Generalizing Morita duality: a homological approach

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Abstract

Let R and S be arbitrary associative rings. Given a bimodule ${}_RW_S$, we denote by $\Delta_?$ and $\Gamma_?$ the functors $\mathrm{Hom}_?(-,W)$ and $\mathrm{Ext}^1_?(-,W)$, where $? = R$ or S. The functors Δ_R and Δ_S are right adjoint with the evaluation maps δ as unities. A module M is Δ-reflexive if δ_M is an isomorphism. In this paper we give, for a weakly cotilting bimodule ${}_RW_S$, the notion of Γ-reflexivity. We construct large abelian subcategories $\mathcal{M}_R$ and $\mathcal{M}_S$ where the functors Γ_R and Γ_S are left adjoint and a "Cotilting theorem" holds.

Introduction

In this paper R and S will be associative rings with unity and ${}_RW_S$ will be a bimodule. We denote by Δ_R and Δ_S the contravariant functors

$$\mathrm{Hom}_R(-,W) : R\text{-Mod} \to \text{Mod-}S \quad \text{and} \quad \mathrm{Hom}_S(-,W) : \text{Mod-}S \to R\text{-Mod}$$

and by Γ_R and Γ_S the contravariant functors

$$\mathrm{Ext}^1_R(-,W) : R\text{-Mod} \to \text{Mod-}S \quad \text{and} \quad \mathrm{Ext}^1_S(-,W) : \text{Mod-}S \to R\text{-Mod}.$$

For each left R-module (resp. right S-module) M, we denote by $\delta_M : M \to \Delta_S\Delta_R M$ (resp. $\delta_M : M \to \Delta_R\Delta_S M$) the evaluation map. These maps define natural transformations δ between the identity functor $1_{R\text{-Mod}}$ and $\Delta_S\Delta_R$ and between the identity functor $1_{\text{Mod-}S}$ and $\Delta_R\Delta_S$, which are the unities of the right adjoint pair (Δ_R, Δ_S). A module M is said to be Δ-*reflexive* if δ_M is an isomorphism.

The bimodule ${}_RW_S$ defines a *Morita duality* ([14, 2]) if the classes of Δ-reflexive modules contain the rings and are finitely closed, i.e. closed with respect to submodules, factor modules and finite direct sums. This happens if and only if ${}_RW_S$ is a *Morita bimodule*, i.e. it is balanced and ${}_RW$ and W_S are injective cogenerators ([1, Theorem 24.1]). Morita bimodules are "rare": B. J. Müller has proved ([15]) that there exists a Morita bimodule ${}_RW_S$ if and only if both the regular module ${}_RR$ and the minimal cogenerator of R-Mod are linearly compact. For an extensive introduction to Morita duality, including various recent results, see [19].

Let ${}_RW_S$ be an arbitrary bimodule. The subcategories $\mathrm{Cogen}\,{}_RW$ and $\mathrm{Cogen}\,W_S$ of left R- and right S-modules cogenerated by W are the classes of modules M such that δ_M is a monomorphism; they contain the classes of Δ-reflexive modules. Outside these classes the functors Δ_R and Δ_S are not faithful: there we will consider the contribution of their derived functors Γ_R and Γ_S.

In order that the functors Δ_R, Δ_S and Γ_R, Γ_S play a major role in R-Mod and Mod-S, we require that on both sides the injective dimension of W is less than or equal to 1 and, to avoid overlaps of the two functors, that the functors Γ_R and Γ_S vanish on modules cogenerated by W. Such a bimodule ${}_RW_S$ will be called *weakly cotilting* (see page 6). A Morita bimodule is clearly a weakly cotilting bimodule, since $\mathrm{Cogen}\,{}_RW = \mathrm{Ker}\,\Gamma_R$ and $\mathrm{Cogen}\,W_S = \mathrm{Ker}\,\Gamma_S$ are the whole categories of modules. Interesting examples of weakly cotilting bimodules exist, also in the commutative case: if R is a maximal valuation domain, the regular bimodule ${}_R R_R$ is weakly cotilting (see Example 2.1).

The word "cotilting" appears for the first time in [12] for modules over finite dimensional algebras. Next, in [4], cotilting modules over noetherian rings are considered. In [5], a "Cotilting theorem" for modules over arbitrary rings is given: it is a dual form of the celebrated Brenner and Butler theorem, known also as the "Tilting theorem". Recently (see [8, 10] and in particular [7, 9]) the theory has been developed further.

Notation: we denote by Δ^2_{SR} (resp. Δ^2_{RS}) and by Γ^2_{SR} (resp. Γ^2_{SR}) the compositions $\Delta_S\Delta_R$ (resp. $\Delta_R\Delta_S$) and $\Gamma_S\Gamma_R$ (resp. $\Gamma_S\Gamma_R$). Writing Δ, Γ, Δ^2, Γ^2, $\Delta\Gamma$, $\Gamma\Delta$, ... as well as simply "module" we intend that we are indifferently working with left R- or right S- modules.

In this paper we try to understand, in the whole categories of modules, the behaviour and the relationships among the functors Γ^2, the identity functors and Δ^2: the zero left derived functor of Δ^2 will have a key role to relate them (see Theorem 1.2). It leads us to a natural definition of Γ-reflexivity (Definition 2.5), whereas in the literature (see [4, 15]) this problem is solved only inside special classes of modules. Hence we construct naturally abelian subcategories $\mathcal{M}_R$ and $\mathcal{M}_S$ where a Cotilting theorem (see Corollary 2.10) can be proved: they are the classes of left R- and right S- modules where the left derived maps $L_0\delta$ and $L_1\delta$ of the evaluation map δ are natural equivalences. For the first time, as far as we know, the existence of a local adjunction between the functors Γ_R and Γ_S is studied and proved.

1 Deriving the functor Δ^2

Let ${}_R W_S$ be a bimodule. Consider a projective resolution

$$\dots \stackrel{d_1}{\to} P_1 \stackrel{d_0}{\to} P_0 \ (\stackrel{\varepsilon}{\to} M\ ) \to 0$$

of a left R-module M. Applying the covariant functor Δ^2_{SR} we obtain the complex

$$\dots \stackrel{\Delta^2_{SR}(d_1)}{\to} \Delta^2_{SR}P_1 \stackrel{\Delta^2_{SR}(d_0)}{\to} \Delta^2_{SR}P_0 \stackrel{\Delta^2_{SR}(d_{-1})}{\to} 0.$$

The n-th left derived functor $L_n\Delta^2_{SR}$ is defined by

$$L_n\Delta^2_{SR}(M) = [\operatorname{Ker}\Delta^2_{SR}(d_{n-1})]/[\operatorname{Im}\Delta^2_{SR}(d_n)].$$

The augmentation ε yields a map $\Delta^2_{SR}(P_0) \to \Delta^2_{SR}M$ thus defining a natural map $\beta : L_0\Delta^2_{SR} \to \Delta^2_{SR}$. Denoted by δ the unity of the right adjoint pair (Δ_R, Δ_S), we have the following commutative diagram of functors and natural maps

$$\begin{array}{ccc} L_0 1_{R\text{-Mod}} & \xrightarrow{\cong} & 1_{R\text{-Mod}} \\ \downarrow{\scriptstyle L_0\delta} & & \downarrow{\scriptstyle \delta} \\ L_0\Delta^2_{SR} & \xrightarrow{\beta} & \Delta^2_{SR} \end{array}$$

In the sequel the natural map $L_0\delta$ will be denoted simply by $\delta^{(0)}$.

