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Dynamic stability of elastically constrained beams: an exact approach

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1. Introduction

The dynamic stability of mechanical systems, according to the definition of Bolotin (1964), represents a specific aspect of the stability of the motion.

Several works followed the first pioneering studies of Bolotin (1964). The main object of these works was to obtain a quantitative description of the phenomenon, taking into account that if one tries to solve the problem by means of continuum models, the study becomes particularly difficult also for rather simple mechanical systems.

In fact, the majority of works on the dynamic stability of structures have been founded on a finite element approach (Brown *et al.*, 1968). Some improvements, always using the finite element method, can be found in Shastry and Rao (1984), where shear deformation and rotatory inertia of the beam were taken into account, and in Shastry and Rao (1986), where the effect of distributed axial load was included.

In this paper we analyse the fundamental aspect of the dynamic stability of elastic beams with elastic constraints and suggest a calculation procedure to find the exact solution in practical cases. This approach may also be useful for comparison purposes, when a complex system must be analysed using finite elements. A solution of static stability of this problem can be found in Bonvizini (1969) and in computational form in Contri (1989).

2. Governing equations

Let us consider the problem of the transverse and axial vibrations of a straight beam loaded by a periodical longitudinal forcing function $F(t)$. To characterize the problem we assume initially the beam as simply supported, with uniform cross-section along its length, and deformable by bending and axial stress (Figure 1). The theory of small displacements and strains is assumed valid. Subsequently we will remove the restriction on boundary conditions.

The motion is described by the following equations:

$$\frac{\partial^2 \xi}{\partial x^2} EA - m \frac{\partial^2 \xi}{\partial t^2} = 0$$

$$EJ \frac{\partial^4 \eta}{\partial x^4} - N(x) \frac{\partial^2 \eta}{\partial x^2} - \frac{\partial N}{\partial x} \frac{\partial \eta}{\partial x} + m \frac{\partial^2 \eta}{\partial t^2} = 0 \tag{1}$$

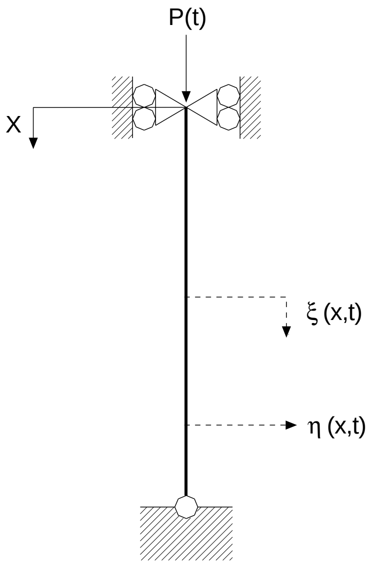

Figure 1. Beam loaded by a periodical forcing function *F*(*t*)

where ξ is the axial displacement component, η the transverse one, x the abscissa, t the time, m the mass per unit length, E the Young's modulus, A the area of the cross-section, J the moment of inertia, $N(x)$ the axial stress resultant.

The initial conditions are:

$$\begin{cases} \xi(x,0) = \xi_1(x,0) \\ \dfrac{d\xi}{dt}(x,0) = \dot{\xi}_1(x,0) \\ \eta(x,0) = \eta_1(x,0) \\ \dfrac{d\eta}{dt}(x,0) = \dot{\eta}_1(x,0) \end{cases} \tag{2}$$

and the boundary conditions:

$$\begin{cases} N(0,t) = -P(t) \\ \xi(l,t) = 0 \\ \eta(0,t) = 0 \quad ; \quad \dfrac{\partial^2\eta}{\partial x^2}(0,t) = 0 \\ \eta(l,t) = 0 \quad ; \quad \dfrac{\partial^2\eta}{\partial x^2}(l,t) = 0 \end{cases} \tag{3}$$

The solution is:

$$\begin{cases} \xi(x,t) = f(x,t) \\ \eta(x,t) = 0 \end{cases} \tag{4}$$

If a small perturbation to the initial conditions does not cause a large variation of the functions $\xi(x,t)$ and $\eta(x,t)$ describing the motion, the same motion is defined stable.

Moreover, if the axial deformability is neglected, we can write the equation of motion in the following form:

$$EJ\frac{\partial^4\eta}{\partial x^4} + P(t)\frac{\partial^2\eta}{\partial x^2} + m\frac{\partial^2\eta}{\partial t^2} = 0 \tag{5}$$

in fact, equilibrium requires that the axial force is spatially uniform and equal to $N(x,t) = -P(t)$.

The hypothesis of axial undeformability is allowed when the beam is stiffer with respect to axial stresses than for bending stresses (in the case of slender columns) and when the frequency of the external force is not near the longitudinal natural frequencies of the beam.

