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A three-dimensional stress recovery procedure for composite materials

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Abstract

A three-dimensional stress recovery procedure is described in this paper. As usual, in the finite element method, we suppose to know a good approximation of the nodal values of the displacements; starting from the nodal displacements, we evaluate the strain components at Gauss points of a 27-node prismatic element, then the stress components at the same points. Stresses are finally projected to corner nodes with a smoothing procedure. The repeated application of such a procedure to a set of adjacent elements allows the construction of the stress field in a finite region of any deformed body. The proposed method finds its most natural application in the field of composite materials as shown in the numerical applications which reveal a good concordance of results with an equivalent three-dimensional finite element analysis. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords: Three-dimensional stress recovery; Composite materials; Layered beam

1. Introduction

Recovery of stresses is an important procedure in the study of composite structures (but also in normal homogeneous structures), in particular for through-thickness shear stresses. If we analyse a structure with the finite element method, the stresses obtained at the end of the analysis do not possess, generally, inter-element continuity because of the nature of the assumed displacement variation [1]. Experience has shown that the nodes, which are generally the most interesting points, appear to be points where stresses are known with bad approximation [2]. To obtain acceptable results for stresses, with finite element calculations, a nodal averaging or projection process is necessary; in particular, through error estimation [3], a good smoothed stress field is obtained.

In the field of composite structures several different analytical approximated methods are known to compute the displacement fields of such structures subjected to external loadings [4–6].

A full three-dimensional finite element analysis of composite structures would generally require a large number of elements because of the possible fine scale material distribution. This is the reason why displacements of composite structures are often computed by means of the introduction of an equivalent homogeneous structure. In this case it may be difficult to evaluate the stresses in the real materials starting from the knowledge of the displacements, e.g. by integrating the equilibrium equations, transverse shear stresses may show discontinuities at interlayer surfaces or the boundary conditions for stresses may not be satisfied. Several "unsmearing" procedures have been proposed for particular situations, e.g. Refs [7, 8] for structures which show periodicity in the transverse section. In Ref. [9], two-dimensional situations were handled with an analytical explicitly calculated projection matrix. Two-dimensional situations are however limited because they do not allow the handling of torsion and bending at the same time; we extend here the method to the more general case of three-dimensional situations, again with the use of an explicitly calculated smoothing matrix. Moreover, since some stress components have to be discontinuous in composite struc-

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tures, the local smoothing procedure presented in Ref. [1] is adopted, generalised and applied in the present paper.

Also, the more sophisticated super-convergent patch recovery could be used to obtain improved stress distributions [10–13].

The three-dimensional stress recovery procedure used here is described in Section 2, while in Section 3 some numerical applications are shown in which it is possible to observe a good accordance between the results of our procedure and those of a three-dimensional analysis.

2. Three-dimensional Stress Recovery Procedure

The three-dimensional stress recovery procedure presented in this paper can be applied to any kind of solid continuum material as long as the displacement field is known with sufficient accuracy. However, it finds its most natural application in the study of composite materials.

The solution of equilibrium equations gives the values of the displacements which are the starting point of our procedure. The goal of the method is to obtain the stress distribution in any region of a specific material (and, therefore, if needed, also in the whole structure) starting from the knowledge of the displacement field. We construct a local finite element discretisation (Fig. 1) in the region under investigation with at least one element per material component. The displacement field is then imposed to the nodes to compute the Gauss point stresses and their nodal projection.

We restrict now our attention to a single element of the local discretisation. Supposing that the displacements in 27 nodes of a prismatic three-dimensional discretisation are given (this does not mean that the problem has to be solved with 27-node three-dimensional elements), these are imposed to a 27-node Lagrangian element. In this way, a quadratic distribution of each displacement component can be obtained in the element and, therefore, a linear distribution of their derivatives.

By means of the usual differential rules, strains are obtained in the Gauss points located in the known positions and finally the constitutive law gives the stresses in the same points.

At this point of the procedure, stresses should be extrapolated to nodes by means of a smoothing procedure. Considering eight Gauss points (see Fig. 2), if σ_i ($i = 1, 2, \ldots, 8$) are the unsmoothed stresses in the integration points and $g(x,y,z)$ is the conventional smoothing function [1], the extrapolated stress values at the corner nodes of the three-dimensional solid element can be obtained by minimising the functional:

$$\chi = \sum_{i=1}^{8} (\sigma_i - g(x_i, y_i, z_i))^2, \tag{1}$$

where the smoothing function g has the following representation:

$$g(x_i, y_i, z_i) = \sum_{j=1}^{8} \tilde{N}_j(x_i, y_i, z_i)\tilde{\sigma}_j. \tag{2}$$

In Eq. (2), $\tilde{N}_j$ are linear smoothing shape functions and $\tilde{\sigma}_j$ the unknown smoothed nodal stresses.