Lemma 1.1. *Given the solid part of the commutative diagram*

$$\begin{array}{ccccccc}
L & \xrightarrow{\psi} & M & \longrightarrow & A & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle\vartheta} & & \downarrow{\scriptstyle\alpha} & & \\
L & \xrightarrow{\varphi} & N & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\beta} & & \\
 & & C & = & C & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns, there are unique maps α and β such that the diagram commutes. With these maps the second column is exact; moreover, if ϑ is monic, then so is α.

Proof. It follows by diagram chasing. ∎

Assuming that $\operatorname{Cogen} W_S \subseteq \operatorname{Ker}\Gamma_S$, it is possible to calculate the left derived functors of Δ^2_{SR} working with short exact sequences. The i-th differentiation operator d_i factorizes through its image K_i; let $d_i = \lambda_i \circ \mu_i$ such a factorization. Applying Δ_R to $0 \to K_{i+1} \overset{\lambda_{i+1}}{\to} P_{i+1} \overset{\mu_i}{\to} K_i \to 0$ we get $0 \to \Delta_R K_i \overset{\Delta_R(\mu_i)}{\to} \Delta_R P_{i+1} \to C \to 0$ where C is the cokernel of $\Delta_R(\mu_i)$. Since $C \leq \Delta_R K_{i+1}$ and $\operatorname{Im}\Delta_R \subseteq \operatorname{Cogen} W_S$, $\Delta^2_{SR}(\mu_i)$ is surjective. Therefore

$$L_n\Delta^2_{SR}(M) = [\operatorname{Ker}\Delta^2_{SR}(d_{n-1})]/[\operatorname{Im}\Delta^2_{SR}(\lambda_n \circ \mu_n)] = [\operatorname{Ker}\Delta^2_{SR}(d_{n-1})]/[\operatorname{Im}\Delta^2_{SR}(\lambda_n)].$$

The following theorem describes how the functors 1_R, Δ^2_{SR}, Γ^2_{SR} and $L_0\Delta^2_{SR}$ are related on the whole category of left R-modules.

Theorem 1.2. *Let ${}_RW_S$ be a bimodule such that* $\operatorname{Cogen} W_S \subseteq \operatorname{Ker}\Gamma_S$*. Then there exists a natural map α such that*

$$\begin{array}{ccccccccc}
 & & & & 1_{R\text{-Mod}} & & & & \\
 & & & & \downarrow{\scriptstyle\delta^{(0)}} & \searrow{\scriptstyle\delta} & & & \\
0 & \longrightarrow & \Gamma^2_{SR} & \xrightarrow{\alpha} & L_0\Delta^2_{SR} & \xrightarrow{\beta} & \Delta^2_{SR} & \longrightarrow & 0
\end{array}$$

is a commutative diagram with exact row of functors and natural maps. In particular, on the subcategory $\operatorname{Ker}\Gamma_R$ *(resp.* $\operatorname{Ker}\Delta_R$*) β (resp. α) is a natural isomorphism.*

Proof. About the triangle involving the natural maps $\delta^{(0)}$, δ and β we have discussed above. Let us prove the existence of the wished natural map α. Consider an exact sequence

$$(\#) \qquad 0 \to K \overset{\lambda_0}{\to} P \overset{\varepsilon}{\to} M \to 0$$

with P projective. Denote by I the $\operatorname{Im}\Delta_R(\lambda_0)$ and by $i : I \to \Delta_R K$, $p : \Delta_R P \to I$ the morphisms factorizing $\Delta_R(\lambda_0)$. Applying the functors Δ_R and hence Δ_S to $(\#)$, we obtain the exact sequences

$$0 \to \Delta_S I \xrightarrow{\Delta_S(p)} \Delta^2_{SR}P \to \Delta^2_{SR}M \to 0 \qquad 0 \to \Delta_S\Gamma_R M \to \Delta^2_{SR}K \xrightarrow{\Delta_S(i)} \Delta_S I \overset{\partial}{\to} \Gamma^2_{SR}M \to 0$$

and hence, after an application of Lemma 1.1, we have the commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
0 & \to & \Delta_S\Gamma_R M & \to & \Delta^2_{SR}K & \xrightarrow{\Delta_S(i)} & \Delta_S I & \longrightarrow & \Gamma^2_{SR}M & \to & 0\\
 & & \| & & \| & & \downarrow{\scriptstyle \Delta_S(p)} & & \downarrow{\scriptstyle \alpha_M} & & \\
0 & \to & \Delta_S\Gamma_R M & \to & \Delta^2_{SR}K & \xrightarrow{\Delta^2_{SR}(\lambda_0)} & \Delta^2_{SR}P & \xrightarrow{\rho} & (L_0\Delta^2_{SR})M & \to & 0\\
 & & & & & & \downarrow{\scriptstyle \Delta^2_{SR}(\varepsilon)} & & \downarrow{\scriptstyle \beta_M} & & \\
 & & & & & & \Delta^2_{SR}M & = & \Delta^2_{SR}M & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & &
\end{array}
$$

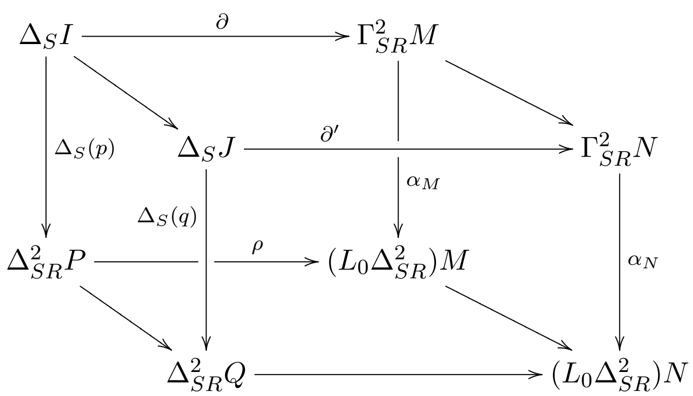
Now it remains to see that α is natural. Consider a morphism $f : M \to N$ of left R-modules and the commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \to & K & \xrightarrow{\lambda} & P & \xrightarrow{\varepsilon} & M & \to & 0\\
 & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle f} & & \\
0 & \to & H & \xrightarrow{\mu} & Q & \xrightarrow{\eta} & N & \to & 0
\end{array}
$$

