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On space-time scaling of cumulated rainfall fields

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Abstract. In the study of space-time rainfall it is particularly important to establish characteristic properties to guide both theoretical and modeling research efforts. In the present paper, new observational analyses on the scaling properties of time-evolving cumulated rainfall fields are presented, and a theoretical framework for their interpretation is introduced. It is found that the time evolution of the spatial organization of a cumulated rainfall field produces scaling relationships of spatial variance versus time and characteristic values for the scaling exponent. The reproduction of these values constitutes a basic requirement for spatial-temporal field generators in order to model important properties of real rainfall fields. It is then shown, on theoretical grounds, what properties the instantaneous rainfall intensity fields must obey in order to reproduce the experimental observations and how the size of the observation domain affects the scaling relationships. Some current stochastic models of space-time precipitation are finally discussed and analyzed in the light of the tools introduced, to show under what circumstances the models considered give acceptable results. Furthermore, it is shown that the assumption of an exponential time correlation function, used in many current rainfall models, is not compatible with observations.

1. Introduction

The characterization of the scaling properties of rainfall fields is an important research area in hydrology and hydrometeorology due to the deep implications of such properties both from a theoretical and a practical stand point. On one hand it is important to establish firm results on the scaling character of rainfall on which to base theoretical investigations. On the other hand these results are very useful in the practical statistical description of precipitation and in the implementation and testing of stochastic rainfall models.

Statistical and scaling properties of precipitation time series have been extensively addressed [e.g., *Waymire and Gupta*, 1981a, b, c; *Kedem and Chiu*, 1987; *Zawadzki*, 1987]. Models of point rainfall are known to capture at least some of the observed features quite well [*LeCam*, 1961; *Rodriguez-Iturbe et al*., 1987, 1988; *Cowpertwait*, 1994; *Katz and Parlange*, 1995]. On the contrary, the spatial-temporal structure of evolving rainfall fields [e.g., *Austin and Houze*, 1972; *Zawadzki*, 1973; *Houze*, 1981; *Lovejoy and Mandelbrot*, 1985; *Crane*, 1990] appears to be more controversial, and its stochastic modeling has seen the development of at least three quite different approaches.

A first approach exploits self-affinity relationships to produce rain-rate fields through an iterative random cascade process [*Gupta and Waymire*, 1993; *Tessier et al*., 1993; *Over and Gupta*, 1996]. A fundamental problem with this approach resides in the difficulty of incorporating the description of the temporal evolution of rain-rate fields and perhaps a link of clear physical nature on the parameters of the cascade process.

A second approach makes use of generators of random space-time functions [*Matheron*, 1973; *Mejia and Rodriguez-Iturbe*, 1974; *Bell*, 1987; *Gutjahr*, 1989; *Bellin and Rubin*, 1996] to generate fields with specified spatial-temporal covariance structures. These models are quite flexible and can incorporate constraints to observed deterministic values at arbitrary positions [e.g., *Rubin and Bellin*, 1997] but require the detailed specification of the correlation structure of the random function both in space and time. Further, there does not seem to be a general consensus as to what this structure should be in general. Finally, these models are more algorithmic schemes for generating space-time fields than attempts to provide a representation of rainfall processes within a physical framework.

The third approach is based on the stochastic modeling of physical processes occurring during a precipitation event [e.g., *Bras and Rodriguez-Iturbe*, 1976; *Waymire et al*., 1984; *Rodriguez-Iturbe et al*., 1986; *Cox and Isham*, 1988; *Cowpertwait*, 1995]. This approach describes the arrival process of storms and, within a storm, the arrival process of rain cells. These are assigned an intensity with a given distribution. These models have the clear advantage of directly describing observed rainfall mechanisms but still require the assignment of probability distributions, and their defining parameters, for many storm properties (storm interarrival, storm duration, cell interarrival, duration and intensity, etc.). Furthermore, the functional form of some important characters (intensity distribution at a point, correlation structure, etc.) are often difficult to derive, making model fitting and validation difficult also as a consequence of the large number of parameters involved.

In this framework it is important to establish properties that a synthetic rainfall field should obey on the basis of observed precipitation properties. Thus new observational analyses of the scaling properties of time-evolving rainfall fields are important as well as any theoretical framework for their interpretation.

In this paper we provide new experimental evidence on the statistical characters of cumulated rainfall fields. It is seen that the observed characters, as captured by the exponent of the scaling relationship of spatial variance versus time, constitute a basic requirement for spatial-temporal field generators in order to reproduce important properties of real precipitation fields. We then discuss, specifically and in general, the properties of some generators of space-time correlated fields to show under what circumstances these requirements are met and what modeling assumptions are compatible with observations.

2. Cumulated Fields

We study the deposition process in which the field $h(\mathbf{x}, t)$ is the superposition over time of the field $i(\mathbf{x}, \tau)$:

$$h(\mathbf{x}, t) = \int_0^t i(\mathbf{x}, \tau)\, d\tau \qquad (1)$$

In the case of the study of precipitation, $h(\mathbf{x}, t)$ is the cumulated field of rainfall depths (dimensions [L]), while $i(\mathbf{x}, \tau)$ is the field of instantaneous rainfall intensities (dimensions [L/T]), thought of as a realization of an underlying stochastic process.

One way to characterize the degree of spatial irregularity of the rainfall deposition process is through the "roughness" of the surface of the cumulated rainfall field, given by the spatial variance:

$$\sigma^2(t) = \frac{1}{V(B_R)} \int_{B_R} [h(\mathbf{x}, t) - \mu(t)]^2\, d\mathbf{x} \qquad (2)$$

where B_R is the arbitrary domain considered (with characteristic length R) and $V(B_R)$ is its measure; also,

$$\mu(t) = \frac{1}{V(B_R)} \int_{B_R} h(\mathbf{x}, t)\, d\mathbf{x} \qquad (3)$$

is the spatial mean of the field.

The spatial variance of $h(\mathbf{x}, t)$ may be linked to the covariance structure of the increment fields. After some manipulation (see the appendix), the expected value of the sample spatial variance evaluated over B_R is

$$E[\sigma^2](t) = E\left\{\frac{1}{V(B_R)} \int_{B_R} \left[h(\mathbf{x}', t) - \frac{1}{V(B_R)} \int_{B_R} h(\mathbf{x}'', t)\, d\mathbf{x}''\right]^2 d\mathbf{x}'\right\}$$
$$= \frac{1}{V(B_R)} \int_{B_R} \int_0^t \int_0^t E[i(\mathbf{x}', \tau')i(\mathbf{x}', \tau'')]\, d\mathbf{x}'\, d\tau'\, d\tau''$$
$$- \frac{1}{V^2(B_R)} \int_{B_R} \int_{B_R} \int_0^t \int_0^t E[i(\mathbf{x}', \tau')i(\mathbf{x}'', \tau'')]\, d\mathbf{x}'\, d\mathbf{x}''\, d\tau'\, d\tau'' \qquad (4)$$

where (1) has been used.