For χ to be a minimum:

$$\frac{\partial \chi}{\partial \tilde{\sigma}_j} = 0, \tag{3}$$

hence, for three-dimensional brick elements with sampling points at the eight points of a $2 \times 2 \times 2$ Gauss rule, the following matricial expression is obtained:

$$\begin{bmatrix} \sum_{i=1}^{8} \tilde{N}_1(x_i, y_i, z_i)\tilde{N}_1(x_i, y_i, z_i) & \cdots & \sum_{i=1}^{8} \tilde{N}_1(x_i, y_i, z_i)\tilde{N}_8(x_i, y_i, z_i) \\ \vdots & & \vdots \\ \sum_{i=1}^{8} \tilde{N}_8(x_i, y_i, z_i)\tilde{N}_1(x_i, y_i, z_i) & \cdots & \sum_{i=1}^{8} \tilde{N}_8(x_i, y_i, z_i)\tilde{N}_8(x_i, y_i, z_i) \end{bmatrix} \cdot \begin{bmatrix} \tilde{\sigma}_1 \\ \vdots \\ \vdots \\ \tilde{\sigma}_8 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{8} \tilde{N}_1(x_i, y_i, z_i)\sigma_i \\ \vdots \\ \sum_{i=1}^{8} \tilde{N}_8(x_i, y_i, z_i)\sigma_i \end{bmatrix}. \tag{4}$$

Calling $\tilde{\mathbf{N}}$ the matrix of the smoothing shape functions calculated in the $2 \times 2 \times 2$ Gauss points:

$$\tilde{\mathbf{N}} = \begin{bmatrix} \tilde{N}_1(x_1, y_1, z_1) & \tilde{N}_2(x_1, y_1, z_1) & \cdots & \tilde{N}_8(x_1, y_1, z_1) \\ \tilde{N}_1(x_2, y_2, z_2) & \tilde{N}_2(x_2, y_2, z_2) & \cdots & \tilde{N}_8(x_2, y_2, z_2) \\ \vdots & \vdots & & \vdots \\ \tilde{N}_1(x_8, y_8, z_8) & \tilde{N}_2(x_8, y_8, z_8) & \cdots & \tilde{N}_8(x_8, y_8, z_8) \end{bmatrix}, \tag{5}$$

Eq. (4) becomes:

$$\tilde{\mathbf{N}}^T\tilde{\mathbf{N}}\tilde{\boldsymbol{\sigma}} = \tilde{\mathbf{N}}^T\boldsymbol{\sigma}. \tag{6}$$

By inverting the matrix $\tilde{\mathbf{N}}$ we obtain a direct relation between the smoothed stresses at corner nodes of a three-dimensional solid element and the unsmoothed stresses at the $2 \times 2 \times 2$ Gauss points of the same element:

$$\tilde{\boldsymbol{\sigma}} = \tilde{\mathbf{N}}^{-1}\sigma. \tag{7}$$

In Ref. [14] a direct procedure is shown which avoids to calculate the inverse of matrix $\tilde{\mathbf{N}}$ of the smoothing shape functions by substituting the local coordinates x_i, y_i, z_i (i = 1, 2,..., 8) of the Gauss points with their inverse $1/x_i$, $1/y_i$, $1/z_i$.

For prismatic elements in which the Jacobian determinant is constant, the matrix $\tilde{\mathbf{N}}^{-1}$ is given by

$$\tilde{\mathbf{N}}^{-1} = \begin{bmatrix} a & b & c & b & b & c & d & c \\ b & a & b & c & c & b & c & d \\ c & b & a & b & d & c & b & c \\ b & c & b & a & c & d & c & b \\ b & c & d & c & a & b & c & b \\ c & b & c & d & b & a & b & c \\ d & c & b & c & c & b & a & b \\ c & d & c & b & b & c & b & a \end{bmatrix}, \tag{8}$$

where

$$a = \frac{5 + 3\sqrt{3}}{4}, \quad b = \frac{-(\sqrt{3} + 1)}{4},$$

$$c = \frac{\sqrt{3} - 1}{4}, \quad d = \frac{5 - 3\sqrt{3}}{4}.$$

It has to be observed that in Eq. (7) the vector $\tilde{\boldsymbol{\sigma}}$ indicates the eight stress components σ_{ij} (with the same indices) acting on the nodes as function of the relevant components of the Gauss points. Therefore, the knowledge of the full three-dimensional tensional state in the eight nodes requires the application of Eq. (7) to the six stress components.

3. Numerical Examples

The nodal displacements are interpolated in the Lagrangian 27-node element with quadratic shape functions, therefore quadratic distributions of displacements, or linear distributions of strains, are exactly represented in the element; then the linear elastic constitutive relation gives the exact stresses in the eight Gauss points. Finally, Eq. (7) gives the nodal stresses as linear functions of Gauss point stresses, consequently linear distributions of stresses can be exactly represented by the present method, whereas more complex stress fields are approximated by means of a least squares fit.