We study the behaviour of the structure applying a force composed of a static and a harmonic component: $P(t) = P_s + P_d \cos\theta t$, seeking a solution in the form:

$$\eta(x,t) = f_k(t)\sin\frac{k\pi x}{l} \qquad (k = 1, 2, 3, \ldots) \tag{6}$$

where $f_k(t)$ are unknown functions of time and l is the length of the beam. Equation (6) satisfies implicitly the boundary conditions of the problem. Substituting equation (6) in equation (5) we obtain:

$$\left[m\frac{d^2f_k}{dt^2} + EJ\frac{k^4\pi^4f_k}{l^4} - (P_0 + P_d\cos\theta t)\frac{k^2\pi^2f_k}{l^2}\right]\sin\frac{k\pi x}{l} = 0 \tag{7}$$

After some substitutions we find the Mathieu-Hill differential equation governing the motion, which is of second order, linear, homogeneous, with variable coefficients:

$$\frac{d^2f}{dt^2} + \Omega^2(1 - 2\mu\phi(t))f = 0 \tag{8}$$

$\phi(t)$ is a periodic function with a period T, Ω is the frequency of the free vibrations of the beam loaded by a constant longitudinal force P_0 (natural frequency of vibration of the beam):

$$\Omega_k = \omega_k\sqrt{1 - \frac{P_0}{P_{*k}}} \tag{9}$$

and

$$\omega_k = \frac{k^2\pi^2}{l^2}\sqrt{\frac{EJ}{m}} \tag{10}$$

is the kth frequency of the free vibrations of an unloaded beam.

Then:

$$P_{*k} = \frac{k^2\pi^2 EJ}{l^2} \tag{11}$$

represents the kth Euler buckling load and μ_k is a quantity called the excitation parameter (Hansen, 1985; McLachlan, 1957; Pedersen, 1983):

$$\mu_k = \frac{P_d}{2(P_{*k} - P_0)} \tag{12}$$

Notice that equation (8) is independent of the spatial portion of the solution and, as a result, is equally valid for elastically restrained beams.

Since equation (8) is identical for all the forms of vibrations, i.e. it is identical for all k, we omitted the indices of Ω_k and μ_k.

Mathieu-Hill equations are encountered in various areas of physics and engineering. Several problems in theoretical physics lead to similar equations, in particular the problem of wave propagation in a medium of a periodic structure. This equation is also encountered in the problem of motion of electrons in a crystal lattice, in the investigation of stability of oscillatory processes in non-linear systems, in the theory of parametric excitation of electrical oscillations and in other branches of the theory of oscillations. Certain problems of celestial mechanics and cosmology also lead to Mathieu-Hill equations, particularly the theory of the motion of the moon. Hence it is not surprising that a lot of works, in the past and also today, have been devoted to the investigation of Mathieu-Hill equations (Bolotin, 1984; McLachlan, 1957).

In the problem considered, we look for a periodic solution, with a period $2T$; we will assume that this function can be represented in the form of the converging Fourier series:

$$f(t) = \sum_{k=1,3,5}^{\infty}\left(a_k \sin\frac{k\theta t}{2} + b_k \cos\frac{k\theta t}{2}\right). \tag{13}$$

Substituting the series in the Mathieu-Hill equations and equating the coefficients of identical $\sin(k\theta t)/2$ and $\cos(k\theta t)/2$ terms, we obtain the following system of homogeneous linear algebraic equations whose unknowns a_k and b_k:

$$\left(1+\mu-\frac{\theta^2}{4\Omega^2}\right)a_1-\mu a_3=0,$$

$$\left(1-\frac{k^2\theta^2}{4\Omega^2}\right)a_k-\mu\left(a_{k-2}+a_{k+2}\right)\ldots(k=3,5,7\ldots),$$

$$\left(1+\mu-\frac{\theta^2}{4\Omega^2}\right)b_1-\mu b_3=0,$$

$$\left(1-\frac{k^2\theta^2}{4\Omega^2}\right)b_k-\mu\left(b_{k-2}+b_{k+2}\right)\ldots(k=3,5,7\ldots). \tag{14}$$

We observe that the first system contains a_k coefficients only, and the second b_k coefficients only.

