On a Characterization of Linear Compactness

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Introduction

It was shown by Manes ([17]) and later by Herrlich, Salicrup and Strecker ([14]), that a notion of compactness can be given in arbitrary categories provided with a factorization structure for single morphisms. This generalization of compactness was based on the characterization of the compact topological spaces by the Kuratowski-Mrowka closed projection theorem. It is pointed out in [4] and [9] that this generalization can be strengthened replacing factorization structures by closure operators, introduced in [3] (see also [6] and [8]), and shown to provide as a particular case the factorization structures. The notion of C-compactness with respect to a closure operator C, thus obtained, is thoroughly studied in [4], [5] and [9], for regular C in [1], for factorization structures in the category of abstract modules in [10] and for non abelian groups in [11], [12], [13].

The aim of the present work is to extend to categories of topological modules the study of C-compactness started in [5] in the case of abstract modules. Our main results concern regular closure operators as in [1]. Extending the main result of [10] we characterize C-compact modules for a regular weakly hereditary closure operator C by properties of the C-separated quotients. In the case when C is the usual Kuratowski closure operator K and the category is that of linearly topologized modules, we obtain a new characterization of the well known class of linearly compact modules as K-compact modules.

In order to make the paper selfcontained we give all necessary definitions and properties of closure operators in categories of topological modules in §1. In §2 we introduce a weaker notion of C-compactness (Definition 2.1) and we show that it coincides with C-compactness under certain conditions (Theorem 2.4). In section §3 we give the characterization of the linearly compact modules (Theorem 3.3) and obtain most of the known properties of the linearly compact modules as a consequence of this categorical characterization (Corollary 2.5). In §4 we introduce the notation of total C-density which is known to be of great importance in the theory of topological groups for $C = K$. We show that total C-density is preserved by taking inverse images along C-preserving morphisms, generalizing a result of [2] regarding the case of compact topological groups (Corollary 4.4). This gives a technical result (Theorem 4.5) applied in [20] to resolve a long-standing problem on the existence of the Warner topology. Other applications will be given in [21].

1 Closure operators and compactness

Let R_ρ be a topological ring and R_ρ-TM be the category of all left topological modules over R_ρ and continuous module-homomorphisms. In this paper we shall be concerned only with full subcategories $\mathcal{X}$ of the category R_ρ-TM closed with respect to products and quotients taken in R_ρ-TM. For each topological modules X in $\mathcal{X}$ we denote by $S(X)$ the set of all topological submodules of X.

Definition 1.1 ([3]) *A closure operator C on $\mathcal{X}$ is a collection of maps $C_X : S(X) \to S(X)$ for each $X \in \mathcal{X}$ such that*

i) for every $M \in S(X)$, the C-closure of M in X, $C_X(M)$, contains as topological submodule M, i.e. $M \leq C_X(M)$;

ii) C is a monotone operator, i.e. if $M \leq M'$ are topological submodules of X, then $C_X(M) \leq C_X(M')$;

iii) C is a "continuous" operator, i.e. for each morphism $f : X \to Y$ in the category $\mathcal{X}$, given $M \leq X$ we have $f(C_X(M)) \leq C_Y(f(M))$.

A topological submodule Y of X is said to be

- *C-closed* in X if $C_X(Y) = Y$;
- *C-dense* in X if $C_X(Y) = X$.

Following [3] we call a closure operator C

- *idempotent* if for each X in $\mathcal{X}$ and $M \in S(X)$ we have

  $$C_X(M) = C_X(C_X(M)),$$

  i.e. each C-closure is a C-closed topological submodule of X;

- *weakly hereditary* if given X in $\mathcal{X}$, for each $M \in S(X)$ we have

  $$C_{C_X(M)}(M) = C_X(M),$$

  i.e. each topological submodule of X is C-dense in its closure;