The previous considerations are confirmed in the two following simple tests. Two examples taken from De Saint Venant beam theory are utilised to check the correctness of the Fortran routines developed. A prismatic cantilever beam with rectangular cross-section is subjected to bending moment, the displacements in 27 points corresponding to the nodes of a Lagrangian element have then be given as input to our routine

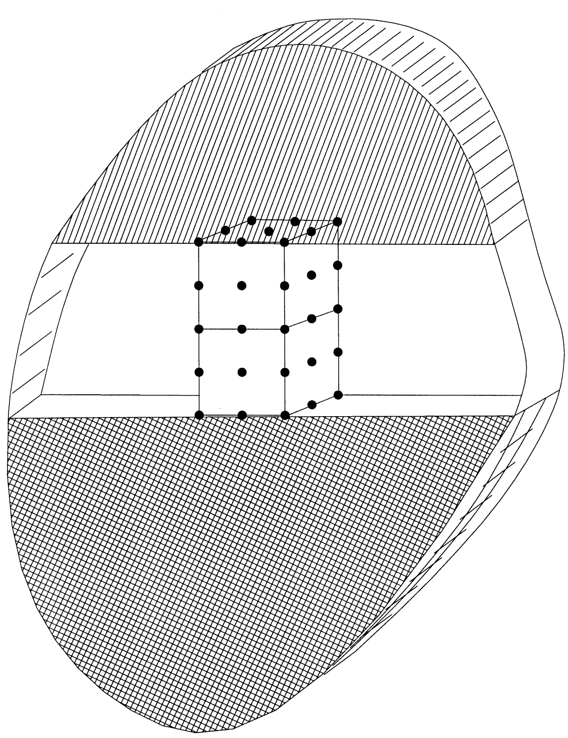

Fig. 1. Local finite element discretisation with 27 node prismatic elements.

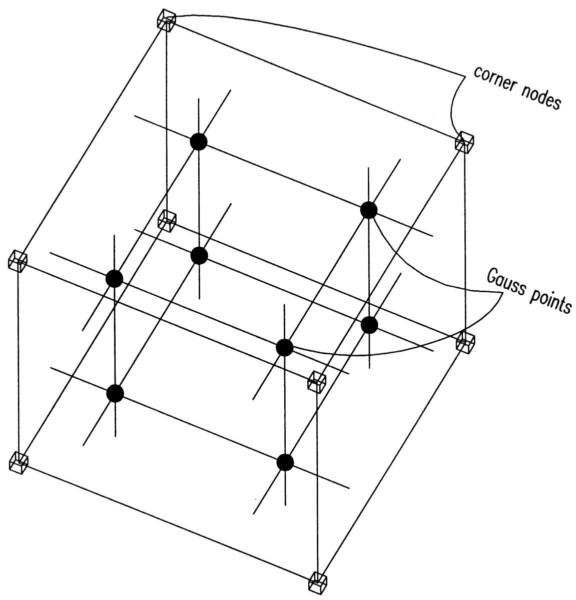

Fig. 2. Corner nodes and Gauss points of the element.

Table 1
The geometrical and mechanical characteristics

Total length: L = 90.0	Total width: B = 10.0
Total height: H = 11.0	Thicknesses:
	$h_1 = 2.0$
	$h_2 = 1.0$
Young's moduli:	Poisson ratios:
$E_1 = 2.1E + 06$	$\nu_1 = 0.294$
E_2 = variable	$\nu_2 = 0.333$

which computed the exact nodal stresses. A similar computation was carried out for a prismatic cantilever beam with elliptical cross-section subjected to torsional moment; also in this case we obtained the exact nodal stresses.

Finally, the numerical application of a composite cantilever beam with rectangular cross-section under bending and torsional moments is illustrated. A moderately thick beam composed of two types of layers each made of a different material is examined [9], [15]. In total the beam is composed of seven layers: four of type 1 and three of type 2. The geometrical and mechanical characteristics are given in Table 1.

The Young's modulus of the layers of type 2 may vary to give different E_1/E_2 ratios. In particular, we show the results for two typical cases:

(1) $E_2 = E_1/2 = 1.05E + 06$;

(2) $E_2 = E_1/100 = 2.1E + 04$.

Let us consider a system of coordinates with x axis coincident with the longitudinal axis of the layered beam and y and z axes belonging to the plane of the cross-section of the beam.

The displacement field for a generic load case is obtained utilising a rough discretisation (816 nodes), with the general purpose code ADINA [16].

First, we consider the beam with one end fixed and the other end loaded by a transverse vertical force P = 10.0 (see Fig. 3). In Fig. 4a and b ($E_1/E_2 = 2$) and Fig. 5a and b ($E_1/E_2 = 100$) we show the distribution of stresses σ_{xx} and τ_{xz} obtained with our three-dimensional stress recovery procedure at x/L = 0.5 cross-section. For this analysis, we have considered a local finite element discretisation with one 27-node prismatic element for each layer of the beam. The values of the stresses obtained with our procedure are compared with the values of the same stresses found with a separate three-dimensional finite element analysis made with the general purpose program ADINA [16]. In this three-dimensional analysis we have used a fine discreti-

