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When is a vector field injective?

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Abstract

In this paper we define a notion of injectivity for a vector field over a Riemannian manifold and we give a sufficient condition for it. The proof is an extension of a well known Theorem stating that a map from an open convex set of the euclidean space $\mathcal{R}^n$ into $\mathcal{R}^n$ is a diffeomorphism onto its image provided that the bilinear operator associated to its differential is (positive or negative) definite everywhere. In the second part of the paper, we show that these results have a natural extension to the tangent bundle, with straightforward applications to second order differential equations.

1 Introduction and main Theorem

The problem considered in this paper is related to the general problem of stating the conditions under which a map f between given spaces has an inverse. Usually the spaces considered are euclidean finite dimensional ones or infinite dimensional Banach spaces (see e.g. [3] for a survey). Notice that in both cases one is dealing with vector space structures.

The prototype for global invertibility results on finite dimensional vector spaces is Theor.1.1 below. The globalization of the Legendre Transformation in Mechanics uses this theorem; also, in Thermodynamics, if a convexity property on the constitutive functions holds, then the internal energy setting and the free energy setting are related and, in some cases, the absence of convexity denotes new phenomena, like phase transitions.

In this paper we address ourselves to the case of a vector field X on a finite dimensional manifold. In this framework X takes values on a vector bundle structure, therefore the determinations of X at different points of the manifold are not directly comparable anymore, as in vector spaces. Thus, for a Riemannian manifold, we introduce a definition of injectivity for a vector field using parallel transport along geodesic of the manifold (Def. 1.3 of g-injectivity) and then we give a sufficient condition for g-injectivity that involves the definiteness of the *deformation tensor* $\mathcal{D}^X$ of X (Theor. 1.3). The proof is still modelled on the one of

Theorem 1.1, but the condition stated in Theor. 1.3 depends strongly on the given Riemannian metric (Prop. 1.2).

In the second part of the paper, we show that the above results have a natural extension to the tangent bundle manifold, with straightforward applications to second order differential equations, for example of mechanical kind.

To begin with, we recall here the aforementioned Theorem and its proof to make the paper self-contained and because the proof gives some hints for its generalization. With a minor abuse of language, we denote with $\mathrm{d}f$ the symmetric bilinear operator $h \longmapsto h^t \mathrm{d}f\ h$.

Theorem 1.1 (see [2], p. 136) *Let f be a continuously differentiable map of a open convex set Ω of $\mathcal{R}^n$ into $\mathcal{R}^n$. Suppose that $\mathrm{d}f$ is (positive or negative) definite in Ω. Then f is a diffeomorphism of Ω on its image.*

Proof. It is not restrictive to suppose that $\mathrm{d}f$ is positive definite in Ω. To every x, y in Ω, with $x \neq y$, let $c(\lambda) = \lambda y + (1-\lambda)x,\ \lambda \in [0,1]$, be the segment whose endpoints are x and y respectively. We define the map

$$\phi(\lambda) = f(c(\lambda)) \cdot (y - x), \qquad \lambda \in [0,1],$$

where dot denotes the euclidean scalar product in $\mathcal{R}^n$. If $\phi(0) \neq \phi(1)$, then the components of $f(x)$ and $f(y)$ along $y - x$ are different, therefore $f(x) \neq f(y)$ and f is injective.

To see this, note that

$$\phi(1) - \phi(0) = \int_0^1 \frac{d\phi(\lambda)}{d\lambda} d\lambda = \int_0^1 (y-x)^t \mathrm{d}f(c(\lambda))(y-x) d\lambda > 0$$

because $\mathrm{d}f$ is positive definite on the convex (hence) connected set Ω. Since $\mathrm{d}f$ has maximum rank, f is locally a diffeomorphism. Moreover, f is injective and hence f is a global diffeomorphism. □

Among the key entries of the above proof there are the convexity and injectivity notions and we turn to them first. We consider a Riemannian manifold (M, g) and a vector field X on it. In the following we will use only the Levi-Civita connection associated to g, defined as the (unique) torsion-free connection on M whose associated parallel transport is an isometry.

A minimal requirement to extend the above Theorem is that the open subset Ω of M is *geodesically connected*, that is to every couple of points in Ω there exists a geodesic curve contained in Ω between those points. This notion, which is used in e.g. [1], is too weak for our aims, since we will need uniqueness of the aforementioned geodesic. Among the definitions used in literature, let us quote the one in [4], p.355,

Definition 1.1 *The open subset Ω of the Riemannian manifold M is* strictly geodesically convex *if to every $x, y \in \Omega$ there exists a unique geodesic contained in Ω whose endpoints are x and y and such that $d(x, y) = d_\Omega(x, y)$.*

Here d is the distance function induced by the metric g by the standard infimum procedure and d_Ω is its restriction to Ω.

