

A Cesàro average of Goldbach numbers

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Abstract. Let Λ be the von Mangoldt function and

$$r_G(n) = \sum_{m_1+m_2=n} \Lambda(m_1)\Lambda(m_2)$$

be the counting function for the Goldbach numbers. Let $N \geq 2$ be an integer. We prove that

$$\begin{aligned} \sum_{n \leq N} r_G(n) \frac{(1 - \frac{n}{N})^k}{\Gamma(k+1)} &= \frac{N^2}{\Gamma(k+3)} - 2 \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(\rho+k+2)} N^{\rho+1} \\ &+ \sum_{\rho_1} \sum_{\rho_2} \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1+\rho_2+k+1)} N^{\rho_1+\rho_2} + \mathcal{O}_k(N^{1/2}), \end{aligned}$$

for $k > 1$, where ρ , with or without subscripts, runs over the non-trivial zeros of the Riemann zeta-function $\zeta(s)$.

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1 Introduction

We continue our recent work on the number of representations of an integer as a sum of primes. In [8] we studied the *average* number of representations of an integer as a sum of two primes, whereas in [7] we considered individual integers. In this paper we study a Cesàro weighted *explicit* formula for Goldbach numbers and the goal is similar to the one in [8], that is, we want to obtain the expected main term and one or more terms that depend explicitly on the zeros of the Riemann zeta-function, with a small error. Letting

$$r_G(n) = \sum_{m_1+m_2=n} \Lambda(m_1)\Lambda(m_2),$$

the main result of the paper is the following theorem.

Theorem 1.1. *Let N be a positive integer. We have*

$$\begin{aligned} \sum_{n \leq N} r_G(n) \frac{(1 - \frac{n}{N})^k}{\Gamma(k+1)} &= \frac{N^2}{\Gamma(k+3)} - 2 \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(\rho+k+2)} N^{\rho+1} \\ &+ \sum_{\rho_1} \sum_{\rho_2} \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1+\rho_2+k+1)} N^{\rho_1+\rho_2} \\ &+ \mathcal{O}_k(N^{1/2}), \end{aligned} \quad (1.1)$$

for $k > 1$, where ρ , with or without subscripts, runs over the non-trivial zeros of the Riemann zeta-function $\zeta(s)$.

We remark that the double series over zeros in (1.1) converges absolutely for $k > 1/2$, and it seems reasonable to believe that the stated equality holds for the same values of k , possibly with a weaker error term, although the bound $k > 1$ appears in several places of the proof and it seems to be the limit of the method.

The result in [8] is the case $k = 0$ of (1.1) under the assumption of the Riemann Hypothesis (RH); there we only get the first sum over zeros and the error term is $\mathcal{O}(N(\log N)^3)$. The proof in [8] depends on RH in just one place; it is not hard to get an unconditional version of such a result with an error term $o(N^2)$. The technique here is completely different and RH has no consequences on the lower bound for the size of k .

Similar averages of arithmetical functions are common in the literature, see, e.g., Chandrasekharan-Narasimhan [2] and Berndt [1] who built on earlier classical works. In their setting the generalized Dirichlet series associated to the arithmetical function satisfies a suitable functional equation and this leads to an asymptotic formula containing Bessel functions of real order. In our case we have no functional equation, and Bessel functions are naturally replaced by Gamma functions. In [9] we developed further the present technique to deal with the case $p + m^2$, where Bessel functions with complex order arise.

The most interesting explicit formula in Goldbach's problem has recently been given by Pintz [13]. It is too complicated to be reproduced here, but we remark that in his formula, which deals with individual values of $r_G(n)$, the summation is over zeros of suitable Dirichlet L -functions, whereas in an average problem like the present one, only the zeros of the Riemann ζ -function are relevant. The same phenomenon occurs in our papers [7, 8].

The method we will use is based on a formula due to Laplace [10], namely

$$\frac{1}{2\pi i} \int_{(a)} v^{-s} e^v dv = \frac{1}{\Gamma(s)}, \quad (1.2)$$

where $\Re(s) > 0$ and $a > 0$, see, e.g., [4, p. 238, formula 5.4 (1)]. In the following we will need the general case of (1.2) which can be found in de Azevedo Pribitkin [3, formulae (8) and (9)]:

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{iDu}}{(a + iu)^s} du = \begin{cases} \frac{D^{s-1}e^{-aD}}{\Gamma(s)} & \text{if } D > 0, \\ 0 & \text{if } D < 0, \end{cases} \tag{1.3}$$

which is valid for $\sigma = \Re(s) > 0$ and $a \in \mathbb{C}$ with $\Re(a) > 0$, and

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{(a + iu)^s} du = \begin{cases} 0 & \text{if } \Re(s) > 1, \\ \frac{1}{2} & \text{if } s = 1, \end{cases} \tag{1.4}$$

for $a \in \mathbb{C}$ with $\Re(a) > 0$ (for the case $s = 1$ the principal value of the integral has to be taken). Formulae (1.3)–(1.4) enable us to write averages of arithmetical functions by means of line integrals as we will see in Section 2 below. We recall that Walfisz, see [16, Chapter X], used (1.2) in the particular case where $s \in \mathbb{N}$, $s \geq 2$.