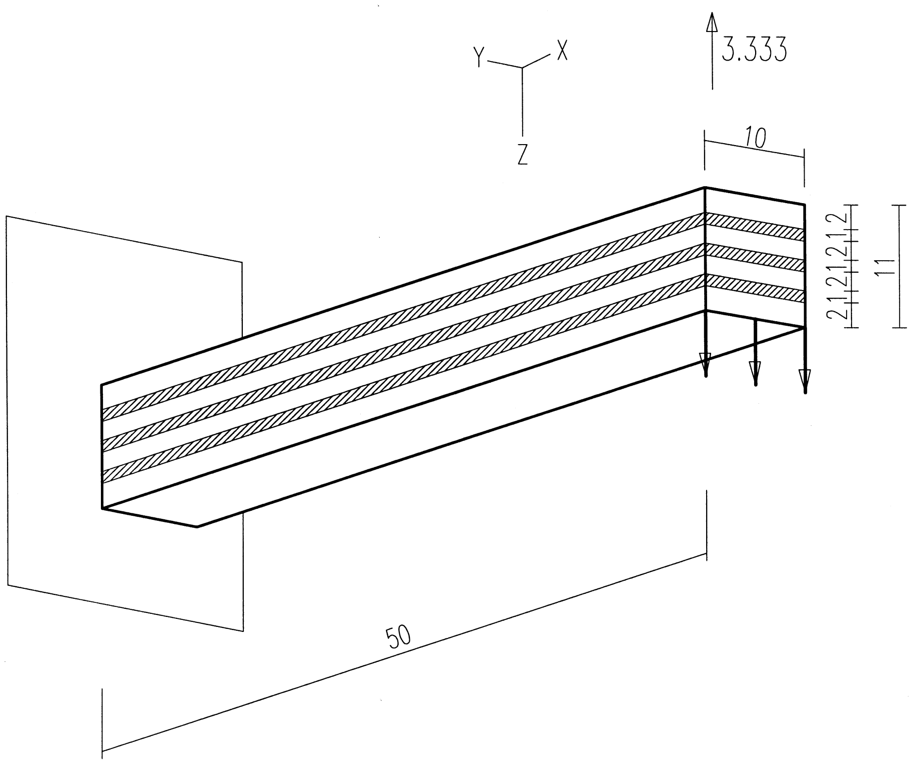

Fig. 3. First load case (bending).

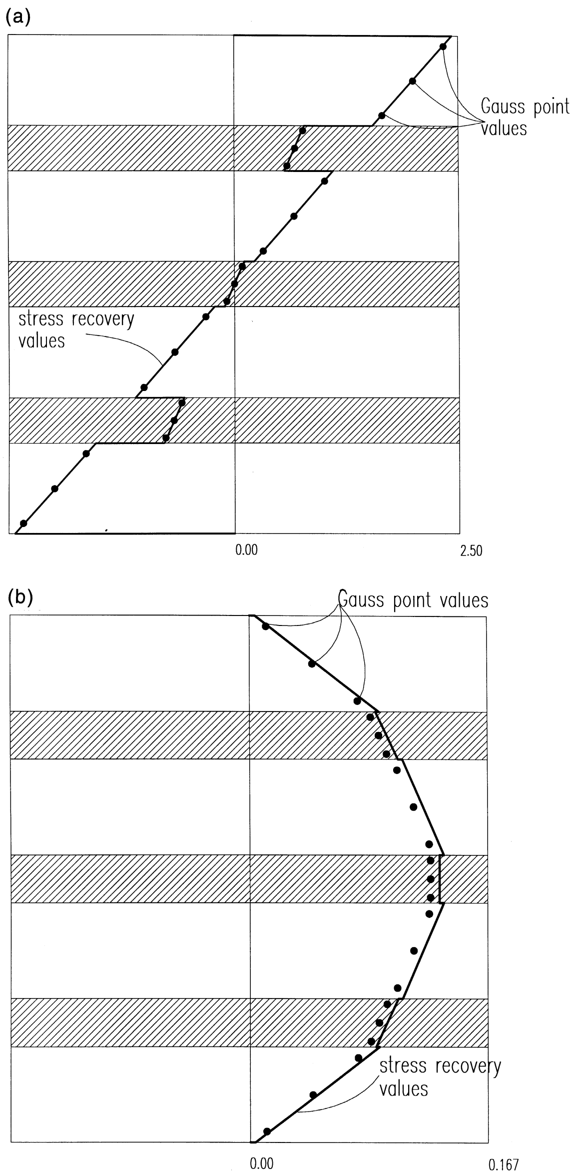

Fig. 4. (a) Distribution of stresses σ_{xx} at $x/L = 0.5$ cross-section ($E_1/E_2 = 2$). (b) Distribution of stresses τ_{xz} at $x/L = 0.5$ cross-section ($E_1/E_2 = 2$). Dots come from a full three-dimensional analysis.

sation with 27-node three-dimensional-solid elements (4545 nodes) and an integration scheme with 27 Gauss points; in Figs. 4 and 5 we also show the values of σ_{xx} and τ_{xz} at these integration points. The stress recovery procedure gives two values of stress for each layer, at its top and at its bottom; in Figs. 4 and 5 such values are linearly connected whereas the stress values obtained with the more complex three-dimensional solution are given in a discrete manner at the Gauss point level.

Clearly, σ_{xx} values have discontinuities at layer interfaces, while τ_{xz} values are almost continuous on the cross-section. We can note that the normal stress values obtained with our procedure concide and the tangential stress values are almost coincident with the values obtained with ADINA.

Now, we consider the same beam with one end fixed and the other end loaded with a torsional moment $M_t = 50.0$ (see Fig. 6).