with P and Q projective modules. Applying Δ_R we have the following commutative diagrams with exact rows

$$
(*) \qquad
\begin{array}{ccccccccc}
0 & \to & \Delta_R M & \to & \Delta_R P & \xrightarrow{p} & I & \to & 0\\
 & & \uparrow{\scriptstyle \Delta_R(f)} & & \uparrow{\scriptstyle \Gamma_R(\varphi)} & & \uparrow & & \\
0 & \to & \Delta_R N & \to & \Delta_R Q & \xrightarrow{q} & J & \to & 0
\end{array}
\qquad
\begin{array}{ccccccccc}
0 & \to & I & \to & \Delta_R K & \to & \Gamma_R M & \to & 0\\
 & & \uparrow & & \uparrow{\scriptstyle \Delta_R(\varphi')} & & \uparrow{\scriptstyle \Gamma_R(f)} & & \\
0 & \to & J & \to & \Delta_R H & \to & \Gamma_R N & \to & 0
\end{array}
$$

where I and J are the images of $\Delta_R(\lambda)$ and $\Delta_R(\mu)$. Applying Δ_S we obtain the diagram



The back and the front square commute by definition of α_M and α_N. The top square commutes by the naturality of the connecting homomorphisms (see [17, Theorem 6.4]). The left hand square commutes by diagram $(*)$ and the bottom square commutes by definition of $L_0\Delta^2_{SR}$. Thus, since ∂ is an epimorphism, an easy diagram chase shows that the right hand square commutes. Thus α is natural. ∎

The first and the second claims of the following proposition suggest the forthcoming assumptions on the injective dimension of ${}_RW$.

Proposition 1.3. *If* $\operatorname{Cogen} W_S \subseteq \operatorname{Ker}\Gamma_S$*, then*

1. *on the subcategory* $\operatorname{Ker}\operatorname{Ext}^2_R(-,W)$*, the functors* $L_1\Delta^2_{SR}$ *and* $\Delta_S\Gamma_R$ *are naturally isomorphic;*
2. *if* $\operatorname{Ext}^i_R(M,W)=0$ *for* $i=2,3,\dots,n+1$*, then* $(L_n\Delta^2_{SR})M=0$*.*

Proof. 1. Let $f:M\to N$ be a morphism of left R-modules. Consider $P^\bullet$ and $Q^\bullet$ projective resolutions of M and N with augmentations ε and ε', and differentiation operators d and d'. Denote by $F:P^\bullet\to Q^\bullet$ the map of complexes over f and by K_i (resp. K'_i) the image of d_i (resp. d'_i). Consider the diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \xrightarrow{\lambda_1} & P_1 & \xrightarrow{\mu_1} & K_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle F_1} & & \downarrow & & \\ 0 & \longrightarrow & K'_1 & \xrightarrow{\lambda'_1} & Q_1 & \xrightarrow{\mu'_1} & K'_0 & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & K_0 & \xrightarrow{\lambda_0} & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle F_0} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & K'_0 & \xrightarrow{\lambda'_0} & Q_0 & \xrightarrow{\mu'_0} & N & \longrightarrow & 0 \end{array}$$

Since $\operatorname{Ext}^2_R(M,W)=0=\operatorname{Ext}^2_R(N,W)$, we have $\Gamma_R K_0=0=\Gamma_R K'_0$. Denoted by I (I') the image of $\Delta_R(\lambda_0)$ ($\Delta_R(\lambda'_0)$), applying Δ_R we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta_R K_0 & \xrightarrow{\Delta_R(\mu_0)} & \Delta_R P_1 & \xrightarrow{\Delta_R(\lambda_1)} & \Delta_R K_1 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle \Delta_R(F_1)} & & \uparrow & & \\ 0 & \longrightarrow & \Delta_R K'_0 & \longrightarrow & \Delta_R Q_1 & \longrightarrow & \Delta_R K'_1 & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta_R M & \longrightarrow & \Delta_R P_0 & \longrightarrow & I & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \Delta_R(f)} & & \uparrow{\scriptstyle \Gamma_R(F_0)} & & \uparrow & & \\ 0 & \longrightarrow & \Delta_R N & \longrightarrow & \Delta_R Q_0 & \longrightarrow & I' & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & \Delta_R K_0 & \longrightarrow & \Gamma_R M & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \Gamma_R(f)} & & \\ 0 & \longrightarrow & I' & \longrightarrow & \Delta_R K'_0 & \longrightarrow & \Gamma_R N & \longrightarrow & 0 \end{array}$$

Applying Δ_S we obtain the diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta^2_{SR} K_1 & \xrightarrow{\Delta^2_{SR}(\lambda_1)} & \Delta^2_{SR} P_1 & \xrightarrow{\Delta^2_{SR}(\mu_0)} & \Delta^2_{SR} K_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \Delta^2_{SR}(F_1)} & & \downarrow & & \\ 0 & \longrightarrow & \Delta^2_{SR} K'_1 & \longrightarrow & \Delta^2_{SR} Q_1 & \longrightarrow & \Delta^2_{SR} K'_0 & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \Delta_S I & \xrightarrow{\vartheta} & \Delta^2_{SR} P_0 & \longrightarrow & \Delta^2_{SR} M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \Delta^2_{SR}(F_0)} & & \downarrow & & \\ 0 & \longrightarrow & \Delta_S I' & \xrightarrow{\vartheta'} & \Delta^2_{SR} Q_0 & \longrightarrow & \Delta^2_{SR} N & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Delta_S\Gamma_R M & \longrightarrow & \Delta^2_{SR} K_0 & \xrightarrow{\eta} & \Delta_S I & \longrightarrow & \Gamma^2_{SR} M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Delta_S\Gamma_R(f)} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Gamma^2_{SR}(f)} & & \\ 0 & \longrightarrow & \Delta_S\Gamma_R N & \longrightarrow & \Delta^2_{SR} K'_0 & \xrightarrow{\eta'} & \Delta_S I' & \longrightarrow & \Gamma^2_{SR} N & \longrightarrow & 0 \end{array}$$

Then, since $\Delta^2_{SR}(d_0)=\vartheta\circ\eta\circ\Delta^2_{SR}(\mu_0)$ we have

$$(L_1\Delta^2_{SR})M \cong \operatorname{Ker}[\vartheta\circ\eta\circ\Delta^2_{SR}(\mu_0)]/[\operatorname{Im}\Delta^2_{SR}(\lambda_1)] \cong \Delta_S\Gamma_R M,$$

$(L_1\Delta^2_{SR})N \cong \Delta_S\Gamma_R N$ and $(L_1\Delta^2_{SR})(f)\cong\Delta_S\Gamma_R(f)$.

