Modulational Instability and Complex Dynamics of Confined Matter-Wave Solitons

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(Received 28 January 2003; published 22 August 2003)

We study the formation of bright solitons in a Bose-Einstein condensate of ^{7}Li atoms induced by a sudden change in the sign of the scattering length from positive to negative, as reported in a recent experiment [Nature (London) **417**, 150 (2002)]. The numerical simulations are performed by using the Gross-Pitaevskii equation with a dissipative three-body term. We show that a number of bright solitons is produced and this can be interpreted in terms of the modulational instability of the time-dependent macroscopic wave function of the Bose condensate. In particular, we derive a simple formula for the number of solitons that is in good agreement with the numerical results. We find that during the motion of the soliton train in an axial harmonic potential the number of solitonic peaks changes in time and the density of individual peaks shows an intermittent behavior.

DOI: 10.1103/PhysRevLett.91.080405 PACS numbers: 03.75.Kk, 32.80.Pj, 42.50.Vk

The recent experimental observation of dark [1] and bright [2,3] solitons in Bose-Einstein condensates (BECs) has renewed the interest in the intriguing dynamical phenomena of nonlinear matter waves [4]. In the experiment of Strecker *et al.* [3] soliton trains have been formed by making a stable condensate of ^{7}Li atoms with a large positive scattering length a_s using a Feshbach resonance and then switching a_s to a negative value. The formation of the soliton train has been interpreted as due to quantum mechanical phase fluctuations of the bosonic field operator [5]. By imposing a suitable space dependent pattern in the initial phase of the BEC and then using the 1D time-dependent Gross-Pitaevskii equation (GPE), Al Khawaja *et al.* [5] have reproduced the formation of the soliton train.

In this Letter we show that the formation and subsequent evolution of a soliton train can be adequately investigated by using the classical (mean-field) time-dependent 3D GPE with a dissipative term which takes into account the three-body recombination process [6]. The multisoliton configuration is obtained without imprinting the initial wave function with a fluctuating phase. We show that the soliton train is produced by the modulational instability (MI) of the evolving phase of the BEC (see also [7]). MI is a nonlinear wave phenomenon in which an exponential growth of small perturbations results from the interplay between nonlinearity and anomalous dispersion. MI has been previously studied for waves in fluids [8], in plasma physics [9], in nonlinear optics [10], and also in the context of the superfluid-insulator transition of a BEC trapped in a periodic potential [11]. Here we find that the number of bright solitons induced by MI is given by a simple analytical formula which reproduces our numerical simulations. By investigating the long time evolution of the soliton train under the action of a harmonic potential of frequency ω_z in the traveling axial direction, we find that the center of mass motion is periodic with frequency ω_z but the density of each solitonic peak strongly changes in time.

At zero temperature the macroscopic wave function $\psi(\mathbf{r}, t)$ of a BEC made of ^{7}Li atoms can be modeled by the following dissipative time-dependent Gross-Pitaevskii equation:

$$\left[i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 - U - gN|\psi|^2 + i\gamma N^2|\psi|^4 \right]\psi = 0, \quad (1)$$

where $g = 4\pi\hbar^2 a_s/m$, a_s is the s-wave scattering length, m is the atomic mass, N is the number of condensed atoms, and γ is the strength of the dissipative three-body term [6]. At $t = 0$ the condensate wave function is normalized to one. Following the experiment of Strecker *et al.* [3], the external potential $U(\mathbf{r})$ is given by

$$U(\mathbf{r}) = \frac{m}{2}[\omega_\perp^2(x^2 + y^2) + \chi^2\omega_z^2 z^2] + V_L(z), \quad (2)$$

where $V_L(z)$ is the box optical potential that initially confines the condensate in the longitudinal direction. The harmonic confinement is anisotropic with $\omega_\perp = 2\pi \times 800$ Hz and $\omega_z = 2\pi \times 4$ Hz. χ is a parameter that modifies the intensity of confinement in the axial direction.