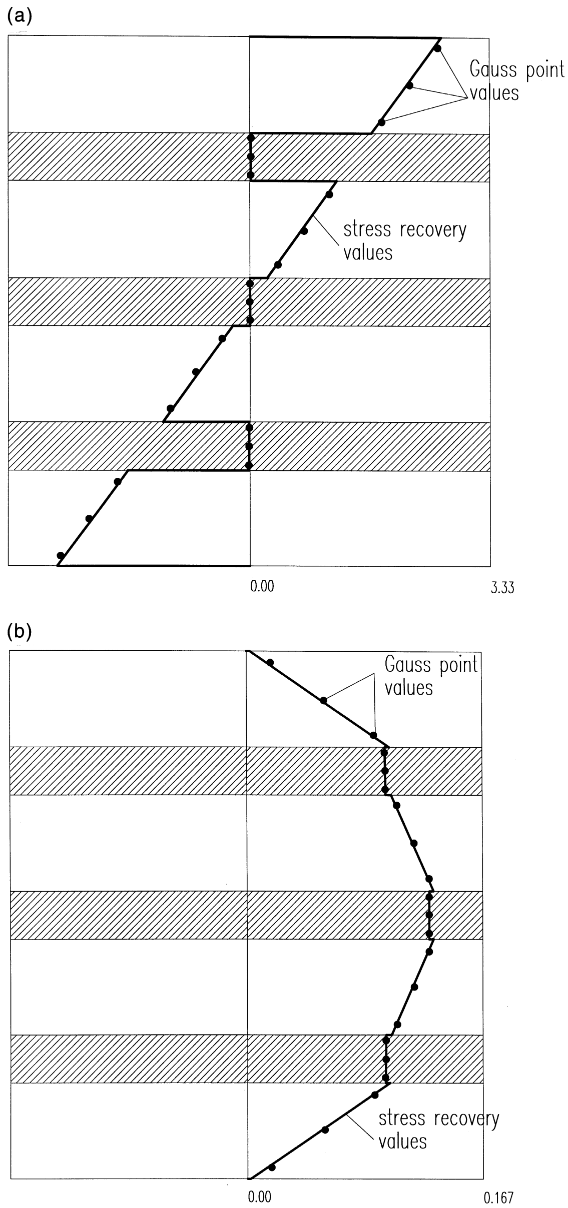

Fig. 5. (a) Distribution of stresses σ_{xx} at $x/L = 0.5$ cross-section ($E_1/E_2 = 100$). (b) Distribution of stresses τ_{xz} at $x/L = 0.5$ cross-section ($E_1/E_2 = 100$). Dots come from a full three-dimensional analysis.

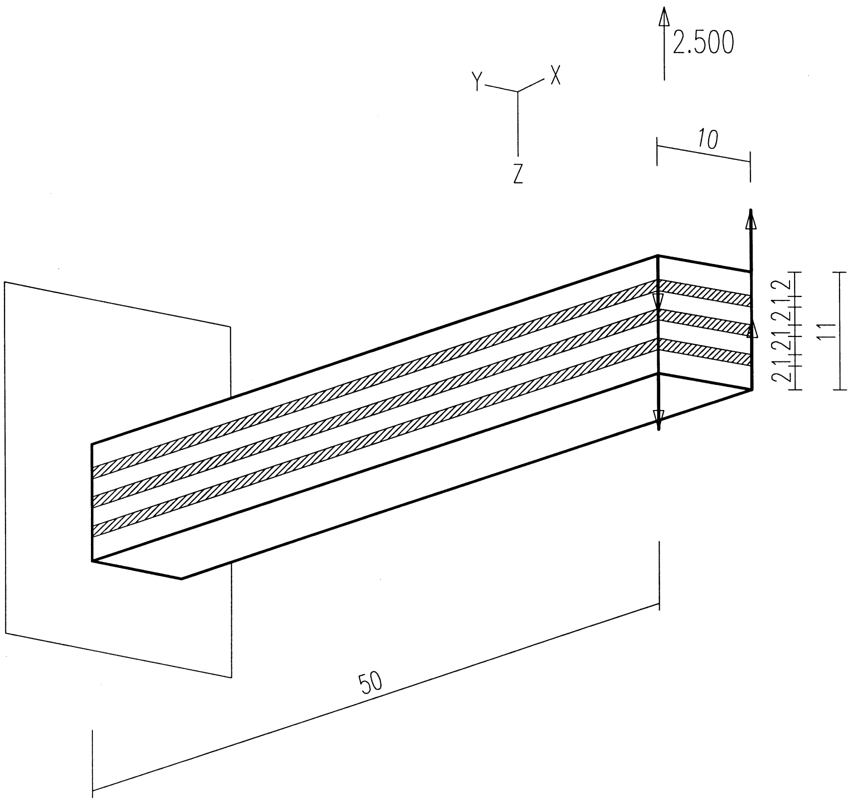

Fig. 6. Second load case (torsion).