We combine this approach with line integrals with the classical methods dealing with infinite sums, exploited by Hardy & Littlewood (see [5] and [6]) and by Linnik [11]. In particular, in [5, Section 2.5] there is a sort of “explicit formula” for a function related to $\psi(x) - x$.

2 Settings

Let

$$\tilde{S}(z) = \sum_{m \geq 1} \Lambda(m)e^{-mz}, \tag{2.1}$$

where $z = a + iy$ with $y \in \mathbb{R}$ and real $a > 0$. We recall that the Prime Number Theorem (PNT) is equivalent, via Lemma 4.1 below, to the statement

$$\tilde{S}(a) \sim a^{-1} \quad \text{for } a \rightarrow 0+, \tag{2.2}$$

which is classical: for the proof see for instance [6, Lemma 9]. By (2.1) we have

$$\tilde{S}(z)^2 = \sum_{n \geq 1} r_G(n)e^{-nz}.$$

Hence, for $N \in \mathbb{N}$ with $N > 0$ and $a > 0$ we have

$$\frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \tilde{S}(z)^2 dz = \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \sum_{n \geq 1} r_G(n)e^{-nz} dz. \tag{2.3}$$

Since

$$\sum_{n \geq 1} |r_G(n)e^{-nz}| = \tilde{S}(a)^2 \asymp a^{-2}$$

by (2.2), where $f \asymp g$ means $g \ll f \ll g$, we can exchange the series and the line integral in (2.3) provided that $k > 0$. In fact, if $z = a + iy$, taking into account the estimate

$$|z|^{-1} \asymp \begin{cases} a^{-1} & \text{if } |y| \leq a, \\ |y|^{-1} & \text{if } |y| \geq a, \end{cases} \tag{2.4}$$

we have

$$|e^{Nz} z^{-k-1}| \asymp e^{Na} \begin{cases} a^{-k-1} & \text{if } |y| \leq a, \\ |y|^{-k-1} & \text{if } |y| \geq a, \end{cases}$$

and hence, recalling (2.2), we obtain

$$\int_{(a)} |e^{Nz} z^{-k-1}| \left| \sum_{n \geq 1} r_G(n)e^{-nz} \right| |dz| \ll a^{-2} e^{Na} \left(a^{-k} + \frac{a^{-k}}{k} \right),$$

but only for $k > 0$. Using (1.3) for $n \neq N$ and (1.4) for $n = N$, we see that for $k > 0$ the right-hand side of (2.3) is as follows:

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \sum_{n \geq 1} r_G(n)e^{-nz} dz \\ &= \sum_{n \geq 1} r_G(n) \left[\frac{1}{2\pi i} \int_{(a)} e^{(N-n)z} z^{-k-1} dz \right] \\ &= \sum_{n \leq N} r_G(n) \frac{(N-n)^k}{\Gamma(k+1)}. \end{aligned}$$

Remark. It is important to notice that the previous computation reveals that we cannot get rid of the Cesàro weight in our method since, for $k = 0$, it is not clear if the integral at the right-hand side of (2.3) converges absolutely or not. In fact, if we could prove (1.1) for $k = 0$, assuming the RH we could easily derive the main result of [8] with an error term $\mathcal{O}(N)$, and this seems to be quite unreachable in the present state of knowledge. See the concluding remarks in the latter paper for an explanation.

Summing up, for $k > 0$ we have

$$\sum_{n \leq N} r_G(n) \frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \tilde{S}(z)^2 dz, \tag{2.5}$$

where $N \in \mathbb{N}$ with $N > 0$, $a > 0$. This is the fundamental relation for the method.

3 Inserting zeros

In this section we need $k > 1$. By Lemma 4.1 below we have

$$\tilde{S}(z) = \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) + E(y, a),$$

where $E(y, a)$ satisfies (4.3). Hence

$$\tilde{S}(z)^2 = \left(\frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right)^2 + E(y, a)^2 + 2E(y, a) \left(\frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right).$$

We have

$$\left| \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right| \leq \tilde{S}(a) + |E(y, a)| \ll a^{-1} + |E(y, a)|$$

by (2.2) again, so that

$$\tilde{S}(z)^2 = \left(\frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right)^2 + \mathcal{O}(|E(y, a)|^2 + |E(y, a)|a^{-1}). \tag{3.1}$$

Recalling (2.4) and (4.3), we have

$$\int_{(a)} |E(y, a)|^2 |e^{Nz}| |z|^{-k-1} |dz| \ll_k e^{Na} a^{-k+1}.$$

Choosing $a = 1/N$, the error term is $\ll_k N^{k-1}$ for $k > 1$. For $a = 1/N$, by (2.4) and (4.3), the second remainder term in (3.1) gives a contribution

$$\ll N \int_{(1/N)} \left| E\left(y, \frac{1}{N}\right) \right| |e^{Nz}| |z|^{-k-1} |dz| \ll_k N^{k+1/2}.$$

With a little effort we can give an explicit dependence on k for the implicit constants in the last two estimates, showing that the condition $k > 1$ is indeed necessary.