Introducing the covariance function

$$c(\mathbf{x}', \mathbf{x}'', \tau', \tau'') = E[i(\mathbf{x}', \tau')i(\mathbf{x}'', \tau'')] - E[i(\mathbf{x}', \tau')]E[i(\mathbf{x}'', \tau'')] \qquad (5)$$

(4) may be expressed in the following form:

$$E[\sigma^2](t) = \frac{1}{V^2(B_R)} \int_{B_R}\int_{B_R} \int_0^t \int_0^t [c(\mathbf{x}', \mathbf{x}', \tau', \tau'') - c(\mathbf{x}', \mathbf{x}'', \tau', \tau'') + E[i(\mathbf{x}', \tau')] \cdot \{E[i(\mathbf{x}', \tau'')] - E[i(\mathbf{x}'', \tau'')]\}]\, d\mathbf{x}'\, d\mathbf{x}''\, d\tau'\, d\tau'' \qquad (6)$$

If the stochastic process underlying $i(\mathbf{x}, \tau)$ is homogeneous and stationary (an important but reasonable hypothesis in the case of rainfall [e.g., *Barancourt et al.*, 1992; *Crane*, 1990]), the covariance function only depends on the spatial and temporal lags:

$$c(\mathbf{r}, \tau) = E[i(\mathbf{x} + \mathbf{r}, t + \tau)i(\mathbf{x}, t)] - \mu_{i,1}^2 \qquad (7)$$

where $\mathbf{r} = \mathbf{x}'' - \mathbf{x}'$, $\tau = \tau'' - \tau'$, and $\mu_{i,1}$ is the expected value of $i(\mathbf{x}, t)$. In this case, (6) becomes

$$E[\sigma^2](t) = \frac{1}{V^2(B_R)} \cdot \int_{B_R} \int_{B_R} \int_0^t \int_0^t [c(\mathbf{0}, \tau) - c(\mathbf{r}, \tau)]\, d\mathbf{x}'\, d\mathbf{x}''\, d\tau'\, d\tau'' \qquad (8)$$

In this homogenous and stationary framework the expected value of the spatial mean of $h(\mathbf{x}, t)$ is found to be

$$E[\mu](t) = E\left[\frac{1}{V(B_R)} \int_{B_R} h(\mathbf{x}, t)\, d\mathbf{x}\right] = \frac{1}{V(B_R)} \int_{B_R} \int_0^t E[i(\mathbf{x}, \tau)]\, d\mathbf{x}\, d\tau = E[i(\mathbf{0}, 0)]t = \mu_{i,1}t \qquad (9)$$

Equation (9) is a one-to-one relationship between time and $E[\mu](t)$, thus showing that time can be surrogated by the spatial mean of the cumulated field. This situation will be exploited in the analysis of actual data, in which the increment process is not stationary, being rather intermittent in time. It is thus convenient to study the "conditional process" defined only when rainfall is actually greater than zero. This can be done by studying the scaling of the spatial variance versus the spatial mean of the cumulated field, which automatically keeps track only of those time steps in which precipitation is actually occurring.

Let us consider the important case in which $i(\mathbf{x}, t)$ is uncorrelated in time. Then

$$c(\mathbf{0}, \tau) = \begin{cases} 0 & \text{if } \tau' \neq \tau'' \\ \mu_{i,2} - \mu_{i,1}^2 & \text{if } \tau' = \tau'' \end{cases}$$
$$c(\mathbf{r}, \tau) = \begin{cases} 0 & \text{if } \tau' \neq \tau'' \\ c(\mathbf{r}) - \mu_{i,1}^2 & \text{if } \tau' = \tau'' \end{cases} \qquad (10)$$

where $\mu_{i,2}$ is the second moment of the probability distribution of $i(\mathbf{x}, t)$ and $c(\mathbf{r})$ is its now solely spatial covariance. The

