}, q_1^{(m)})\}_{m\in\mathbb{N}} \in \mathcal{K}\times\mathcal{K}$ such that

$$\sup_{\gamma\in G(q_0^{(m)},q_1^{(m)})} \ \sup_{t\in[0,1]} |\gamma(t)| > m.$$

By compactness, there exists a sub-sequence $\{(q_0^{(m_l)}, q_1^{(m_l)})\}_{l\in\mathbb{N}}$ converging to $(\bar{q}_0, \bar{q}_1) \in \mathcal{K}\times\mathcal{K}$.
Thus, between $\bar{q}_0$ and $\bar{q}_1$, there exists a finite number of geodesics $\bar{\gamma}_\alpha \in H^1, \alpha = 1, ..., J < +\infty$, which by Sobolev are continuous, hence

$$\sup_{t\in[0,1]} \ \sup_{\alpha=1,\ldots,J} |\bar{\gamma}_\alpha(t)| = \max_{t\in[0,1]} \ \max_{\alpha=1,\ldots,J} |\bar{\gamma}_\alpha(t)| = c < +\infty$$

which is a contradiction.

If we suppose as in Remark 4, instead of finiteness, that all geodesics joining pairs of points of $\mathcal{K}$ are contained in a unique compact set $\mathcal{G} \subset H^1$ —see $(**)'$— by Sobolev we find

$$\sup_{t\in[0,1]} \ \sup_{\gamma\in\mathcal{G}} |\bar{\gamma}(t)| = \sup_{\gamma\in\mathcal{G}} \|\gamma\|_{C^0} \le c_0 \sup_{\gamma\in\mathcal{G}} \|\gamma\|_{H^1} \le c_0\, c_1 < +\infty$$

where $c_0, c_1 > 0$ are suitable constants.

As announced in the statement, we take definitively $T = 1$. The generating function $S(q_0, q_1, u)$ so obtained is precisely a (generalized, with auxiliary parameters) World Function of the geodesic system. This is due to the *invariance* of the solutions of the geodesic equations with respect to general *affine* transformations of the evolution parameter

$$t = a\,\bar{t} + b, \quad \text{for every fixed} \quad a > 0,\ b \in \mathbb{R}.$$

In such a case it is simple to see that $(H = \frac{1}{2} g_{ij}^{-1}(q) p_i p_j)$

$$\gamma(t) = (q(t), p(t)) \text{ solves } \frac{d\gamma}{dt}(t) = J\nabla H(\gamma(t))$$

$$\text{if and only if}$$

$$\bar{\gamma}(\bar{t}) = (\bar{q}(\bar{t}), \bar{p}(\bar{t})) := (q(t(\bar{t})), p(t(\bar{t}))a) \text{ solves } \frac{d\bar{\gamma}}{d\bar{t}}(\bar{t}) = J\nabla H(\bar{\gamma}(\bar{t})).$$

Finally, by recalling that S is obtained by reduction of Hamilton-Helmholtz Action Functional A, from the invariance (cf. the original formula by Synge in the Introduction, and recalling also the Legendre transformation)

$$T\int_{t=0}^{t=T} (p \cdot \dot{q} - H)|_{\gamma} dt = (\bar{t}_{fin} - \bar{t}_{in}) \int_{\bar{t}_{in}}^{\bar{t}_{fin}} (p \cdot \dot{q} - H)|_{\bar{\gamma}} d\bar{t},$$

$$\bar{t}_{in} = -\frac{b}{a}, \;\; \bar{t}_{fin} = \frac{T-b}{a},$$

we have that (see Lemma 10),

$$S(q_0, q_1, u) = \int_{t=0}^{t=1} (p \cdot \dot{q} - H) dt\Big|_{\gamma = h(q_0, q_1, u + f(q_0, q_1, u))}$$

represents exactly the (generalized, with auxiliary parameters) Synge World Function. □

Proof of Corollary 5.
From the very definition of $S(q_0, q_1, u)$, along the geodesics, it holds that

$$S = \int_0^1 (p \cdot \dot{q} - H) dt\Big|_{\gamma = h(q_0, q_1, u + f(q_0, q_1, u))} = \int_0^1 L dt\Big|_{\gamma = h} = \int_0^1 H dt\Big|_{\gamma = h},$$

so,

$$S(q_0, q_1, u) = H(q_1, \frac{\partial S}{\partial q_1}(q_0, q_1, u)), \qquad \frac{\partial S}{\partial u}(q_0, q_1, u) = 0.$$

in particular: sgn(H)=sgn(S). For $\frac{\partial S}{\partial u}(q_0, q_1, u) = 0$, we compute:

$$H(q_1, \frac{\partial \mathcal{S}}{\partial q_1}(q_0, q_1, u)) = \frac{1}{2} g^{ij}(q_1) \frac{|e| \frac{\partial S}{\partial q_1^i} \frac{\partial S}{\partial q_1^j}}{|S|} = |e| \frac{H}{|H|} = |e| \text{sgn} H,$$

so finally: $H = e$.

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