Hence, by (2.5)–(3.1) and the last two estimates, we have

$$\begin{aligned} & \sum_{n \leq N} r_G(n) \frac{(N-n)^k}{\Gamma(k+1)} \\ &= \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} \left(\frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right)^2 dz + \mathcal{O}_k(N^{k+1/2}) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-3} dz - \frac{1}{\pi i} \int_{(1/N)} e^{Nz} z^{-k-2} \sum_{\rho} z^{-\rho} \Gamma(\rho) dz \\
 &\quad + \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} \sum_{\rho_1} \sum_{\rho_2} z^{-\rho_1-\rho_2} \Gamma(\rho_1) \Gamma(\rho_2) dz \\
 &\quad + \mathcal{O}_k(N^{k+1/2}).
 \end{aligned} \tag{3.2}$$

Interchanging the series with the integrals (see Sections 5–6 for a proof that this is permitted when $k > 1$), by (3.2) we get that

$$\begin{aligned}
 &\sum_{n \leq N} r_G(n) \frac{(N-n)^k}{\Gamma(k+1)} \\
 &= \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-3} dz - \frac{1}{\pi i} \sum_{\rho} \Gamma(\rho) \int_{(1/N)} e^{Nz} z^{-k-2-\rho} dz \\
 &\quad + \frac{1}{2\pi i} \sum_{\rho_1} \sum_{\rho_2} \Gamma(\rho_1) \Gamma(\rho_2) \int_{(1/N)} e^{Nz} z^{-k-1-\rho_1-\rho_2} dz \\
 &\quad + \mathcal{O}_k(N^{k+1/2}) \\
 &= I_1 + I_2 + I_3 + \mathcal{O}_k(N^{k+1/2}),
 \end{aligned} \tag{3.3}$$

say. We now evaluate I_1 , I_2 and I_3 . Using (1.2) and putting $s = Nz$, we immediately get

$$\begin{aligned}
 I_1 &= \frac{N^{k+2}}{2\pi i} \int_{(1)} e^s s^{-k-3} ds = \frac{N^{k+2}}{\Gamma(k+3)}, \\
 I_2 &= -\frac{1}{\pi i} \sum_{\rho} \Gamma(\rho) N^{k+\rho+1} \int_{(1)} e^s s^{-k-2-\rho} ds \\
 &= -2 \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(\rho+k+2)} N^{k+\rho+1},
 \end{aligned}$$

and

$$I_3 = \sum_{\rho_1} \sum_{\rho_2} \frac{\Gamma(\rho_1) \Gamma(\rho_2)}{\Gamma(\rho_1 + \rho_2 + k + 1)} N^{k+\rho_1+\rho_2}.$$

Theorem 1.1 follows inserting I_1 , I_2 and I_3 in (3.3) and dividing through by N^k , provided that $k > 1$. The proof that the double sum over zeros converges absolutely for $k > 1/2$ is given in Section 7 below.

4 Lemmas

We recall some basic facts in complex analysis. First, if $z = a + iy$ with $a > 0$, we see that for complex w we have

$$|z^{-w}| = |z|^{-\Re(w)} \exp\left(\Im(w) \arctan\left(\frac{y}{a}\right)\right). \tag{4.1}$$

We also recall that, uniformly for $x \in [x_1, x_2]$, with x_1 and x_2 fixed, and for $|y| \rightarrow +\infty$, by the Stirling formula we have

$$|\Gamma(x + iy)| \sim \sqrt{2\pi} e^{-\pi|y|/2} |y|^{x-1/2}, \tag{4.2}$$

see, e.g., Titchmarsh [15, Section 4.42]. We will need the Hardy–Littlewood–Linnik formula (see, e.g., Languasco & Zaccagnini [7]): we notice that here $y \in \mathbb{R}$, while in [7] we had the restricted range $y \in [-1/2, 1/2]$. Hence there are some modifications to be made. We will follow the proof in Linnik [11] (see also [12, equation (4.1)]).

Lemma 4.1. *Let $z = a + iy$, where $a > 0$ and $y \in \mathbb{R}$. Then*

$$\tilde{S}(z) = \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) + E(a, y)$$

where $\rho = \beta + iy$ runs over the non-trivial zeros of $\zeta(s)$ and

$$E(a, y) \ll |z|^{1/2} \begin{cases} 1 & \text{if } |y| \leq a, \\ 1 + \log^2\left(\frac{|y|}{a}\right) & \text{if } |y| > a. \end{cases} \tag{4.3}$$

Proof. Following the line of Hardy and Littlewood, see [5, Section 2.2], [6, Lemma 4] and of [11, Section 4], we have that

$$\tilde{S}(z) = \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) - \frac{\zeta'}{\zeta}(0) - \frac{1}{2\pi i} \int_{(-1/2)}^{\zeta'} \frac{\zeta'}{\zeta}(w) \Gamma(w) z^{-w} dw. \tag{4.4}$$

