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ANALYTICAL FORMULATION OF FAST FLYBY TRAJECTORIES AROUND A TRIAXIAL BODY

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Using the Born approximation as the unperturbed solution, the relative trajectory of the Two-Body problem is obtained by a straightforward application of the variation of parameters technique. This is done both for the case of two point-masses and the case in which the primary is a finite rigid body, whose potential can be described by the MacCullagh's approximation. In both instances the resulting analytical formulation is expressed directly in terms of simple functions of the time, without recourse to the use of Kepler's equation. Although valid as an approximation for all three types of Keplerian motion, this solution is particularly useful in modeling high speed flyby trajectories. An evaluation of the closeness of the present approximate analytical formulation to the exact solution is given. For the two point-masses case the true trajectory is obtained from hyperbolic motion, for the case of an extended primary, the solution is obtained from numerical integration. The theory presented in this contribution has been developed mainly as a quick computational tool for application to covariance studies. In particular, it can be applied to assess the sensitivity of flyby trajectory determination as a sensing tool for the determination of the position and mass of the disturbing body.

In this respect it is also possible to envision the application of this analytical formulation (restricted to the perturbations induced by the Keplerian term) to the determination of these parameters by using real observational data. Another possible application is the determination of the moments of inertia of a body from the analysis of the flyby radio tracking data. As an example, given the high accuracy expected from the tracking in K band during the Cassini mission, it is expected that the determination of the moment of inertia of Titan is possible by the combined analysis of Earth-based doppler tracking data and synthetic aperture data of its surface.

INTRODUCTION

It is of interest in the design of radio experiments in interplanetary missions to have a quick tool to perform covariance or sensitivity analyses. One application in point is the use of accurate radiometric tracking data to measure the mass of small bodies by analysing the behaviour of a spacecraft during a flyby.[11] More recently proposals have been made to use the same basic technique to also measure the irregularity of the mass distribution at least in terms of the second order inertia tensor of larger bodies like Titan, which will be visited repeatedly during the Cassini tour of the Saturnian system.[10] Previous suggestions in this sense were made for the Galileo mission.[4] It has also been suggested that both the monopole and the quadrupole parameters be determined in a combined manner in order to tie the orbit of the attracting center–Titan, in this case–to the accurately known orbit of the

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spacecraft.[5] In practice, it has been suggested to improve the orbital parameters of Titan by analysing the perturbations it inflicts on the Cassini spacecraft. It is appealing to compare this line of action with the approach that Leverrier and Adams took in order to determine the orbital elements of Neptune by looking at the unexplained perturbations in the observed orbit of Uranus.

It is clear that in the analysis of observational data the best and most complete models of the physics and dynamics involved in the experiments should be used. However, at the design level and in order to assess the quality of the expected results some coarser modelling may be acceptable. In the context of a sensitivity analysis of the radiometric experiments briefly recalled above, it has been suggested to use analytical tools based on the simplest representation of a flyby orbit: uniform motion along a straight line,[10] also known as the Born approximation, which represents the solution to the problem of motion in the presence of no mass. Subsequently a more refined approach was introduced whereby the reference trajectory more closely represents the actual orbit.[1] This was accomplished by applying the method of variation of parameters to the Born approximation.

It is the purpose of the present contribution to attack the problem in much the same way with the difference that we apply the method of perturbations of the coordinates, without recourse to the use of Herrick's perturbed derivatives[8] and parametrisations of the orbit based on the osculating trihedron used in.[1] We apply our method both to monopole and quadrupole perturbations, which in this scheme are additive. The perturbation equations can be solved explicitly in terms of logarithmic and rational functions of the time. This is a useful feature, since no introduction of awkward intermediate variables, like the eccentric anomaly in the Two-Body problem, is involved.

THE EQUATIONS OF MOTION

The equations of motion of a massless particle orbiting in the potential field $\mathcal{U}$ of an extended non-spherical body are

$$\ddot{\mathbf{r}} = \nabla \mathcal{U} \tag{1}$$

The force field can be taken for simplicity as due to a central newtonian *monopole* term $\mathbf{f}_m$ and a *quadrupole* term $\mathbf{f}_q$ stemming from the second degree expansion of the potential in spherical harmonics, so that in a reference frame centered on the center of mass of the extended body we have

$$\nabla \mathcal{U} = \mathbf{f}_m \left(\mathbf{r}, t\right) + \mathbf{f}_q \left(\mathbf{r}, t\right). \tag{2}$$

The usual approach to solving the equations (1) is to first consider the monopole term leading to the well-known keplerian solutions. Then the quadrupole term is taken into account in a perturbation scheme based on a keplerian reference orbit. It is the purpose of this contribution to take a different approach and consider the full force model (2) as a perturbation term. The small parameter for the perturbation expansion is ultimately related to the deflection angle in the case of the monopole term and to the dynamical ellipticities for the quadrupole terms. The perturbation scheme cannot then be based on one single parameter, but as a justification we will assume that the two parameters at play are of the same order of smallness.

The present work aims at developing a solution to the equations of motion

$$\ddot{\mathbf{r}} = \mathbf{f}_m\left(\mathbf{r}, t\right) + \mathbf{f}_q\left(\mathbf{r}, t\right) \tag{3}$$

in the special case of fast flyby trajectories by adopting the method of perturbations in the coordinates, as will be explained in the following sections.

THE METHOD OF PERTURBATIONS OF THE COORDINATES

The equation (3) can be solved by applying the well-known method of the perturbations of the coordinates[3].[6] Considering that a fast flyby trajectory closely resembles a straight line swept by a particle almost insensitive to the presence of an attracting body, it is of interest to adopt the point of view of the Born approximation of the Old Quantum Theory.[2] The reference or unperturbed trajectory $\mathbf{r}^*\left(t\right)$ therefore satisfies the equation of motion

$$\ddot{\mathbf{r}}^* = \mathbf{0}\,, \tag{4}$$

subject to the initial conditions $\mathbf{r}^*\left(t_0\right) = \mathbf{r}_0$, $\mathbf{v}^*\left(t_0\right) = \mathbf{v}_0$, where it is convenient to choose t_0 as the epoch of pericenter passage.

The solution to this equation, which we'll refer to as the Born solution, is then

$$\begin{cases} \mathbf{r}^*\left(t\right) = \mathbf{r}_0 + \mathbf{v}_0\left(t - t_0\right)\,, \\ \mathbf{v}^*\left(t\right) = \mathbf{v}_0, \end{cases} \tag{5}$$

where $\mathbf{r}_0$ and $\mathbf{v}_0$ constant, mutually orthogonal vectors.