- *hereditary* if given X in $\mathcal{X}$, for each M, $N \in S(X)$ with $N \leq M$ we have

  $$C_M(N) = C_X(N) \cap M.$$

The usual topological closure in $\mathcal{X}$ is obviously a closure operator; it is called the Kuratowski operator and denoted by K. It is easy to verify that K is idempotent and hereditary. Let C be a closure operator on $\mathcal{X}$. A module $X \in \mathcal{X}$ is *C-separated* if 0 is C-closed in X. We denote by $\Delta_{\mathcal{X}}(C)$ the subcategory of $\mathcal{X}$ consisting of C-separated topological modules. It is easy to see that a topological module X is C-separated if and only if the diagonal Δ_X is C-closed in $X \times X$. Obviously, for $C = K$, $\Delta_{\mathcal{X}}(K)$ is the subcategory of Hausdorff topological modules of $\mathcal{X}$. Given $X \in \mathcal{X}$, a submodule M of X is K-closed if and only if $X/M \in \Delta_{\mathcal{X}}(K)$. In general M is C-closed whenever $X/M \in \Delta_{\mathcal{X}}(C)$. An idempotent closure operator C is said to be *regular* if for each $X \in \mathcal{X}$ a submodule M of X is C-closed if and only if $X/M \in \Delta(C)$.

Remark 1.2 *It can be shown that* $\Delta_{\mathcal{X}}(C)$ *is an extremally epireflective subcategory of* $\mathcal{X}$*, i.e. a subcategory closed with respect to products, subobjects and refinement (see [15]). Given a subcategory* $\mathcal{A}$ *of* $\mathcal{X}$*, for each* $M \subseteq X \in \mathcal{O}b(\mathcal{X})$*, consider the class of all morphisms* $f_\alpha : X \to A_\alpha$ *with* A_α *in* $\mathcal{A}$ *such that* $f(M) = 0$*. We define a closure operator setting* $(C_{\mathcal{A}})_X(M) := \cap_\alpha \mathrm{Ker}(f_\alpha)$*. The closure operators of this type are precisely the regular operators defined as in [3], [6] and [8]. Associating to each closure operator* C *the subcategory* $\Delta_{\mathcal{X}}(C)$ *and to each subcategory* $\mathcal{A}$ *of* $\mathcal{X}$ *the correspondent regular operator* $(C_{\mathcal{A}})_X$ *we obtain a Galois correspondence. It is a Galois equivalence between regular operators and extremally epireflective subcategories.*

Fix a closure operator C on $\mathcal{X}$.

Definition 1.3 ([4]) *A morphism* $f : X \to Y$ *in* $\mathcal{X}$ *is* C*-preserving, if for each* $H \leq X$*, the equality* $f(C_X(H)) = C_Y(f(H))$ *holds.*

Clearly, if C is idempotent, a morphism $f : X \to Y$ is C-preserving if and only if for each C-closed topological submodule L of X, $f(L)$ is C-closed in Y.
The following properties of C-preserving morphisms will be used in the sequel.

Proposition 1.4 *Let* C *be a closure operator of* $\mathcal{X}$*; then*

i) the class of C*-preserving morphisms is closed under composition;*

ii) if $g \circ f$ *is* C*-preserving and* f *is surjective, then also* g *is* C*-preserving;*

iii) if $g \circ f$ *is* C*-preserving and* g *is injective, then* f *is* C*-preserving;*

iv) if C *is weakly hereditary, then for each* C*-closed submodule* $Y \leq X$*, the inclusion* $m : Y \hookrightarrow X$ *is* C*-preserving.*

Proof. *i)* is obvious.
ii) Denote by h the composition $g \circ f : X \to Y \to Z$ and take any $M \in S(Y)$. Then $M_1 = f^{-1}(M) \in S(X)$ satisfies $f(M_1) = M$ since f is surjective. Then $C_Z(g(M)) = C_Z(h(M_1))$. Since h is C-preserving we get