2. Let us consider the long exact sequence

$$\cdots \longrightarrow (L_n\Delta^2_{SR})P_0 = 0 \longrightarrow (L_n\Delta^2_{SR})M \longrightarrow (L_{n-1}\Delta^2_{SR})K_0 \longrightarrow (L_{n-1}\Delta^2_{SR})P_0 = 0 \longrightarrow \cdots$$

We proceed by induction on $n\geq 2$. Let $n=2$: since $\operatorname{Ext}^3_R(M,W)=0$, we have $\operatorname{Ext}^2_R(K_0,W)=0$ and hence, by 1., $(L_1\Delta^2_{SR})K_0=\Delta_S\Gamma_R K_0$. Being $\operatorname{Ext}^2_R(M,W)=0$, then $\Gamma_R K_0=0$. Therefore $(L_2\Delta^2_{SR})M=(L_1\Delta^2_{SR})K_0=0$. Next, let $n>2$: if $\operatorname{Ext}^i_R(M,W)=0$, $2\leq i\leq n+1$, then $\operatorname{Ext}^i_R(K_0,W)=0$, $1\leq i\leq n$. By inductive hypothesis $(L_{n-1}\Delta^2_{SR})K_0=0$. Therefore $(L_n\Delta^2_{SR})M=0$. ∎

In the next section we will study modules M such that $\delta^{(0)}_M$ is an isomorphism; we have the following

Proposition 1.4. *If* $\operatorname{Cogen} W_S \subseteq \operatorname{Ker} \Gamma_S$*, then for each module* M *in* Mod-S *the maps* $\delta^{(0)}_{\Delta M}$ *and* $\delta^{(0)}_{\Gamma M}$ *are both monomorphisms.*

Proof. Since for each module cogenerated by W, the evaluation map is injective, $\delta^{(0)}_{\Delta M}$ is a monomorphism by Theorem 1.2. Next, consider an exact sequence $0 \to K \overset{i}{\to} P \to M \to 0$ with P projective. Applying Δ we obtain the exact sequences

$$0 \to \Delta M \to \Delta P \to I \to 0 \quad 0 \to I \to \Delta K \overset{\varphi}{\to} \Gamma M \to 0.$$

Applying the functor $L_0\Delta^2$ we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} \Delta P & \xrightarrow{\Delta(i)} & \Delta K & \xrightarrow{\varphi} & \Gamma M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \delta_{\Delta P}} & & \downarrow{\scriptstyle \delta_{\Delta K}} & & \downarrow{\scriptstyle \delta^{(0)}_{\Gamma M}} & & \\ \Delta^3 P & \xrightarrow{\Delta^3(i)} & \Delta^3 K & \xrightarrow{(L_0\Delta^2)(\varphi)} & (L_0\Delta^2)\Gamma M & \longrightarrow & 0 \end{array}$$

Let $\delta^{(0)}_{\Gamma M}(x) = 0$ with $x \in \Gamma M$; consider $y \in \Delta K$ such that $x = \varphi(y)$. Since $[(L_0\Delta^2)(\varphi) \circ \delta_{\Delta K}](y) = 0$, there exists $z \in \Delta^3 P$ such that $\delta_{\Delta K}(y) = \Delta^3(i)(z)$. Thus

$$y = [\Delta(\delta_K) \circ \delta_{\Delta K}](y) = [\Delta(\delta_K) \circ \Delta^3(i)](z) = [\Delta(i) \circ \Delta(\delta_P)](z)$$

belongs to $\operatorname{Im} \Delta(i)$ and hence $x = \varphi(y) = 0$. ∎

2 The Cotilting Theorem

A left R-module W is said to be *weakly cotilting* if

(i) $\operatorname{id}_R W \leq 1$,

(ii) $\operatorname{Ext}^1_R(W^\alpha, W) = 0$ for each cardinal α.

These conditions (i) and (ii) are equivalent to say that $\operatorname{Cogen}_R W \subseteq \operatorname{Ker} \Gamma_R$ and $\operatorname{id}_R W \leq 1$. It is easy to see that any faithful left R-module ${}_R W$ such that $\operatorname{Cogen}_R W \subseteq \operatorname{Ker} \Gamma_R$ is weakly cotilting. A weakly cotilting module ${}_R W$ is *cotilting* (see [8, Definition 1.6], [6, §2]) if and only if

for all M in R-Mod, if $\operatorname{Hom}_R(M, W) = 0 = \operatorname{Ext}^1_R(M, W)$, then $M = 0$.

In the sequel of the paper we suppose always that ${}_R W_S$ is a *weakly cotilting bimodule*, i.e. both ${}_R W$ and W_S are weakly cotilting.

Example 2.1. Consider a complete almost maximal Prüfer domain R (e.g. a maximal valuation domain). By [3, Proposition 4.2] $\operatorname{id} R \leq 1$ and, by [11, Theorem 3.1], $\operatorname{Ext}^1_R(F, R) = 0$ for each torsion-free R-module F: in particular $\operatorname{Ext}^1_R(R^\alpha, R) = 0$ for each cardinal α. Therefore the regular bimodule ${}_R R_R$ is weakly cotilting. Observe that if R is not a Dedekind domain, it is not noetherian.

The simmetry of the setting suggests to denote simply by Δ^2 and by Γ^2 both the compositions $\Delta_S \circ \Delta_R$ and $\Delta_R \circ \Delta_S$, and $\Gamma_S \circ \Gamma_R$ and $\Gamma_S \circ \Gamma_R$; we will write also Δ, Γ, $\Delta\Gamma$, $\Gamma\Delta$, ... as well "module" to intend that we are indifferently working with left R- or right S- modules.

Proposition 2.2. *For each module* M *we have the following commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Rej}_W M & \xrightarrow{i_M} & M & \xrightarrow{p_M} & [M/\operatorname{Rej}_W M] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta^{(0)}_{\operatorname{Rej}_W M}} & & \downarrow{\scriptstyle \delta^{(0)}_M} & & \downarrow{\scriptstyle \delta_{[M/\operatorname{Rej}_W M]}} & & \\ 0 & \longrightarrow & \Gamma^2 \operatorname{Rej}_W M & \xrightarrow{(L_0\Delta^2)(i_M)} & (L_0\Delta^2)M & \xrightarrow{(L_0\Delta^2)(p_M)} & \Delta^2[M/\operatorname{Rej}_W M] & \longrightarrow & 0 \\ & & \cong \downarrow{\scriptstyle \Gamma^2(i_M)} & & \| & & {\scriptstyle \Delta^2(p_M)} \uparrow \cong & & \\ 0 & \longrightarrow & \Gamma^2 M & \xrightarrow{\alpha_M} & (L_0\Delta^2)M & \xrightarrow{\beta_M} & \Delta^2 M & \longrightarrow & 0 \end{array}$$