The perturbed problen is again defined by the equation

$$\ddot{\mathbf{r}} = \mathbf{f}_m\left(\mathbf{r}, t\right) + \mathbf{f}_q\left(\mathbf{r}, t\right)\ . \tag{6}$$

Define $\delta\mathbf{r}$ as the difference between the perturbed and the unperturbed positions, i.e.,

$$\delta\mathbf{r} = \mathbf{r} - \mathbf{r}^*\,, \tag{7}$$

from which

$$\delta\ddot{\mathbf{r}} = \ddot{\mathbf{r}} - \ddot{\mathbf{r}}^* = \mathbf{q}\left(\mathbf{r}, t\right)\ , \tag{8}$$

where $\mathbf{q}$ indicates the perturbing acceleration,

$$\mathbf{q}\left(\mathbf{r}, t\right) = \mathbf{f}_m\left(\mathbf{r}, t\right) + \mathbf{f}_q\left(\mathbf{r}, t\right)\ . \tag{9}$$

Equation (8) represents a system of ODEs of the sixth order. We look for a solution in terms of the well-known classical scheme of perturbations of the coordinates,[3].[6] Application of this method entails expressing the disturbing force as a function of the time only, i.e., a reference orbit must be assumed along which to compute the perturbing terms. We have therefore

$$\delta\ddot{\mathbf{r}} = \mathbf{p}\left(t\right) \tag{10}$$

with

$$\mathbf{p}\left(t\right)=\mathbf{f}_{m}\left(\mathbf{r}^{*},t\right)+\mathbf{f}_{q}\left(\mathbf{r}^{*},t\right). \tag{11}$$

Defining the state vector $\delta\mathbf{x}=\left(\begin{array}{cc}\delta\mathbf{r} & \delta\dot{\mathbf{r}}\end{array}\right)^{T}$, the system (8) can be transformed into the system of first order ODEs

$$\left\{\begin{array}{l}\frac{d}{dt}\delta\mathbf{r}=\delta\dot{\mathbf{r}},\\ \frac{d}{dt}\delta\dot{\mathbf{r}}=\mathbf{p}\left(t\right),\end{array}\right. \tag{12}$$

or more compactly into

$$\delta\dot{\mathbf{x}}=\mathbf{A}\,\delta\mathbf{x}+\mathbf{Q}\left(t\right)\ , \tag{13}$$

where

$$\mathbf{A}=\left(\begin{array}{cc}\mathbf{0} & \mathbf{I}\\ \mathbf{0} & \mathbf{0}\end{array}\right), \tag{14}$$

and

$$\mathbf{Q}\left(t\right)=\left(\begin{array}{c}\mathbf{0}\\ \mathbf{p}\left(t\right)\end{array}\right), \tag{15}$$

and $\mathbf{0}$ and $\mathbf{I}$ are respectively the 3×3 zero and unit matrices. The initial conditions are

$$\delta\mathbf{x}\left(t_{0}\right)=\left(\begin{array}{c}\delta\mathbf{r}\left(t_{0}\right)\\ \delta\dot{\mathbf{r}}\left(t_{0}\right)\end{array}\right)=\left(\begin{array}{c}\mathbf{r}\left(t_{0}\right)-\mathbf{r}_{0}\\ \dot{\mathbf{r}}\left(t_{0}\right)-\mathbf{v}_{0}\end{array}\right). \tag{16}$$

The solution of the non-homogeneous linear differential equation (13) can easily be expressed in the form

$$\delta\mathbf{x}\left(t\right)=\Phi\left(t,t_{0}\right)\delta\mathbf{x}\left(t_{0}\right)+\int_{t_{0}}^{t}\Phi\left(t,\tau\right)\mathbf{Q}\left(\tau\right)\,d\tau\,, \tag{17}$$

where

$$\Phi\left(t,t_{0}\right)=R(t)R^{-1}(t_{0})\,, \tag{18}$$

is the state transition matrix and $R(t)$ is a fundamental matrix of solutions. The matrix $R(t)$ can be obtained as the matrix of partial derivatives of the unperturbed solution with respect to the integration constants. In the present case

$$R(t)=\frac{\partial\mathbf{x}^{*}}{\partial\mathbf{x}_{0}}=\left(\begin{array}{cc}\mathbf{I} & \mathbf{I}\left(t-t_{0}\right)\\ \mathbf{0} & \mathbf{I}\end{array}\right) \tag{19}$$

and since the inverse R^{-1} only differs from R in the sign of the off-diagonal term, we easily find

$$\Phi\left(t,\tau\right)=R(t)R^{-1}(\tau)\;=\left(\begin{array}{cc}\mathbf{I} & \mathbf{I}\left(t-\tau\right)\\ \mathbf{0} & \mathbf{I}\end{array}\right). \tag{20}$$

Assuming osculating initial conditions, which implies $\delta\mathbf{x}\left(t_{0}\right)=\mathbf{0}$, equation (17) provides the quadrature formulas

$$\delta\mathbf{r}(t)=\int_{t_{0}}^{t}\left(t-\tau\right)\mathbf{p}\left(\tau\right)d\tau\,, \tag{21}$$

$$\delta\dot{\mathbf{r}}(t)=\int_{t_{0}}^{t}\mathbf{p}\left(\tau\right)d\tau\,. \tag{22}$$

Since definite integration is a linear operator, we find upon substitution of $\mathbf{p}\,(\tau)$ from (11), that

$$\delta\mathbf{r}(t) = \delta\mathbf{r}_m(t) + \delta\mathbf{r}_q(t)\,, \tag{23}$$

$$\delta\dot{\mathbf{r}}(t) = \delta\dot{\mathbf{r}}_m(t) + \delta\dot{\mathbf{r}}_q(t)\,, \tag{24}$$

where

$$\delta\mathbf{r}_m(t) = \int_{t_0}^{t} (t-\tau)\,\mathbf{f}_m\left(\mathbf{r}^*,\tau\right)d\tau, \qquad \delta\dot{\mathbf{r}}_m(t) = \int_{t_0}^{t} \mathbf{f}_m\left(\mathbf{r}^*,\tau\right)d\tau \tag{25}$$

and

$$\delta\mathbf{r}_q(t) = \int_{t_0}^{t} (t-\tau)\,\mathbf{f}_q\left(\mathbf{r}^*,\tau\right)d\tau, \qquad \delta\dot{\mathbf{r}}_q(t) = \int_{t_0}^{t} \mathbf{f}_q\left(\mathbf{r}^*,\tau\right)d\tau \tag{26}$$

represent the contributions stemming from the monopole and quadrupole terms.

It is important to note from that within this perturbation scheme the monopole and quadrupole perturbations behave linearly. We now then proceed to the separate evaluation of each contribution.