$$C_Z(g(M)) = h(C_X(M_1)) = g(f(C_X(M_1))) \leq g(C_Y(f(M_1))) = g(C_Y(M)).$$

In this way we have proved that $C_Z(g(M)) \leq g(C_Y(M))$. The other inclusion is always true in view of property *iii)* of Definition 1.1.
iii) On the notation of *ii)* assume that h is C-preserving and g is injective. Let $N \in S(X)$. To prove that $C_Y(f(N)) = f(C_X(N))$ it suffices to check the inclusion $C_Y(f(N)) \subseteq f(C_X(N))$ in view of Definition 1.1, *iii)*. Since g is injective it suffices to show that $g(C_Y(f(N))) \subseteq g(f(C_X(N)))$. The right-hand side is equal to $h(C_X(N))$ which coincides with $C_Z(h(N))$ since h is C-preserving . By property *iii)* of Definition 1.1 $g(C_Y(f(N))) \leq C_Z(g(f(N))) = C_Z(h(N))$.
iv) For $M \in S(Y)$ obviously $C_Y(M) \subseteq C_X(M)$ holds by Definition 1.1, *iii)* applied to M. Since C is weakly hereditary $C_X(M) = C_{C_X(M)}(M) \leq C_Y(M)$ in view of the obvious inclusion $C_X(M) \leq C_X(Y) = Y$. ■

Definition 1.5 *A topological module* X *belonging to* $\mathcal{X}$ *is* C*-compact if for every* Y *in* $\mathcal{X}$ *the projection* $X \times Y \to Y$ *is* C*-preserving.*

Now we give two results regarding categorical properties of C-compactness. Although similar results are given in other situations (see [1], [4], [5],[10], [11], [14], [17]), they are not covered by those already existing, so we give a proof following, more or less, the same ideas.

Proposition 1.6 *Let C be a closure operator of $\mathcal{X}$ and $X \in \mathcal{X}$ be a C-compact object. Then:*

a) for every surjective morphism $f : X \to Y$ also Y is C-compact;

b) if C is weakly hereditary and Y is a C-closed topological submodule of X, then also Y is C-compact.

Proof. *a)* Let Z be a topological module in $\mathcal{X}$ and let $p : X \times Z \to Z$ and $q : Y \times Z \to Z$ be the canonical homomorphisms. Then we have $p = q \circ (f \times 1_Z)$. Since $f \times 1_Z$ is surjective, q is C-preserving by Proposition 1.6, *ii)*. Hence Y is C-compact.
b) Let Z be a topological module in $\mathcal{X}$ and let $p : X \times Z \to Z$ and $q : Y \times Z \to Z$ be the canonical homomorphisms. By Proposition 1.6, *iv)*, $\iota : Y \times Z \hookrightarrow X \times Z$ is C-preserving and we have $q = p \circ \iota$. By Proposition 1.6, *i)*, q is C-preserving. Therefore Y is C-compact. ■

Theorem 1.7 *If X is C-compact, then every morphism $f : X \to Y$ is C-preserving, provided $Y \in \Delta_{\mathcal{X}}(C)$.*

Proof. Consider the morphism $\phi : X \to X \times Y$ defined by $\phi(x) = (x, f(x))$. Then the composition of ϕ with the projection $p : X \times Y \to Y$ gives f. Since p is C-preserving by the C-compactness of X, it suffices to prove that ϕ is also C-preserving. Take a topological submodule $L \leq X$, then $\phi(C_X(L)) \subseteq C_{X \times Y}(\phi(L))$. We have to show the opposite inclusion. Note that $\phi(L) \subseteq Graph(f)$, which is a C-closed topological submodule of $X \times Y$ since the diagonal Δ_Y in $Y \times Y$ is C-closed and $Graph(f) = h^{-1}(\Delta_Y)$, where $h : X \times Y \to Y \times Y$ is defined by $h(x, y) = (f(x), y)$. Consequently, $C_{X \times Y}(\phi(L)) \subseteq Graph(f) \cap (C_X(L) \times Y) = \phi(C_X(L))$. ■

2 The weak C-compactness

Let C be a closure operator on the category $\mathcal{X}$.