Moreover

1. *the squares on the left are pullback: in particular* $\operatorname{Ker}\delta^{(0)}_{\operatorname{Rej}_W M} \cong \operatorname{Ker}\delta^{(0)}_M$*;*

2. $\operatorname{Coker}\delta^{(0)}_{\operatorname{Rej}_W M}$ *belongs to* $\operatorname{Ker}\Gamma$ *if and only if* $\operatorname{Coker}\delta^{(0)}_M$ *belongs to* $\operatorname{Ker}\Gamma$*;*

3. *the squares on the right are pushout if and only if* $\delta^{(0)}_{\operatorname{Rej}_W M}$ *is surjective.*

Proof. The second row of the diagram, except for the injectivity of $(L_0\Delta^2)(i_M)$, is obtained applying $L_0\Delta^2$ to the first row: remember that, by Theorem 1.2, the functor $L_0\Delta^2$ is naturally isomorphic to Γ^2 and Δ^2 on $\operatorname{Ker}\Delta$ and $\operatorname{Ker}\Gamma$, respectively. The third row is part of Theorem 1.2. The commutativity of the top squares follows by the naturality of $\delta^{(0)}$. The maps $\Gamma^2(i_M)$ and $\Delta^2(p_M)$ are clearly isomorphisms and $\delta_{[M/\operatorname{Rej}_W M]}$ is a monomorphism. Then we have to verify only that $\alpha_M \circ \Gamma^2(i_M) = (L_0\Delta^2)(i_M)$ and $\Delta^2(p_M)\circ\beta_M = (L_0\Delta^2)(p_M)$. Let us see the first equality; the second one is obtained in a similar way. Given a projective resolution $P^\bullet$ of $\operatorname{Rej}_W M$ and one $Q^\bullet$ of M, consider the map $F : P^\bullet \to Q^\bullet$ over the inclusion i_M. We have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0 & \xrightarrow{\lambda_0} & P_0 & \longrightarrow & \operatorname{Rej}_W M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle F_0} & & \downarrow{\scriptstyle i_M} & & \\
0 & \longrightarrow & H_0 & \xrightarrow{\mu_0} & Q_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Applying Δ we get

$$\begin{array}{ccccccccccc}
 & & 0 & \longrightarrow & \Delta P_0 & \xrightarrow{\Delta(\lambda_0)} & \Delta K_0 & \longrightarrow & \Gamma\operatorname{Rej}_W M & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \Gamma(i_M)} & & \\
0 & \longrightarrow & \Delta M & \longrightarrow & \Delta Q_0 & \xrightarrow{\Delta(\mu_0)} & \Delta H_0 & \longrightarrow & \Gamma M & \longrightarrow & 0
\end{array}$$

Denote by J the image of $\Delta(\mu_0)$, by $q : \Delta Q_0 \to J$ the canonical projection and by $\rho : J \to \Delta P_0$ the induced morphism such that $\rho\circ q = \Delta(F_0)$. Applying Δ to the last diagram and $L_0\Delta^2$ to $0 \to H_0 \to Q_0 \to M \to 0$ we have the commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Delta\Gamma\operatorname{Rej}_W M & \longrightarrow & \Delta^2 K_0 & \longrightarrow & \Delta^2 P_0 & \longrightarrow & \Gamma^2\operatorname{Rej}_W M = (L_0\Delta^2)\operatorname{Rej}_W M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Delta\Gamma(i_M)} & & \downarrow & & \downarrow{\scriptstyle \Delta(\rho)} & & \downarrow{\scriptstyle \Gamma^2(i_M)} & & \\
0 & \longrightarrow & \Delta\Gamma_R M & \longrightarrow & \Delta^2 H_0 & \longrightarrow & \Delta J & \longrightarrow & \Gamma^2 M & \longrightarrow & 0 \\
 & & & & & & \downarrow{\scriptstyle \Delta(q)} & & \downarrow{\scriptstyle \alpha_M} & & \\
0 & \longrightarrow & (L_1\Delta^2)M & \longrightarrow & (L_0\Delta^2)H_0 & \longrightarrow & \Delta^2 Q_0 & \longrightarrow & (L_0\Delta^2)M & \longrightarrow & 0
\end{array}$$

Looking at the first and the third rows of the diagram, since $\Delta(q)\circ\Delta(\rho) = \Delta(\rho\circ q) = \Delta^2(F_0)$, we have $\alpha_M \circ \Gamma^2_{SR}(i_M) = (L_0\Delta^2_{SR})(i_M)$. In particular we obtain that $(L_0\Delta^2_{SR})(i_M)$ is a monomorphism. Properties 1. and 3. follow by [18, 10.3, 10.6]. The Snake Lemma (see [16, 11.3]) give us the exact sequence

$$0 \to \operatorname{Coker}\delta^{(0)}_{\operatorname{Rej}_W M} \to \operatorname{Coker}\delta^{(0)}_M \to \operatorname{Coker}\delta_{[M/\operatorname{Rej}_W M]} \to 0.$$

By [9, Lemma 1.1, d)] $\operatorname{Coker}\delta_{[M/\operatorname{Rej}_W M]}$ belongs to $\operatorname{Ker}\Gamma$. Therefore, since $\operatorname{Ker}\Gamma$ is closed under submodules, also property 2. is easily proved. ■

Corollary 2.3. *The following conditions are equivalent*

1. $\delta^{(0)}_M$ *is an isomorphism,*

2. $\delta^{(0)}_{\mathrm{Rej}_W M}$ *and* $\delta^{(0)}_{[M/\mathrm{Rej}_W M]}$ *are isomorphisms.*

In such a case $\delta^{(0)}_{\Gamma^2 M}$ *and* $\delta^{(0)}_{\Delta^2 M}$ *are isomorphisms.*

Proof. The equivalence of 1. and 2. follows easily by Proposition 2.2. If 2. is satisfied, then, again by Proposition 2.2, we have $\mathrm{Rej}_W M \cong \Gamma^2 M$ and $M/\mathrm{Rej}_W M \cong \Delta^2 M$. ∎

Proposition 2.4. *A module M is Δ-reflexive if and only if* $\delta^{(0)}_M$ *and* β_M *are isomorphisms.*

Proof. Since $\delta_M = \beta_M \circ \delta^{(0)}_M$ the sufficiency is clear. Suppose $\delta_M = \beta_M \circ \delta^{(0)}_M$ an isomorphism; looking at the diagram of Proposition 2.2, this happen if and only if $\delta^{(0)}_{[M/\mathrm{Rej}_W M]} \circ p_M$ is an isomorphism. Now, since p_M is surjective, $\delta^{(0)}_{[M/\mathrm{Rej}_W M]} \circ p_M$ is an isomorphism if and only if both p_M and $\delta^{(0)}_{[M/\mathrm{Rej}_W M]}$ are isomorphisms. Therefore $\mathrm{Rej}_W M = 0$ and hence $\delta^{(0)}_M = \delta^{(0)}_{[M/\mathrm{Rej}_W M]}$ and β_M are isomorphisms. ∎

The above proposition suggests the following

Definition 2.5. We say that a module M is Γ-reflexive if and only if $\delta^{(0)}_M$ and α_M are isomorphisms.