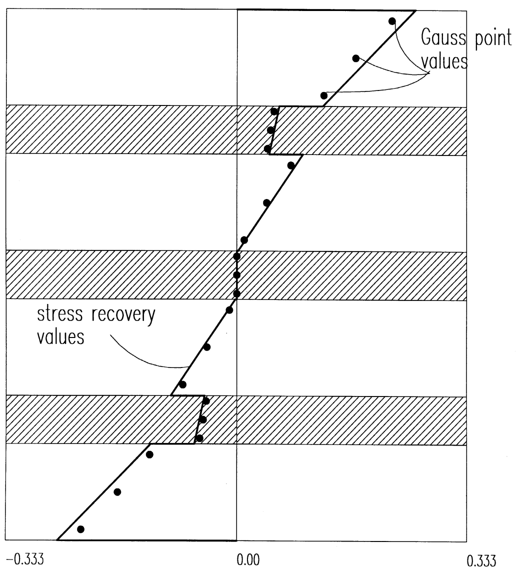

Fig. 7. Distribution of stresses τ_{xy} at x/L = 0.5 cross-section (E_1/E_2 = 2). Dots come from a full three-dimensional analysis.

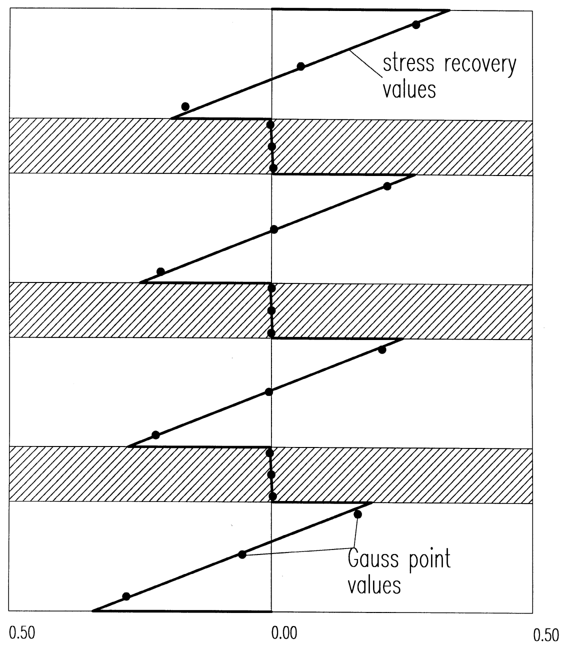

Fig. 8. Distribution of stresses τ_{xy} at x/L = 0.5 cross-section (E_1/E_2 = 100). Dots come from a full three-dimensional analysis.

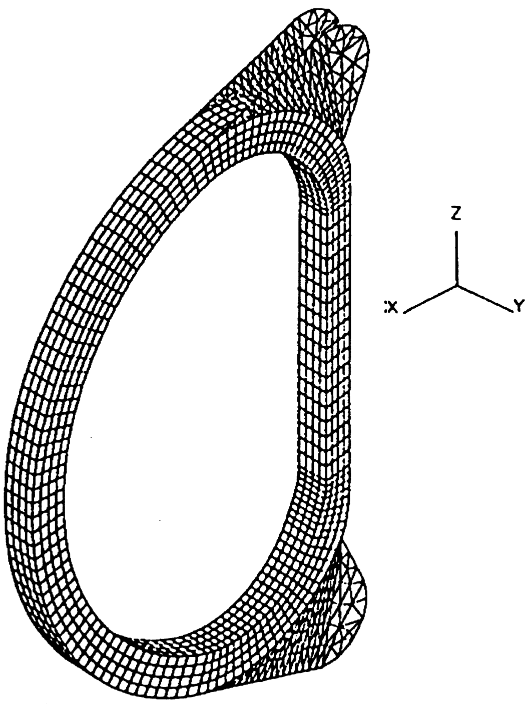

Fig. 9. Rough finite element discretization of the homogenised coil.

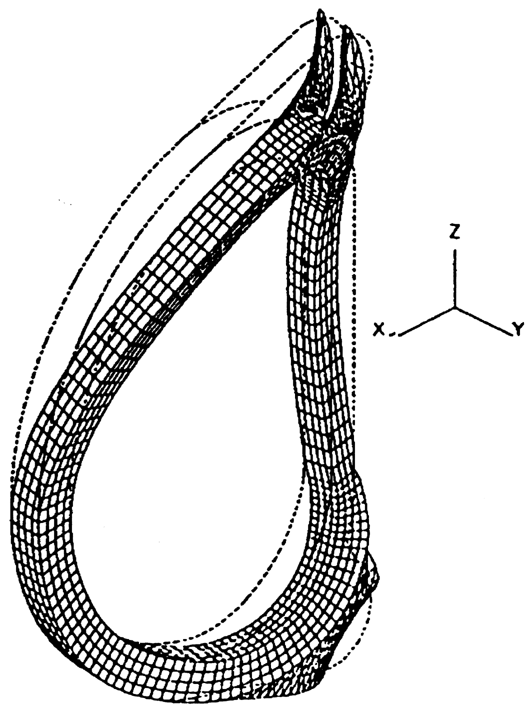

Fig. 10. Deformed configuration of the homogenised coil subjected to magnetic loads.