As known, the system of linear homogeneous equations presents solutions different from zero only in the case in which the determinant composed of the coefficients of the system is equal to zero. This also holds in the case where the system contains an infinite number of unknowns. Thus, the necessary condition for the existence of the periodic solution of Mathieu-Hill equations is that the obtained determinants of the homogeneous systems are equal to zero. Combining the two conditions under the ± sign, we obtain:

$$\begin{vmatrix} \left(1\pm\mu-\frac{\theta^2}{4\Omega^2}\right) & -\mu & 0 & \ldots\ldots \\ -\mu & \left(1-\frac{9\theta^2}{4\Omega^2}\right) & -\mu & \ldots\ldots \\ 0 & -\mu & \left(1-\frac{25\theta^2}{4\Omega^2}\right) & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} \tag{15}$$

This equation, relating the frequencies of the external loading with the natural frequencies of the beam and the magnitude of the external force, is called the equation of boundary frequencies, where boundary frequencies are understood to be the frequencies of external loading θ, corresponding to the boundaries of the regions of instability (Bolotin, 1964; Hansen, 1985; McLachlan, 1957). Equation (14) makes it possible to find the regions of instability that are bounded by the periodic solutions with a period $2T$.

To determine the regions of instability bounded by the periodic solutions with a period T, we proceed in a similar way (Bolotin, 1964).

3. Exact integration of the equations of motion

Consider now the problem of transverse vibrations of the beam restrained by translational and rotational springs at the ends (see Figure 2); the equation of motion for a simply supported beam was found in the form of equation (7).

To generalize the boundary conditions, we consider again the differential equation of the transverse free vibration of a beam, in the absence of damping effects:

$$m\frac{d^2\eta}{dt^2}+EJ\frac{d^4\eta}{dx^4}=0 \qquad (16)$$

A method to solve this equation consists in the separation of the variables; thus we can assume that the transverse displacement varies with x and t by means of two different functions, i.e.:

$$\eta(x,t)=\phi(x)Y(t) \qquad (17)$$

where $\phi(x)$ is a function of the distance along the axis of the beam defining the deformed shape of the beam when it vibrates, and $Y(t)$ defines the entity of the vibration as a function of time.

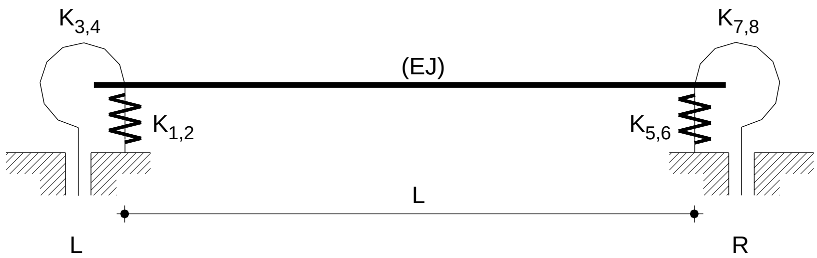

Figure 2.
Beam with translational and rotational springs at the end sections

Substituting equation (17) in equation (16) we obtain:

$$m\phi(x)\frac{\partial^2 Y(t)}{\partial t^2}+EJY(t)\frac{\partial^4\phi(x)}{\partial x^4}=0 \qquad (18)$$

We can rewrite the latter equation in the following form:

$$\frac{EJ}{m}\frac{1}{\phi(x)}\frac{\partial^4\phi(x)}{\partial x^4}=-\frac{1}{Y(t)}\frac{\partial^2 Y(t)}{\partial t^2}=\text{constant}=\omega^2 \qquad (19)$$

In this way we have obtained a function of x at the left-hand side and a function of t at the right-hand side. Since both terms must be constant, we can write:

$$EJ\frac{d^4\phi(x)}{dx^4}=\omega^2 m\phi(x) \qquad (20)$$

$$Y''(t)+\omega^2 Y(t)=0 \qquad (21)$$

The last equation can be compared with the equations of free vibration of the systems with one or two degrees of freedom. It is clear that the time dependent function can be represented by:

$$Y(t) = A\cos\omega t + B\sin\omega t \tag{22}$$

Let us consider now equation (20) that gives a relation between θ, the bending stiffness and the mass of the beam; with the notation:

$$\alpha^4 = \frac{\omega^2 m}{EJ} \tag{23}$$

it can be verified that the general solution of equation (23) becomes:

$$\phi(x) = A_1 \cos\alpha x + A_2 \sin\alpha x + A_3 \cosh\alpha x + A_4 \sinh\alpha x \tag{24}$$

where A_1, A_2, A_3, A_4 are coefficients depending on the boundary conditions of the beam.

The boundary conditions of the beam of Figure 2 can be given in homogeneous form as follows (see Figure 3):

$$
\begin{aligned}
\text{at}\quad & x = -L/2 && k_2 V_L = -k_1\,\phi(-L/2)\\
& x = -L/2 && k_4 M_L = -k_3\,\phi(-L/2)\\
& x = L/2 && k_6 V_R = -k_5\,\phi(L/2)\\
& x = L/2 && k_8 M_R = -k_7\,\phi(L/2)
\end{aligned} \tag{25}
$$

Each spring stiffness is represented by the ratio $ki/ki+1$, hence homogeneous conditions are represented by $ki = 0$, $ki + 1 = 1$ and $ki = 1$, $ki + 1 = 0$. This notation is more efficient for computational purposes.