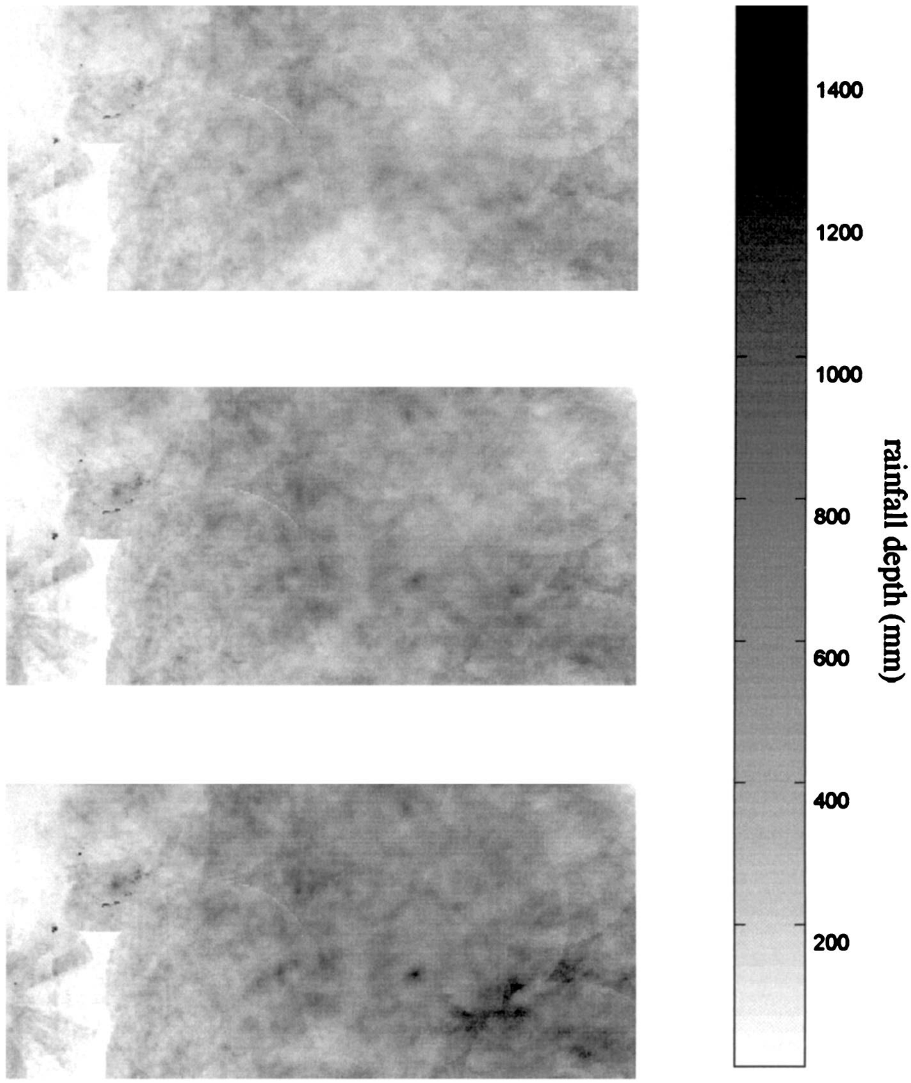

Figure 1. Cumulated fields at different times for a sequence of daily data starting May 1, 1996. It is clear how differences between total depths at different positions are growing in time.

expected value of the variance is therefore from (8) (see the appendix):

$$E[\sigma^2](t) = \frac{t}{V^2(B_R)} \int_{B_R} \int_{B_R} [\mu_{i,2} - c(\mathbf{r})] \, d\mathbf{x}' \, d\mathbf{x}'' \quad (11)$$

i.e., the variance scales as time to the first power. This result shows that a nontrivial scaling (i.e., an exponent different from unity) of the variance may be obtained only by adding a temporally correlated process. Moreover, since the integral on the right-hand side of (11) depends on what portion of the spatial covariance of $i(\mathbf{x}, t)$ is seen by the domain, variances computed on domains of different sizes will not collapse onto one another, and the amount by which variances from different domains are shifted is controlled by the shape of the covariance function.

Equation (11) has an interesting parallel in a result established in the context of one-dimensional self-similar signal analysis [*Cox*, 1984; *Mandelbrot*, 1983]. The value of the exponent (Hurst exponent) in the relation

$$\{\mathrm{Var}\,[z(t + \tau) - z(t)]\}^{1/2} \propto \tau^{H_1} \quad (12)$$

is in fact known to characterize the correlation properties of the process z. The subscript "1" is used to distinguish this one-dimensional setting from the space-time framework of the following discussion. In this one-dimensional framework a value of $H_1 = 0.5$ is the sign that the increments of z are uncorrelated, while values of $H_1 > 0.5$ are an indication of the positive correlation of time fluctuations and of long-range memory in the signal.

3. Experimental Observations

The data used in the analyses presented below were obtained from the Arkansas-Red Basin River Forecast Center (http://info.abrfc.noaa.gov). The fields analyzed are composites of radar and rain gauge measurements and have a resolution of 4 km, and their size is 335 × 159 pixels. Only daily data have been considered in the analyses. As an example, Figure 1 shows the cumulated field at time $t = 120$ days (mean = 351

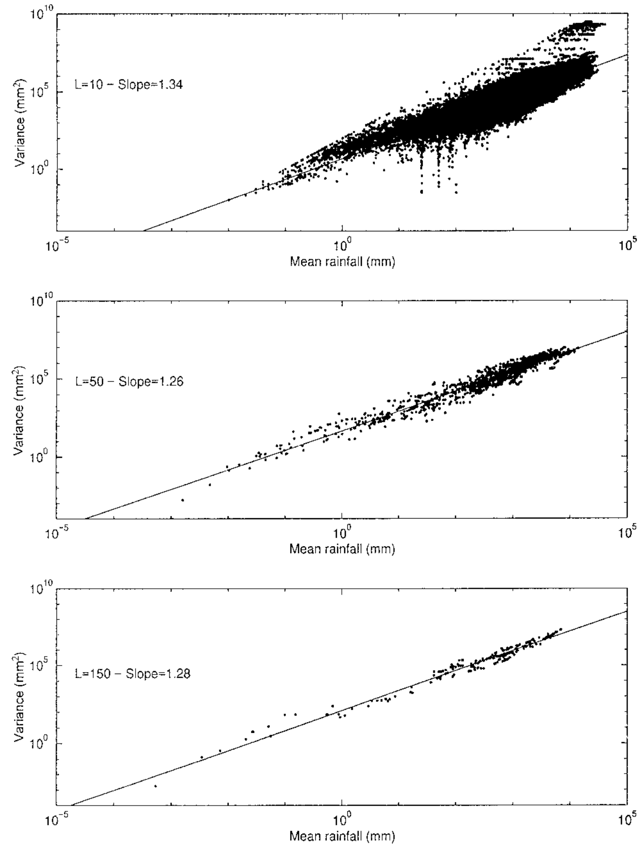

Figure 2. Scaling of cumulated fields: November 1995 to March 1996. The dependence of the variance on the spatial mean is markedly power law at all scales of observation.

mm), $t = 150$ days (mean = 456 mm), and $t = 180$ days (mean = 502 mm) obtained from a sequence of daily data starting on May 1, 1996.

The sequences (both in winter and summer) of cumulated fields have been analyzed by computing the time-dependent spatial mean and variance, which, for a discrete domain of side L are defined as

$$m(t) = \frac{1}{L^2} \sum_{k=1}^{L^2} h(\mathbf{x}_k, t)$$
$$s^2(t) = \frac{1}{L^2 - 1} \sum_{k=1}^{L^2} [h(\mathbf{x}_k, t) - m(t)]^2 \qquad (13)$$

Time may now be substituted by the mean rainfall $m(t)$ by incrementing the sums in (13) only when at least one $h(\mathbf{x}_k, t)$ in the domain is larger than zero and by indexing the value of s^2 with the value of $m(t)$, thus obtaining pairs of values (m, s^2).

Given a subdivision of the domain into subgrids of side L, these computations are carried out on each of these subgrids. This procedure, for a fixed value of L, gives a large number of data points (m, s^2) as daily precipitation fields are successively added over time. The results from such a procedure are shown in Figure 2 for subgrids of side $L = 10, 50, 150$ pixels. The data cover the period November 1995 to March 1996.

Notice that the data points follow a power law relationship

$$s = \alpha t^H \qquad (14)$$

with relatively low scatter (recall that in Figure 2, $m(t)$ surrogates time). The consistency of the slope estimates for different grid sizes is also rather satisfactory, giving exponents between 1.26 and 1.43.