Now we estimate the integral in (4.4). Let $c > 0$ be a positive constant to be chosen later. Writing $w = -1/2 + it$, we have

$$\left| \left(\frac{\zeta'}{\zeta} \right) (w) \right| \ll \log(|t| + 2),$$

$|z^{-w}| = |z|^{1/2} \exp(t \arctan(y/a))$ by (4.1) and, for $|t| > c$,

$$\Gamma(w) \ll |t|^{-1} \exp\left(-\frac{\pi}{2}|t|\right)$$

by (4.2). Letting $L_c = \{-1/2 + it : |t| > c\}$, we have

$$\int_{L_c} \frac{\zeta'}{\zeta}(w)\Gamma(w)z^{-w} dw \ll |z|^{1/2} \int_{L_c} \frac{\log |t|}{|t|} \exp\left(-\frac{\pi}{2}|t| + t \arctan\left(\frac{y}{a}\right)\right) dt.$$

If $ty \leq 0$, we call η the quantity

$$\eta = \frac{\pi}{2} + \left| \arctan\left(\frac{y}{a}\right) \right| \in \left[\frac{\pi}{2}, \pi \right).$$

If $|y| \leq a$, we define η as

$$\eta = \frac{\pi}{2} - \arctan\left(\frac{y}{a}\right) > \frac{\pi}{2} - \arctan(1) = \frac{\pi}{4}.$$

In the remaining case ($|y| > a$ and $ty > 0$) we set

$$\eta = \arctan\left(\frac{a}{|y|}\right) \gg \frac{a}{|y|}.$$

Now fix c such that $c\eta < 1$ (e.g., $c = \frac{1}{\pi}$ is allowed). Letting $u = \eta t$, we get

$$\begin{aligned} \int_{L_c} \frac{\zeta'}{\zeta}(w)\Gamma(w)z^{-w} dw &\ll |z|^{1/2} \int_{c\eta}^{+\infty} \frac{\log u}{ue^u} du + |z|^{1/2} \log(1/\eta) \int_{c\eta}^{+\infty} \frac{du}{ue^u} \\ &= J_1 + J_2, \end{aligned} \tag{4.5}$$

say. We remark that $0 \leq u^{-1} \log u \leq e^{-1}$ for $u \geq 1$, since the maximum of the function $u^{-1} \log u$ is attained at $u = e$. Since

$$0 \leq \int_1^{+\infty} \frac{\log u}{ue^u} du \ll 1 \quad \text{and} \quad \left| \int_{c\eta}^1 \frac{\log u}{ue^u} du \right| \leq \int_{c\eta}^1 \frac{-\log u}{u} du \ll \log^2\left(\frac{1}{\eta}\right),$$

we have that

$$J_1 \ll |z|^{1/2} \log^2\left(\frac{1}{\eta}\right).$$

For J_2 it is sufficient to remark that

$$0 \leq J_2 \leq |z|^{1/2} \log\left(\frac{1}{\eta}\right) \left(\int_{c\eta}^1 \frac{du}{u} + \int_1^{+\infty} e^{-u} du \right) \ll |z|^{1/2} \log^2\left(\frac{1}{\eta}\right).$$

Inserting the last two estimates in (4.5), recalling the definition of η and remarking that the integration over $|t| \leq c$ gives immediately a contribution $\ll 1$, we obtain that the integral in (4.4) is dominated by the right-hand side of (4.3) and the lemma is proved. □

In the next sections we will need to perform several times a set of similar computations; so we collected them in the following two lemmas.

Lemma 4.2. *Let $\beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta-function and $\alpha > 1$ be a parameter. The series*

$$\sum_{\rho: \gamma > 0} \gamma^{\beta-1/2} \int_1^{+\infty} \exp\left(-\gamma \arctan \frac{1}{u}\right) \frac{du}{u^{\alpha+\beta}}$$

converges provided that $\alpha > 3/2$. For $\alpha \leq 3/2$ the series does not converge. The result remains true if we insert in the integral a factor $(\log u)^c$, for any fixed $c \geq 0$.

Proof. Setting $y = \arctan(1/u)$, for any real $\gamma > 0$ we have

$$\begin{aligned} \int_1^{+\infty} \exp\left(-\gamma \arctan \frac{1}{u}\right) \frac{du}{u^{\alpha+\beta}} &= \int_0^{\pi/4} \exp(-\gamma y) \frac{(\sin y)^{\alpha+\beta-2}}{(\cos y)^{\alpha+\beta}} dy \\ &\ll_{\alpha} \int_0^{\pi/4} \exp(-\gamma y) y^{\alpha+\beta-2} dy \\ &\leq \gamma^{1-\alpha-\beta} (\Gamma(\alpha-1) + \Gamma(\alpha)), \end{aligned}$$

since $0 < \beta < 1$. This shows that the series over γ converges for $\alpha > 3/2$. For $\alpha = 3/2$ essentially the same computation shows that the integral is $\gg \gamma^{-1/2-\beta}$ and it is well known that in this case the series over zeros diverges. The other assertions are proved in the same way. \square