Since also:

$$
\begin{aligned}
\text{at}\quad & x = -L/2 && \phi''(-L/2) = -M_L/EJ\\
& x = -L/2 && \phi'''(-L/2) = -V_L/EJ\\
& x = L/2 && \phi''(L/2) = -M_R/EJ\\
& x = L/2 && \phi'''(L/2) = -V_R/EJ
\end{aligned} \tag{26}
$$

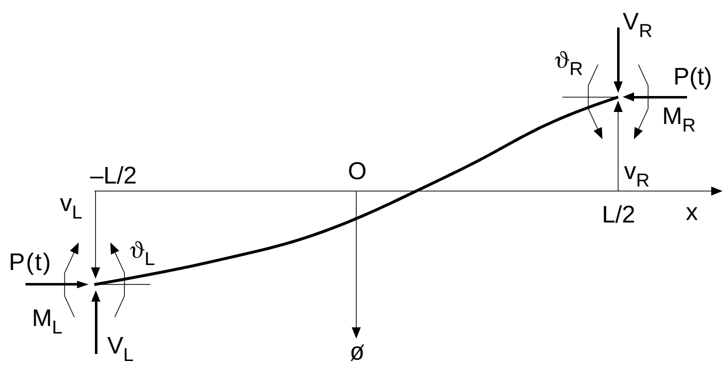

Figure 3.
Deformed configuration of the beam

Substituting in equation (25) we obtain a system with eight equations with eight unknowns. The determinant of the matrix of the coefficients must be set equal to zero.

$$\begin{aligned} eqdr = 2\,(& (k1\ k3 - k2\ k4)\ (k5\ k7 - k6\ k8) \\ & - k1\ k3\ k5\ k7 \cos[2\ a] \cosh[2\ a] - k2\ k4\ k5\ k7 \cos[2\ a] \cosh[2\ a] \\ & + 2\ k1\ k4\ k6\ k7 \cos[2\ a] \cosh[2\ a] + 2\ k2\ k3\ k5\ k8 \cos[2\ a] \cosh[2\ a] \\ & - k1\ k3\ k6\ k8 \cos[2\ a] \cosh[2\ a] - k2\ k4\ k6\ k8 \cos[2\ a] \cosh[2\ a] \\ & - k2\ k3\ k5\ k7 \cosh[2\ a] \sin[2\ a] + k1\ k4\ k5\ k7 \cosh[2\ a] \sin[2\ a] \\ & + k1\ k3\ k6\ k7 \cosh[2\ a] \sin[2\ a] + k2\ k4\ k6\ k7 \cosh[2\ a] \sin[2\ a] \\ & - k1\ k3\ k5\ k8 \cosh[2\ a] \sin[2\ a] - k2\ k4\ k5\ k8 \cosh[2\ a] \sin[2\ a] \\ & - k2\ k3\ k6\ k8 \cosh[2\ a] \sin[2\ a] + k1\ k4\ k6\ k8 \cosh[2\ a] \sin[2\ a] \\ & - k2\ k3\ k5\ k7 \cos[2\ a] \sinh[2\ a] - k1\ k4\ k5\ k7 \cos[2\ a] \sinh[2\ a] \\ & + k1\ k3\ k6\ k7 \cos[2\ a] \sinh[2\ a] + k2\ k4\ k6\ k7 \cos[2\ a] \sinh[2\ a] \\ & + k1\ k3\ k5\ k8 \cos[2\ a] \sinh[2\ a] + k2\ k4\ k5\ k8 \cos[2\ a] \sinh[2\ a] \\ & - k2\ k3\ k6\ k8 \cos[2\ a] \sinh[2\ a] - k1\ k4\ k6\ k8 \cos[2\ a] \sinh[2\ a] \\ & + 2\ k2\ k3\ k6\ k7 \sin[2\ a] \sinh[2\ a] - 2\ k1\ k4\ k5\ k8 \sin[2\ a] \sinh[2\ a]\ ) \\ & = 0 \end{aligned} \tag{27}$$

