REAL AND COMPLEX SYMPLECTIC STRUCTURES. APPLICATION TO CONCENTRATION IN DEGREE FOR MICROFUNCTIONS AT THE BOUNDARY

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§0. Introduction.

Let X be a complex analytic manifold, let $M \subset X$ be a real analytic hypersurface, and denote by σ the symplectic form on the cotangent bundle $\pi : T^*X \to X$. Assume that the conormal bundle $T^*_M X$ is symplectic with respect to the imaginary part of σ.

In this paper, we discuss the link between the (real) symplectic form $\sigma|_{T^*_M X}$, and the classical Levi form L_M on the complex tangent plane $T^{\mathbf{C}}M = TM \cap iTM$. In particular, we prove that if M is the boundary of a strictly pseudoconvex (or pseudoconcave) domain, and if S is a submanifold of M such that $\Sigma = S \times_M T^*_M X$ is regular involutive in $T^*_M X$, then S is generic in X.

Define $\widetilde{\Sigma}$ to be the union of the complexifications of the bicharacteristic leaves of Σ. For $p \in \Sigma$, set $\lambda_S(p) = T_p T^*_S X$, $\lambda_0(p) = T_p \pi^{-1}\pi(p)$. In §2 we show the relationship between $\lambda_S \cap i\lambda_S$ and the null-space of $L_M|_{T^{\mathbf{C}}S}$. We then show that:

$$\dim_{\mathbf{R}}(T\widetilde{\Sigma} \cap \lambda_0) = 1 + \dim_{\mathbf{C}}(\lambda_S \cap i\lambda_S).$$

Denote by $s^{\pm}(M)$ (resp. $s^{\pm}(S)$) the numbers of positive and negative eigenvalues of L_M (resp. $L_M|_{T^{\mathbf{C}}S}$), and set $\gamma(S) = \dim_{\mathbf{C}}(\lambda_S \cap i\lambda_S \cap \lambda_0)$. Denote by $\mu_W(\mathcal{O}_X)$ the Sato microlocalization of the sheaf $\mathcal{O}_X$ of holomorphic functions along a real submanifold $W \subset X$. In §3 we assume that the dimension of $\lambda_S \cap i\lambda_S$ is constant respect to p. Hence $\widetilde{\Sigma} = T^*_W X$ for a suitable submanifold W of X, and we show that $\mu_W(\mathcal{O}_X)$ is concentrated in degree $d = 1 + \dim_{\mathbf{C}}(\lambda_S \cap i\lambda_S) + s^-(S) - \gamma(S)$. When $s^-(M) - s^-(S) = \dim_{\mathbf{C}}(\lambda_S \cap i\lambda_S) - \gamma(S)$, we also show that there is a natural morphism $\mu_W(\mathcal{O}_X) \to \mu_M(\mathcal{O}_X)$ which induces an injective morphism between the cohomology groups in degree d. This is, in a generalized sense, a theorem of concentration in degree for microfunctions at the boundary.

§1. Symplectic forms and Levi forms.

Let X be a complex manifold of dimension n, $\pi : T^*X \to X$ the cotangent bundle, ∂ the holomorphic differential on X. Let $X^{\mathbf{R}}$ denote the real underlying manifold to X and $d = \partial + \bar{\partial}$ its real differential. We shall always identify $T^*(X^{\mathbf{R}}) \xrightarrow{\sim} (T^*X)^{\mathbf{R}}$. (If ϕ is a C^1 function on $X^{\mathbf{R}}$, the identification is given by $d\phi(x) \mapsto \partial\phi(x)$.) Let ω be the (complex) canonical 1-form on T^*X, $\sigma = \partial\omega$ the canonical

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2-form, H: $T^*T^*X \to TT^*X$ the corresponding Hamiltonian isomorphism. We shall consider the induced one-forms $\omega^{\mathbf{R}} = 2\operatorname{Re}\omega$, $\omega^I = 2\operatorname{Im}\omega$ and two-forms $\sigma^{\mathbf{R}} = 2\operatorname{Re}\sigma$, $\sigma^I = 2\operatorname{Im}\sigma$ on $T^*X^{\mathbf{R}}$.