We first calculate the ground-state wave function of the condensate with a positive scattering length $a_s = 100a_B$, where $a_B = 0.53$ Å is the Bohr radius, by using Eq. (1) with imaginary time and $\gamma = 0$. The numerical code implements in cylindric symmetry (z, r) a finite-difference splitting method (spatial grid with 400×100 points) with a predictor-corrector algorithm to treat the nonlinear term [12]. We choose a condensate with longitudinal width $L = 284.4$ μm and $N = 10^4$ atoms. Then we use the ground-state wave function as an initial condition for the time evolution of Eq. (1) with $a_s = -3a_B$,

$V_L(z) = 0$, and $\chi = 0$. Following [6] we choose $\gamma = 1.77 \times 10^{-11}$ $(\hbar\omega_z)/a_z^6$. γ is important during the collapse, because it fixes the critical density at which the compression ceases.

In Fig. 1 we plot the probability density in the longitudinal direction $\rho(z) = \int dxdy|\psi(x, y, z)|^2$ of the evolving wave function. The initially homogeneous condensate shows the formation of five peaks. Note that initially the phase of the condensate is set equal to zero. After the formation, the peaks start to separate each other showing a repulsive force between them. Eventually each peak evolves with small shape oscillations but without appreciable dispersion. The formation of these solitonic peaks can be explained as due to MI of the time-dependent wave function of the BEC, driven by imaginary Bogoliubov excitations [5]. The Bogoliubov elementary excitations ϵ_k of the static BEC $\Phi(\mathbf{r})$ can be found from Eq. (1) by looking for solutions of the form $\psi(\mathbf{r}, t) = e^{-i\mu t/\hbar}[\Phi(\mathbf{r}) + u_k(\mathbf{r})e^{-i\epsilon_k t/\hbar} + v_k^*(\mathbf{r})e^{i\epsilon_k t/\hbar}]$, and keeping terms linear in the complex functions $u(\mathbf{r})$ and $v(\mathbf{r})$. Neglecting the dissipative term γ, in the quasi-1D limit [13] one finds $\epsilon_k = \sqrt{\hbar^2k^2/(2m)[\hbar^2k^2/(2m) + 2g_{1D}n]}$, where $g_{1D} = g/(2\pi a_\perp^2)$ with $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$, $n = N/L$ is the linear density, and L is the length of the condensate. By suddenly changing the scattering length a_s to a negative value, the excitation frequencies corresponding to $k < k_c = \sqrt{16\pi|g_{1D}|n}$ become imaginary and, as a result, small perturbations grow exponentially in time. It is easy to find that the maximum rate of growth is at $k_0 = k_c/\sqrt{2}$. The wavelength of this mode is $\lambda_0 = 2\pi/k_0$ and the ratio L/λ_0 gives an estimate of the number N_s of bright solitons which are generated: $N_s = \sqrt{N|a_s|L/(\pi a_\perp)}$.

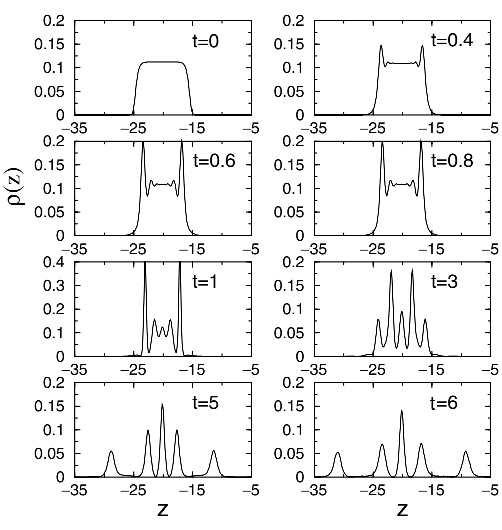

FIG. 1. Axial density profile $\rho(z)$ of 10^4 BEC ^{7}Li atoms from dissipative 3D GPE. For $t < 0$ the scattering length is $a_s = 100a_B$, while for $t \geq 0$ we set $a_s = -3a_B$ with a_B the Bohr radius and $V_L = 0$. External harmonic potential given by Eq. (2) with $\chi = 0$. Length z in units $a_z = (\hbar/m\omega_z)^{1/2}$, density ρ in units $1/a_z$, and time t in units ω_z^{-1}.