THE MONOPOLE PERTURBATION

The Perturbed Born Approximation

In order to compute the solutions(25) we first note that the position formula can also be rewritten as

$$\delta\mathbf{r}_m(t) = t\,\delta\dot{\mathbf{r}}_m(t) - \int_{t_0}^{t} \tau\,\mathbf{f}_m\left(\mathbf{r}^*,\tau\right)d\tau \tag{27}$$

so that the definite integrals involved are

$$\int_{t_0}^{t} \mathbf{f}_m\left(\mathbf{r}^*,\tau\right)d\tau \qquad \text{and} \qquad \int_{t_0}^{t} \tau\,\mathbf{f}_m\left(\mathbf{r}^*,\tau\right)d\tau. \tag{28}$$

We now express $\mathbf{f}_m\left(\mathbf{r}^*,t\right)$ in terms of t. Upon introduction of the constant $s = v_0/r_0$ and by noting that $\mathbf{r}_0\cdot\mathbf{v}_0 = 0$, absolute value of $\mathbf{r}^*\,(t)$ can be written

$$r^*\,(t) = r_0\sqrt{1 + s^2\left(t-t_0\right)^2}\,, \tag{29}$$

so that after recalling that $\mathbf{f}_m\left(\mathbf{r}^*,\tau\right) = -\mu\mathbf{r}^*/r^{*3}$, we have

$$\delta\dot{\mathbf{r}}_m(t) = -\mu\int_{t_0}^{t} \frac{\mathbf{r}_0 + \mathbf{v}_0\left(\tau-t_0\right)}{r_0^3\left[1+s^2\left(\tau-t_0\right)^2\right]^{3/2}}\,d\tau. \tag{30}$$

After introducing the parameter ε defined by

$$\varepsilon = \frac{\mu}{r_0\,v_0^2}, \tag{31}$$

the final expressions for the position and velocity perturbations $\delta\mathbf{r}_m(t)$ e $\delta\dot{\mathbf{r}}_m(t)$ due to the monopole term are

$$
\begin{aligned}
\delta\mathbf{r}_m(t) = &-\frac{\varepsilon}{r_0}\mathbf{r}_0\left(\sqrt{r_0^2+v_0^2\left(t-t_0\right)^2}-r_0\right) \\
&-\varepsilon\,\mathbf{v}_0\left\{\left(t-t_0\right)-\frac{r_0}{v_0}\ln\left(\frac{1}{r_0}\left(v_0\left(t-t_0\right)+\sqrt{r_0^2+v_0^2\left(t-t_0\right)^2}\right)\right)\right\}\,, \qquad (32)
\end{aligned}
$$

$$
\delta\dot{\mathbf{r}}_m(t) = -\varepsilon\,\frac{v_0^2}{r_0}\frac{\left(t-t_0\right)}{\sqrt{r_0^2+v_0^2\left(t-t_0\right)^2}}\mathbf{r}_0-\varepsilon\,\mathbf{v}_0\left(1-\frac{r_0}{\sqrt{r_0^2+v_0^2\left(t-t_0\right)^2}}\right)\,. \qquad (33)
$$

Recalling that $\delta\mathbf{r} = \mathbf{r} - \mathbf{r}^*$ it follows that the perturbed trajectory is finally

$$
\mathbf{r}_m(t) = \mathbf{r}_0 + \mathbf{v}_0\left(t-t_0\right) + \delta\mathbf{r}_m(t)\,, \qquad (34)
$$

$$
\dot{\mathbf{r}}_m(t) = \mathbf{v}_0 + \delta\dot{\mathbf{r}}_m(t). \qquad (35)
$$

It follows that by collecting the initial position and velocity vectors, the solution takes the explicit form

$$
\begin{aligned}
\mathbf{r}_m(t) = &\left(1-\frac{\varepsilon}{r_0}\left(\sqrt{r_0^2+v_0^2\left(t-t_0\right)^2}-r_0\right)\right)\mathbf{r}_0 \\
&+\left\{\left(1-\varepsilon\right)\left(t-t_0\right)+\varepsilon\frac{r_0}{v_0}\ln\left(\frac{1}{r_0}\left(v_0\left(t-t_0\right)+\sqrt{r_0^2+v_0^2\left(t-t_0\right)^2}\right)\right)\right\}\mathbf{v}_0\,, \qquad (36)
\end{aligned}
$$

$$
\dot{\mathbf{r}}_m(t) = -\varepsilon\,\frac{v_0^2}{r_0}\frac{\left(t-t_0\right)}{\sqrt{r_0^2+v_0^2\left(t-t_0\right)^2}}\mathbf{r}_0+\left(1-\varepsilon\left(1-\frac{r_0}{\sqrt{r_0^2+v_0^2\left(t-t_0\right)^2}}\right)\right)\mathbf{v}_0\,. \qquad (37)
$$

Comparison with the Keplerian Solution

It is of interest at this point to assess the goodness of the perturbed Born approximation found in the previous section. It will be sufficient for this to compare the expression for the deflection angle δ and the relationship between the speeds at pericenter v_0 and at infinity v_∞. We start from the expression for the keplerian eccentricity of a hyperbolic orbit

$$
e = 1 + \frac{r_0 v_\infty^2}{\mu}\,, \qquad (38)
$$

where r_0 is the pericenter distance, which leads to the expression

$$
\sin\frac{\delta}{2} = \frac{1}{e} \qquad (39)
$$

for the deflection angle δ. In the keplerian case we also have that v_∞ is related to v_0 by the expression

$$
v_\infty^2 = v_0^2 - \frac{2\mu}{r_0} = v_0^2\left(1-2\varepsilon\right)\,. \qquad (40)
$$

where we have used the definition (31) for the parameter ε. If we expand to second order in ε we obtain

$$v_\infty \cong v_0 \left(1 - \varepsilon - \frac{1}{2}\varepsilon^2\right) . \tag{41}$$

Carrying out a similar expansion for the deflection angle δ, we find

$$\sin\frac{\delta}{2} = \frac{\varepsilon}{1-\varepsilon} \cong \varepsilon\left(1+\varepsilon\right) . \tag{42}$$

We now turn to the determination of the analogous quantities from the Born approximation model. To obtain the speed at infinity, consider the limit of the velocity for ever increasing distances, i.e. for $t \to \infty$. From equation (37) we get

$$\mathbf{v}_\infty = \lim_{t\to\infty} \dot{\mathbf{r}}(t) = -\varepsilon\,\frac{v_0}{r_0}\mathbf{r}_0 + \left(1-\varepsilon\right)\mathbf{v}_0 \tag{43}$$

and consequently

$$v_\infty = v_0\left(1 - 2\varepsilon + 2\varepsilon^2\right)^{1/2} \cong v_0\left(1 - \varepsilon + \varepsilon^2\right) \tag{44}$$

where we have retained terms up to the second order ε. The comparison of this expression with the corresponding keplerian expression (41) shows agreement to first order in ε.