For each module M such that $\delta^{(0)}_M$ is an isomorphism we define a morphism $\gamma_M : \Gamma^2 M \to M$, setting $\gamma_M = {\delta^{(0)}_M}^{-1} \circ \alpha_M$.

$$\begin{array}{ccccccccc} & & & & M & & & & \\ & & & \overset{\gamma_M}{\nearrow} & \cong\downarrow \delta_M^{(0)} & \overset{\delta_M}{\searrow} & & & \\ 0 & \longrightarrow & \Gamma^2_{SR}M & \xrightarrow{\alpha_M} & [L_0\Delta^2_{SR}]M & \xrightarrow{\beta_M} & \Delta^2_{SR}M & \longrightarrow & 0 \end{array}$$

The maps γ_M define a natural transformation γ between Γ^2 and the identity functor restricted to the class of modules where $\delta^{(0)}$ is a natural equivalence. Then a module M is Γ-reflexive if and only if γ_M is defined and it is an isomorphism; in such a case $M = \mathrm{Rej}_W M$ belongs to $\mathrm{Ker}\,\Delta$.

Let us consider the subcategories

- $\mathcal{M}_0$ of all modules M such that $\delta^{(0)}_M$ is an isomorphism,
- $\mathcal{M}_1$ of all modules M such that $\delta^{(1)}_M := L_1\delta_M$ is an isomorphism,
- $\mathcal{M} = \mathcal{M}_0 \cap \mathcal{M}_1$.

Since $\delta^{(1)}$ is a natural map between the zero functor (the first derived of the identity functor) and $L_1\Delta^2 \cong \Delta\Gamma$ (see Proposition 1.3), $\mathcal{M}_1 = \mathrm{Ker}\,\Delta\Gamma$ and it is the largest subcategory where the functor $L_0\Delta^2$ is exact. It is interesting to observe that the subcategory of Δ-reflexive modules like all of these subcategories $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{M}$ are defined through the evaluation map δ.

Clearly (see Proposition 2.4 and Definition 2.5) the Δ-reflexive and the Γ-reflexive modules belong to $\mathcal{M}_0$. In fact the Δ-reflexive modules belong to $\mathcal{M}$, since $\Gamma\Delta = 0$. The next theorem shows as each module in $\mathcal{M}_0$ is an extension of a Γ-reflexive module by a Δ-reflexive module.

Theorem 2.6. *For each module $M \in \mathcal{M}_0$ the sequence*

$$0 \to \Gamma^2 M \overset{\gamma_M}{\to} M \overset{\delta_M}{\to} \Delta^2 M \to 0$$

is exact, ΔM and $\Delta^2 M$ are Δ-reflexive and $\Gamma^2 M$ is Γ-reflexive.

Proof. The short exact sequence follows by Theorem 1.2 and the above definition of the map γ. Applying Δ to it, we obtain the long exact sequence of right S-modules

$$0 \to \Delta^3 M \overset{\Delta(\delta_M)}{\to} \Delta M \overset{\Delta(\gamma_M)}{\to} \Delta\Gamma^2 M \to \Gamma\Delta^2 M = 0 \to \Gamma M \overset{\Gamma(\gamma_M)}{\to} \Gamma^3 M \to 0.$$

Since $\Delta(\delta_M) \circ \delta_{\Delta_M} = 1_{\Delta M}$, $\Delta(\delta_M)$ and $\delta_{\Delta M}$ are isomorphisms and $\Delta\Gamma^2 M = 0$. Then ΔM and $\Delta^2 M$ are Δ-reflexive. Since $\Delta\Gamma^2 M = 0$, $\alpha_{\Gamma^2 M}$ is an isomorphism. Also, by Corollary 2.3, $\Gamma^2 M \cong \operatorname{Ker} \delta_M = \operatorname{Rej}_W M$ implies $\delta^{(0)}_{\Gamma^2 M}$ is an isomorphism, thus $\Gamma^2 M$ is Γ-reflexive. ■

Corollary 2.7. *1. A module M is Δ-reflexive if and only if $M \in \operatorname{Ker}\Gamma \cap \mathcal{M}_0$.*

2. A module M is Γ-reflexive if and only if $M \in \operatorname{Ker}\Delta \cap \mathcal{M}_0$.

3. The functors Δ_R and Δ_S send objects in $\mathcal{M}_0$ to objects in $\operatorname{Ker}\Gamma \cap \mathcal{M}_0$, inducing a duality between the full subcategories $\operatorname{Ker}\Gamma \cap \mathcal{M}_0$.

4. The pair $(\operatorname{Ker}\Delta \cap \mathcal{M}_0, \operatorname{Ker}\Gamma \cap \mathcal{M}_0)$ is a torsion theory in $\mathcal{M}_0$.

5. The class $\mathcal{M}_0$ is closed under finite direct sums and direct summands of modules in $\mathcal{M}_0$ and images, cokernels and pushout of morphisms in $\mathcal{M}_0$.

Proof. 1., 2. and 3. follow immediately by Theorem 2.6.

4. There are no non zero homomorphisms between Γ-reflexive and Δ-reflexive objects. For, let $M \in \operatorname{Ker}\Delta \cap \mathcal{M}_0$ and $N \in \operatorname{Ker}\Gamma \cap \mathcal{M}_0$ and f a morphism of M to N; since $N \cong \Delta^2 N$, there exists a monomorphism $\varphi : N \to W^\alpha$ for some cardinal α. Since $M \cong \Gamma^2 M$ and $\Delta\Gamma^2 M = 0$, $\varphi \circ f = 0$ and hence $f = 0$. Moreover these classes are maximal in $\mathcal{M}_0$, with respect to this property: if $L \in \mathcal{M}_0$ (resp. $M \in \mathcal{M}_0$) and $\operatorname{Hom}(L, M) = 0$ for each $M \in \operatorname{Ker}\Gamma \cap \mathcal{M}_0$ (resp. for each $L \in \operatorname{Ker}\Delta \cap \mathcal{M}_0$), by Theorem 2.6 $\delta_L = 0$ (resp. $\gamma_M = 0$) and hence $L \cong \Gamma^2 L$ belongs to $\operatorname{Ker}\Delta$ (resp. $M \cong \Delta^2 M$ belongs to $\operatorname{Ker}\Gamma$).