Definition 2.1 *A topological module $X \in \mathcal{X}$ is said to be C-weakly compact (briefly C_w-compact) if for each morphism $\phi : X \to Y$ with $Y \in \Delta(C)$, $\phi(X)$ is C-closed in Y.*

By Theorem 1.7 each C-compact topological module is C_w-compact. In the sequel we consider short exact sequence $0 \to X_1 \xrightarrow{f} X \xrightarrow{g} X_2 \to 0$ in $\mathcal{X}$, where f is a topological embedding and g is open. We say that a class $\mathcal{C} in \mathcal{X}$ is closed with respect to extensions if $X \in \mathcal{C}$ whenever X_1, $X_2 \in \mathcal{C}$.

Proposition 2.2 *The category of C_w-compact topological modules is closed with respect to surjective images and extensions.*

Proof. Closedness with respect to surjective images is trivial, so that we check only the second assertion. Consider the exact sequence

$$0 \to Z \to X \to X/Z \to 0$$

with Z and X/Z topological modules C_w-compact. Let $f : X \to Y$ be an arbitrary morphism of the category $\mathcal{X}$ with $Y \in \Delta(C)$. By hypothesis $f(Z)$ is C_Y-closed, hence $Y/f(Z) \in \Delta(C)$. Denoted by p the projection $Y \to Y/f(Z)$, obviously $p \circ f$ induces a morphism $\phi : X/Z \to Y/f(Z)$. Now $f(X) = p^{-1}(\phi(X/Z))$ is C_Y-closed; for, being X/Z C_w-compact, $\phi(X/Z)$ is $C_{Y/f(Z)}$-closed. ■

Lemma 2.3 *A topological module X in $\mathcal{X}$ is C_w-compact if and only if $X/C_X(0)$ is C_w-compact.*

Proof. If X is C_w-compact then Proposition 2.2 applies. Suppose $X_1 = X/C_X(0)$ is C_w-compact. Let $f : X \to Y$ be an arbitrary morphism of the category $\mathcal{X}$ with $Y \in \Delta(C)$. Then $f(C_X(0)) \leq C_Y(f(0)) = C_Y(0) = 0$ since Y is C-separated. Then there exists a continuous homomorphism $g : X_1 \to Y$ such that $f = gh$, where $h : X \to X_1$ is the canonical homomorphism. Obviously $f(X) = g(X_1)$, so that $f(X)$ is C-closed by our assumption on X_1. ■

Now, we can give the central result of this section.

Theorem 2.4 *If C is regular, a topological module X in $\mathcal{X}$ is C-compact if and only if it is C_w-compact.*

Proof. We have observed that a C-compact object is C_w-compact. Let X be a C_w-compact topological module. We have to show that for each $Y \in \mathcal{X}$, the projection $p : X \times Y \to Y$ is C-preserving. Since C is regular this is equivalent to the fact that for each C-closed topological submodule Z of $X \times Y$, the topological quotient $Y/p(Z)$ is C-separated. Consider the natural morphism $\phi : X + Y \to (X + Y)/Z$. Since X is C_w-compact, the submodule $\phi(X) = \phi(X + Z)$ of the C-separated quotient $X + Y/Z$ is C-closed. Hence the quotient $[(X + Y)/Z]/\phi(X)$ is C separated. Our aim is to show that it is topologically isomorphic to $Y/p(Z)$. In fact, the submodule $\phi(X)$ of $(X + Y)/Z$ is algebraically isomorphic to $X + Z/Z$, so that $[(X + Y)/Z]/\phi(X) \cong (X + Y)/(X + Z)$. In view of the equalities $X + Y = X \oplus Y = X \times Y$ and $X + Z = X + p(Z) = X \oplus p(Z) = X \times p(Z)$ we get $[(X + Y)/Z]/\phi(X) \cong (X \times Y)/(X \times p(Z)) \cong Y/p(Z)$. ■

The following corollary adds new properties of the class of C-compact or C_w-compact objects.

Corollary 2.5 *Let C be a regular closure operator.*
a) The category of C-compact objects is closed with respect to extensions and contains all modules with no C-closed submodules beyond X. A module X is C-compact iff $X/C_X(0)$ is C-compact.
b) If C is also weakly hereditary, then the category of C_w-compact objects is closed with respect to C-closed subobjects.