Consider now Figure 1. It is easy to see that the sine of half the deflection angle is given by the ratio of the component $v_{\infty x}$ of the velocity at infinity along the direction of the pericenter and the speed at infinity v_∞

$$\sin\frac{\delta}{2} = \frac{v_{\infty x}}{v_\infty} . \tag{45}$$

Since equation (37) shows that $v_{\infty x} = -\varepsilon\, v_0$, we find

$$\sin\frac{\delta}{2} = \frac{\varepsilon}{\left[1 + 2\varepsilon\left(\varepsilon - 1\right)\right]^{1/2}} \cong \varepsilon\left(1-\varepsilon\right) , \tag{46}$$

where we have made use of (44) and expanded to second order in ε. Again we notice that the agreement between the exact keplerian expression and the perturbed Born approximation extends to first order terms.

The perturbed Born approximation has been implemented in a S/W tool, which was used to study parametrically the closeness of this approximation to the exact, keplerian solution. The case of the flyby of the Cassini spacecraft at Titan was considered first. We recall here that the gravitational parameter of Titan is $GM_T = 8.978173 \times 10^{12}$ m^{3}s^{-2}, that the design flyby distance is $r_0 = 4075$ km and that the relative speed at closest approach is around 5.9 km/s. This implies a deflection angle δ of approximately $7.75°$. Figure 2 shows the errors with respect the keplerian solution in the x and y coordinates and its absolute value in the flyby reference frame. The perturbed Born approximation and the keplerian solution share the dynamical state at closest approach, which takes place on the x-axis. The error on the

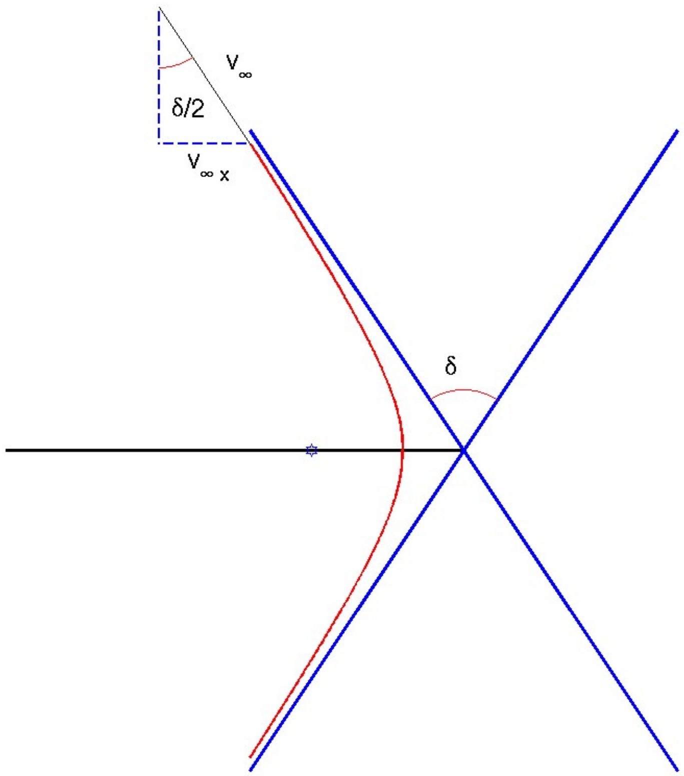

Figure 1 Schematics of a hyperbolic flyby showing the deflection angle δ and the velocity at infinity v$_\infty$.

ordinate axis is scaled by the pericenter distance r_0. The time interval of the comparison covers almost entirely flyby event within the activity sphere of Titan. It can be seen that for this case the perturbed Born approximation is good at the level of a few percentage points. Considering that this is an extreme case in the panorama of artificial flybys in the Solar System, the performance of the approximation is rather good. As another extreme case at the opposite end, we considered the NEAR flyby of Mathilde. The small mass of the asteroid coupled with a relative velocity of 9.93 km/s implies that the deflection angle is about 0.11 μrad. In this case the perturbed Born approximation matches almost perfectly (at the level of a few μmeters) the keplerian solution.

THE QUADRUPOLE PERTURBATION

In this section we address the solution of the perturbation equations for the non-sphericity of the attracting body as described by its second order inertia tensor. In so doing we adopt the description of the potential according to the well-known MacCullagh approximation. We assume that the underlying physics of the flyby event can be adequately described by keeping the orientation of the attracting body fixed. This is supported experimentally by the fact that most minor bodies of the Solar System are in slow rotation and by the high

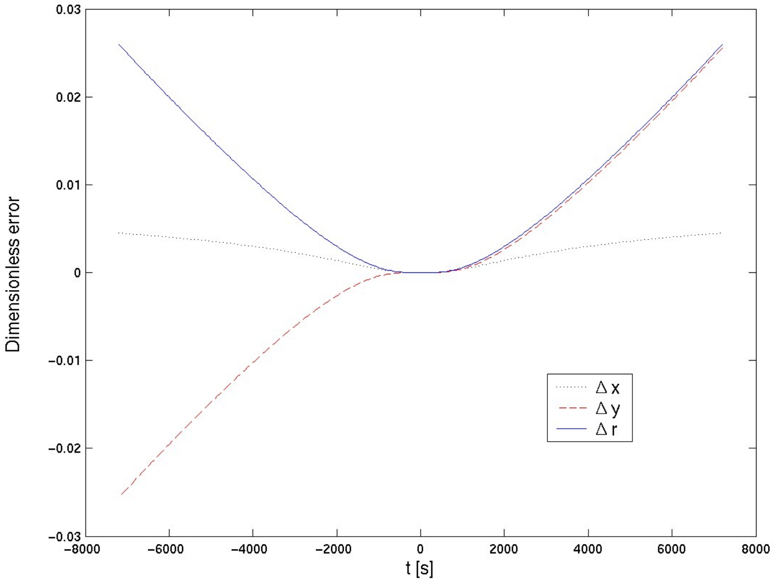

Figure 2 Errors of the perturbed Born approximation in the case of a close Titan equatorial flyby (distances scaled by r_0).

relative velocity of the encounter.