Figure 3 shows the results of the above procedure applied to the data collected between May and October 1996. The fit to a power law is again good and characterized by low scatter. The consistency of the estimates of the slopes is quite remarkable at all scales of observation, yielding values between 1.20 and 1.25.

The scatter in Figures 2 and 3 is not unexpected since each data point represents a single realization of a cumulated rainfall field. Equations (6), (8), and (9), however, are relations between the expected values of the quantities involved. It is therefore necessary to obtain estimates of such expected values by computing averages over many realizations. Figures 4 and 5 are obtained by binning the data in Figures 2 and 3 and by computing, within each bin, the average of both the spatial variance and mean. The power law dependence of $E[\sigma^2]$ on $E[\mu]$ is very clear with a good agreement of slope estimates from different spatial scales.

These results hold for what has been referred to as the "conditional" rainfall process, from which values of zero rainfall have been removed and all the remaining measurements have been put in the same, continuous, sequence. With this approach, two values which end up near each other in the sequence may not have a significant correlation when they belong to different events. To address this problem we have

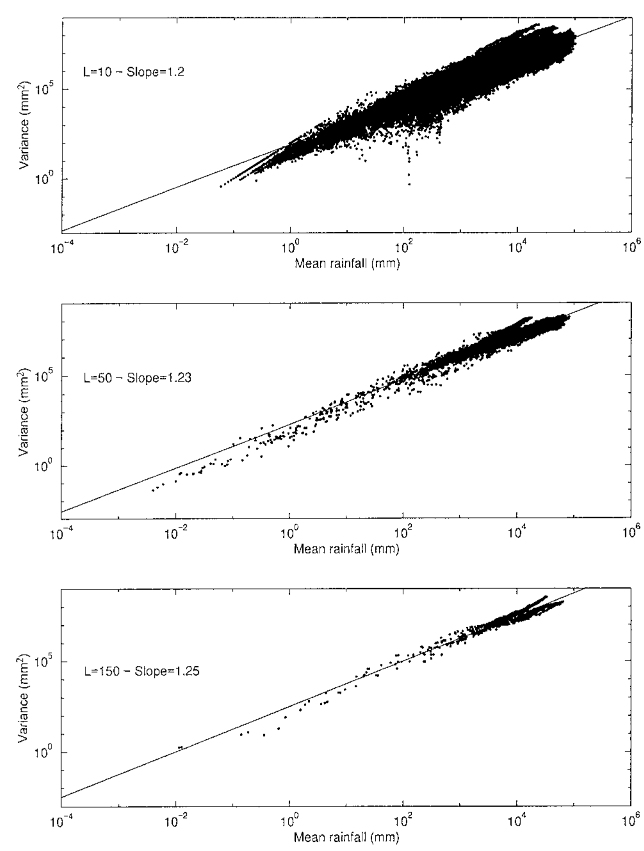

Figure 3. Scaling of cumulated fields: May–October 1996. The dependence of the variance on the spatial mean is markedly power law at all scales of observation.

repeated the analyses of variance versus mean by letting the cumulated field grow only as long as when at least one rain-rate value in the grid is larger than zero. If, at any step, rainfall intensity is zero everywhere in the grid, the cumulated field is reset to zero. By this procedure the time-intermittency problem is avoided, as before, and two successive rainfall values always belong to the same event thus avoiding the risk of introducing spurious time correlations. The result of this procedure is shown in Figure 6, where it is clear that a power law dependence of variance versus mean holds and the slope estimates are consistent at all scales and with the results obtained through the previous methods.

The binning procedure described above has been applied also to the values (m, s^2) obtained from single events. The result is shown in Figure 7, where, again, a power law form is consistently found at all scales of aggregation.

The fact that the scaling exponent $2H$ of $E[\sigma^2]$ versus $E[\mu]$ has been consistently estimated employing different methods on the series considered convincingly shows that the value of H is nontrivially >0.5, falling between $H = 0.60$ and $H = 0.74$. It was suggested above, and it will be more clearly shown in the following, how such values of the exponent H are the mark of long-range correlation in the system. It is important to notice that the "long-memory" of the rain fields may be justified by the positive feedback of local rainfall recycling [*Rodriguez-Iturbe et al.*, 1991].

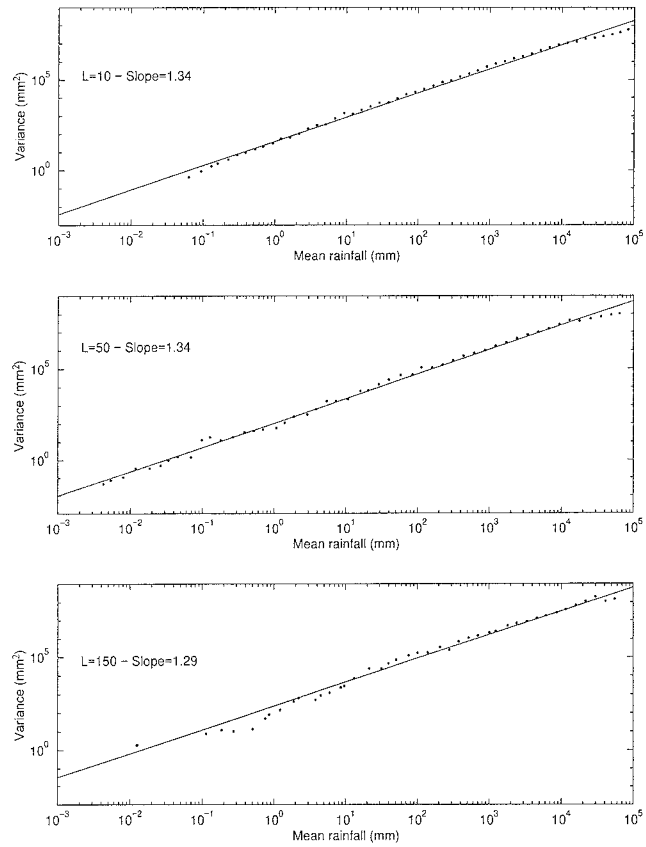

Figure 5. Scaling of cumulated fields: May–October 1996. The range covered by experimental values of m has been divided into subintervals within which the mean of m and s^2 was computed. The ensemble estimates thus obtained exhibit a remarkable power law form at all scales of observation.