Lemma 4.3. *Let $\alpha > 1$, $z = a + iy$, $a \in (0, 1)$ and $y \in \mathbb{R}$. Let further $\rho = \beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta-function. We have*

$$\sum_{\rho} |\gamma|^{\beta-1/2} \int_{\mathbb{Y}_1 \cup \mathbb{Y}_2} \exp\left(\gamma \arctan \frac{y}{a} - \frac{\pi}{2} |\gamma| \right) \frac{dy}{|z|^{\alpha+\beta}} \ll_{\alpha} a^{-\alpha},$$

where $\mathbb{Y}_1 = \{y \in \mathbb{R} : y\gamma \leq 0\}$ and $\mathbb{Y}_2 = \{y \in [-a, a] : y\gamma > 0\}$. The result remains true if we insert in the integral a factor $(\log(|y|/a))^c$, for any fixed $c \geq 0$.

Proof. We first work on \mathbb{Y}_1 . By symmetry, we may assume that $\gamma > 0$. For any $y \in (-\infty, 0]$ we have

$$\gamma \arctan\left(\frac{y}{a}\right) - \frac{\pi}{2} |\gamma| \leq -\frac{\pi}{2} |\gamma|$$

and hence the quantity we are estimating becomes

$$\begin{aligned} \sum_{\rho: \gamma > 0} \gamma^{\beta-1/2} \exp\left(-\frac{\pi}{2} \gamma\right) \int_{-\infty}^0 \frac{dy}{|z|^{\alpha+\beta}} &\ll_{\alpha} \sum_{\rho: \gamma > 0} \gamma^{\beta-1/2} \exp\left(-\frac{\pi}{2} \gamma\right) a^{1-\alpha-\beta} \\ &\ll_{\alpha} a^{-\alpha}, \end{aligned}$$

using $0 < \beta < 1$, standard zero-density estimates and (2.4). We now consider the integral over \mathbb{Y}_2 . Again by symmetry we can assume that $\gamma > 0$ and so we get

$$\sum_{\rho:\gamma>0} \gamma^{\beta-1/2} \int_0^a \exp\left(\gamma\left(\arctan \frac{y}{a} - \frac{\pi}{2}\right)\right) \frac{dy}{|z|^{\alpha+\beta}} \ll_{\alpha} a^{-\alpha}$$

arguing as above. The other assertions are proved in the same way. □

5 Interchange of the series over zeros with the line integral

We need $k > 1/2$ in this section. We need to establish the convergence of

$$\sum_{\rho} |\Gamma(\rho)| \int_{(1/N)} |e^{Nz}| |z|^{-k-1} |z^{-\rho}| |dz|. \tag{5.1}$$

By (4.1) and the Stirling formula (4.2), we are left with estimating

$$\sum_{\rho} |\gamma|^{\beta-1/2} \int_{\mathbb{R}} \exp\left(\gamma \arctan(Ny) - \frac{\pi}{2} |\gamma|\right) \frac{dy}{|z|^{k+1+\beta}}. \tag{5.2}$$

We have just to consider the case $\gamma y > 0$, $|y| > 1/N$ since in the other cases the total contribution is $\ll_k N^{k+1}$ by Lemma 4.3 with $\alpha = k + 1$ and $a = 1/N$. By symmetry, we may assume that $\gamma > 0$. We have that the integral in (5.2) is

$$\ll N^k \sum_{\rho:\gamma>0} N^{\beta} \gamma^{\beta-1/2} \int_1^{+\infty} \exp\left(-\gamma \arctan \frac{1}{u}\right) \frac{du}{u^{k+1+\beta}}.$$

For $k > 1/2$ this is $\ll_k N^{k+1}$ by Lemma 4.2. This implies that the integrals in (5.2) and in (5.1) are both $\ll_k N^{k+1}$ and hence this exchange step is fully justified.

6 Interchange of the double series over zeros with the line integral

We need $k > 1$ in this section. Arguing as in Section 5, we first need to establish the convergence of

$$\sum_{\rho_1} |\Gamma(\rho_1)| \int_{(1/N)} \left| \sum_{\rho_2} \Gamma(\rho_2) z^{-\rho_2} \right| |e^{Nz}| |z|^{-k-1} |z^{-\rho_1}| |dz|. \tag{6.1}$$

Using the PNT and (4.3), we first remark that

$$\left| \sum_{\rho} z^{-\rho} \Gamma(\rho) \right| \ll \begin{cases} N & \text{if } |y| \leq 1/N, \\ |z|^{-1} + |z|^{1/2} \log^2(2N|y|) & \text{if } |y| > 1/N. \end{cases} \tag{6.2}$$

By symmetry, we may assume that $\gamma_1 > 0$. By estimates (6.2), (2.4) and (4.1), for $y \in (-\infty, 0]$ we are first led to estimate

$$\sum_{\rho_1 : \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \left(\int_{-1/N}^0 N^{k+2+\beta_1} dy + \int_{-\infty}^{-1/N} \frac{dy}{|y|^{k+2+\beta_1}} + \int_{-\infty}^{-1/N} \log^2(2N|y|) \frac{dy}{|y|^{k+1/2+\beta_1}} \right) \ll_k N^{k+2},$$

by the same argument used in the proof of Lemma 4.3 with $\alpha = k + 1/2$ and $a = 1/N$.