MacCullagh's Form of the Potential

The quadrupole potential can be described by MacCullagh's formula, which in a reference frame centered on the center of mass of the body reads

$$\mathcal{U}_{MC} = \frac{G}{2r^3}\left(A + B + C - 3I_{\mathbf{r}}\right) , \tag{47}$$

where A, B and C are the principal moments of inertia and $I_{\mathbf{r}}$ is the moment of inertia of the body around the line connecting its center O to the field point P identified by $\mathbf{r} = (x, y, z)$. If we take the reference frame aligned with the principal axes, then we can express $I_{\mathbf{r}}$ as a function of the principal moments to yield

$$\mathcal{U}_{MC}\left(\mathbf{r}\right) = \frac{G}{2r^3}\left\{(B-A)\left[1-3\left(\frac{y}{r}\right)^2\right] + (C-A)\left[1-3\left(\frac{z}{r}\right)^2\right]\right\}. \tag{48}$$

The acceleration is then

$$\begin{aligned}\frac{\partial}{\partial \mathbf{r}}\mathcal{U}_{MC} = &-\frac{3G}{2\,r^5}\left\{(B-A)\left[1-5\left(\frac{y}{r}\right)^2\right] + (C-A)\left[1-5\left(\frac{z}{r}\right)^2\right]\right\}\mathbf{r}\\ &-\frac{3G}{r^5}\left[(B-A)\;y\,\mathbf{j} + (C-A)\;z\,\mathbf{k}\right],\end{aligned} \tag{49}$$

where $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors along the y and z axes.

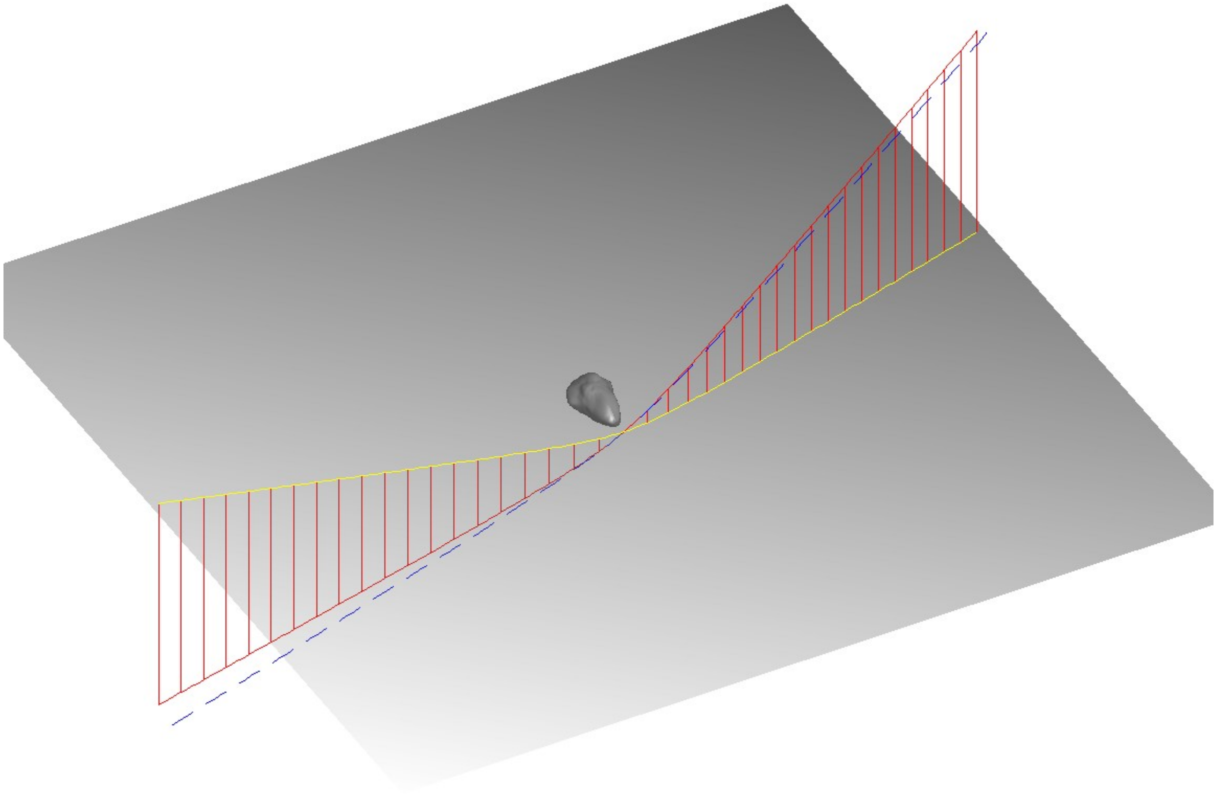

Figure 3 Schematics of a minor body flyby showing for the same conditions at closest approach the keplerian hyperbolic orbit (dashed line) and the orbit perturbed by the quadrupole field (continuous line). The projection of the perturbed orbit onto the equatorial plane is also shown. The inclination at closes approach is 30 deg.

Solution of the Perturbation Equations

The perturbation equations are (26), or

$$\delta\mathbf{r}_q(t) = \int_{t_0}^{t} (t-\tau)\,\mathbf{f}_q\left(\mathbf{r}^*,\tau\right)d\tau, \tag{50}$$

$$\delta\dot{\mathbf{r}}_q(t) = \int_{t_0}^{t} \mathbf{f}_q\left(\mathbf{r}^*,\tau\right)d\tau, \tag{51}$$

with $\mathbf{f}_q\left(\mathbf{r}^*,\tau\right)$ given by (49). The manipulations necessary to obtain the final expressions are rather tedious, so we give here only the end results for the perturbed position and velocity

$$\mathbf{r}_q\left(t\right) = \mathbf{r}_0 + \mathbf{v}_0\left(t-t_0\right) + \delta\mathbf{r}_q(t) = \left(1+\delta r_{\mathbf{r}_0}\right)\mathbf{r}_0 + \left[\left(t-t_0\right)+\delta r_{\mathbf{v}_0}\right]\mathbf{v}_0 + \delta r_{\mathbf{j}}\,\mathbf{j} + \delta r_{\mathbf{k}}\,\mathbf{k}\,, \tag{52}$$

$$\dot{\mathbf{r}}_q\left(t\right) = \mathbf{v}_0 + \delta\dot{\mathbf{r}}_q(t) = \delta\dot{r}_{\mathbf{r}_0}\mathbf{r}_0 + \left(1+\delta\dot{r}_{\mathbf{v}_0}\right)\mathbf{v}_0 + \delta\dot{r}_{\mathbf{j}}\,\mathbf{j} + \delta\dot{r}_{\mathbf{k}}\,\mathbf{k}\,, \tag{53}$$

where

$$\delta r_{\mathbf{r}_0} = \frac{1}{r_0^5}\left\{\begin{array}{c} D_1\mathcal{J}_0 + D_2\left[R_{12}^2\mathcal{L}_0 + 2sR_{12}R_{22}\mathcal{L}_1 + s^2R_{22}^2\mathcal{L}_2\right] \\ +D_3\ \left[R_{13}^2\mathcal{L}_0 + 2sR_{13}R_{23}\mathcal{L}_1 + s^2R_{23}^2\mathcal{L}_2\right] \end{array}\right\} \tag{54}$$