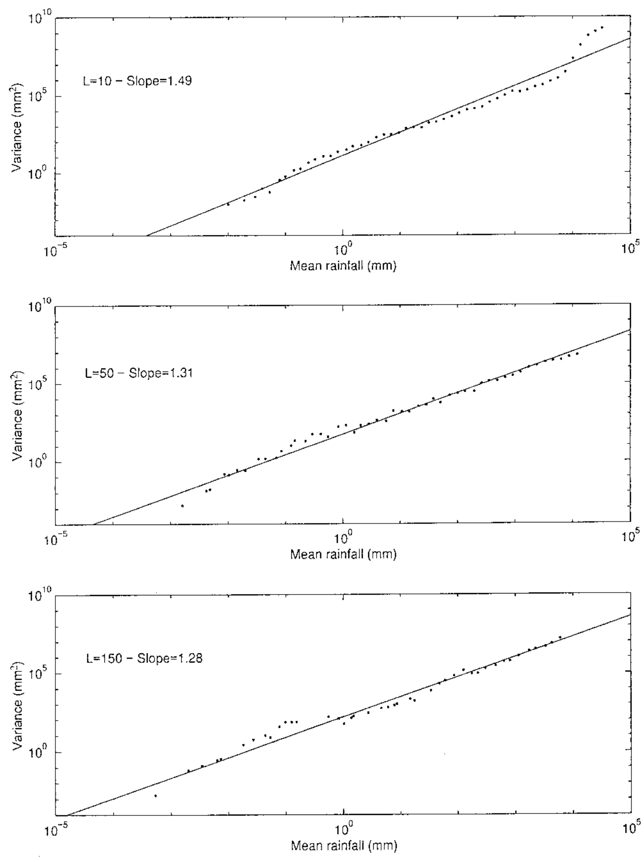

Figure 4. Scaling of cumulated fields: November 1995 to March 1996. The range covered by experimental values of m has been divided into subintervals within which the mean of m and s^2 was computed. The ensemble estimates thus obtained exhibit a remarkable power law form at all scales of observation.

The scaling of the variance with time or with mean deposited rainfall, which is the same in the present context, is a useful tool to represent and analyze the spatial-temporal statistical characters of precipitation. It is a way of capturing the degree of singularity of the process, characterizing how differences in space of deposited rainfall grow over time. The exponent H is thus an important property of a spatial-temporal field, and, in order to understand the link between H and the physical and statistical characters of rainfall, it is useful to consider some examples of spatial-temporal stochastic models to put the experimental findings in a clearer perspective and help understand their implications.

4. On Stochastic Deposition Models

The simplest example of a deposition process is the one in which a unit depth is deposited independently at each site at each time step with probability f. The probability density function (independent of time and space) of $i(\mathbf{x}, t)$ is $p(i) = f\delta(i - 1) + (1 - f)\delta(i)$ (where $\delta(i)$ is Dirac's Delta), which gives $\mu_{i,1} = f$, $\mu_{i,2} = f$, and $c(\mathbf{r}) = \mu_{i,1}^2 = f^2$. By use of (11), one finds

$$E[\sigma^2](t) = tf(1 - f)$$

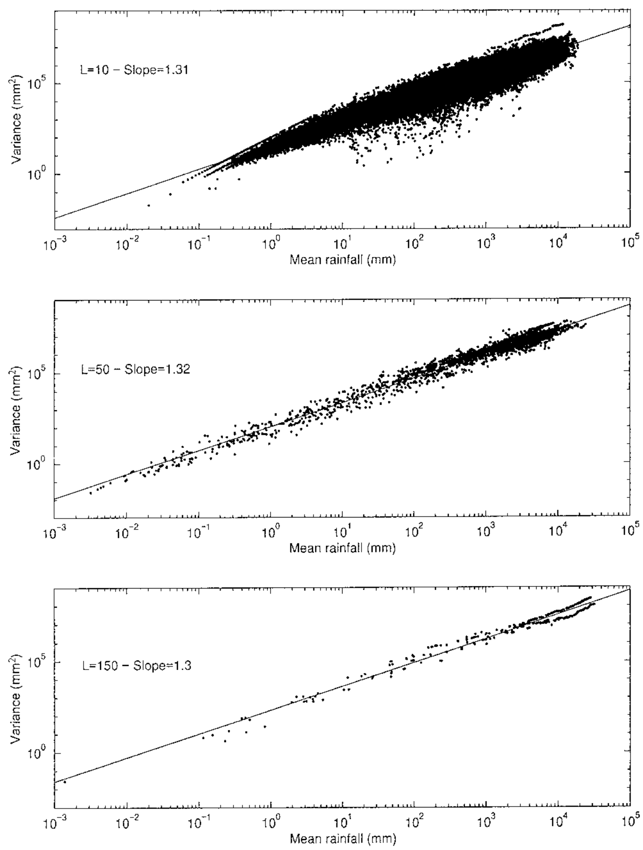

Figure 6. Scaling of cumulated fields: May–October 1996. The variance refers to cumulated fields computed only within the same event.

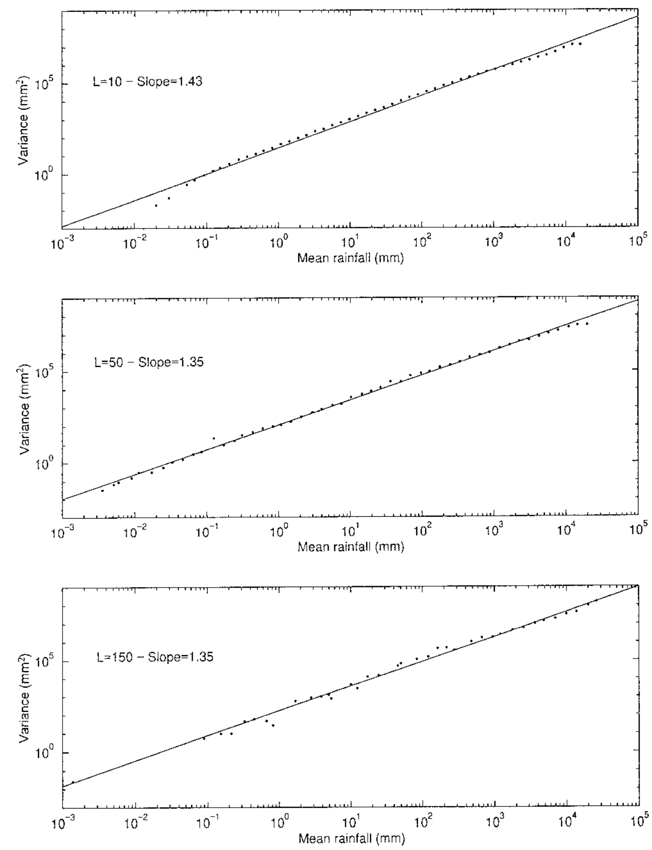

Figure 7. Scaling of the variance computed within single events: May–October 1996. The range covered by experimental values of m has been divided into subintervals within which the mean of m and s^2 was computed.

that is, the variance grows linearly with time. Notice that since the variogram is constant, the expected value of the variance computed over domains of different size will coincide. This is an elementary known result in the literature on deposition processes [*Barabasi and Stanley*, 1995].