This definition is too restrictive for our aims; we simply ask for existence and uniqueness of a geodesic curve contained in Ω between every couple of points in Ω or, equivalently, we state the following

Definition 1.2 (g-convexity) *The open subset Ω of M is a* g-convex *set if to every $x \in \Omega$ there exists a domain $D_x \subset T_x\Omega$ such that the exponential mapping $\exp_x : D_x \longrightarrow \Omega$ is a* bijection *on Ω.*

The g-convexity requirement is motivated by the following

Definition 1.3 (g-injectivity) *A vector field $X : \Omega \longrightarrow TM$ over a g-convex subset Ω of the Riemannian manifold M is* g-injective *if to every $x, y \in \Omega$,*

$$x \neq y \Rightarrow X(y) \neq Tp(\gamma)X(x)$$

where $Tp(\gamma)$ is the parallel transport operator of the Levi-Civita connection along the unique geodesic γ with endpoints x and y.

This definition of injectivity is originated by the only conceivable way to compare vectors attached to different points of a manifold, i.e. by parallel transport. As a simple, but important consequence, we have that a g-convex subset Ω of a Riemannian manifold is *simply connected* because every smooth loop in Ω can be homotopically retracted to a point z along the geodesic arcs joining z with every point of the loop.

The following Theorem in [5] vol. 3, p. 248, gives a sufficient condition for g-convexity.

Theorem 1.2 *Let M be a complete, simply connected smooth manifold with negative sectional curvature. Then every couple of points in M is connected by a unique minimal geodesic and M is diffeomorphic to the euclidean space.*

Manifolds of positive curvature may have g-convex subsets; take the spherical cap of a S^2 sphere as an example.
Now things are in order to give the vector field version of the above Theorem 1.1.

Theorem 1.3 *Let (M, g) be a Riemannian manifold, Ω a g-convex subset of M and X a vector field on Ω. If the* deformation tensor $\mathcal{D}^X$,

$$\mathcal{D}^X_{ij} := \frac{1}{2}[g_{il}(\nabla_j X)^l + g_{jm}(\nabla_i X)^m],$$

is definite in Ω, then X is g-injective.

Proof. Given $x \neq y$ in Ω, for $s \in [0, l]$ we define

$$\phi_\gamma(s) = g(\gamma(s))(X(\gamma(s)), \dot{\gamma}(s)),$$

where γ is the arc of geodesic between x and y and l is its arc-length. Accordingly,

$$\phi_\gamma(0) = g(x)(X(x), \dot{\gamma}(0)) = g(y)(Tp(\gamma)X(x), \dot{\gamma}(l)), \quad \text{and } \phi_\gamma(l) = g(y)(X(y), \dot{\gamma}(l)).$$

The second equality above comes from the property that the parallel trasport is an isometry for the Levi-Civita connection and that geodesic curves are auto-parallel. Hence

$$\phi_\gamma(0) \neq \phi_\gamma(l) \Rightarrow X(y) \neq Tp(\gamma)X(x)$$

and g-injectivity follows. To apply the same argument used in the proof of Theorem 1.1, we calculate (remind that $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ for a geodesic)

$$\begin{aligned}
\frac{d\phi_\gamma}{ds}(s) &= \frac{d}{ds} g(\gamma(s))(X(\gamma(s)), \dot{\gamma}(s)) \\
&= g(\gamma(s))(\nabla_{\dot{\gamma}} X, \dot{\gamma}) + g(\gamma(s))(X, \nabla_{\dot{\gamma}}\dot{\gamma}) \\
&= g(\gamma(s))(\nabla_{\dot{\gamma}} X, \dot{\gamma}) \\
&= (g(\gamma(s))\nabla X)(\dot{\gamma}, \dot{\gamma}) = g_{kj}(\nabla_i X)^k \dot{\gamma}^i \dot{\gamma}^j \\
&= \frac{1}{2}[g_{il}(\nabla_j X)^l + g_{jm}(\nabla_i X)^m]\dot{\gamma}^i\dot{\gamma}^j = \mathcal{D}^X_{ij}\dot{\gamma}^i\dot{\gamma}^j.
\end{aligned}$$

Hence

$$\phi_\gamma(l) - \phi_\gamma(0) = \int_0^l \mathcal{D}^X_{ij}\dot{\gamma}^i\dot{\gamma}^j ds \neq 0. \qquad \square$$

The Theorem gives a condition which is sufficient for g-injectivity. In the following we study how g-injectivity depends on the particular metric g on M. To this, we will need the following Definition and Proposition.