$$\delta r_{\mathbf{v}_0} = \frac{1}{r_0^5}\left\{\begin{array}{c} D_1\mathcal{J}_1 + D_2\left[R_{12}^2\mathcal{L}_1 + 2sR_{12}R_{22}\mathcal{L}_2 + s^2R_{22}^2\mathcal{L}_3\right] \\ +D_3\ \left[R_{13}^2\mathcal{L}_1 + 2sR_{13}R_{23}\mathcal{L}_2 + s^2R_{23}^2\mathcal{L}_3\right] \end{array}\right\} \tag{55}$$

$$\delta r_{\mathbf{j}} = \frac{D_4}{r_0^4}\left[R_{12}\mathcal{J}_0 + sR_{22}\mathcal{J}_1\right] \tag{56}$$

$$\delta r_{\mathbf{k}} = \frac{D_5}{r_0^4}\left[R_{13}\mathcal{J}_0 + sR_{23}\mathcal{J}_1\right] \tag{57}$$

$$\delta \dot{r}_{\mathbf{r}_0} = \frac{1}{r_0^5}\left\{\begin{array}{c} D_1\mathcal{I}_{005} + D_2\left[R_{12}^2\mathcal{I}_{007} + 2sR_{12}R_{22}\mathcal{I}_{017} + s^2R_{22}^2\mathcal{I}_{027}\right] \\ +D_3\ \left[R_{13}^2\mathcal{I}_{007} + 2sR_{13}R_{23}\mathcal{I}_{017} + s^2R_{23}^2\mathcal{I}_{027}\right] \end{array}\right\} \tag{58}$$

$$\delta \dot{r}_{\mathbf{v}_0} = \frac{1}{r_0^5}\left\{\begin{array}{c} D_1\mathcal{I}_{015} + D_2\left[R_{12}^2\mathcal{I}_{017} + 2sR_{12}R_{22}\mathcal{I}_{027} + s^2R_{22}^2\mathcal{I}_{037}\right] \\ +D_3\left[R_{13}^2\mathcal{I}_{017} + 2sR_{13}R_{23}\mathcal{I}_{027} + s^2R_{23}^2\mathcal{I}_{037}\right] \end{array}\right\} \tag{59}$$

$$\delta \dot{r}_{\mathbf{j}} = \frac{D_4}{r_0^4}\left[R_{12}\mathcal{I}_{005} + sR_{22}\mathcal{I}_{015}\right] \tag{60}$$

$$\delta \dot{r}_{\mathbf{k}} = \frac{D_5}{r_0^4}\left[R_{13}\mathcal{I}_{005} + sR_{23}\mathcal{I}_{015}\right] \tag{61}$$

The many quantities appearing in these formulation are defined next. The R_{ik}'s are the elements of the rotation matrix R that maps coordinates from the principal axes frame to the *flyby frame* defined by the three unit vectors along the pericenter, the velocity at pericenter and the normal to the instantaneous orbit plane defined by these two vectors. Explicitly,

$$R = \begin{pmatrix} \hat{\mathbf{r}}_0\cdot\mathbf{i} & \hat{\mathbf{r}}_0\cdot\mathbf{j} & \hat{\mathbf{r}}_0\cdot\mathbf{k} \\ \hat{\mathbf{v}}_0\cdot\mathbf{i} & \hat{\mathbf{v}}_0\cdot\mathbf{j} & \hat{\mathbf{v}}_0\cdot\mathbf{k} \\ \hat{\mathbf{n}}\cdot\mathbf{i} & \hat{\mathbf{n}}\cdot\mathbf{j} & \hat{\mathbf{n}}\cdot\mathbf{k} \end{pmatrix}. \tag{62}$$

The constants D_i, $i = 1, ..., 5$,are defined as

$$D_1 = -\frac{3G}{2}\left(B + C - 2A\right), \quad D_2 = \frac{15G}{2}\left(B - A\right), \quad D_3 = \frac{15G}{2}\left(C - A\right), \tag{63}$$

$$D_4 = -3G\left(B - A\right), \quad D_5 = -3G\left(C - A\right). \tag{64}$$

Finally, the quantities denoted by $\mathcal{I}_{ijk}$ are the definite integrals

$$\mathcal{I}_{ijk}\left(t\right) = \int_{t_0}^{t} \frac{\tau^i\left(\tau - t_0\right)^j}{\left(1 + s^2\left(\tau - t_0\right)^2\right)^{k/2}}d\tau, \tag{65}$$

which are computed in the Appendix for the required combinations of the indices i, j, k.

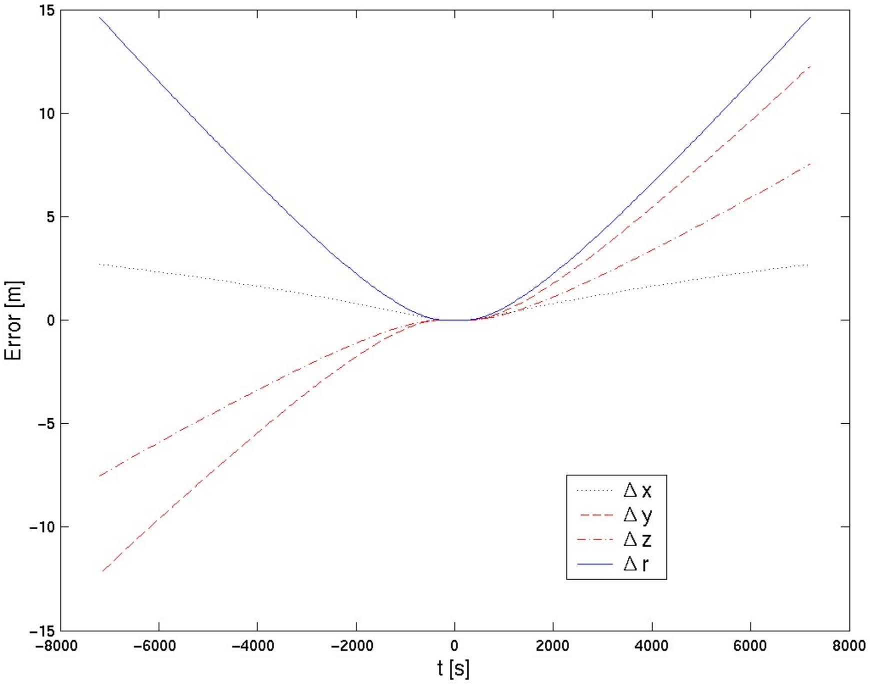

Figure 4 Errors in meters for the perturbed Born quadrupole approximation in the case of a close Titan flyby. Inclination at (equatorial) closest approach is $30\circ$.