The next simplest case is that of a space-correlated "rainfall" field with no correlation in time. Let us consider the following generation scheme [*Cox and Isham*, 1988]: Storms are rectangular cells whose arrival is given by a Poisson process in space and time with rate λ (i.e., the number of new storms at a given time step is Poisson distributed with mean λA, A being the area of the simulation domain). Every cell has a random size, l, with exponential distribution; the rain depth, d, has a spatially homogeneous exponential distribution. These random variables are independent from each other. Cells can overlap in space and their lifetime is fixed to one temporal step. This implies that the resulting "rain-rate" fields are not correlated in time. The random fields obtained through this procedure have been used to construct sequences of cumulated fields.

Figure 8 shows the scaling of the standard deviation, s, with time, t, for the model just described. The simulations are carried out on a lattice of size 128×128 pixels. The cells have an average side length $\langle l \rangle = 3$ pixels, and the mean of the exponentially distributed intensity is $\langle i \rangle = 5$. The value adopted for the parameter λ allows an average of 100 new storms per time step. The plots use time rather than mean rainfall on the x axis because the intermittency is negligible in these examples as the probability of zero rainfall at any time is indistinguishable form zero.

A well-defined power law as in (14) is observed. As expected, $H = 0.5$ since the increments $i(\mathbf{x}, t)$ are uncorrelated in time.

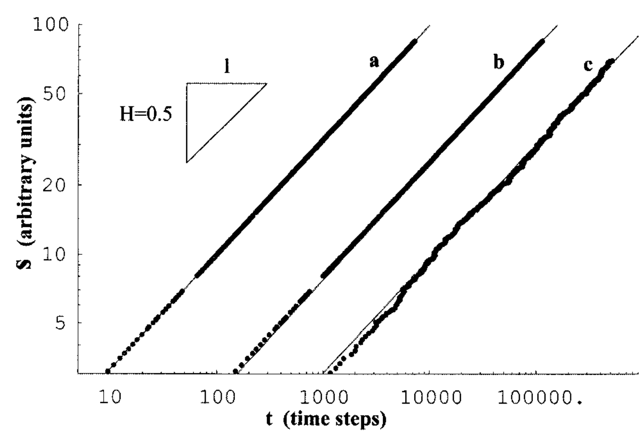

Figure 8. Examples of the scaling of the spatial standard deviation, s, in time-uncorrelated deposition processes on a 128×128 pixels domain. In one case the increment fields are obtained through a two-dimensional Poisson process, and s is an ensemble estimate on subgrids of side (a) $L = 128$ and (b) $L = 32$. In the other, $i(\mathbf{x}, \tau)$ is generated through (c) a random cascade model.

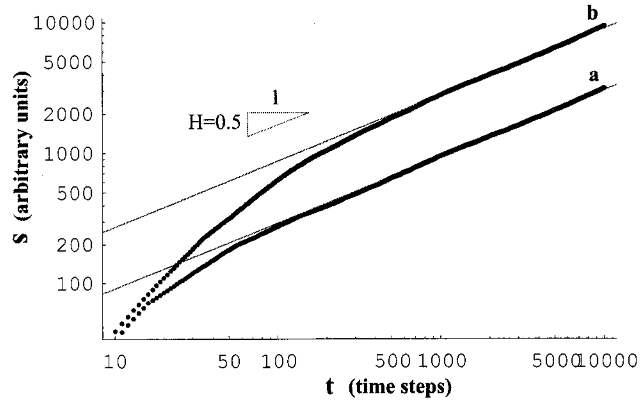

Figure 9. Scaling of the standard deviation with time in a two-dimensional deposition process whose increments are realizations of a Poissonian model for storm arrival with correlation in time. Storms have an exponentially distributed duration, with (a) mean equal to 10 and (b) 30 steps.

The same exponent is consistently found, regardless of the size of the subgrids considered in the ensemble estimation procedure. Also, notice that curves deriving from different subgrid sizes do not collapse onto one another; that is, α in (14) is a function of the size of the domain. This is in agreement with (11) since the variogram is not constant as in the previous case.

Similar results are found whenever the analysis is applied to cumulated fields obtained by summing time-uncorrelated fields. Figure 8 also shows an example of the scaling of s with time for a cumulated field whose increments are obtained through six steps of a two-dimensional random cascade with a branching rate $b = 2$. The generator, W, is chosen according to the so-called Beta model [*Novikov and Stewart*, 1964] ($P(W = 0) = 1 - b^{-\beta}$ and $P(W = b^{\beta}) = b^{-\beta}$ and $\beta = 0.15$). Again, the value $H = 0.5$ is found, in agreement with the analytical results.

We next consider random fields, $i(\mathbf{x}, t)$, correlated both in space and time, generated with a Poisson process in which storms are allowed to have a duration, D, longer than a single time step. Storms have a constant (and uniform) intensity, i, throughout their lives and may overlap both in time and in space. This model, introduced by *Cox and Isham* [1988], yields fields whose covariance function, if the storm variables D, i, l (cell size) are independent and D is exponentially distributed with parameter ν, is given by

$$c(\mathbf{r}, \tau) = \mu_{i,2} e^{-\nu|\tau|} \left(1 + \frac{cr}{2}\right) e^{-cr} - \mu_{i,1}^2 \qquad (15)$$

where $r = |\mathbf{r}|$ and $1/c$ is the mean extent of a storm. The covariance exponentially tends to zero and the scheme is thus capable of inducing a short-lived correlation in time.