Proof. Let X be a module such that no submodule of X except X itself is C-closed. Then every morphism $f : X \to Y$ with $Y \in \Delta(C)$ has trivial image, thus X is C_w-compact and consequently C-compact by Theorem 2.4. The rest follows by Proposition 2.2 and b) of Proposition 1.6. ■

3 Linear Compactness

In this section we fix $\mathcal{X}$ to be the category *R-LT* of linearly topologized modules and continuous module-homomorphisms. Let us consider M_τ in *R–LT*, where τ is the linear topology on M. Denote by $\mathcal{J} = \{J_\lambda \leq M : \lambda \in \Lambda\}$ a base for the filter of τ-neighborhoods of zero in M consisting of open submodules. Setting

$$\lambda \geq \mu \text{ if } J_\lambda \leq J_\mu,$$

we define a partial order on the set Λ; for each $\lambda \geq \mu$, λ, $\mu \in \Lambda$, denote by $\phi_\mu^\lambda : M/J_\lambda \to M/J_\mu$ the morphism defined by $x + J_\lambda \mapsto x + J_\mu$. The inverse limit $\varprojlim_\lambda M/J_\lambda$ of the family of modules $\{M/J_\lambda : \lambda \in \Lambda\}$ and the family of morphisms $\{\phi_\mu^\lambda : \lambda \geq \mu \in \Lambda\}$ is the Hausdorff completion of M_τ. For each element $(a_\lambda + J_\lambda)_{\lambda\in\Lambda}$ of $\varprojlim_\lambda M/J_\lambda$ consider the system

$$(*) \qquad\qquad x \equiv a_\lambda \pmod{J_\lambda},\ \lambda \in \Lambda.$$

Given a finite subset F of Λ, there exists $\lambda_F \in \Lambda$ with $\lambda_F \geq \lambda$ for each $\lambda \in F$. Now each element of the coset $a_{\lambda_F} + J_{\lambda_F}$ solves the finite system

$$x \equiv a_\lambda \ (\mathrm{mod}\ J_\lambda), \quad \lambda \in F,$$

hence the system $(*)$ is finitely solvable. Conversely, let us consider a finitely solvable system

$$(**) \qquad\qquad x \equiv b_\theta \pmod{I_\theta},\ \theta \in \Theta,$$

with $I_\theta \leq M$, $b_\theta \in M$ for each $\theta \in \Theta$. It is not restrictive to suppose $\mathcal{I} := \{I_\theta : \theta \in \Theta\}$ closed with respect to finite intersections (if not, enlarge $(**)$ to an equivalent system with this property). The inclusion between submodules defines again the partial order

$$\theta_1 \geq \theta_2 \Leftrightarrow I_{\theta_1} \leq I_{\theta_2}.$$

Let $\theta_1 \geq \theta_2$ and denote by c the solution of

$$\begin{cases} x & \equiv b_{\theta_1} \pmod{I_{\theta_1}} \\ x & \equiv b_{\theta_2} \pmod{I_{\theta_2}} \end{cases} ;$$

since $I_{\theta_1} \leq I_{\theta_2}$, we have

$$b_{\theta_1} - b_{\theta_2} = (c - b_{\theta_2}) - (c - b_{\theta_1}) \in I_{\theta_2}.$$

Then the element $(b_\theta + I_\theta)_{\theta\in\Theta}$ of $\prod M/I_\theta$ belongs to $\varprojlim_\theta M/I_\theta$.

Thus, for each family $\mathcal{I} := \{I_\xi \leq M : \xi \in \Xi\}$, closed with respect to finite intersection, there is a bijective correspondence between the finitely solvable systems of congruences modulo I_ξ, $\xi \in \Xi$, and the elements of $\varprojlim_\xi M/I_\xi$.

Proposition 3.1 *Let M_τ be a linearly topologized module and let $(I_\theta)_{\theta\in\Theta}$ be a filter base of neighborhoods of zero. The canonical morphism*

$$\phi : M \to \varprojlim_\theta M/I_\theta, \quad m \mapsto (m + I_\theta)_{\theta\in\Theta},$$

is surjective if and only if for each family $\{a_\theta : \theta \in \Theta\}$ of elements of M the system

$$x \equiv a_\theta \pmod{I_\theta},\ \theta \in \Theta$$

is solvable whenever it is finitely solvable.