We show in Fig. 7 ($E_1/E_2 = 2$) and Fig. 8 ($E_1/E_2 = 100$) an analogous comparison of stresses. In this case, the most significant stresses are τ_{xy} values, i.e. the tangential stresses belonging to the cross-section plane (clearly, these values must have discontinuities at layer interfaces). Also in this case, we obtain a good agree-

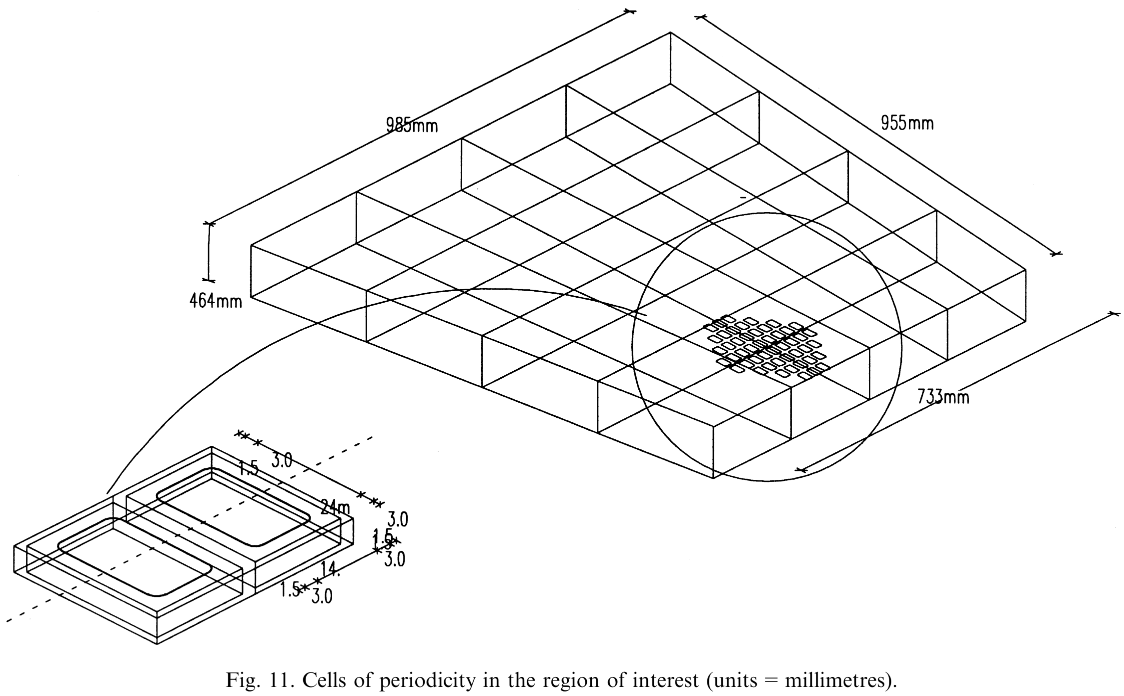

Fig. 11. Cells of periodicity in the region of interest (units = millimetres).