By isolating the terms in sin, cosh, etc.:

$$\begin{aligned} eqdr = 2\,(& (k1\ k3 - k2\ k4)\ (k5\ k7 - k6\ k8) \\ & + \cos[2\ a] \cosh[2\ a]\ \{- k5\ k7\ (k1\ k3 + k2\ k4) - k6\ k8\ (k1\ k3 + k2\ k4)\} \\ & + 2 \cos[2\ a] \cosh[2\ a]\ \{k1\ k4\ k6\ k7 + k2\ k3\ k5\ k8\} \\ & + \cosh[2\ a] \sin[2\ a]\ \{k6\ k7\ (k1\ k3 + k2\ k4) + k5\ k7\ (k1\ k4 - k2\ k3) \\ & - k5\ k8\ (k1\ k3 + k2\ k4) + k6\ k8\ (k1\ k4 - k2\ k3)\} \\ & + \cos[2\ a] \sinh[2\ a]\ \{- k5\ k7\ (k1\ k4 + k2\ k3) + k6\ k7\ (k1\ k3 + k2\ k4) \\ & - k6\ k8\ (k1\ k4 + k2\ k3) + k5\ k8\ (k1\ k3 + k2\ k4)\} \\ & + 2 \sin[2\ a] \sinh[2\ a]\ \{k2\ k3\ k6\ k7 - k1\ k4\ k5\ k8\}) = 0 \end{aligned} \tag{28}$$

If we put in evidence the terms *ki*:

$$\begin{aligned} eqdr = 2\,(& (k1\ k3 - k2\ k4)\ (k5\ k7 - k6\ k8) \\ & + \cos[2\ a] \cosh[2\ a]\ \{(k1\ k3 + k2\ k4)\ (- k5\ k7 - k6\ k8)\} \\ & + 2 \cos[2\ a] \cosh[2\ a]\ \{k1\ k4\ k6\ k7 + k2\ k3\ k5\ k8\} \\ & + \cosh[2\ a] \sin[2\ a]\ \{(k1\ k3 + k2\ k4)\ (k6\ k7 - k5\ k8) \\ & + (k1\ k4 - k2\ k3)\ (k5\ k7 + k6\ k8)\} \\ & + \cos[2\ a] \sinh[2\ a]\ \{(k1\ k4 + k2\ k3)\ (- k5\ k7 - k6\ k8) \\ & + (k1\ k3 + k2\ k4)\ (k6\ k7 + k5\ k8)\} \\ & + 2 \sin[2\ a] \sinh[2\ a]\ \{k2\ k3\ k6\ k7 - k1\ k4\ k5\ k8\}) = 0 \end{aligned} \tag{29}$$

Then we order the equation:

$$\begin{aligned} eqdr = 2\,(& (k1\ k3 - k2\ k4)\ (k5\ k7 - k6\ k8) \\ & + \cos[2\ a] \cosh[2\ a]\ \{(k1\ k3 + k2\ k4)\ (- k5\ k7 - k6\ k8)\} \\ & + \cosh[2\ a] \sin[2\ a]\ \{(k1\ k3 + k2\ k4)\ (- k5\ k8 + k6\ k7) \\ & + (k1\ k4 - k2\ k3)\ (k5\ k7 + k6\ k8)\} \\ & + \cos[2\ a] \sinh[2\ a]\ \{(k1\ k4 + k2\ k3)\ (- k5\ k7 - k6\ k8) \\ & + (k1\ k3 + k2\ k4)\ (k6\ k7 + k5\ k8)\} \end{aligned}$$

$$+ 2 \cos[2 a] \cosh[2 a] \{k1\ k4\ k6\ k7 + k2\ k3\ k5\ k8\}$$
$$+ 2 \sin[2 a] \sinh[2 a] \{k2\ k3\ k6\ k7 - k1\ k4\ k5\ k8\}) = 0 \quad (30)$$

and write the products and sums of ki in a more compact form:

$$\begin{aligned} &k1\ k3 + k2\ k4 = kl && k1\ k4 = k_{14} \\ & && k2\ k3 = k_{23} \\ &k5\ k7 + k6\ k8 = kr && k5\ k8 = k_{58} \\ & && k6\ k7 = k_{67} \end{aligned} \quad (31)$$

$$\begin{aligned} eqdr = 2\ (&(k1\ k3 - k2\ k4)\ (k5\ k7 - k6\ k8) \\ &- \cos[2 a] \cosh [2 a] \{kl\ kr\} \\ &+ \cosh[2 a] \sin [2 a] \{kl\ (- k_{58} + k_{67}) + kr\ (k_{14} - k_{23})\} \\ &+ \cos[2 a] \sinh [2 a] \{kl\ (k_{58} + k_{67}) - kr\ (k_{14} + k_{23})\} \\ &+ 2 \cos[2 a] \cosh [2 a] \{k_{14}\ k_{67} + k_{23}\ k_{58}\} \\ &+ 2 \sin[2 a] \sinh[2 a] \{k_{23}\ k_{67} - k_{14}\ k_{58}\}) = 0 \end{aligned} \quad (32)$$