5. The closure under finite direct sums is a consequence of the additivity of $L_0\Delta^2$. Let $f : M \to N$ a morphism with M, $N \in \mathcal{M}_0$. Consider the commutative diagrams with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} f & \longrightarrow & M & \longrightarrow & \operatorname{Im} f & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta^{(0)}_{\operatorname{Ker} f}} & & {\scriptstyle \delta^{(0)}_M}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \delta^{(0)}_{\operatorname{Im} f}} & & \\
 & & (L_0\Delta^2)\operatorname{Ker} f & \longrightarrow & (L_0\Delta^2)M & \longrightarrow & (L_0\Delta^2)\operatorname{Im} f & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Im} f & \longrightarrow & N & \longrightarrow & N/\operatorname{Im} f & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta^{(0)}_{\operatorname{Im} f}} & & {\scriptstyle \delta^{(0)}_N}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \delta^{(0)}_{N/\operatorname{Im} f}} & & \\
 & & (L_0\Delta^2)\operatorname{Im} f & \longrightarrow & (L_0\Delta^2)N & \longrightarrow & (L_0\Delta^2)N/\operatorname{Im} f & \longrightarrow & 0.
\end{array}$$

Since $\delta^{(0)}_M$ and $\delta^{(0)}_N$ are isomorphisms, then $\delta^{(0)}_{\operatorname{Im} f}$ and hence $\delta^{(0)}_{N/\operatorname{Im} f} = \delta^{(0)}_{\operatorname{Coker} f}$ are isomorphisms. If $M_1 \oplus M_2 \in \mathcal{M}_0$, then also the images of the endomorphisms projections belongs to $\mathcal{M}_0$. Finally, the pushout of two morphisms $f : L \to M$ and $g : L \to N$ with L, M, and N in $\mathcal{M}_0$ is the cokernel of the map $L \to M \oplus N$, $l \mapsto (f(l), g(l))$, and hence it belongs to $\mathcal{M}_0$. ■

The adjunction between Δ_R and Δ_S was crucial in proving that the functor Δ sends objects of $\mathcal{M}_0$ to objects which are Δ-reflexive. Lacking such a property it is not even clear if the functor Γ sends objects of $\mathcal{M}_0$ to objects of $\mathcal{M}_0$. The problem is solved in the smaller class $\mathcal{M}$, thanks to the following lemma.

Lemma 2.8. *For each module M in $\mathcal{M}_1$ we have*

$$\Gamma(\delta^{(0)}_M) \circ [\Gamma(\alpha_M)]^{-1} \circ [\alpha_{\Gamma M}]^{-1} \circ \delta^{(0)}_{\Gamma M} = 1_{\Gamma M}.$$

Proof. Let $M \in \mathcal{M}_1$; then $\Gamma(\alpha_M)$ and $\alpha_{\Gamma M}$ are both isomorphisms. Next, consider a short exact sequence $0 \to K \overset{i}{\to} P \overset{p}{\to} M \to 0$ with P projective. Denoting the image of $\Delta(i)$ by I, we have the following diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\ i\ } & P & \xrightarrow{\ p\ } & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_K} & & \downarrow{\scriptstyle \delta_P} & & \downarrow{\scriptstyle \delta_M^{(0)}} & & \\
0 & \longrightarrow & \Delta^2 K & \longrightarrow & \Delta^2 P & \longrightarrow & (L_0\Delta^2)M & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \alpha_M} & & \\
0 & \longrightarrow & \Delta^2 K & \longrightarrow & \Delta I & \longrightarrow & \Gamma^2 M & \longrightarrow & 0
\end{array}$$

Applying Δ to it and $L_0\Delta^2$ to $\Delta K \overset{\partial_1}{\to} \Gamma M \to 0$ we get the following diagram

$$(\#)\qquad \begin{array}{ccc}
\Delta K & \xrightarrow{\ \partial_1\ } & \Gamma M \\
{\scriptstyle \Delta(\delta_K)}\uparrow & & \uparrow{\scriptstyle \Gamma(\delta_M^{(0)})} \\
\Delta^3 K & \xrightarrow{\ \partial_2\ } & \Gamma(L_0\Delta^2)M \\
\| & & \downarrow{\scriptstyle \Gamma(\alpha_M)} \\
\Delta^3 K & \xrightarrow{\ \partial_3\ } & \Gamma^3 M \\
{\scriptstyle \beta_{\Delta K}}\uparrow & & \downarrow{\scriptstyle \alpha_{\Gamma M}} \\
(L_0\Delta^2)\Delta K & \xrightarrow{\ L_0\Delta^2(\partial_1)\ } & (L_0\Delta^2)\Gamma M \\
{\scriptstyle \delta_{\Delta K}^{(0)}}\uparrow & & \uparrow{\scriptstyle \delta_{\Gamma M}^{(0)}} \\
\Delta K & \xrightarrow{\ \partial_1\ } & \Gamma M
\end{array}$$

Its solid part is commutative; let us prove that the whole diagram is commutative. Given an exact sequence $0 \to H_1 \to Q \overset{q}{\to} \Delta K \to 0$ with Q projective, we can construct the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & I & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_1 & \longrightarrow & Q & \xrightarrow{\ q\ } & \Delta K & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle \partial_1} & & \\
0 & \longrightarrow & H_2 & \longrightarrow & Q & \longrightarrow & \Gamma M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & I & \longrightarrow & \Delta K & \xrightarrow{\ \partial_1\ } & \Gamma M & \longrightarrow & 0
\end{array}$$

where $H_2 = q^{-1}(I)$. Applying Δ twice we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Delta^2 H_1 & \longrightarrow & \Delta^2 H_2 & \longrightarrow & \Delta^2 I & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Delta^2 H_1 & \longrightarrow & \Delta^2 Q & \longrightarrow & \Delta^3 K & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \vdots & & \\
0 & \longrightarrow & \Delta^2 H_2 & \longrightarrow & \Delta^2 Q & \longrightarrow & \Gamma^3 M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Delta^2 I & \longrightarrow & \Delta^3 K & \xrightarrow{\partial_3} & \Gamma^3 M & \longrightarrow & 0
\end{array}$$

The dotted arrow $\Delta^3 K \dashrightarrow \Gamma^3 M$ represents the unique mapping such that the middle right square of the diagram commutes. On one hand it is, by construction, $\alpha_{\Gamma M}^{-1} \circ (L_0\Delta^2)(\partial_1) \circ \beta_{\Delta K}^{-1}$; on the other hand, looking at the commutative right bottom square, it must be ∂_3. Therefore, the whole diagram (#) commutes. Now the promised identity follows by

$$\Gamma(\delta_M^{(0)}) \circ [\Gamma(\alpha_M)]^{-1} \circ [\alpha_{\Gamma M}]^{-1} \circ \delta_{\Gamma M}^{(0)} \circ \partial_1 = \Gamma(\delta_M^{(0)}) \circ [\Gamma(\alpha_M)]^{-1} \circ [\alpha_{\Gamma M}]^{-1} \circ (L_0\Delta^2)(\partial_1) \circ \delta_{\Delta K}^{(0)} =$$

$$= \Gamma(\delta_M^{(0)}) \circ [\Gamma(\alpha_M)]^{-1} \circ \partial_3 \circ \beta_{\Delta_K} \circ \delta_{\Delta K}^{(0)} = \Gamma(\delta_M^{(0)}) \circ \partial_2 \circ \delta_{\Delta K} = \partial_1 \circ \Delta(\delta_K) \circ \delta_{\Delta K} = \partial_1$$

and the fact that ∂_1 is epic. ∎

We are ready to present the complete version of our "Cotilting Theorem", knowing better, inside the class $\mathcal{M} = \mathcal{M}_0 \cap \mathcal{M}_1$, the behaviour of the functor Γ.