Let M be a real analytic hypersurface of X defined by the equation $\phi(z) = 0$ at $z_0 \in M$. Let $p = (z_0, \partial\phi(z_0))$, and set:

$$\begin{aligned} T_{z_0}M &= \{v \in T_{z_0}X; \operatorname{Re}\langle v, \partial\phi\rangle = 0\} \\ T^{\mathbf{C}}_{z_0}M &= T_{z_0}M \cap iT_{z_0}M = \{v \in T_{z_0}X; \langle v, \partial\phi\rangle = 0\} \\ \lambda_M(p) &= T_pT^*_MX \\ \lambda_0(p) &= T_p\pi^{-1}\pi(p) \\ \nu(p) &= \mathbf{C}\,\mathrm{H}(\omega(p)). \end{aligned}$$

The morphism:

$$\begin{aligned} M &\to T^*_MX \\ z &\mapsto (z, \partial\phi(z)). \end{aligned}$$

induces the morphism

$$\begin{aligned} \psi : T_{z_0}M &\to \lambda_M(p) \\ v &\mapsto (v, \partial\langle\partial\phi, v\rangle + \partial\langle\bar\partial\phi, \bar v\rangle) \end{aligned}$$

by which $T^{\mathbf{C}}_{z_0}M$ is identified to $\{(v, \partial\langle\bar\partial\phi, \bar v\rangle);\ \langle\partial\phi, v\rangle = 0\}$. Let L_M be the Levi form of M at p. Recall that, if $(z_1, \ldots, z_n)$ is a local system of coordinates at z_0, L_M is the Hermitian form on $T^{\mathbf{C}}_{z_0}M$ represented by the matrix $(\partial_i\bar\partial_j\phi(z_0))_{i,j}$. One immediately checks that:

$$\psi^*(\sigma|_{\psi(T^{\mathbf{C}}_{z_0}M)}) = L_M.$$

Let S be a real analytic submanifold of M with $\operatorname{codim}_X M = 1$, $\operatorname{codim}_M S = r$. We shall set:

$$\gamma(S, p) = \dim_{\mathbf{C}}(\lambda_S(p) \cap i\lambda_S(p) \cap \lambda_0(p)).$$

One says that $S \times_M T^*_MX$ is regular when $\omega|_{S\times_M T^*_MX}\ (= \frac{i}{2}\omega^I|_{S\times_M T^*_MX}) \neq 0$.

Proposition 1.1. *The following assertions are equivalent:*

(1.1) $S \times_M T^*_MX$ *is regular at* p*;*
(1.2) $\dim^{\mathbf{R}}(\lambda_S(p) \cap \nu(p)) = 1$.

Proof. Let S be locally defined by the equations $\phi_i = 0$ $(i = 1, ..., r+1)$, where $\phi_1 \cong \phi$ is a local equation for M, and recall that $p = (z_0, \partial\phi(z_0))$. We have:

$$\psi^*(\omega^I) = -\operatorname{Re}(i\,\partial\phi).$$

Thus:

$$\begin{aligned} S \times_M T^*_MX \quad \text{is regular} \quad &\Leftrightarrow \quad i\,\frac{\partial\phi}{\partial z}(z_0) \notin (T^*_SX)_{z_0} \\ &\Leftrightarrow \quad \dim_{\mathbf{R}}(\lambda_S(p) \cap \nu(p)) = 1. \end{aligned}$$

□

Remark 1.2. *Let codim*$_X S = 2$, *textcodim*$_X M = 1$. *Then* $S \times_M T^*_M X$ *is regular iff* $\gamma(S, p) = 0$ *(i.e. S is "generic").*

We shall assume from now on that

$$T^*_M X \quad \text{is } I\text{-symplectic} \tag{1.3}$$

i.e. that $\sigma^I|_{\lambda_M}$ is non-degenerate. This is equivalent to $\lambda_M(p) \cap i\lambda_M(p) = \{0\}$ or else to

$$L_M \qquad \text{is non-degenerate,}$$

(cf [S1]). In fact via ψ one may easily identify $\lambda_M(p) \cap i\lambda_M(p)$ with the null-space of L_M (cf [D'A-Z1]).

Corresponding to the non-degenerate forms σ^I and L_M we may thus consider two Hamiltonian isomorphisms H^I and $\widetilde{\mathrm{H}}^I$:

$$\begin{array}{ccc} T^*T^*_M X & \xrightarrow{\mathrm{H}^I} & TT^*_M X \\ \psi \downarrow & & \psi \uparrow \\ (T^{\mathbf{C}}M)^* & \xrightarrow{\widetilde{\mathrm{H}}^I} & T^{\mathbf{C}}M. \end{array}$$

Consider the morphism $\pi^*_M : T^*_{z_0} M \to T^*_p T^*_M X$ induced by the projection $\pi_M : T^*_M X \to M$.