Definition 1.4 ([6], p.106) *Given a Riemannian metric g on M and a positive function ρ on M, the metric*

$$\overline{g}(x) = \rho(x)g(x)$$

is a metric conformally related *to g.*

Proposition 1.1 *On a Riemannian manifold endowed with the Levi-Civita connection,*

$$2\mathcal{D}^X = \mathcal{L}_X g,$$

where $\mathcal{L}_X g$ is the Lie derivative of the metric g.

Proof. $\mathcal{L}_X g$ is defined locally as follows in [5] vol. 1, p.201

$$(\mathcal{L}_X g)_{ij} = X^s \frac{\partial g_{ij}}{\partial x^s} + g_{kj}\frac{\partial X^k}{\partial x^i} + g_{ik}\frac{\partial X^k}{\partial x^j}.$$

By the definition of the deformation tensor

$$\begin{aligned} 2\mathcal{D}^X_{ij} &= g_{ik}(\nabla_j X)^k + g_{jl}(\nabla_i X)^l \\ &= g_{ik}(\frac{\partial X^k}{\partial x^j} + \Gamma^k_{lj}X^l) + g_{jl}(\frac{\partial X^l}{\partial x^i} + \Gamma^l_{mi}X^m) \end{aligned}$$

where Γ^k_{lj} are the Christoffel' symbols of the Levi-Civita connection:

$$g_{ik}\Gamma^k_{lj} = \frac{1}{2}(\frac{\partial g_{ij}}{\partial x^l} + \frac{\partial g_{ij}}{\partial x^j} - \frac{\partial g_{lj}}{\partial x^i}).$$

Therefore

$$g_{ik}\Gamma^k_{lj} + g_{jl}\Gamma^l_{mi} = \frac{\partial g_{ij}}{\partial x^l}$$

and the thesis follows. □

This Proposition shows the interplay between a metric g and the related deformation tensor $\mathcal{D}^X$.

Proposition 1.2 *Suppose that on a Riemannian manifold (M, g) the vector field X has a deformation tensor $\mathcal{D}^X \in Sym^+$, that is $\mathcal{D}^X$ is symmetric and positive definite. Then there exists a metric $\overline{g} = \rho g$ conformally related to g such that* $\det \overline{\mathcal{D}}^X = 0$ *in some point of M.*

Proof. In the following, we drop the superscript X from $\mathcal{D}^X$ to ease the notation. We compute

$$\overline{\mathcal{D}} = \frac{1}{2}\mathcal{L}_X\overline{g} = \frac{1}{2}\mathcal{L}_X\rho g = \frac{1}{2}(\mathcal{L}_X\rho)g + \frac{1}{2}\rho\mathcal{L}_X g = \frac{1}{2}(\mathcal{L}_X\rho)g + \rho\mathcal{D}$$

where, as usual,

$$\mathcal{L}_X\rho = X^k\frac{\partial\rho}{\partial x^k} \ .$$

Since $\rho > 0$ in Ω, $\rho\mathcal{D} \in Sym^+$. Let $P = P(x) \in GL(n, \mathcal{R})$ be the matrix of the endomorphism diagonalizing $\rho\mathcal{D}(x)$ and $g(x)$ at x. We denote with starred symbols the diagonal form of $\overline{\mathcal{D}}$, $\mathcal{D}$ and g, and superscript t denotes transposition.

$$\overline{\mathcal{D}}^{\,*} = P^t\ \overline{\mathcal{D}}\ P = \frac{1}{2}(\mathcal{L}_X\rho)P^t g P + \rho P^t\mathcal{D}P = \frac{1}{2}(\mathcal{L}_X\rho)g^* + \rho\mathcal{D}^*$$

Therefore

$$\det\overline{\mathcal{D}} = 0 \ \Leftrightarrow\ \det\overline{\mathcal{D}}^{\,*} = 0 \Leftrightarrow \prod_{i=1}^n(\frac{1}{2}(\mathcal{L}_X\rho)g^*_{ii} + \rho\mathcal{D}^*_{ii}) = 0.$$