Theorem 2.9. *For each module $M \in \mathcal{M}$ the sequence*

$$0 \to \Gamma^2 M \overset{\gamma_M}{\to} M \overset{\delta_M}{\to} \Delta^2 M \to 0$$

is exact, ΔM and $\Delta^2 M$ are Δ-reflexive, ΓM and $\Gamma^2 M$ are Γ-reflexive.

Proof. We have only to prove that ΓM is Γ-reflexive, the rest following by Theorem 2.6. Applying Theorem 1.2 to ΓM we have the short exact sequence

$$0 \to \Gamma^3 M \overset{\alpha_{\Gamma M}}{\to} (L_0\Delta^2)\Gamma M \to \Delta^2 \Gamma M = 0;$$

hence $\alpha_{\Gamma M}$ is an isomorphism. Since $\Gamma(\delta_M^{(0)})$ is an isomorphism, by Lemma 2.8 also $\delta_{\Gamma M}^{(0)}$ is an isomorphism and hence ΓM is Γ-reflexive. ∎

Corollary 2.10 (The Cotilting Theorem). *1. The functors Δ_R and Δ_S send objects in $\mathcal{M}$ to objects in* $\operatorname{Ker}\Gamma \cap \mathcal{M} = \operatorname{Ker}\Gamma \cap \mathcal{M}_0$*, inducing a duality between the full subcategories* $\operatorname{Ker}\Gamma \cap \mathcal{M}$.

2. *The functors Γ_R and Γ_S send objects in $\mathcal{M}$ to objects in* $\operatorname{Ker}\Delta \cap \mathcal{M}$*, inducing a duality between the full subcategories* $\operatorname{Ker}\Delta \cap \mathcal{M}$.

3. *The pair* $(\operatorname{Ker}\Delta \cap \mathcal{M}, \operatorname{Ker}\Gamma \cap \mathcal{M})$ *is a torsion theory in $\mathcal{M}$.*

4. *The class $\mathcal{M}$ is closed under extensions and direct summands of modules in $\mathcal{M}$ and images, kernels, cokernels, pullback and pushout of morphisms in $\mathcal{M}$: in particular, it is an abelian subcategory of the category of left R- or right S- modules.*

5. *The functors Γ_R and Γ_S are left adjoint in $\mathcal{M}$ with the natural maps γ as counities.*

Proof. 1. If $M \in \operatorname{Ker}\Gamma \cap \mathcal{M}_0$, then by Theorem 2.6 $M \cong \Delta^2 M$. Therefore $\Delta\Gamma M \cong \Delta\Gamma\Delta^2 M = 0$, so, since $\mathcal{M}_1 = \operatorname{Ker}\Delta\Gamma$, M belongs to $\mathcal{M}$. Now the claim follows by Corollary 2.7, 3.

2. follows by Theorems 2.9 and 2.6.

3. follows by 2 and Corollary 2.7, 4.

4. Consider an exact sequence $0 \to J \to H \to K \to 0$ with J, $K \in \mathcal{M}$; applying $L_0\Delta^2$ we obtain the exact sequence

$$0 \to (L_0\Delta^2)J \to (L_0\Delta^2)H \to (L_0\Delta^2)K \to 0.$$

Since $\delta_J^{(0)}$ and $\delta_K^{(0)}$ are isomorphisms, also $\delta_H^{(0)}$ is an isomorphism. Applying $L_1\Delta^2$ we have the exact sequence

$$0 = (L_1\Delta^2)J \to (L_1\Delta^2)H \to (L_1\Delta^2)K = 0;$$

therefore $H \in \mathcal{M}$ and $\mathcal{M}$ is closed under extensions. Next, observe that given an exact sequence $0 \to A \to B \to C \to 0$ with $B \in \mathcal{M}$, it is $A \in \mathcal{M}$ if and only if $C \in \mathcal{M}$: applying $L_0\Delta^2$ to $0 \to A \to B \to C \to 0$ we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccccccccc}
 & & & & & & 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & & & & & \downarrow{\scriptstyle \delta_A^{(0)}} & & \cong\downarrow{\scriptstyle \delta_B^{(0)}} & & \downarrow{\scriptstyle \delta_C^{(0)}} & & \\
0 & \longrightarrow & \Delta\Gamma A & \longrightarrow & \Delta\Gamma B = 0 & \longrightarrow & \Delta\Gamma C & \longrightarrow & (L_0\Delta^2)A & \longrightarrow & (L_0\Delta^2)B & \longrightarrow & (L_0\Delta^2)C & \longrightarrow & 0
\end{array}$$

We have $\Delta\Gamma L = 0$ and $\delta_L^{(0)}$ is an isomorphism if and only if $\Delta\Gamma N = 0$ and $\delta_N^{(0)}$ is an isomorphism. We can then continue the proof of Corollary 2.7, 5., claiming that if N belongs to $\mathcal{M}$ then $N/\operatorname{Im} f$ belongs to $\mathcal{M}$. Therefore, for what we have seen, $\operatorname{Im} f$ and hence $\operatorname{Ker} f$ belong to $\mathcal{M}$. In particular direct summands, pullback and pushout of morphisms in $\mathcal{M}$ are in $\mathcal{M}$.

5. By Lemma 2.8 we obtain $\gamma_{\Gamma M} \circ \Gamma(\gamma_M) = 1_{\Gamma M}$. Therefore we conclude by [18, 45.5]. ■

Remark 2.11. By the discussion preceeding Theorem 2.6 and Corollary 2.10, 4., all modules M, such that there exists an exact sequence $0 \to A \to B \to M \to 0$ with A and B which are Δ-reflexive, belong to $\mathcal{M}$ (cf. with the class $\mathcal{C}$ in [7, 9, 13]). If ${}_RW_S$ is a faithfully balanced weakly cotilting bimodule, then all finitely generated modules cogenerated by W are Δ-reflexive and hence they belong to $\mathcal{M}$. Therefore, again by Corollary 2.10, 4., all finitely presented modules are in $\mathcal{M}$. Moreover, by Proposition 1.4, finitely generated submodules of modules in $\operatorname{Im}\Gamma$ (resp. $\operatorname{Im}\Gamma \cap \mathcal{M}_1$) belong to $\mathcal{M}_0$ (resp. $\mathcal{M}$).

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