Proposition 1.3. *For* $\theta \in T^*_{z_0} M$, *we have*

$$\mathrm{H}^I(\pi^*_M(\theta))|_{\psi(T^{\mathbf{C}}M)} = \psi(v) \tag{1.4}$$

(in the identification $\lambda_M(p) \tilde{\to} \lambda^*_M(p)$ *given by* $w \mapsto \sigma(w, \cdot)$*), where* $v \in T^{\mathbf{C}}_{z_0} M$ *is the unique solution of*

$$\partial\langle\bar\partial\phi, \bar v\rangle = \frac{i}{2}\theta|_{T^{\mathbf{C}}M} \tag{1.5}$$

(in the identification $T^{\mathbf{C}}M \tilde{\to} T^{\mathbf{C}}M^*$ *given by* $v \mapsto \langle v, \cdot\rangle$*).*

Proof. The vectors v which satisfy (1.4) must verify for every $u \in T^{\mathbf{C}}_{z_0} M$:

$$\begin{aligned} \frac{1}{2}[\langle\theta, u\rangle + \langle\bar\theta, \bar u\rangle] &= \sigma(\psi(v), \psi(u)) \\ &= -[i\langle\partial\langle\bar\partial\phi, \bar v\rangle, u\rangle - i\langle\partial\langle\bar\partial\phi, \bar u\rangle, v\rangle] \\ &= -[i\langle\partial\langle\bar\partial\phi, \bar v, \rangle, u\rangle + \overline{i\langle\partial\langle\bar\partial\phi, \bar v\rangle, u\rangle}], \end{aligned}$$

and hence:

$$\mathrm{Re}\langle\frac{i}{2}\theta, \cdot\rangle|_{T^{\mathbf{C}}M} = \mathrm{Re}\langle\partial\langle\bar\partial\phi, \bar v\rangle, \cdot\rangle|_{T^{\mathbf{C}}M}$$

Reasoning in the same way for iu we get the conclusion. $\square$

Let us remark now that (when (1.3) holds):

$$\dim_{\mathbf{R}}(\lambda_S(p) \cap \nu(p)) = 1 \quad \Leftrightarrow \quad i\,\mathrm{H}^I(\pi^*(p)) \notin \lambda_S(p),$$

where $\pi^* : T^*_{z_0} X \to T^*_p T^* X$ denotes the map associated to $\pi : T^* X \to X$. We may then correspondingly rephrase the equivalent conditions of Proposition 1.1. We also have

Proposition 1.4. *Let $S \subset M \subset X$ with $codim_X M = 1$ and with L_M being positive or negative definite at p. Then*

$$S \times_M T^*_M X \text{ is regular involutive} \quad \Rightarrow \quad \gamma(S,p) = 0. \tag{1.6}$$

Proof. Let S be locally defined by the equations $\phi_i = 0$ $(i = 1, ..., r+1)$, where $\phi_1 \cong \phi$ is a local equation for M. We have already seen that:

(1.7)
$$\begin{aligned} S \times_M T^*_M X \text{ is regular} \quad &\Leftrightarrow \quad i\,\partial\phi \notin \mathbf{R}\partial\phi_1 + \cdots + \mathbf{R}\partial\phi_{r+1} \\ &\Leftrightarrow \quad \partial\phi_2|_{T^{\mathbf{C}}M}, \ldots, \partial\phi_{r+1}|_{T^{\mathbf{C}}M} \text{ are } \mathbf{R}\text{-independent.} \end{aligned}$$

Let v_i solve (1.4) for $\theta_i = \partial\phi_i$. Let χ and ψ be functions on M which vanish on S and let $u, v \in \underset{i}{\oplus}\mathbf{R}v_i$ be such that: $\psi(u) = \mathrm{H}^I(\pi^*_M(d\chi))$, $\psi(v) = \mathrm{H}^I(\pi^*_M(d\psi))$. Owing to Proposition 1.3, we have:

$$\{\chi, \psi\}_I = -i\, L_M(\bar{u} \wedge v),$$

($\{\cdot,\cdot\}_I$ denoting the Poisson bracket). Hence:

$$\{\chi,\psi\}_I = 0\ \forall \chi,\psi \text{ (i.e. } S \times_M T^*_M X \text{ is involutive)} \quad \Rightarrow \quad u \neq i\,v\ \forall u, v. \tag{1.8}$$