The above product vanishes at a fixed $\overline{x} \in M$ if, for some $i' \in \{1, \dots n\}$,

$$\mathcal{L}_X \rho + \varphi_{i'} \rho = 0 \tag{1.1}$$

where $\varphi_{i'}(\overline{x}) = 2\mathcal{D}^*_{i'i'}(\overline{x})(g^*_{i'i'}(\overline{x}))^{-1} > 0$. We define a conformal deformation ρ in this way: $x = (x^1, \dots, x^n)$ are local coordinates on a open set U containing $\overline{x}$ and $\xi(x)$ is a 'bump' function at $\overline{x}$, that is a smooth map with compact support B contained in U, with $\xi(\overline{x}) = 1$, and ξ is vanishing ouside B. Then we define ρ to be

$$\rho(x) = \exp\big(\frac{-\varphi_{i'}(\overline{x})\xi(x)X(\overline{x}) \cdot (x - \overline{x})}{|X(\overline{x})|^2}\big) \qquad \forall x \in U$$

and $\rho(x) = 1$ outside U. It is easy to check that ρ above is globally defined on M, it is positive and it satisfies (1.1) at $\overline{x}$. □

2 An extension to TM

In this Section we show that the previous results and definitions for (M, g), Ω and X have an extension to TM. Given a Riemannian manifold (M, g) referred to local coordinates x, the *complete lift* of the metric g is the metric g^C on TM with fibered coordinates (x, u), defined as (see [8], p. 38)

$$g^C = (\partial g_{ij}) du^i du^j + 2 g_{ij} du^i dx^j,$$

where, for a function f on M, its complete lift ∂f is the function on TM defined as $(\partial f)(x, u) = (\partial_i f) u^i$.

The following theorem illustrates the relation between geodesics of the Levi-Civita connection on (M, g) and on (TM, g^C). Here $\tau : TM \longrightarrow M$ is the tangent bundle fibration.

Theorem 2.1 ([8], p. 58) *The curve $\hat{\gamma}$ on TM is a geodesic of the Levi-Civita connection on TM associated to the metric g^C if and only if its projection $\tau(\hat{\gamma}) = \gamma$ is a geodesic on M and the vector field W associated to $\hat{\gamma}$, i.e. $\hat{\gamma} = (\gamma, W)$, is a* Jacobi field *along γ.*

We recall that Jacobi fields of a connection are vector fields $W = W^i \partial_i$ defined along a geodesic γ that are solutions of the second-order Jacobi equation, linear on W (see [5], vol.1, §36),

$$\nabla^2_{\dot{\gamma}} W + \mathrm{Riem}(\dot{\gamma}, W)\dot{\gamma} = 0$$

where Riem is the curvature of the Levi-Civita connection on M (a linear operator). Notice also that the Jacobi equation is the variation equation of the geodesic equation along a fixed solution.

For the aforementioned extension we have to strenghten our previous Definition 1.2 of g-convexity with the following

Definition 2.1 (strong g-convexity) *The open subset* Ω *of* M *is a* strongly g-convex *set if to every* $x \in \Omega$ *there exists a domain* $D_x \subset T_x\Omega$ *such that the exponential mapping* $\exp_x : D_x \longrightarrow \Omega$ *is a* diffeomorphism *on* Ω.

Obviously, strong g-convexity implies g-convexity.

Remark. The proof of the Theorem below is based on the following general fact: given an o.d.e., its flow $x = x(t, y)$ and a solution $x = x(t, y_0)$, then the differential of the flow $\xi(t, y_0, \xi_0) := D_y x(t, y_0)\xi_0$ is the flow of the variation equation along the given solution $x = x(t, y_0)$. We will use this fact in connection with the Jacobi equation, which is the variation equation associated to the geodesic equation. We will show that if the exponential mapping is a diffeomorphism, as in the above Definition of strong g-convexity, then there is a one-to-one correspondence between the boundary problem and the initial value problem for the Jacobi equation related to a given geodesic.