Proof. The system

$$x \equiv a_\theta \pmod{I_\theta},\ \theta \in \Theta$$

is finitely solvable if and only if $(a_\theta + I_\theta)_{\theta\in\Theta} \in \varprojlim_\theta M/I_\theta$. Next it is solvable if and only if there exists $m \in M$ such that $(m - a_\theta) \in I_\Theta$ for each $\theta \in \Theta$, i.e. $(a_\theta + I_\theta)_{\theta\in\Theta} = (m + I_\theta)_{\theta\in\Theta} = \phi(m)$. ■

Definition 3.2 *([16]) A linearly topologized module M_τ is linearly compact if for each family $(I_\theta)_{\theta\in\Theta}$ of closed submodule of M_τ and each family a_θ of elements of M, the system*

$$x \equiv a_\theta \pmod{I_\theta},\ \theta \in \Theta$$

is solvable if and only if it is finitely solvable.

We prove now that linear compactness coincides with K-compactness.

Theorem 3.3 *For each M_τ in R–LT the following conditions are equivalent:*

i) M_τ is linearly compact;

ii) each finitely solvable system

$$x \equiv a_\theta \pmod{I_\theta},\ \theta \in \Theta,$$

with I_θ open submodules of M_τ and a_θ in M, is solvable;

iii) for each linear topology σ coarser than τ, the Hausdorff completion of M_σ is equal to the topological quotient of M_σ with respect to the σ-closure of zero;

iv) M_τ is K_w-compact;

v) M_τ is K-compact.

Proof. i) $\Rightarrow$ ii) follows from the definition of linear compactness.

ii) $\Rightarrow$ iii) : let $\mathcal{J} := \{J_\xi : \xi \in \Xi\}$ be a filter base of neighborhoods of zero in M_σ; clearly the J_ξ are τ-open, hence, by Proposition 3.1, the canonical morphism $M_\sigma \to \varprojlim_{\xi} M/J_\xi$ is surjective.

iii) $\Rightarrow$ iv) : consider a morphism $\psi : M_\tau \to N_\nu$ with N_ν a K-separated, i.e. Hausdorff, linearly topologized module. Let us denote by σ the weak topology with respect to the morphism ψ. Now $M/\mathrm{Ker}(\psi)$, endowed with the quotient topology of σ, is complete; then $\mathrm{Im}(\psi)$ is also complete, hence closed in N_ν.

iv) $\Leftrightarrow$ v) : follows by Theorem 2.4, since K is a regular closure operator.

iv) $\Rightarrow$ i) : let

$$(*) \qquad\qquad x \equiv a_\xi \pmod{J_\xi},\ \xi \in \Xi$$

a finitely solvable system with the J_ξ closed submodules of M_τ. For each ξ there exist open submodules $I_{\xi,i}$ such that $J_\xi = \bigcap_{i\in I} I_{\xi,i}$. It is easy to verify that *(*)* is equivalent to the finitely solvable system

$$(**) \qquad\qquad x \equiv a_\xi \pmod{I_{\xi,i}},\ \xi \in \Xi,\ i \in I.$$

As we have seen before, it is not restrictive to suppose $\{I_{\xi,i} : \xi \in \Xi,\ i \in I\}$ a filter-base of neighborhoods of zero for a linear topology σ on M, coarser than τ. By hypothesis, the image of the canonical morphism $\phi : M_\sigma \to \varprojlim_{\xi,i} M/I_{\xi,i}$ is closed, hence ϕ is surjective. Then there exists $m \in M$ such that

$$(m + I_{\xi,i})_{\xi,i} = (a_\xi + I_{\xi,i})_{\xi,i},$$

i.e. the system *(**)*, hence also the equivalent system *(*)*, is solvable. ■

Corollary 3.4 *Linear compactness is preserved under taking closed submodules, surjective images and extensions.*

4 Total C-density

Now we will see some properties of C-preserving morphisms, which permit to characterize the class of dense submodules with respect to a modified notion of C-density. This helps to understand better the class of dense subobjects also in the abstract setting (see [21] and [9], Chapter 5). These results are applied in other forthcoming papers of the authors (see [20] and [21]).