Now we introduce the functions **Q**, **R**, **T**, **S**, **C**:

$$\mathbf{Q} = - \cos[2 a] \cosh[2 a]$$
$$\mathbf{R} = \cosh[2 a] \sin[2 a]$$
$$\mathbf{T} = \cos[2 a] \sinh[2 a]$$
$$\mathbf{C} = + 2 \cos[2 a] \cosh[2 a]$$
$$\mathbf{S} = 2 \sin[2 a] \sinh[2 a] \quad (33)$$

At the end of the manipulations we obtain the solution:

$$\begin{aligned} eqrd = &(k1\ k3 - k2\ k4)\ (k5\ k7 - k6\ k8) \\ &+ \mathbf{Q} \{kl\ kr\} + \mathbf{R} \{kl\ (-k_{58} + k_{67}) + kr\ (k_{14} - k_{23})\} + \mathbf{T} \{kl\ (k_{58} + k_{67}) \\ &- kr\ (k_{14} + k_{23})\} + \mathbf{C} \{k_{14}\ k_{67} + k_{23}\ k_{58}\} + \mathbf{S} \{k_{23}\ k_{67} - k_{14}\ k_{58}\} = 0 \end{aligned} \quad (34)$$

and making the positions:

$$\mathbf{b} = (k1\ k3 - k2\ k4)\ (k5\ k7 - k6\ k8)$$
$$\mathbf{c} = \{kl\ kr\}$$
$$\mathbf{d} = \{kl\ (-k_{58} + k_{67}) + kr\ (k_{14} - k_{23})\}$$
$$\mathbf{e} = \{kl\ (k_{58} + k_{67}) - kr\ (k_{14} + k_{23})\}$$
$$\mathbf{f} = \{k_{14}\ k_{67} + k_{23}\ k_{58}\}$$
$$\mathbf{g} = \{k_{23}\ k_{67} - k_{14}\ k_{58}\} \quad (35)$$

we can write a compact expression of the exact solution:

$$\mathbf{b} + \mathbf{c}\ \mathbf{Q} + \mathbf{d}\ \mathbf{R} + \mathbf{e}\ \mathbf{T} + \mathbf{f}\ \mathbf{C} + \mathbf{g}\ \mathbf{S} = 0 \quad (36)$$

whereby, **c**, **d**, **e**, **f**, **g** are functions of the spring constants only.

Obviously equation (36) can be easily treated by a numerical tool, e.g. Mathematica or similar programs. The goal is to find the first eigenvalue of the transcendental equation and consequently the corresponding boundary frequency domain, which is given by equation (15) since the time dependent solution has not been changed by the boundary conditions, hence equations (8)-(15) remain valid. The eigenmodes can be found using equation (24).

4. Applications

In relation to the first three modes of vibration of the beam we have determined the regions of dynamic instability, for comparison purposes with known solutions.

We have solved the equation of boundary frequencies in an exact form (see equation (35)) in the case of a beam with generical elastic constraints (Figure 2).

We show some of the results obtained with this procedure, introducing mass characteristics, moment of inertia of the constant cross-section, Young modulus of the material and length of the structure as input data. We have considered the case of a HEA 200 beam. The input data of the problem are:

$EJ = 2{,}100{,}000^{*}2{,}003/100^{2}$

$m = 61.3$

$l = 7$

In the following pictures we show, for different values of the coefficients k_i of the springs, the regions of dynamic instability in (P_d,θ) diagrams.

The following examples show the beam subjected to different boundary conditions. In Figure 4 the case of fixed end sections is analysed. It can be compared with the case shown in Figure 5 where elastic rotational springs constraints are applied to the same sections. The increasing of the stiffness gives a translation of the stability regions towards the higher frequencies. The same conclusions are gained from analysis of Figures 6 and 7 where the boundary conditions of rotational and translational built and elastic springs respectively are shown. In Figures 8 and 9 the stability regions of a beam fixed and elastically constrained at one end and free at the other are shown, while in