(In fact if $v = i\,u$, then $\sigma(\mathrm{H}\,u, \mathrm{H}\,v) = i\,\sigma^I(\mathrm{H}\,u, i\,\mathrm{H}\,u) = -2i\,L_M(\bar{u} \wedge u) \neq 0$ since L_M definite (> 0 or < 0).) We also remark that

$$v \mapsto \langle \partial\langle\bar{\partial}\phi, \bar{v}\rangle, \cdot\rangle|_{T^{\mathbf{C}}M} \quad \text{is injective,}$$

due to (1.4). By (1.7), (1.8), this implies that $\partial\phi_i|_{T^{\mathbf{C}}M}$, $i \geq 2$ are $\mathbf{C}$-independent. This is in turn equivalent to the fact that $\partial\phi_i$, $i \geq 1$ are $\mathbf{C}$-independent.

§2. Degenerate R-Lagrangian submanifolds.

Let X be a complex manifold of dimension n, and let $\bar{X}$ be the complex conjugate manifold to X. We shall identify $X^{\mathbf{R}}$ to the diagonal $X \times_X \bar{X}$. The manifold $X \times \bar{X}$ is then a natural complexification of $X^{\mathbf{R}}$.

Let M be a C^ω-hypersurface of $X^{\mathbf{R}}$ defined at $z_0 \in M$ by the equation $\phi(z) = 0$, and set $p = (z_0, \partial\phi(z_0))$. We shall identify $\mathbf{C} \otimes_{\mathbf{R}} T_{z_0}M$ with $\{(u, \bar{v}) \in T_{z_0}X \times T_{z_0}\bar{X}; \langle\partial\phi, u\rangle + \langle\bar{\partial}\phi, \bar{v}\rangle = 0\}$ and consider:

$$\begin{aligned} \psi^{\mathbf{C}}:\ \mathbf{C} \otimes_{\mathbf{R}} T_{z_0}M &\to \lambda_M(p) + i\,\lambda_M(p) \\ (u, \bar{v}) &\mapsto (u; \partial\langle\partial\phi, u\rangle + \partial\langle\bar{\partial}\phi, \bar{v}\rangle). \end{aligned} \tag{2.1}$$

Let $S \subset M$ be a submanifold with $\mathrm{codim}_M S = r$. Consider the null space:

$$\mathrm{NS}(L_M|_{T^{\mathbf{C}}_{z_0}S}) = \{v \in T^{\mathbf{C}}_{z_0}S; \partial\langle\bar{\partial}\phi, \bar{v}\rangle \in \pi^*((T^*_S X)_{z_0})\}.$$

Using $TX \hookrightarrow TX \times_{TX} T\bar{X}$ and ψ, we get

Proposition 2.1. *We have an identification*

$$\lambda_S(p) \cap i\lambda_S(p) \cong NS(L_M|_{T^{\mathbf{C}}_{z_0}S}) \oplus (\lambda_S(p) \cap i\lambda_S(p) \cap \lambda_0(p)). \tag{2.2}$$

Proof. We shall often omit the indices z_0 and p in the following. The projection $\mathcal{R} : H\pi^*(T^*_S X + iT^*_S X) \longrightarrow H\pi^*(T^*_S X)$ is well defined modulo $\lambda_S \cap i\lambda_S \cap \lambda_0$. It is easy to check that (cf also [D'A-Z1]):

$$\frac{\lambda_S \cap i\lambda_S}{\lambda_S \cap i\lambda_S \cap \lambda_0} = \{\psi^{\mathbf{C}}(v) - 2\mathcal{R}(\partial\langle\bar\partial\phi, \bar v\rangle);\ v \in T^{\mathbf{C}}S, \partial\langle\bar\partial\phi, \bar v\rangle \in H\pi^*(T^*_S X + iT^*_S X)\}.$$

Then (2.2) follows. □

We suppose all through this section that $T^*_M X$ is I-symplectic and that $S \times_M T^*_M X$ is a regular involutive submanifold of $T^*_M X$. We also set $\Sigma = S \times_M T^*_M X$, and define $\widetilde\Sigma$ to be the union of the complexifications of the bicharacteristic leaves of Σ. This is a germ of $\mathbf{R}$-Lagrangian submanifold of $T^*X^{\mathbf{R}}$.