The quantities $\mathcal{J}_0$ and $\mathcal{J}_1$ are related to these integrals through

$$\mathcal{J}_0 = t\,\mathcal{I}_{005} - \mathcal{I}_{105} \tag{66}$$

$$\mathcal{J}_1 = t\,\mathcal{I}_{015} - \mathcal{I}_{115} \tag{67}$$

while $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ are given by

$$\mathcal{L}_k = t\,\mathcal{I}_{0k7} - \mathcal{I}_{1k7}\,, \qquad k = 0, 1, 2, 3. \tag{68}$$

Evaluation of the approximation

In order to assess the goodness of the approximation found in this section, the Cassini flyby of Titan was again considered. This time the potential model was extended to include the Stokes coefficients $C_{20} = -8.413 \times 10^{-5}$ and $C_{22} = 3.107 \times 10^{-5}$, which were derived from the appropriate moments of inertia. The inclination of the flyby orbit at closest approach, which occurs on the equatorial plane on the x-axis, was taken as $30°$. The same speed of 5900 m/s at pericenter used for the monopole approximation was kept. Figure 4 shows the errors in the position in the three coordinates in the principal axes frame and the absolute value of the error. The errors are here shown in meters.

THE COMBINED SOLUTION

For completeness we provide the formulation for the combined perturbing actions of the monopole and quadrupole terms of the potential in both the flyby and the principal axes frames. In the former case we have

$$\begin{aligned}\mathbf{r} &= \left[r_0 + \alpha\left(t\right) + \gamma\left(t\right) + R_{12}\epsilon\left(t\right) + R_{13}\zeta\left(t\right)\right]\hat{\mathbf{r}}_0 \\ &\quad + \left[v_0\left(t-t_0\right) + \beta\left(t\right) + \delta\left(t\right) + R_{22}\epsilon\left(t\right) + R_{23}\zeta\left(t\right)\right]\hat{\mathbf{v}}_0 \\ &\quad + \left[R_{32}\epsilon\left(t\right) + R_{33}\zeta\left(t\right)\right]\hat{\mathbf{n}}\,, \end{aligned} \tag{69}$$

for the position and

$$\begin{aligned}\mathbf{v} &= \left[\dot{\alpha}\left(t\right) + \dot{\gamma}\left(t\right) + R_{12}\dot{\epsilon}\left(t\right) + R_{13}\dot{\zeta}\left(t\right)\right]\hat{\mathbf{r}}_0 \\ &\quad + \left[v_0 + \dot{\beta}\left(t\right) + \dot{\delta}\left(t\right) + R_{22}\dot{\epsilon}\left(t\right) + R_{23}\dot{\zeta}\left(t\right)\right]\hat{\mathbf{v}}_0 \\ &\quad + \left[R_{32}\dot{\epsilon}\left(t\right) + R_{33}\dot{\zeta}\left(t\right)\right]\hat{\mathbf{n}}\,. \end{aligned} \tag{70}$$

for the velocity. Here

$$\alpha\left(t\right) = -\varepsilon\left(\sqrt{r_0^2 + v_0^2\left(t-t_0\right)^2} - r_0\right)\,, \tag{71}$$

$$\beta\left(t\right) = -\varepsilon\left\{v_0(t-t_0) - r_0\ln\left(\frac{1}{r_0}\left(v_0\left(t-t_0\right) + \sqrt{r_0^2 + v_0^2\left(t-t_0\right)^2}\right)\right)\right\}\,, \tag{72}$$

$$\gamma\left(t\right) = \frac{1}{r_0^4}\left\{\begin{array}{c} D_1\mathcal{J}_0 + \left(D_2R_{12}^2 + D_3\,R_{13}^2\right)\mathcal{L}_0 \\ +2s\left(D_2R_{12}R_{22} + D_3R_{13}R_{23}\right)\mathcal{L}_1 + s^2\left(D_2R_{22}^2 + D_3R_{23}^2\right)\mathcal{L}_2 \end{array}\right\}, \tag{73}$$

$$\delta\left(t\right) = \frac{s}{r_0^4}\left\{\begin{array}{c} D_1\mathcal{J}_1 + \left(D_2R_{12}^2 + D_3R_{13}^2\right)\mathcal{L}_1 \\ +2s\left(D_2R_{12}R_{22} + D_3\,R_{13}R_{23}\right)\mathcal{L}_2 + s^2\left(D_2R_{22}^2 + D_3R_{23}^2\right)\mathcal{L}_3 \end{array}\right\}, \tag{74}$$

$$\epsilon\left(t\right) = \frac{D_4}{r_0^4}\left[R_{12}\mathcal{J}_0 + sR_{22}\mathcal{J}_1\right]\,, \tag{75}$$

$$\zeta\left(t\right) = \frac{D_5}{r_0^4}\left[R_{13}\mathcal{J}_0 + sR_{23}\mathcal{J}_1\right]\,. \tag{76}$$

The derivatives $\dot{\alpha}\left(t\right), \dot{\beta}\left(t\right), \ldots \dot{\zeta}\left(t\right)$ that appear in the expression (70) for the velocity can be obtained by direct differentiation of the expressions (71) to (76). In carrying out the differentiations advantage may be taken of the rules

$$\dot{\mathcal{J}}_k = \mathcal{I}_{0k5} \tag{77}$$

$$\dot{\mathcal{L}}_k = \mathcal{I}_{0k7} \tag{78}$$

according to which the integrals defining $\mathcal{J}_k$ and $\mathcal{L}_k$ in (66), (67) and (68) can be treated as constants. This is due to the fact that

$$t\,\dot{\mathcal{I}}_{0k5} = \dot{\mathcal{I}}_{1k5}. \tag{79}$$

In the principal axes frame, on the other hand, we find

$$\begin{aligned}\mathbf{r} &= \left[R_{11}\iota\left(t\right)+R_{21}\kappa\left(t\right)\right]\mathbf{i}\\ &+\left[R_{12}\iota\left(t\right)+R_{22}\kappa\left(t\right)+\epsilon\left(t\right)\right]\mathbf{j}\\ &+\left[R_{13}\iota\left(t\right)+R_{23}\kappa\left(t\right)+\zeta\left(t\right)\right]\mathbf{k}\end{aligned} \tag{80}$$

and

$$\begin{aligned}\mathbf{v} &= \left[R_{11}\dot{\iota}\left(t\right)+R_{21}\dot{\kappa}\left(t\right)\right]\mathbf{i}\\ &+\left[R_{12}\dot{\iota}\left(t\right)+R_{22}\dot{\kappa}\left(t\right)+\dot{\epsilon}\left(t\right)\right]\mathbf{j}\\ &+\left[R_{13}\dot{\iota}\left(t\right)+R_{23}\dot{\kappa}\left(t\right)+\dot{\zeta}\left(t\right)\right]\mathbf{k}\,,\end{aligned} \tag{81}$$

where the functions ι, κ are defined as

$$\iota\left(t\right) = r_0+\alpha\left(t\right)+\gamma\left(t\right), \tag{82}$$

$$\kappa\left(t\right) = v_0\left(t-t_0\right)+\beta\left(t\right)+\delta\left(t\right). \tag{83}$$