As predicted by this formula, Fig. 2 shows that the number of peaks grows with the scattering length. The predicted number N_s of solitons is in very good agreement with the numerical results: $N_s = 2.60$ for $a_s = -1a_B$, $N_s = 4.51$ for $a_s = -3a_B$, and $N_s = 5.82$ for $a_s = -5a_B$. The period $\tau_0 = \hbar/\mathrm{Im}(\tilde{\epsilon}_{k_0})$ associated with the most unstable mode is given by $\tau_0 = mL^2/(\pi\hbar N_s^2)$, in rough agreement with the numerically estimated formation time of the strongest fluctuations in the number of peaks belonging to the soliton train.

In the experiment of Strecker *et al.* [3] there are initially about 10^5 atoms. In this case our numerical (see below) and analytical results predict the formation of about $N_s = 15$ solitons, that is, twice the number experimentally detected. We have verified that the number of solitons does not depend on the dissipation constant γ of Eq. (1), but by increasing γ the densities of solitons are reduced and their widths are increased. Note that in this system recombination processes have not been investigated experimentally and these phenomena can affect the imaging process of soliton train densities.

We have also verified that the number of solitons increases by increasing Δt, the delay time between the removal of end caps and the change of scattering length (in Figs. 1 and 2 it is $\Delta t = 0$). For instance, with the data and units of Fig. 2, $N_s = 6$ for $\Delta t = 0.3$ and $N_s = 7$ for $\Delta t = 0.6$. That is a simple consequence of the enlargement of the axial width of the BEC, in full agreement with the experimental results [3].

In order to clarify the role of phases in the interaction between bright solitons we consider a simpler model: two solitons without dissipation. It has been found that the

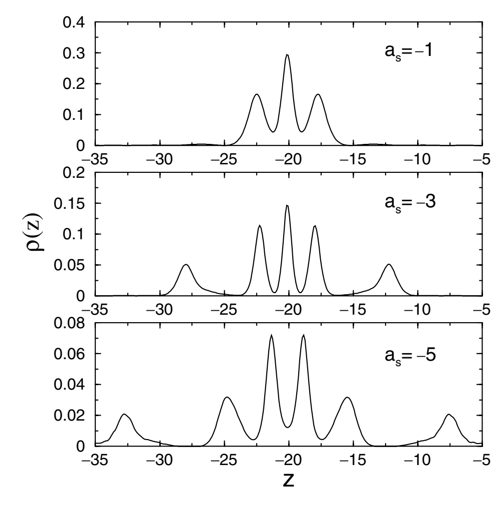

FIG. 2. Axial density profile $\rho(z)$ of 10^4 BEC ^{7}Li atoms at $t = 4.6$ from dissipative 3D GPE for three values of the final scattering length a_s. External harmonic potential given by Eq. (2) with $\chi = 0$. Scattering length a_s in units of the Bohr radius a_B. Units as in Fig. 1.

effective interaction between two bosonic matter waves depends on their phase difference ϕ, being proportional to $a_s \cos(\phi)$ [5,14]. The axial wave function $f(z)$ of a bright soliton under transverse harmonic confinement can be analytically determined [14] by using the nonpolynomial Schrödinger equation (NPSE), an effective 1D equation we have recently derived from the 3D GPE [15]. In [14,15] we have shown that 1D GPE reproduces bright solitons of 3D GPE only when the BEC is strongly cigar shaped. Instead, NPSE always reproduces 3D GPE solitons with great accuracy. As the initial condition for the numerical solution of the time-dependent NPSE (spatial grid of 10^4 points) we choose $\psi(z) = [f(z - z_0) + f(z + z_0)e^{i\phi}]/\sqrt{2}$, where ϕ is the phase difference of the two neighboring solitons centered in $-z_0$ and z_0.