On the other hand, for $y > 0$ we split the range of integration into the disjoint union $(0, 1/N] \cup (1/N, +\infty)$. By (6.2), (2.4) and Lemma 4.3 with $\alpha = k + 1$ and $a = 1/N$, on the first interval we have

$$N \sum_{\rho_1 : \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_0^{1/N} \exp\left(\gamma_1 \left(\arctan(Ny) - \frac{\pi}{2}\right)\right) \frac{dy}{|z|^{k+1+\beta_1}} \ll_k N^{k+2}.$$

On the other interval, again by (2.4), we have to estimate

$$\begin{aligned} & \sum_{\rho_1 : \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_{1/N}^{+\infty} \exp\left(-\gamma_1 \arctan \frac{1}{Ny}\right) \frac{y^{-1} + y^{1/2} \log^2(2Ny)}{y^{k+1+\beta_1}} dy \\ &= N^k \sum_{\rho_1 : \gamma_1 > 0} N^{\beta_1} \gamma_1^{\beta_1 - 1/2} \int_1^{+\infty} \exp\left(-\gamma_1 \arctan \frac{1}{u}\right) \\ & \quad \times \frac{Nu^{-1} + u^{1/2} N^{-1/2} \log^2(2u)}{u^{k+1+\beta_1}} dy. \end{aligned}$$

Lemma 4.2 with $\alpha = k + 1/2$ shows that the last term is $\ll_k N^{k+2}$. This implies that the integral in (6.1) is $\ll_k N^{k+2}$ provided that $k > 1$ and hence we can exchange the first summation with the integral in this case.

To exchange the second summation we have to consider

$$\sum_{\rho_1} |\Gamma(\rho_1)| \sum_{\rho_2} |\Gamma(\rho_2)| \int_{(1/N)} |e^{Nz}| |z|^{-k-1} |z^{-\rho_1}| |z^{-\rho_2}| |dz|. \tag{6.3}$$

By symmetry, we can consider $\gamma_1, \gamma_2 > 0$ or $\gamma_1 > 0, \gamma_2 < 0$.

Assuming $\gamma_1, \gamma_2 > 0$, for $y \leq 0$ we have

$$\gamma_j \arctan(Ny) - \frac{\pi}{2} \gamma_j \leq -\frac{\pi}{2} \gamma_j, \quad j = 1, 2,$$

and, by (4.1), the corresponding contribution to (6.3) is

$$\sum_{\rho_1 : \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp\left(-\frac{\pi}{2} \gamma_1\right) \sum_{\rho_2 : \gamma_2 > 0} \gamma_2^{\beta_2 - 1/2} \exp\left(-\frac{\pi}{2} \gamma_2\right) \times \left(\int_{-\infty}^0 \frac{dy}{|z|^{k+1+\beta_1+\beta_2}}\right) \ll_k N^{k+2},$$

using standard zero-density estimates and (2.4). On the other hand, for $y > 0$ we split the range of integration into $(0, 1/N] \cup (1/N, +\infty)$. On the first interval we have

$$\sum_{\substack{\rho_1 : \gamma_1 > 0 \\ \rho_2 : \gamma_2 > 0}} \gamma_1^{\beta_1 - 1/2} \gamma_2^{\beta_2 - 1/2} \times \int_0^{1/N} \exp\left((\gamma_1 + \gamma_2) \left(\arctan(Ny) - \frac{\pi}{2}\right)\right) \frac{dy}{|z|^{k+1+\beta_1+\beta_2}} \ll_k N^{k+2},$$

arguing as above. With similar computations and a substitution, on the other interval we have a contribution

$$N^k \sum_{\rho_1 : \gamma_1 > 0} N^{\beta_1} \gamma_1^{\beta_1 - 1/2} \sum_{\rho_2 : \gamma_2 > 0} N^{\beta_2} \gamma_2^{\beta_2 - 1/2} \times \int_1^{+\infty} \exp\left(-(\gamma_1 + \gamma_2) \arctan \frac{1}{u}\right) \frac{du}{u^{k+1+\beta_1+\beta_2}}.$$

Arguing as in the proof of Lemma 4.2, the integral on the right is as follows:

$$\int_1^{+\infty} \exp\left(-(\gamma_1 + \gamma_2) \arctan \frac{1}{u}\right) \frac{du}{u^{k+1+\beta_1+\beta_2}} \asymp (\gamma_1 + \gamma_2)^{-k-\beta_1-\beta_2}.$$

The inequality

$$\frac{\gamma_1^{\beta_1 - 1/2} \gamma_2^{\beta_2 - 1/2}}{(\gamma_1 + \gamma_2)^{\beta_1 + \beta_2}} \leq \frac{1}{\gamma_1^{1/2} \gamma_2^{1/2}} \tag{6.4}$$

shows that it is sufficient to consider

$$N^k \sum_{\rho_1 : \gamma_1 > 0} \sum_{\rho_2 : \gamma_2 > 0} N^{\beta_1 + \beta_2} \frac{1}{\gamma_1^{1/2} \gamma_2^{1/2} (\gamma_1 + \gamma_2)^k} \ll N^{k+2} \sum_{\rho_1 : \gamma_1 > 0} \frac{\log \gamma_1}{\gamma_1^k}$$

and the last series over zeros converges for $k > 1$.