In more details, we denote with

$$x = \gamma(t, \gamma_0, \dot\gamma_0)$$

the solution of the geodesic equation with initial data $\gamma_0, \dot\gamma_0$, and we denote with

$$\exp_{\gamma_0}(\dot\gamma_0) = \gamma(1, \gamma_0, \dot\gamma_0)$$

and

$$D\exp_{\gamma_0}(\dot\gamma_0)v = \frac{\partial\gamma}{\partial\dot\gamma_0}(1, \gamma_0, \dot\gamma_0)v =: Bv$$

the exponential mapping and its differential respectively. The flow associated to the geodesic equation reduced to the first order is

$$\begin{pmatrix} x \\ \dot x \end{pmatrix} = \begin{pmatrix} \gamma(t, \gamma_0, \dot\gamma_0) \\ \frac{\partial}{\partial t}\gamma(t, \gamma_0, \dot\gamma_0) \end{pmatrix}$$

and its differential with respect to $(\gamma_0, \dot\gamma_0)$, which is the flow of the Jacobi equation, is

$$\begin{pmatrix} w \\ \dot w \end{pmatrix}(t, w_0, \dot w_0) = \begin{pmatrix} \frac{\partial\gamma}{\partial\gamma_0} & \frac{\partial\gamma}{\partial\dot\gamma_0} \\ \frac{\partial^2\gamma}{\partial\gamma_0\partial t} & \frac{\partial^2\gamma}{\partial\dot\gamma_0\partial t} \end{pmatrix}(t, \gamma_0, \dot\gamma_0)\begin{pmatrix} w_0 \\ \dot w_0 \end{pmatrix}.$$

Now we derive the relation between initial value problem and boundary value problem. For fixed boundary data w_0 and w_1, one has that

$$w_1 = \frac{\partial\gamma}{\partial\gamma_0}(1, \gamma_0, \dot\gamma_0)w_0 + \frac{\partial\gamma}{\partial\dot\gamma_0}(1, \gamma_0, \dot\gamma_0)\dot w_0 = \frac{\partial\gamma}{\partial\gamma_0}w_0 + D\exp_{\gamma_0}(\dot\gamma_0)\dot w_0,$$

hence

$$w_1 - \frac{\partial\gamma}{\partial\gamma_0}w_0 = B\dot w_0.$$

We thus can state that: if the exponential mapping is a diffeomorphism (hence $B = D\exp_\gamma$ has maximum rank for every $(\gamma, \dot\gamma) \in T\Omega$), then the assignment of boundary conditions (w_0, w_1) for the Jacobi equation determines uniquely an initial data assignment $(w_0, \dot w_0)$ by the above linear relation.

The above Remark allows us to prove the following Proposition.

Proposition 2.1 *A subset Ω of M is strongly g-convex if and only if $T\Omega$ is a g^C-convex subset of TM.*

Proof. Let Ω be strongly g-convex. Given two points (x_0, w_0) and (x_1, w_1) on $T\Omega$ we have to prove the existence and uniqueness of a geodesic arc between the given endpoints and contained in $T\Omega$. By the above characterization of geodesics of g^C, it is enough to show that there exist i) a unique geodesic γ of g in Ω between x and y and ii) a unique Jacobi field W along γ with prescribed boundary values w_0 and w_1. Assertion i) holds trivially, and since for a strongly g-convex set Ω the differential of the exponential mapping has maximum rank, assertion ii) is true by the above Remark.

Conversely, let $T\Omega$ be a g^C-convex subset of TM. As an hypothesis for reduction ad absurdum, suppose that on $T\Omega$ there are points (x_0, w_0) and (x_1, w_1) such that there are at least two geodesic curves contained in Ω whose endpoints are x_0 and x_1. Then, along one of these geodesics, say γ, the boundary problem with data (w_0, w_1) for the (variation) Jacobi equation cannot be solved, otherwise $T\Omega$ fails to be g^C-convex. This implies that the differential of the exponential mapping (i.e. the linear operator B in Remark above) has a nontrivial kernel and hence there exists more than one geodesic in $T\Omega$ with the same endpoints $w_0 = w_1 = 0$, against the hypothesis. Therefore, Ω is g-convex since to every $x \in \Omega$ there exists a domain $D_x \subset T_x\Omega$ such that the exponential mapping is a bijection on Ω; moreover, the differential of exp has maximum rank, hence it is a diffeomorphism. □

The definition of the complete lift g^C of a metric g and the above characterization of g^C-convex subsets of TM are the only tools needed for the aforementioned extension. Now Definition 1.3 and the content and the proof of Theorem 1.3 for $T\Omega \subset TM$ repeat word-for-word those for $\Omega \subset M$.

3 An example involving special vector fields

Our aim here is to apply some of the previous results to second order differential equations. We recall some preliminary definitions from [8].