We finally remark that the solutions (69)-(70) and (80)-(81) can be used with some advantage for software implementation. In fact the present form allows to easily switch on or off the monopole and/or the quadrupole perturbations as follows

- if $\alpha=\beta=\gamma=\delta=\epsilon=\zeta\equiv 0$ provides the Born trajectory $\mathbf{r}\left(t\right)=\mathbf{r}_0+\mathbf{v}_0\left(t-t_0\right)$,
- if $\gamma=\delta=\epsilon=\zeta\equiv 0$ gives the perturbed Born approximation,
- if $\alpha=\beta\equiv 0$ provides the Born trajectory perturbed by the quadrupole terms.

CONCLUSIONS

Using the Born approximation as the unperturbed solution, the relative trajectory of the Two-Body problem was obtained by a straightforward application of the technique of perturbations in the coordinates. This was done both for the case of two point-masses and the case in which the primary is a finite rigid body, whose potential is described by MacCullagh's quadrupole approximation. In both instances the resulting analytical formulation is expressed directly in terms of simple functions of the time, without recourse to the use of Kepler's equation. Although valid as an approximation for all three types of Keplerian motion, this solution is particularly useful in modeling high speed flyby trajectories. An evaluation of the closeness of the present approximate analytical formulation to the exact solution showed that even in the case of relatively slow flybys associated with deflection angles of around 7°, the error in position reaches only up to 3% (in terms of lengths scaled by the distance of the pericenter) at the boundary of the sphere of influence.

The theory presented in this contribution has been developed mainly as a quick computational tool for application to covariance studies. In particular, it can be applied to assess the sensitivity of flyby trajectory determination as a sensing tool for the determination of the position or mass of the disturbing body. In this respect it is also possible to envision the application of this analytical formulation (restricted to the perturbations induced by the Keplerian term) to the determination of these parameters by using real observational data. Another possible application is the determination of the moments of inertia of a body from the analysis of the radio tracking data during a flyby. As an example, given the high accuracy expected from the tracking in K band during the Cassini mission, it is expected that the determination of the moment of inertia of Titan is possible by the combined analysis of Earth-based doppler tracking data of the spacecraft and synthetic aperture radar data of its surface.

APPENDIX - A TABLE OF INTEGRALS

We address here the integrals which appear in the solution of the quadrupole perturbation problem of Section . Since a general formula appears to be unavailable (e.g., see[7]), we provide an exhaustive list as follows:

$$\mathcal{I}_{005} = \frac{1}{3}\left(t-t_0\right)\left[\frac{1}{\left[1+s^2\left(t-t_0\right)^2\right]^{3/2}} + \frac{2}{\sqrt{1+s^2\left(t-t_0\right)^2}}\right], \tag{84}$$

$$\mathcal{I}_{015} = \frac{1}{3s^2}\left[1-\frac{1}{\left[1+s^2\left(t-t_0\right)^2\right]^{3/2}}\right], \tag{85}$$

$$\begin{aligned} \mathcal{I}_{105} = {} & \frac{1}{3s^2}\left[1-\frac{1}{\left[1+s^2\left(t-t_0\right)^2\right]^{3/2}}\right] \\ & +\frac{t_0}{3}\left(t-t_0\right)\left[\frac{1}{\left[1+s^2\left(t-t_0\right)^2\right]^{3/2}} + \frac{2}{\sqrt{1+s^2\left(t-t_0\right)^2}}\right], \end{aligned} \tag{86}$$

$$\mathcal{I}_{115} = \frac{1}{3s^2}\left[t_0+\frac{s^2\left(t-t_0\right)^3-t_0}{\left[1+s^2\left(t-t_0\right)^2\right]^{3/2}}\right], \tag{87}$$

$$\mathcal{I}_{007} = \frac{1}{15}\frac{\left(t-t_0\right)\left[15+4s^2\left(t-t_0\right)^2\left[2s^2\left(t-t_0\right)^2+5\right]\right]}{\left[1+s^2\left(t-t_0\right)^2\right]^{5/2}}, \tag{88}$$

$$\mathcal{I}_{017} = \frac{1}{5s^2}\left[1-\frac{1}{\left[1+s^2\left(t-t_0\right)^2\right]^{5/2}}\right], \tag{89}$$

$$\mathcal{I}_{027} = \frac{\left[5+2s^2\left(t-t_0\right)^2\right]\left(t-t_0\right)^3}{15\left[1+s^2\left(t-t_0\right)^2\right]^{5/2}}, \tag{90}$$

$$\begin{aligned}
\mathcal{I}_{037} &= \frac{1}{15s^4}\left[2-\frac{2+5s^2\left(t-t_0\right)^2}{\left[1+s^2\left(t-t_0\right)^2\right]^{5/2}}\right], && (91)\\
\mathcal{I}_{107} &= \frac{1}{5s^2}+\frac{-3+s^2t_0\left(t-t_0\right)\left[15+4s^2\left(t-t_0\right)^2\left[5+2s^2\left(t-t_0\right)^2\right]\right]}{15s^2\left[1+s^2\left(t-t_0\right)^2\right]^{5/2}}, && (92)\\
\mathcal{I}_{117} &= \frac{t_0}{5s^2}+\frac{-3t_0+s^2\left(t-t_0\right)^3\left[5+2s^2\left(t-t_0\right)^2\right]}{15s^2\left[1+s^2\left(t-t_0\right)^2\right]^{5/2}}, && (93)\\
\mathcal{I}_{127} &= \frac{2}{15s^4}+\frac{-2+s^2\left(t-t_0\right)^2\left[-5+s^2t_0\left(t-t_0\right)\left[5+2s^2\left(t-t_0\right)^2\right]\right]}{15s^4\left[1+s^2\left(t-t_0\right)^2\right]^{5/2}}, && (94)
\end{aligned}$$

$$\mathcal{I}_{137} = \frac{2t_0}{15s^4}+\frac{-2t_0+s^2\left(t-t_0\right)^2\left[-5t_0+3s^2\left(t-t_0\right)^3\right]}{15s^4\left[1+s^2\left(t-t_0\right)^2\right]^{5/2}}. \qquad (95)$$

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