Assume now $\gamma_1 > 0, \gamma_2 < 0$. For $y \leq 0$ we have

$$\gamma_1 \arctan(Ny) - \frac{\pi}{2}\gamma_1 \leq -\frac{\pi}{2}\gamma_1,$$

and by (2.4) the corresponding contribution to (6.3) is

$$\begin{aligned} &\ll_k \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp\left(-\frac{\pi}{2}\gamma_1\right) \\ &\quad \times \left\{ \sum_{\rho_2: \gamma_2 < 0} |\gamma_2|^{\beta_2 - 1/2} \left[\exp\left(-\frac{\pi}{4}|\gamma_2|\right) \int_{-1/N}^0 N^{k+1+\beta_1+\beta_2} dy \right. \right. \\ &\quad \left. \left. + \int_{-\infty}^{-1/N} \exp\left(-|\gamma_2|\left(\arctan(Ny) + \frac{\pi}{2}\right)\right) \frac{dy}{|y|^{k+1+\beta_1+\beta_2}} \right] \right\} \\ &\ll_k N^{k+2} \end{aligned}$$

for $k > 1/2$, by Lemma 4.2 and standard zero-density estimates.

On the other hand, the case $\gamma_1 > 0, \gamma_2 < 0$ and $y > 0$ can be estimated in a similar way essentially exchanging the role of γ_1 and γ_2 in the previous argument.

This implies that the integral in (6.3) is $\ll_k N^{k+2}$ provided that $k > 1$. Combining the convergence conditions for (6.1)–(6.3), we see that we can exchange both summations with the integral provided that $k > 1$.

7 Convergence of the double sum over zeros

In this section we prove that the double sum on the right of (1.1) converges absolutely for every $k > 1/2$. We need (4.2) uniformly for $x \in [0, k + 3]$ and $|y| \geq T$, where T is large but fixed: this provides both an upper and a lower bound for $|\Gamma(x + iy)|$. Let

$$\Sigma = \sum_{\rho_1} \sum_{\rho_2} \left| \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1 + \rho_2 + k + 1)} \right|,$$

so that, by the symmetry of the zeros of the Riemann zeta-function, we have

$$\begin{aligned} \Sigma &= 2 \sum_{\substack{\rho_1: \gamma_1 > 0 \\ \rho_2: \gamma_2 > 0}} \left| \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1 + \rho_2 + k + 1)} \right| + 2 \sum_{\substack{\rho_1: \gamma_1 > 0 \\ \rho_2: \gamma_2 > 0}} \left| \frac{\Gamma(\rho_1)\Gamma(\bar{\rho}_2)}{\Gamma(\rho_1 + \bar{\rho}_2 + k + 1)} \right| \\ &= 2(\Sigma_1 + \Sigma_2), \end{aligned}$$

say. It is clear that if both Σ_1 and Σ_2 converge, then the double sum on the right-hand side of (1.1) converges absolutely. In order to estimate Σ_1 we choose

a large T and let

$$\begin{aligned} D_0 &= \{(\rho_1, \rho_2) : (\gamma_1, \gamma_2) \in [0, 2T]^2\}, \\ D_1 &= \{(\rho_1, \rho_2) : \gamma_1 \geq T, T \leq \gamma_2 \leq \gamma_1\}, \\ D_2 &= \{(\rho_1, \rho_2) : \gamma_1 \geq T, 0 \leq \gamma_2 \leq T\}, \\ D_3 &= \{(\rho_1, \rho_2) : \gamma_2 \geq T, T \leq \gamma_1 \leq \gamma_2\}, \\ D_4 &= \{(\rho_1, \rho_2) : \gamma_2 \geq T, 0 \leq \gamma_1 \leq T\}, \end{aligned}$$

so that $\Sigma_1 \leq \Sigma_{1,0} + \Sigma_{1,1} + \Sigma_{1,2} + \Sigma_{1,3} + \Sigma_{1,4}$, say, where $\Sigma_{1,j}$ is the sum with $(\rho_1, \rho_2) \in D_j$. Now, D_0 contributes a bounded amount, that depends only on T , and, by symmetry again, $\Sigma_{1,1} = \Sigma_{1,3}$ and $\Sigma_{1,2} = \Sigma_{1,4}$. We also recall inequality (6.4) which is valid for all couples of zeros considered in Σ_1 . Hence by (4.2) we get

$$\begin{aligned} \Sigma_{1,1} &\ll \sum_{\substack{\rho_1 : \gamma_1 \geq T \\ \rho_2 : T \leq \gamma_2 \leq \gamma_1}} \sum \frac{1}{\gamma_1^{1/2} \gamma_2^{1/2} (\gamma_1 + \gamma_2)^{k+1/2}} \\ &\ll \sum_{\rho_1 : \gamma_1 \geq T} \frac{1}{\gamma_1^{k+1}} \sum_{\rho_2 : T \leq \gamma_2 \leq \gamma_1} \frac{1}{\gamma_2^{1/2}} \ll \sum_{\rho_1 : \gamma_1 \geq T} \frac{\log \gamma_1}{\gamma_1^{k+1/2}}. \end{aligned}$$