Proposition 2.2. *We have an identification*

$$T_p\widetilde\Sigma \cap iT_p\widetilde\Sigma \cong \psi(\{v \in T_{z_0}S + iT_{z_0}S;\ \partial\langle\bar\partial\phi, \bar v\rangle \in H\pi^*(T^*_S X_{z_0} + iT^*_S X_{z_0})\}). \tag{2.3}$$

Proof. We have by definition

$$T\widetilde\Sigma = T\Sigma + iT\Sigma^\perp, \tag{2.4}$$

(where $\cdot^\perp$ denotes the symplectic orthogonal in $(T^*_M X,\ \sigma^I)$.) But

$$T\Sigma^\perp = \psi(\{u \in T^{\mathbf{C}}M \cap TS;\ \partial\langle\bar\partial\phi, \bar u\rangle|_{T^{\mathbf{C}}M} \in iH\pi^*(T^*_S X)|_{T^{\mathbf{C}}M}\}). \tag{2.5}$$

Then (2.3) follows immediately. □

Proposition 2.3. *Under the above assumptions, we have*

$$\dim_{\mathbf{C}}(T_p\widetilde\Sigma \cap iT_p\widetilde\Sigma) = r. \tag{2.6}$$

Proof. We have $T\Sigma \cap iT\Sigma = 0$ (due to the fact that $T^*_M X$ is I-symplectic). It follows:

$$\begin{aligned} T\widetilde\Sigma \cap iT\widetilde\Sigma &= (T\Sigma + iT\Sigma^\perp) \cap (T\Sigma^\perp + iT\Sigma) \\ &= T\Sigma^\perp + iT\Sigma^\perp = \mathbf{C} \otimes_{\mathbf{R}} T\Sigma^\perp. \end{aligned}$$

□

For $v \in T\widetilde\Sigma + iT\widetilde\Sigma$ (resp $v \in T\Sigma + iT\Sigma$, resp $v \in \lambda_S + i\lambda_S$) let us denote by $v^{c_{T\widetilde\Sigma}}$ (resp $v^{c_{T\Sigma}}$, resp $v^{c_{\lambda_S}}$) the "conjugate" with respect to $T\widetilde\Sigma$ (resp $T\Sigma$, resp λ_S). This is well defined modulo $T\widetilde\Sigma \cap iT\widetilde\Sigma = (T\widetilde\Sigma + iT\widetilde\Sigma)^\perp$ (resp $T\Sigma \cap iT\Sigma = \{0\}$, resp $\lambda_S \cap i\lambda_S = (\lambda_S + i\lambda_S)^\perp$). We remark now that $T\widetilde\Sigma + iT\widetilde\Sigma = T\Sigma + iT\Sigma$ and thus

$$\sigma(v, w^{c_{T\widetilde\Sigma}})|_{v,w \in (T\widetilde\Sigma + iT\widetilde\Sigma)\cap\lambda_0} \sim \sigma(v, w^{c_{T\Sigma}})|_{v,w\in(T\Sigma+iT\Sigma)\cap\lambda_0} \tag{2.7}$$

(where "$\sim$" means equivalent in signature and rank). On the other hand:

$$\begin{cases} \psi^{\mathbf{C}}(\overline{T^{\mathbf{C}}S}) + (\lambda_S \cap \lambda_0)^{\mathbf{C}} = (\lambda_S + i\lambda_S) \cap \lambda_0 \\ \psi^{\mathbf{C}}(\overline{T^{\mathbf{C}}S}) \cap (\lambda_S \cap \lambda_0)^{\mathbf{C}} = \lambda_S \cap i\lambda_S \cap \lambda_0 \end{cases}$$

and:

$$(T\Sigma + iT\Sigma) \cap \lambda_0 = \psi^{\mathbf{C}}(\overline{T^{\mathbf{C}}S}) \oplus (\lambda_S \cap \lambda_0)^{\mathbf{C}}$$

(where $\cdot^{\mathbf{C}} = \cdot + i\cdot$ and where we make the identification $\overline{T^{\mathbf{C}}S} \hookrightarrow \{0\} \times T\bar{X}$). It follows:

$$\sigma(v, w^{c_{T\Sigma}})|_{v,w\in(T\Sigma+iT\Sigma)\cap\lambda_0} \sim \sigma(v, w^{c_{\lambda_S}})|_{v,w\in(\lambda_S+i\lambda_S)\cap\lambda_0}. \tag{2.8}$$