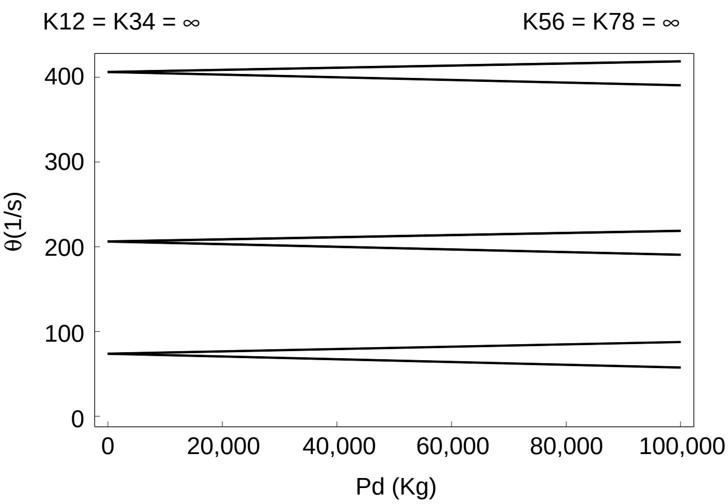

Figure 4. Regions of instability of a beam with fixed end sections

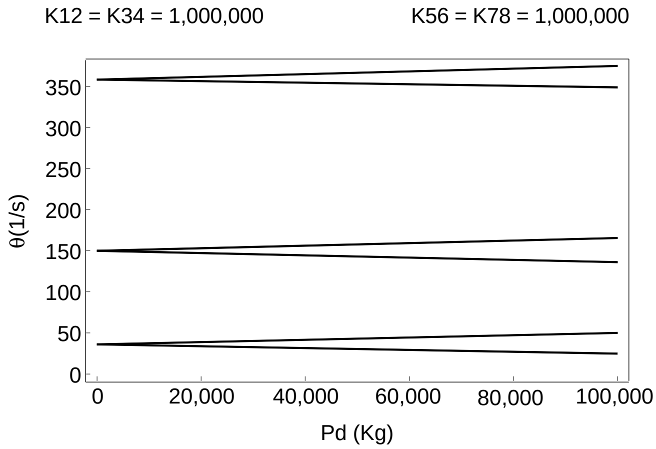

Figure 5.
Regions of instability of a beam with elastic rotational and translational springs at the end sections

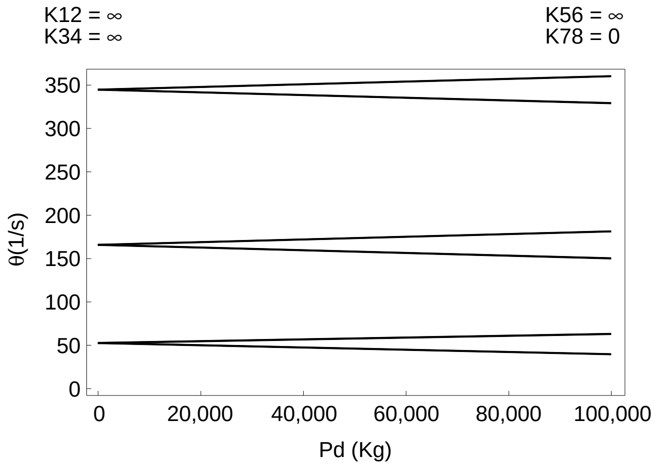

Figure 6.
Regions of instability of a beam fixed at one end section and simply supported at the other

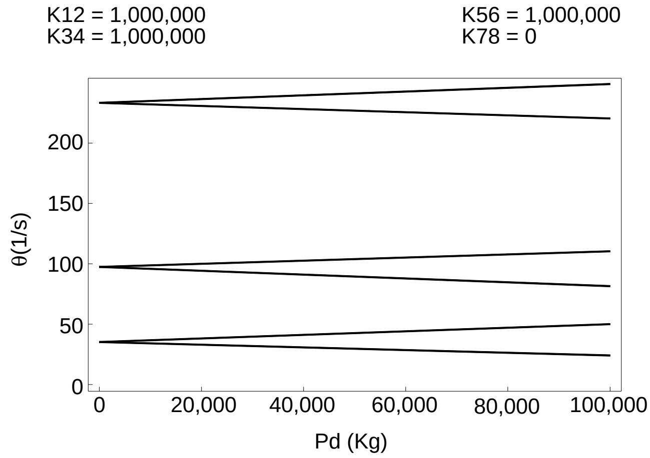

Figure 7.
Regions of instability of a beam with elastic springs at the end sections

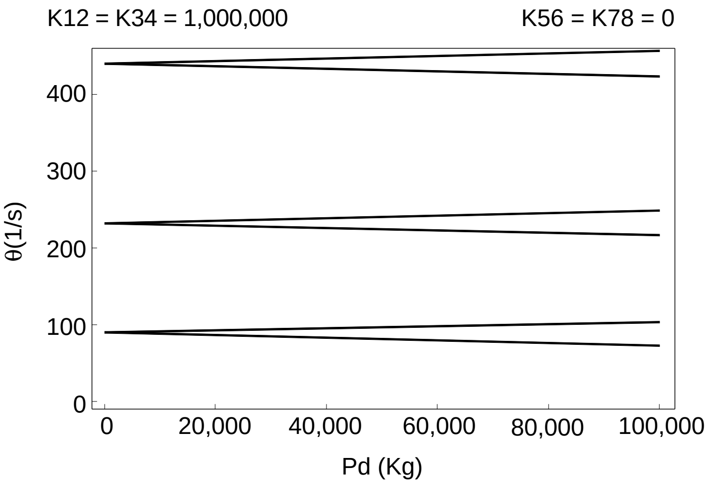

Figure 8.
Regions of instability of a beam elastically constrained at one end section and free at the other