In Fig. 3 we plot the time evolution of N_R/N, where N_R is the number of atoms in the right soliton, for some values of ϕ with $N|a_s|/a_\perp = 1/\sqrt{10}$, $\omega_\perp/\omega_z = 10$, and $\chi = 0$. For $\phi = 0$ the two solitons are attractive and eventually form a static peak which radiates small waves. For $\phi = \pi/4$ the centers of the two solitons do not change with time, but they exchange atoms as in a Josephson junction. Obviously, the amount of atom exchange will depend on the details of the two solitons at initial time (widths and separation distance). We find such a complex exchange of atoms for $0 < \phi < \pi/2$, while for $\phi = \pi/2$ the two solitons are weakly repulsive, their shapes change with time, and eventually two solitons of different densities appear because the Josephson exchange cannot continue when the two solitons separate due to their repulsive interaction. Note that with $\phi = -\pi/2$ the densities of the two peaks are inverted. For larger values of ϕ the dynamics is quite similar to the $\phi = \pi/2$ case, with two solitons of different shapes getting away from each other. Exactly at $\phi = \pi$ parity symmetry of the problem inhibits atom exchange.

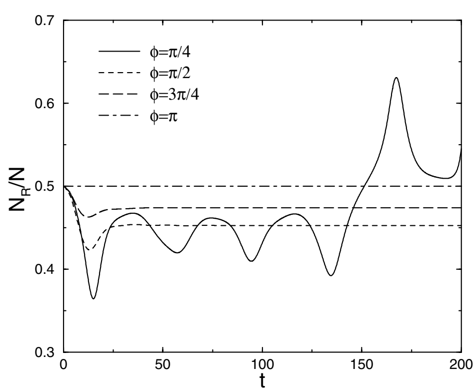

FIG. 3. Ratio N_R/N as a function of time for two neighboring solitons with an initial phase difference ϕ and spatial separation $2z_0 = 6$. $N = N_R + N_L$ is the sum of the number N_R of atoms in the right soliton and the number N_L of atoms in the left soliton. $N|a_s|/a_\perp = 1/\sqrt{10}$. Results from conservative NPSE. External harmonic potential given by Eq. (2) with $\omega_\perp/\omega_z = 10$ and $\chi = 0$. Units as in Fig. 1.

In the soliton train shown in Figs. 1 and 2 the phase difference of adjacent solitons is not imposed *a priori* at $t = 0$, as in Ref. [5], but is self-consistently determined by the GPE. As a consequence, its value is not exactly equal to π and it changes with time. It is interesting to study the long time evolution of the soliton train under axial harmonic confinement as done in the experiment of Strecker *et al.* using 10^5 ^{7}Li atoms [3]. Instead of 3D GPE in this numerical simulation we use the 1D GPE (spatial grid of 10^4 points) with a rescaled effective interaction and dissipative term [15], which allows us to extend our calculations to long times. As explained in [15], 1D GPE is obtained from 3D GPE by imposing a Gaussian wave function of width $a_\perp$ in the transverse direction. Note that we have verified that the 1D GPE results are in good agreement with 3D GPE ones for the set of parameters of the experiment of Strecker *et al.* [3].

In Fig. 4 we show three frames of the traveling train as color contour plots of the density. In each panel the horizontal axis is the z coordinate and the vertical axis is the x coordinate. The center of mass of the soliton train oscillates with the frequency of the harmonic confinement. Moreover, the solitons spread out in the middle and bunch at the turning points in very good agreement with the experimental results under similar conditions [3].

It is interesting to observe that during the time evolution the densities of bright solitons oscillate in an irregular way and, at certain instants, a few solitons practically disappear and reappear after a while. As confirmed in Fig. 5, in the absence of axial confinement ($\chi = 0$), after a transient the peaks, separating each other, become true solitons. In particular, the number N_p of peaks in the train becomes constant and equal to the analytically estimated number N_s of bright solitons. Instead, in the presence of axial harmonic confinement ($\chi = 1$), N_p changes in time. Obviously, in this case the peaks never become true solitons.

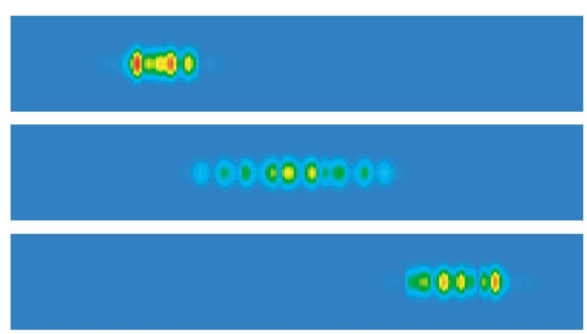

FIG. 4 (color). Moving soliton train of BEC ^{7}Li atoms obtained by solving the dissipative 1D GPE. Initially there are $N = 10^5$ atoms. Three frames: $t = 0.8$ (top), $t = 1.6$ (middle), and $t = 3.8$ (bottom). External harmonic potential given by Eq. (2) with $\chi = 1$. Red color corresponds to highest density. Units as in Fig. 1.