Let τ denote the inertia index for a triple of Lagrangian planes in the sense of [K-S1] and [K-S2]. We recall that $\frac{1}{2}\tau(T\widetilde{\Sigma}, iT\widetilde{\Sigma}, \lambda_0)$ and $n - \dim_{\mathbf{R}}(T\widetilde{\Sigma} \cap \lambda_0) + 2\dim_{\mathbf{C}}(T\widetilde{\Sigma} \cap iT\widetilde{\Sigma} \cap \lambda_0) - \dim_{\mathbf{C}}(T\widetilde{\Sigma} \cap iT\widetilde{\Sigma})$ are respectively the signature and the rank of $\sigma(v, w^{c_{T\widetilde{\Sigma}}})|_{v,w\in(T\widetilde{\Sigma}+iT\widetilde{\Sigma})\cap\lambda_0}$. In the same way $\frac{1}{2}\tau(\lambda_S, i\lambda_S, \lambda_0)$ and $n - 1 - r + 2\gamma(S) - \dim_{\mathbf{C}}(\lambda_S \cap i\lambda_S)$ are signature and rank of $\sigma(v, w^{c_{\lambda_S}})|_{v,w\in\lambda_S\cap i\lambda_S\cap\lambda_0}$ due to [D'A-Z1]. By (2.7), (2.8) the above signatures and ranks have to coincide. In particular

$$\begin{aligned} &\dim_{\mathbf{R}}(T\widetilde{\Sigma} \cap \lambda_0) + \dim_{\mathbf{C}}(T\widetilde{\Sigma} \cap iT\widetilde{\Sigma}) - 2\dim_{\mathbf{C}}(T\widetilde{\Sigma} \cap iT\widetilde{\Sigma} \cap \lambda_0) \\ &\qquad = 1 + r + \dim_{\mathbf{C}}(\lambda_S \cap i\lambda_S) - 2\gamma(S). \end{aligned} \tag{2.9}$$

We also notice that

$$\begin{aligned} T\widetilde{\Sigma} \cap iT\widetilde{\Sigma} \cap \lambda_0 &= H\pi^*\left(T^*_S X \cap iT^*_S X\right)|_{T^{\mathbf{C}}M} \\ &\cong H\pi^*\left(T^*_S X \cap iT^*_S X\right) \\ &= \lambda_S \cap i\lambda_S \cap \lambda_0. \end{aligned} \tag{2.10}$$

Thus by using (2.10) and (2.6) in (2.9) we get at once the following statement.

Theorem 2.4. *We have*

$$\dim_{\mathbf{R}}(T_p\widetilde{\Sigma} \cap \lambda_0) = 1 + \dim_{\mathbf{C}}(\lambda_S(p) \cap i\lambda_S(p)). \tag{2.11}$$

§3. A vanishing theorem for generalized microfunctions at the boundary.

Let X be an open set of $\mathbf{C}^n$, and let M be a C^ω-hypersurface of X. We denote by $\mathrm{D}^b(X)$ the derived category of the category of complexes of sheaves with bounded cohomology. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X and we will consider its microlocalization along M, $\mu_M(\mathcal{O}_X)$ (μ_M being the Sato microlocalization functor). For any C^ω-submanifold W of $X^{\mathbf{R}}$, we shall similarly define $\mu_W(\mathcal{O}_X)$.

Let M be defined in local coordinates by $\phi(z) = 0$ and let $p = (z_0, \partial\phi(z_0))$ with $z_0 \in M$. Let $s^\pm(M,p)$ denote the numbers of positive and negative eigenvalues for the Levi form L_M. Let S be a C^ω-submanifold of M with $\mathrm{codim}_M S = r$. We shall denote by $s^\pm(S,p)$ the corresponding numbers of eigenvalues for $L_M|_{T^{\mathbf{C}}S}$ (cf [D'A-Z1]). Set now $\Lambda = T^*_M X$, $\Sigma = S \times_M T^*_M X$ and assume that Λ is I-symplectic and Σ is regular involutive in Λ. Let us recall that the latter hypothesis implies that $\dim_{\mathbf{R}}(\lambda_S(p)\cap\nu(p))=1$ (recall that ν denotes the Euler vector field), and moreover, if M is the boundary of a strictly pseudoconvex or pseudoconcave domain, it implies that $\gamma(S,p):=\dim_{\mathbf{C}}(\lambda_S(p)\cap i\lambda_S(p)\cap\lambda_0(p))=0$.