Let $X(x, u) = (x, u, X_1(x, u), X_2(x, u))$ be the local representation of a vector field on TM. The vector field is *vertical* if $X_1 \equiv 0$ and it is a *special* vector field (a *spray*) if $X_1(x, u) = u$. Note that a vector field X on M has a canonical vertical lift to a vertical vector field by $X^v(x) = (x, u, 0, X(x))$. Special vector fields define second order autonomous differential equations on M by

$$\dot x = u, \qquad \dot u = X_2(x, u).$$

The spray associated to the geodesic equation will be denoted with

$$Z^i(x,u) = (x^i, u^i, u^i, -\Gamma^i_{jk}(x)u^j u^k).$$

It is known that a connection on TM is the assignment of a sub-bundle HTM complementary to the vertical sub-bundle $VTM = \ker T\tau$. We remember that $T\tau : TTM \longrightarrow TM,\ (x,u,X_1,X_2) \mapsto (x,X_1)$, and $\ker T\tau = VTM = \{(x,u,0,X_2)\}$. It can also be shown that there is only one horizontal special vector field, the geodesic spray Z; as a consequence, every special vector field Y can be (uniquely) decomposed as the sum of Z with a vertical field Y^V, that is

$$Y(x,u) = (x,u,u,-\Gamma uu) + (x,u,0,\Gamma uu + Y_2(x,u)) = Z(x,u) + Y^V(x,u).$$

The relation above has a mechanical interpretation. When (M,g) describes a holonomic system, the special vector field Y describes the dynamics of the system when acted on by a force field Y^V.

The geodesic spray Z has another interesting property: if $\gamma^* = (\gamma, \dot\gamma)$ is the *natural* lift to TM of a geodesic γ of (M,g), then Z is parallel transported along γ^* with respect to the Levi-Civita connection ∇^C associated to g^C, that is

$$\nabla^C_{\dot\gamma^*} Z = 0. \tag{3.1}$$

This latter fact motivates the following definition

Definition 3.1 *A vector field $X : T\Omega \longrightarrow TTM$, where Ω is g-convex, is g^C-injective with respect to the natural lift of geodesics (symbol: g^C_N), if to every $\hat x$, $\hat y$ in $T\Omega$,*

$$\hat x \neq \hat y \ \Rightarrow\ X(\hat y) \neq Tp^C(\gamma^*)X(\hat x),$$

where $Tp^C(\gamma^*)$ is the parallel transport operator of the Levi-Civita connection associated to g^C along the natural lift γ^* of the unique geodesic γ in Ω with endpoints $\pi(\hat x)$ and $\pi(\hat y)$.

Now, as a straightforward consequence of the above property (3.1) of Z and of the above definition we have that

Proposition 3.1 *A vector field X, on a g-convex set Ω, which is g-injective defines a special vector field $Y = Z + X^v$ on $T\Omega$ which is g^C_N-injective.*

Proof. The vector field Y is the sum of the geodesic spray Z with the vertical lift X^v of X, hence Y is a spray. We know (see eqn. (3.1)) that the horizontal spray Z is parallel. To derive the parallel transport formula for a vertical vector field X^v we recall the general formula in [8], p. 60, of parallel transport of a vector field (X_1, X_2) on TM along a path (x,u) with respect to ∇^C,

$$\begin{cases} \frac{dX_1^k}{dt} + \Gamma^k_{ij}\dot x^i X_1^j = 0, \\ \frac{dX_2^k}{dt} + (\partial_h \Gamma^k_{ji} u^h)\dot x^i X_1^j + \Gamma^k_{ij}\dot x^i X_2^j + \Gamma^k_{ij}\dot u^j X_1^i = 0. \end{cases}$$

Now it is easy to see that for the vertical lift X^v of a vector field X, the parallel transport defined by the above formula reduces ($X_1 \equiv 0$) to the parallel transport of X on M. Hence

$$Tp^C(\gamma^*)(Z(x,\dot{x}) + X^v(x)) = Z(y,\dot{y}) + (Tp(\gamma)X(x))^v,$$

and

$$Z(y,\dot{y}) + (Tp(\gamma)X(x))^v \neq Z(y,\dot{y}) + X^v(y) \Leftrightarrow Tp(\gamma)X(x) \neq X(y). \qquad \square$$

Remark. The second order autonomous differential equation related to the above special vector field $Y = Z + X^v$ describes the dynamics of a mechanical system whose kinetic energy is the Riemannian metric g on M and subject to positional force X.

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