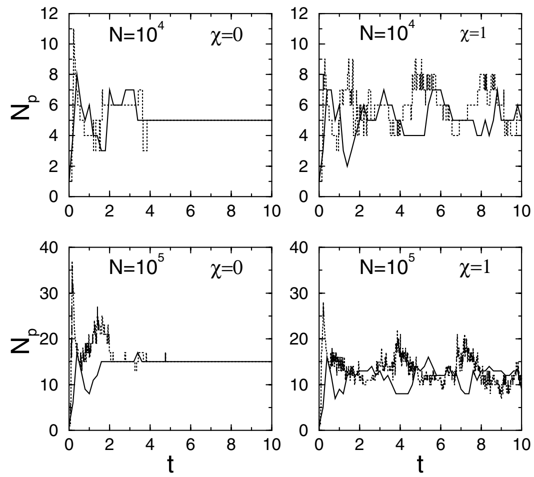

FIG. 5. Number N_p of peaks in the train of BEC ^{7}Li atoms as a function of time t. N is the initial number of atoms. Simulation performed by using the dissipative 1D GPE (dotted line). The solid line is obtained from a smoothed finite-resolution density profile of the train. External harmonic potential given by Eq. (2). Units as in Fig. 1.

In the experiment of Strecker *et al.* [3] missing solitons are frequently observed during the time evolution of the train. Our results strongly suggest that the phenomenon of missing solitons is related to the intermittent dynamics of individual peaks in the train under axial harmonic confinement.

To simulate the finite resolution of the experimental detection and imaging process, we calculate the convolution $\bar{\rho}(z) = \int dz' G(z - z')\rho(z')$ of the axial density profile $\rho(z)$ with a Gaussian $G(z)$ having the width of a single soliton. As shown in Fig. 5, for $\chi = 1$ the number N_p of peaks of the smoothed density $\bar{\rho}(z)$ oscillates in time around a mean value smaller than N_s. In fact, many peaks of the train are too close to be seen by using a finite resolution of the density: that becomes dramatic at the turning points. Thus, the effect of finite resolution could explain the disagreement between the experimentally observed number of solitons and its analytical estimate based on modulational instability.

In Fig. 5 is also shown the behavior of N_p with initially 10^4 atoms. Remarkably, in this case the largest values of N_p are obtained when the train is in the middle of the trap: contrary to the case with initially 10^5 atoms, the solitons bunch in the middle and spread out at the turning points. Such a phenomenon can be qualitatively explained observing that the energy of the repulsive interaction between solitons decreases by reducing the number of atoms [5,14]. It follows that below a critical threshold in the number of atoms the repulsive interaction between solitons is overcome by the potential energy of solitons and the solitons cross in the middle of the trap. We have verified that to get spreading solitons in the middle of the trap with initially 10^4 atoms it is necessary to strongly reduce the axial frequency ($\chi = 1/4$).

In conclusion, we have successfully explained and numerically simulated the dynamical process of soliton train formation induced by modulational instability in the framework of the time-dependent Gross-Pitaevskii equation with a three-body dissipative term. Contrary to the claim of Al Khawaja *et al.* [5], it is not necessary to include quantum phase fluctuations to trigger the formation of the soliton train. We have also investigated the soliton-soliton interaction and found a novel phenomenon: the intermittent dynamics of individual peaks during the time evolution of the soliton train in an axial harmonic potential. Signatures of this phenomenon can be extracted from the data of the experiment of Strecker *et al.* [3]. Because of the intimate connection between atom optics with BECs and light optics we believe that the intermittent dynamics in soliton trains may also be observed with optical solitons in fibers.

L. S. thanks R. G. Hulet for many useful discussions.

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