Define $\widetilde{\Sigma}$ to be the union of the complexifications of the bicharacteristic leaves of Σ. Let us assume

$$\delta(S,p):=\dim_{\mathbf{C}}(\lambda_S(p)\cap i\lambda_S(p))\quad\text{is constant with respect to } p. \tag{3.1}$$

Let us remark that if (3.1) holds, then by (2.11) there exists W with $S\subset W\subset X$ and with $\operatorname{codim}_X W=1+\delta(S,p)$ such that $\widetilde{\Sigma}=T^*_W X$.

We also know from [K-S1, Proposition 11.3.5] that

$$\mu_M(\mathcal{O}_X)\quad\text{is concentrated in degree } 1+s^-(M,p). \tag{3.2}$$

(In fact the assumptions of that Proposition are fulfilled since, $T^*_M X$ being I-symplectic, $s^-(M,p)$ is constant.) Let $S\subset M\subset X$, $\operatorname{codim}_X M=1$, $\operatorname{codim}_M S=r$. Our result goes as follows.

Theorem 3.1. *Let $T^*_M X$ be I-symplectic, let $S\times_M T^*_M X$ be regular involutive in $T^*_M X$, and assume (3.1) to hold. Then*

$$H^j\mu_W(\mathcal{O}_X)=0\quad\textit{for } j\neq 1+\delta(S,p)+s^-(S,p)-\gamma(S,p). \tag{3.3}$$

Proof. (We shall often omit the indices p or $z_0=\pi(p)$ in the following.) We perform a contact transformation χ in T^*X such that

$$\begin{cases} T^*_M X\to T^*_{M'}X & \operatorname{codim}_X M'=1,\quad s^-(M')=0\\ \Sigma\to S'\times_{M'}T^*_{M'}X & \operatorname{codim}_{M'}S'=r,\ \gamma(S')=0\\ T^*_W X(=\widetilde{\Sigma})\to T^*_{W'}X & \operatorname{codim}_X W'=1.\end{cases} \tag{3.4}$$

(The fact that $\gamma(S')=0$ in the second line follows from Proposition 1.4. This, together with $\mathrm{NS}(L_{M'})|_{T^{\mathbf{C}}S'}=\{0\}$ (cf §2), gives $\delta(S')=0$ due to Proposition 2.1. This implies $\operatorname{codim}W'=1$ in the third line of (3.4), due to Theorem 2.4.) Let $s^0(S):=\delta(S)-\gamma(S)$; thus $s^+(S)+s^-(S)+s^0(S)=n-1-r-\gamma(S)=\operatorname{codim}^{\mathbf{C}}_{T^{\mathbf{C}}M}T^{\mathbf{C}}S$. We have

$$s^+(W')\equiv n-1-r, s^0(W')\equiv r \tag{3.5}$$

(since $\delta(W')=\dim_{\mathbf{C}}(\widetilde{\Sigma}\cap i\widetilde{\Sigma})=r$). In particular $s^-(W')=0$ and hence $T^*_{W'}X$ is the (exterior) conormal to the boundary of a (weakly) pseudoconvex domain and therefore by the results of [H1]:

$$\mu_{W'}(\mathcal{O}_X)\quad\text{is concentrated in degree 1.} \tag{3.6}$$

By (3.5) we have $\dfrac{1}{2}\tau(\lambda_{W'},i\lambda_{W'},\lambda_0)=n-1-r$. We recall from §2:

$$\begin{aligned}\sigma(v,w^{c_{\lambda_W}})|_{v,w\in(\lambda_W+i\lambda_W)\cap\lambda_0}&\sim\sigma(v,w^{c_{T\Sigma}})|_{(T\Sigma+iT\Sigma)\cap\lambda_0}\\&\sim\sigma(v,w^{c_{\lambda_S}})|_{(\lambda_S+i\lambda_S)\cap\lambda_0},\end{aligned} \tag{3.7}$$

$$\dim_{\mathbf{C}}(T\widetilde{\Sigma}\cap iT\widetilde{\Sigma})=r, \tag{3.8}$$

$$\dim_{\mathbf{C}}(T\widetilde{\Sigma}\cap iT\widetilde{\Sigma}\cap\lambda_0)=\gamma(S). \tag{3.9}$$