A similar argument proves that

$$\Sigma_{1,2} \ll_{k,T} \sum_{\rho_1 : \gamma_1 \geq T} \gamma_1^{-k-1},$$

since $\Gamma(\rho_2)$ is uniformly bounded, in terms of T , for $(\rho_1, \rho_2) \in D_2$. Summing up, we have

$$\Sigma_1 \ll_{k,T} 1 + \sum_{\rho_1 : \gamma_1 \geq T} \frac{\log \gamma_1}{\gamma_1^{k+1/2}},$$

which is convergent provided that $k > 1/2$. In order to estimate Σ_2 we use a similar argument. Choose a large T and let

$$\begin{aligned} E_0 &= \{(\rho_1, \rho_2) : (\gamma_1, \gamma_2) \in [0, 2T]^2\}, \\ E_1 &= \{(\rho_1, \rho_2) : \gamma_1 \geq 2T, 0 \leq \gamma_2 \leq T\}, \\ E_2 &= \{(\rho_1, \rho_2) : \gamma_1 \geq 2T, T \leq \gamma_2 \leq \gamma_1 - T\}, \\ E_3 &= \{(\rho_1, \rho_2) : \gamma_1 \geq 2T, \gamma_1 - T \leq \gamma_2 \leq \gamma_1\}, \\ E_4 &= \{(\rho_1, \rho_2) : \gamma_2 \geq 2T, \gamma_2 - T \leq \gamma_1 \leq \gamma_2\}, \\ E_5 &= \{(\rho_1, \rho_2) : \gamma_2 \geq 2T, T \leq \gamma_1 \leq \gamma_2 - T\}, \\ E_6 &= \{(\rho_1, \rho_2) : \gamma_2 \geq 2T, 0 \leq \gamma_1 \leq T\}, \end{aligned}$$

so that $\Sigma_2 \leq \Sigma_{2,0} + \Sigma_{2,1} + \Sigma_{2,2} + \Sigma_{2,3} + \Sigma_{2,4} + \Sigma_{2,5} + \Sigma_{2,6}$, say, where $\Sigma_{2,j}$ is the sum with $(\rho_1, \rho_2) \in E_j$. Now, E_0 contributes a bounded amount, that depends only on T , and, by symmetry again, we have $\Sigma_{2,1} = \Sigma_{2,6}$, $\Sigma_{2,2} = \Sigma_{2,5}$ and $\Sigma_{2,3} = \Sigma_{2,4}$. We use (4.2) as above; hence

$$\Sigma_{2,2} \ll \sum_{\substack{\rho_1: \gamma_1 \geq 2T \\ \rho_2: T \leq \gamma_2 \leq \gamma_1 - T}} \frac{\gamma_1^{\beta_1 - 1/2} \gamma_2^{\beta_2 - 1/2} e^{-\pi \gamma_2}}{(\gamma_1 - \gamma_2)^{\beta_1 + \beta_2 + k + 1/2}} \ll_T \sum_{\rho_1: \gamma_1 \geq 2T} \frac{\log \gamma_1}{\gamma_1^{k+1/2}}.$$

The rightmost series over zeros plainly converges for $k > 1/2$. The contribution of zeros in E_1 is treated in a similar fashion, using the uniform upper bound $\Gamma(\rho_2) \ll_T 1$, and is smaller. We now deal with $\Sigma_{2,3}$: by (4.2) we have

$$\begin{aligned} \Sigma_{2,3} &\ll \sum_{\rho_1: \gamma_1 \geq 2T} e^{-\pi \gamma_1/2} \gamma_1^{\beta_1 - 1/2} \\ &\quad \times \sum_{\rho_2: \gamma_1 - T \leq \gamma_2 \leq \gamma_1} e^{-\pi \gamma_2/2} \gamma_2^{\beta_2 - 1/2} \left(\min_{\substack{k+1 \leq x \leq k+3 \\ 0 \leq t \leq T}} |\Gamma(x + it)| \right)^{-1} \\ &\ll_{k,T} \sum_{\rho_1: \gamma_1 \geq 2T} e^{-\pi \gamma_1} \gamma_1^{\beta_1 + 1} \log(\gamma_1 + T), \end{aligned}$$

provided that T is large enough. Here we are using [14, Theorem 9.2] with T large but fixed. The series at the extreme right is plainly convergent.

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