Figure 9 shows the behavior of s for a cumulated Poisson model of storm arrivals with l, i, and D exponentially distributed. Their mean values are $\langle l \rangle = 6$ pixels, $\langle i \rangle = 10$ arbitrary units, and $\langle D \rangle = 1/\nu = 10$ steps and $\langle D \rangle = 30$ steps for case a and b, respectively. The value of λ allows an average of 10 new storms per time step. It may be observed that after a short transience in which the effect of the time correlation is evident, the curves tend to a power law form with exponent $H = 0.5$, as in the case of uncorrelated increments. Notice that the length of this transience is a function of the average life, since the correlation time is proportional to $\langle D \rangle$.

These results agree with analytical predictions. In fact, by substituting (15) into (8) one obtains

$$E[\sigma^2](t) = \frac{2\mu_{i,2} I}{\nu V^2(B_R)} \left(t + \frac{e^{-\nu t}}{\nu} - \frac{1}{\nu}\right) \qquad (16)$$

where

$$I = \int_{B_R} \int_{B_R} \left[1 - \left(1 + \frac{cr}{2}\right) e^{-cr}\right] d\mathbf{x}' \, d\mathbf{x}'' \qquad (17)$$

It is seen that, after an initial nonscaling regime whose duration depends on ν, one has

$$\lim_{t \to \infty} E[\sigma^2](t) \propto t \qquad (18)$$

as shown by the computations.

These analyses are now repeated on fields constructed with the same Poisson model in which the duration of storms is assumed to be power law distributed. A power law distribution, in fact, does induce a long memory in the cumulated field, possibly yielding a nontrivial scaling of the variance.

In order to have an unit mass and finite moments, a lower cut off, ε, has to be specified. The power law probability density function of storm lifetimes may thus be written in the form

$$p(D) = \begin{cases} -1(1 - \beta)\varepsilon^{-(1-\beta)} D^{-\beta} & \text{if } D \geq \varepsilon \\ 0 & \text{if } D < \varepsilon \end{cases} \qquad (19)$$

and

$$\varepsilon = \frac{2 - \beta}{1 - \beta} \langle D \rangle \qquad (20)$$

The $q -$ th moment of distribution (19) is finite as long as $\beta > q + 1$.

The form of the covariance function obtained for this case by use of the procedure described in *Cox and Isham* [1988] is

$$c(\mathbf{r}, \tau) = \mu_{i,2} \left(\frac{|\tau|}{\varepsilon} + 1\right)^{2-\beta} \left(1 + \frac{cr}{2}\right) e^{-cr} - \mu_{i,1}^2 \qquad (21)$$

Figure 10 shows, for various values of the parameters, examples of the scaling of s with time when this scheme is used to generate the increments $i(\mathbf{x}, t)$. To interpret these graphs, one must turn to theory. Using the expression for the covariance function (21) in (8), one finds

$$E[\sigma^2](t) = \frac{2\varepsilon^2 \mu_{i,2} I}{V^2(B_R)(3 - \beta)} \left\{ \frac{1}{4 - \beta} \left[\left(\frac{t}{\varepsilon} + 1\right)^{4-\beta} - 1 \right] - t/\varepsilon \right\} \quad \text{if } \beta \neq 3,\ \beta \neq 4$$

$$E[\sigma^2](t) = \frac{2\varepsilon^2 \mu_{i,2} I}{V^2(B_R)} \left\{ \left(\frac{t}{\varepsilon} + 1\right) \left[\ln\left(\frac{t}{\varepsilon} + 1\right) - 1 \right] + 1 \right\} \quad \text{if } \beta = 3 \qquad (22)$$

$$E[\sigma^2](t) = \frac{2\varepsilon \mu_{i,2} I}{V^2(B_R)} \left\{ t - \varepsilon \left[\ln\left(\frac{t}{\varepsilon} + 1\right) \right] \right\} \quad \text{if } \beta = 4$$

Notice that one needs $\beta > 1$ to ensure existence of the probability density function (19), but that β may be <2. In this case the mean of d does not exist, and thus (20) is meaningless.

From (22), it follows that when $1 < \beta \leq 2$, the square root of expected spatial variance of the cumulated field exhibits an

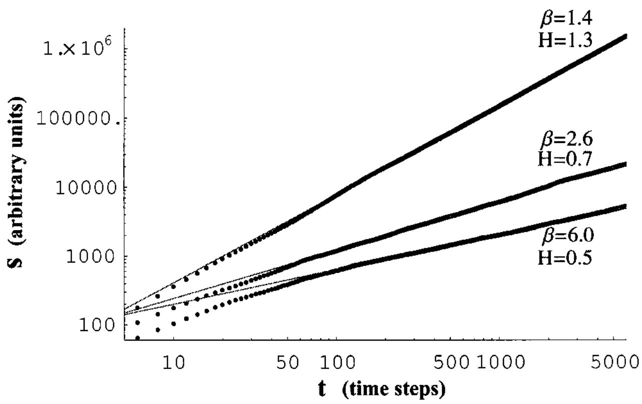

Figure 10. Scaling of the standard deviation s of cumulated fields, $\sigma(t) \propto t^H$, when storm duration D in the Poisson model generating $i(\mathbf{x}, \tau)$ has a power law probability density function $p(D) \propto D^{-\beta}$. If $\beta < 2$, the mean $\langle D \rangle$ does not exist and s scales with an exponent $H > 1$. If $2 < \beta < 3$ $\langle D \rangle$ is finite, the variance $\langle (D - \langle D \rangle)^2 \rangle$ does not exist and $0.5 < H < 1$. When $3 < \beta$, $\langle (D - \langle D \rangle)^2 \rangle$ is finite and $H = 0.5$.

asymptotic nontrivial scaling in time with exponent $H \geq 1$. These values of β cause the nonexistence of the first moment of the distribution of storm durations. When $2 < \beta < 3$, $\{E[\sigma^2](t)\}^{1/2}$ scales asymptotically with an exponent $0.5 < H < 1$. In this case, only the mean of the distribution of storm durations exists while higher moments are not finite. If $\beta = 3$, the growth of the square root of expected spatial variance with time is not power law but asymptotically behaves as $t \log(t)$.

Finally, if $\beta > 3$, $\lim_{t \to \infty} E[\sigma^2](t) \propto t$, and thus $H = 0.5$. In this case the first and second moment of the storm lifetimes distributions exist while the third and higher moments are not finite. These considerations explain the behavior observed in the various cases illustrated in Figure 10.

In summary (see Table 1), nontrivial scaling is to be expected whenever the storm lifetime distribution does not possess a finite variance. The scaling exponent of the standard deviation is $0.5 < H < 1.0$ if the mean is finite while it is otherwise >1.0. In all other cases, either $\{E[\sigma^2](t)\}^{1/2}$ does not scale, or it scales trivially with an exponent $H = 0.5$.