Definition 4.1 *A submodule N of a topological module M is totally C-*dense *if for every C-closed submodule L of M the submodule $N \cap L$ is C-dense in L.*

Total density with respect to the Kuratowski closure operator K is usually considered in (not necessarily abelian) topological groups (see [2] and [7]).

Lemma 4.2 *Let C be a weakly hereditary closure operator. Let $\phi : M \to N$ be a C-preserving homomorphism of topological modules, H be a submodule of N and $H' = \phi^{-1}(H)$. Then:*

a) if H is C-dense in N then H' is C-dense in M;

b) if H is totally C-dense in N, then H' is totally C-dense in M.

Proof. *a)* Set $A = C_M(H')$, then, being ϕ C-preserving, we have

$$\phi(A) = C_N(\phi(H')) = C_N(H) = N.$$

Since $A \subset H' \subset \mathrm{Ker}\phi$, $\phi(A) = N$ yields $A = M$.
b) To check that H' is totally C-dense in M, take a C-dense submodule L of M. Then $\phi(L)$ is a C-closed submodule of N. Thus $\phi(L) \cap H$ is C-dense in $\phi(L)$. Now consider the restriction ϕ_1 of ϕ to A

$$\phi_1 : A \to \phi(A).$$

Since C is weakly hereditary the inclusion $j : A \hookrightarrow M$ is C-preserving, thus $\phi_1 = \phi \circ j$ is C-preserving too. Now $A \cap H' = \phi_1^{-1}(\phi(A) \cap H)$, thus by *a)* $A \cap H'$ is C-dense in A. ■

Remark 4.3 *This generalizes Lemma 2.1 from [7], as well as 3.7.2 from [2] in the case C is the Kuratowski closure operator K.*

Corollary 4.4 *Let C be a weakly hereditary closure operator and let M be a C-compact module. Then for every C-separated module N and every continuous homomorphism $\phi : M \to N$, inverse images under ϕ of totally C-dense submodules of N are totally C-dense submodules of M.*

Proof. It suffices to note that ϕ will be C-preserving, so Lemma 4.2 can be applied. ■

Theorem 4.5 *Let C be a weakly hereditary closure operator, M be a C-separated module, K be a C-compact module of M. If for a submodule V of M, covered by C-compact submodules, the intersection $V \cap H$ is totally C-dense in H, then also V is totally C-dense in $H + V$.*

Proof. To check that V is totally C-dense in $H+V$, take an element $v+h \in V+H$. Denote by V_1 the C-compact submodule of V containing v. Then consider the C-compact submodule $V_1 + H$ of $V + H$. It is C-closed, and $v + h \in V_1 + H$. Therefore, to check that $v+h \in C_M[C_M(\langle v+h\rangle) \cap V]$, it suffices to do it in V_1+H. In other words, we assume that $V + H = V_1 + H$. Consider the homomorphism $\phi : V + H = V_1 + H \to (V_1 + H)/V_1$. Since $V_1 + H$ is C-compact, and V_1 is C-closed (as C-compact in a C-separated module M), ϕ is C-preserving (at least, if C is regular, i.e. $C = C_{\mathcal{Q}}$ for some $\mathcal{Q}$, then $(V_1 + H)/V_1 \in \mathcal{Q}$). Now we can apply Corollary 4.4 to $V = \phi^{-1}(\phi(V))$, since $\phi(V) = \phi(V \cap H)$ is totally C-dense in $(V_1+H)/V_1 \cong H/(V_1 \cap H)$. In fact, by the total C-density of $V \cap H$ in H, it follows that also under $\psi : H \to H/(V_1 \cap H)$, $\psi(V \cap H)$ is totally C dense in $H/(V_1 \cap H)$. Finally note that $H/(V_1 \cap H) \cong (V_1 + H)/V_1$. ■

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