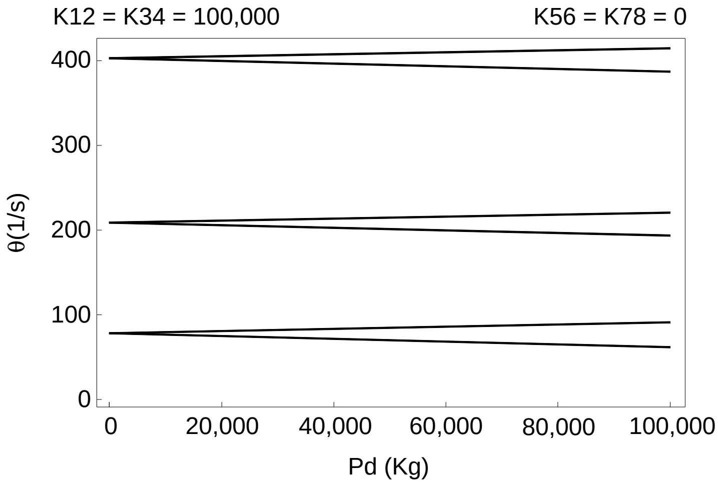

Figure 9.
Regions of instability of a beam elastically constrained at one end section and free at the other

Figures 10 and 11 a beam with translational built and elastic constraints has been considered. We emphasize that such results can be used for a preliminary check when complex structures have to be analysed in a time transient dynamic situation.

5. Conclusions

We have shown a simple and exact procedure that allows the exact solution of the equation of boundary frequencies to be found, i.e. to integrate in an exact form the equation of the motion of beams subject to dynamic loads. With this procedure we have found the regions of instability of beams generically restrained at the end cross-sections.

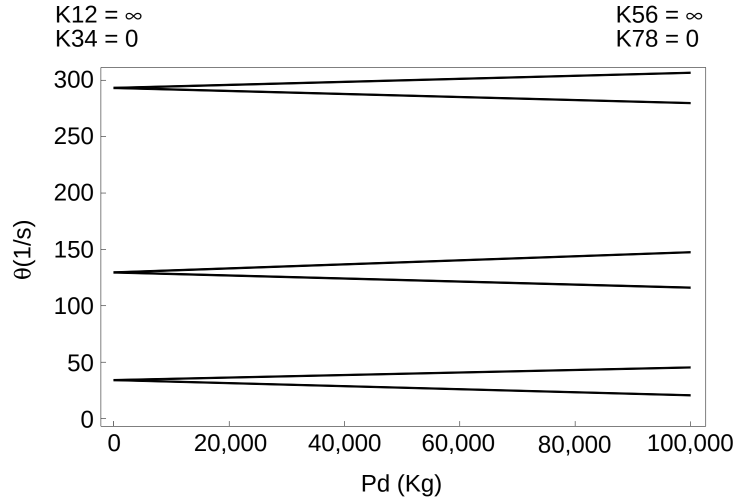

Figure 10.
Regions of instability of a simply supported beam

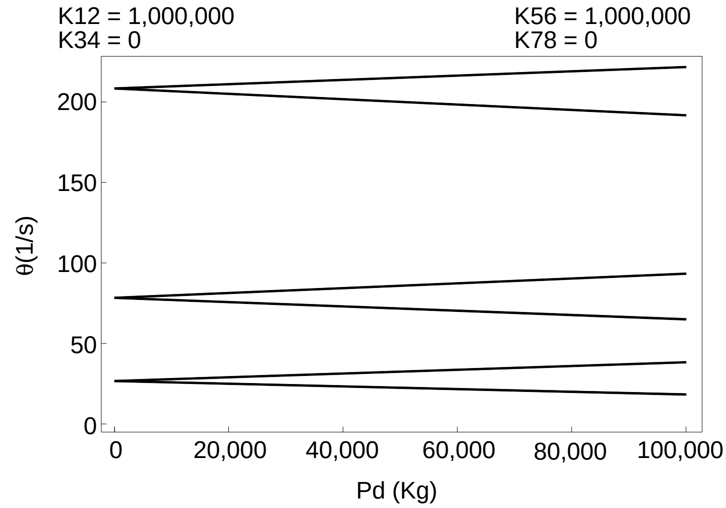

Figure 11.
Regions of instability of beam with elastic translational springs at the end sections

Analysing the results obtained, it can be clearly observed that if the stiffness of the springs increases, the regions of instability shift to higher frequencies of vibration, in accordance with the results obtained previously. The method is a valuable tool for knowing in advance or for verifying the regions of instability of complex structures analysed by a finite element method, reduced to simple beams with springs at the end cross-sections.

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