5. Conclusions

The observational analysis of the scaling characters of cumulated rainfall fields shows that their spatial standard deviation has a power law dependence on time with an exponent ranging from 0.59 to 0.71. The meaning and consequences of this experimental fact seem of foremost importance.

In the present spatial-temporal context, the value of the scaling exponent of the spatial standard deviation found for experimental rainfall fields indicates that the increment fields, $i(\mathbf{x}, \tau)$, are positively time correlated, a quite intuitive result in itself. More importantly, it was shown that not just any temporal correlation of $i(\mathbf{x}, \tau)$ will induce a scaling exponent $H > 0.5$ but that a long-range correlation, or memory, is needed to produce results similar to those found for observed rainfall. In fact, the introduction of an exponential temporal correlation through an arrival process of storms was theoretically shown and computationally validated to produce an asymptotic scaling with exponent $H = 0.5$. This indicates that after a transience at short time lags, the process forgets its history and becomes effectively uncorrelated in a space-time context.

Conversely, the same Poisson model, known to capture many rainfall mechanisms, in which a power law for the duration of storms is used, has been shown to be able to produce a scaling exponent of the standard deviation >0.5 and thus to replicate the observed character of cumulated rainfall fields. This is an important result if one wishes to consider a point process framework which indeed allows for an intuitive physical interpretation of the model parameters.

These considerations suggest that the processes producing rainfall must similarly possess long-range memory and thus cannot be represented through "short-tailed" probability distribution functions. Rather, in the light of the analyses presented, time related power law probability distributions appear to lead to the reproduction of the observed space-time characters of rainfall. The estimate of the exponent H is in fact a very discriminant and robust tool ruling out models in which an exponential, or short-memory, covariance function is used to specify the temporal structure of rainfall fields.

Table 1. Scaling Exponent H of the Spatial Variance of Cumulated Rainfall Fields When Storm Durations are Power Law Distributed With Exponent β

β	Highest Finite Moment	H
$\beta \leq 2$	$q = 0$	$H \geq 1$
$2 < \beta < 3$	$q = 1$	$0.5 < H < 1$
$\beta = 3$	$q = 1$	no scaling
$3 < \beta$	$q > 2$	$H = 0.5$

Appendix

Equation (4) may be obtained as follows:

$$E[\sigma^2](t) = E\left\{\frac{1}{V(B_R)}\int_{B_R}\left[h(\mathbf{x}', t) - \frac{1}{V(B_R)}\int_{B_R} h(\mathbf{x}'', t)\, d\mathbf{x}''\right]^2 d\mathbf{x}'\right\}$$

$$= E\left\{\frac{1}{V(B_R)}\int_{B_R}\left[h^2(\mathbf{x}', t) + \frac{1}{V^2(B_R)}\left(\int_{B_R} h(\mathbf{x}'', t)\, d\mathbf{x}''\right)^2 - \frac{2}{V(B_R)} h(\mathbf{x}', t)\int_{B_R} h(\mathbf{x}'', t)\, d\mathbf{x}''\right] d\mathbf{x}'\right\}$$

$$= \frac{1}{V(B_R)}\int_{B_R} E[h^2(\mathbf{x}', t)]\, d\mathbf{x}' + \frac{1}{V^3(B_R)}\int_{B_R} d\mathbf{x}' \int_{B_R} d\mathbf{x}'' \int_{B_R} d\mathbf{x}'''\, E[h(\mathbf{x}'', t)h(\mathbf{x}''', t)] - \frac{2}{V^2(B_R)}\int_{B_R} d\mathbf{x}' \int_{B_R} d\mathbf{x}''\, E[h(\mathbf{x}', t)h(\mathbf{x}'', t)]$$

Since, in the second term at the right hand side of the last equality, $E[h(\mathbf{x}'', t)h(\mathbf{x}''', t)]$ does not depend on $\mathbf{x}'$, one finds

$$E[\sigma^2](t) = \frac{1}{V(B_R)}\int_{B_R} E[h^2(\mathbf{x}', t)]\, d\mathbf{x}' - \frac{1}{V^2(B_R)}\int_{B_R} d\mathbf{x}' \int_{B_R} d\mathbf{x}''\, E[h(\mathbf{x}', t)h(\mathbf{x}'', t)]$$

Equation (1) may now be used to obtain

$$E[\sigma^2](t) = \frac{1}{V(B_R)} \int_{B_R} E\left[\int_0^t i(\mathbf{x}', \tau')\, d\tau' \int_0^t i(\mathbf{x}', \tau'')\, d\tau''\right] d\mathbf{x}'$$

$$- \frac{1}{V^2(B_R)} \int_{B_R} d\mathbf{x}' \int_{B_R} d\mathbf{x}''\, E\left[\int_0^t i(\mathbf{x}', \tau')\, d\tau' \int_0^t i(\mathbf{x}'', \tau'')\, d\tau''\right]$$

$$= \frac{1}{V(B_R)} \int_{B_R} d\mathbf{x}' \int_0^t d\tau' \int_0^t d\tau''\, E[i(\mathbf{x}', \tau') i(\mathbf{x}', \tau'')]$$

$$- \frac{1}{V^2(B_R)} \int_{B_R} d\mathbf{x}' \int_{B_R} d\mathbf{x}'' \int_0^t d\tau' \int_0^t d\tau''\, E[i(\mathbf{x}', \tau') i(\mathbf{x}'', \tau'')]$$

which is the result of (4).

Equation (11) is obtained by considering that from (10)

$$c(\mathbf{0}, \tau) - c(\mathbf{r}, \tau) = \begin{cases} 0 & \text{if } \tau' \neq \tau'' \\ \mu_{i,2} - c(\mathbf{r}) & \text{if } \tau' = \tau'' \end{cases}$$

Thus, by substitution into (8),

$$\int_0^t d\tau' \int_0^t [c(\mathbf{0}, \tau) - c(\mathbf{r}, \tau)]\, d\tau''$$

$$= \int_0^5 d\tau' \int_0^t [\mu_{i,2} - c(\mathbf{r})]\delta(\tau' - \tau'')\, d\tau''$$

$$= \int_0^t [\mu_{i,2} - c(\mathbf{r})]\, d\tau' = t[\mu_{i,2} - c(\mathbf{r})]$$

where δ( ) is Dirac's Delta.

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