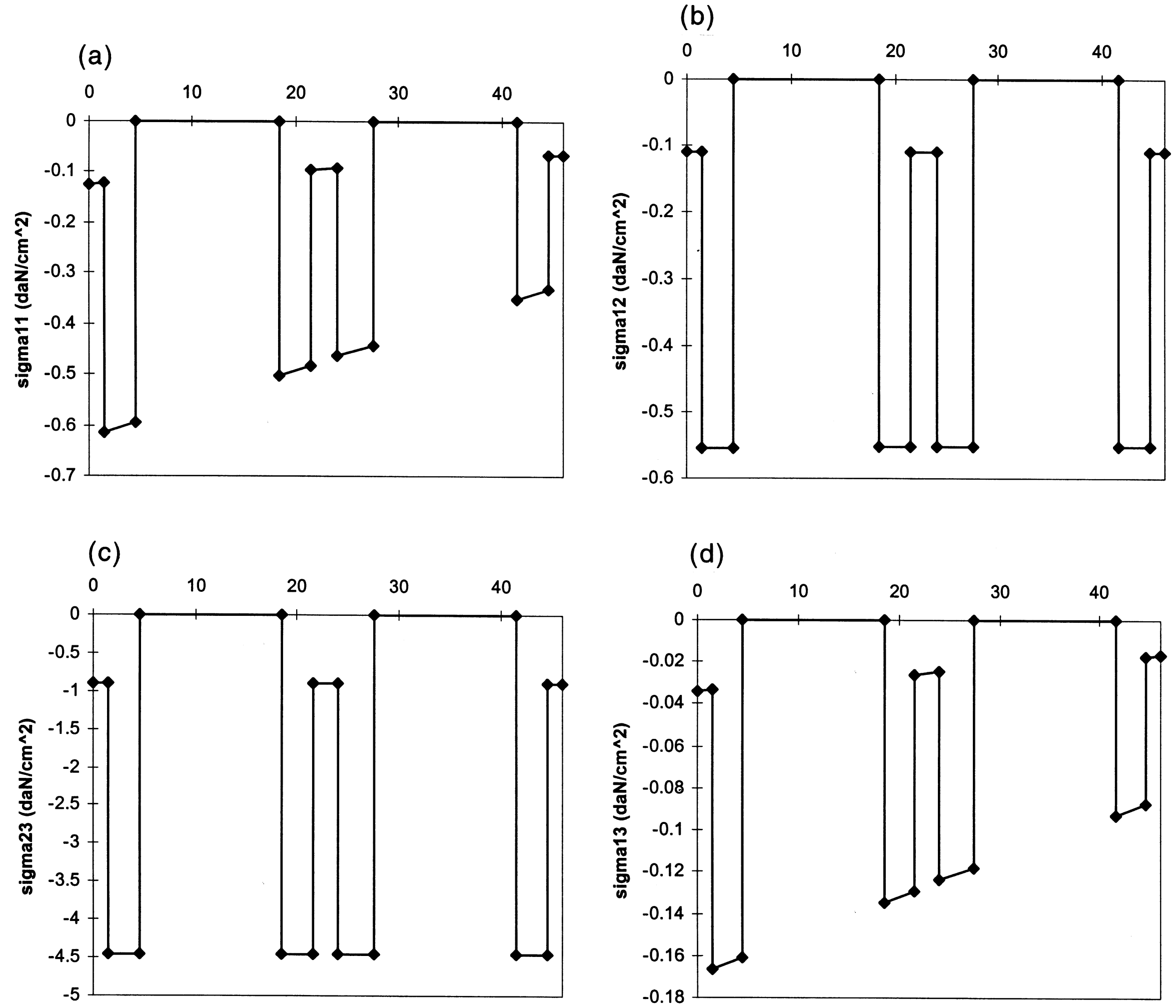

Fig. 12. Distribution of: (a) the normal stress σ_{zz}; (b) tangential stress τ_{xy}; (c) tangential stress τ_{yz}; (d) tangential stress τ_{xz} along two adjacent cells.

ment of results between our three-dimensional recovery procedure and the three-dimensional finite element analysis made with ADINA.

In Fig. 4b and 5b some slight discontinuities in the stress field can be noticed: they can be strongly reduced by the adoption of two elements per layer.

A numerical application referred to as a real size structure is now described. We consider a NET-TF superconducting coil with cable-in-conduit superconductors [7]. The coil is a typical example of composite material with periodic structure. A composite material is called "(spatially) periodic" if it is possible to decompose it in a number of equal elementary components or cells of periodicity. The characteristic size of the cell is assumed to be much smaller than the geometrical dimensions of the structure which is, therefore, composed by a large number of cells.

Now, we concentrate our attention on a cross-section of the coil: the goal is to obtain the three-dimensional distribution of stresses in some cells of this cross-section of the structure subjected to magnetic loads (body forces) [7]. A single cell of periodicity is made of three phases (epoxy, steel and void which is in reality filled rather loosely by a bundle of superconducting strands—see Fig. 11) [7].

The mechanical characteristics of the components of the cell of periodicity are the following:

steel: $E = 2.1E + 06$ daN/cm^2; $\nu = 0.18$;

epoxy: $E = 4.2E + 05$ daN/cm^2; $\nu = 0.18$.

By means of the homogenisation theory [7, 8] a homogeneous anisotropic material, equivalent to the real non-homogeneous periodic one, was computed; in Fig. 9 a rough finite element discretisation of the equivalent homogenised coil is shown. The mechanical analysis of the homogenised superconducting coil can be found in Ref. [7]. This analysis gives us the values of the displacements in the nodes of the discretisation of the homogenised coil. In Fig. 10 the deformed (amplified) configuration of the coil is presented. Starting from the values of these displacements, with usual interpolation procedures, we obtain the displacements at the nodal points of our local discretisation relative to two adjacent cells (Fig. 11). The local discretisation is constructed with three-dimensional Lagrangian 27-node elements.

By means of our stress recovery procedure, we obtain the values of the six components of stresses in the nodes of the local discretisation. The distribution of the normal stress σ_{zz} and of the three tangential stresses τ_{xy}, τ_{yz} and τ_{xz} along two adjacent cells (i.e. along the dotted line of Fig. 11) is shown in Fig. 12a–d. The values compare well with those obtained from the unsmearing procedure of Ref. [8], but it is apparent that the procedure proposed in the present paper is more straightforward and more widely applicable than that presented there.

4. Conclusions

In this paper, a simple stress recovery procedure applicable to three-dimensional continua has been presented. The displacement field is considered as the starting point of the procedure: by imposing these displacements to the nodal points of a local finite element discretisation we can evaluate first the strain and the stress components at integration points of 27-node prismatic elements (with which we construct our local discretisation) and then project stress values to corner nodes.

This procedure overcomes the limits of two-dimensional recovery procedures which do not allow the handling of torsion and bending at the same time. It finds its natural application in the study of composite materials for which, in the majority of cases, some sort of homogenisation approach is necessary. The mechanical analysis of a homogenised material, which is less expensive (from a computational point of view) than the analysis of the real structure, gives the displacement field. Starting from this, the three-dimensional stress distribution in any region of interest of the material can be evaluated.

Numerical applications show the good agreement of results with equivalent three-dimensional finite element analyses and the applicability of the method to real complex composite structures.

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