In particular $s^{\pm}(W) = s^{\pm}(S)$. It follows

$$\begin{aligned}\frac{1}{2}\tau(\lambda_W, i\lambda_W, \lambda_0) &= s^+(W) + s^-(W) - 2s^-(W) \\ &= n - 1 - \delta(S) - r + 2\gamma(S) - 2s^-(S).\end{aligned} \tag{3.10}$$

Let

$$\begin{aligned}d_{W,W'} = \frac{1}{2}[\operatorname{codim} W' - \operatorname{codim} W &- \frac{1}{2}\tau(\lambda_{W'}, i\lambda_{W'}, \lambda_0) \\ &+ \frac{1}{2}\tau(\lambda_W, i\lambda_W, \lambda_0)].\end{aligned} \tag{3.11}$$

We know from [K-S1, Proposition 11.2.8] that the contact transformation χ can be quantized to an isomorphism

$$\chi_*(\mu_W(\mathcal{O}_X)) \cong \mu_{W'}(\mathcal{O}_X)[d_{W,W'}]. \tag{3.12}$$

But

$$\begin{aligned}d_{W,W'} &= \frac{1}{2}[1 - (1 + \delta(S)) - (n - 1 - r) + (n - 1 - \delta(S) + 2\gamma(S) - 2s^-(W)] \\ &= -\delta(S) + \gamma(S) - s^-(S)\end{aligned}$$

□

Lemma 3.2. *The following are equivalent*

$$s^-(M, p) - s^-(S, p) = \delta(S) - \gamma(S, p) \tag{3.13}$$

$$s^+(M, p) - s^+(S, p) = r - \gamma(S, p). \tag{3.14}$$

Proof. We have $(s^+(M) + s^-(M)) - (s^+(S) + s^-(S)) = r + \delta(S) - 2\gamma(S)$. □

Let $S \subset M \subset X$, $\operatorname{codim}_X M = 1$, $\operatorname{codim}_M S = r$.

Proposition 3.3. *Let $T^*_M X$ be I-symplectic, let $S \times_M T^*_M X$ be regular involutive in $T^*_M X$, let (3.1) hold, and let*

$$s^-(M, p) - s^-(S, p) = \delta(S, p) - \gamma(S, p). \tag{3.15}$$

We may then define a natural morphism:

$$\mu_W(\mathcal{O}_X) \to \mu_M(\mathcal{O}_X). \tag{3.16}$$

Proof. We may find a contact transformation on T^*X

$$\begin{cases} T^*_M X \to T^*_{\mathbf{R}^n}\mathbf{C}^n, \\ \Sigma \to W' \times_{\mathbf{R}^n} T^*_{\mathbf{R}^n}\mathbf{C}^n, \quad W' \subset \mathbf{R}^n, \end{cases}.$$

and hence $T^*_W X \to T^*_{W'}\mathbf{C}^n$. Thus the natural morphism

$$\mu_{W'}(\mathcal{O}_X) \to \mu_{\mathbf{R}^n}(\mathcal{O}_X)$$

induces (3.16) (via quantization of χ as in (3.11), (3.12)) if and only if $d_{W',W} = d_{\mathbf{R}^n,M}$ i. e.

$$1 + \delta(S) - (n+r) - \frac{1}{2}\tau(\lambda_W, i\lambda_W, \lambda_0) = 1 - n - \frac{1}{2}\tau(\lambda_M, i\lambda_M, \lambda_0)$$

i. e.

$$\delta(S) - r - (n - 1 - \delta(S) - r + 2\gamma(S)) + 2s^-(S) = -(n-1) + 2s^-(M).$$

The latter is in turn equivalent to (3.15) $\square$

We define now $\mu_{\Lambda\setminus\widetilde{\Sigma}}(\mathcal{O}_X)$ in $\mathrm{D}^b(X)$ to be the third term of a distinguished triangle:

$$\mu_W(\mathcal{O}_X) \to \mu_M(\mathcal{O}_X) \to \mu_{\Lambda\setminus\widetilde{\Sigma}}(\mathcal{O}_X) \overset{+1}{\to}. \tag{3.17}$$

Let $S \subset M \subset X$, $\operatorname{codim}_X M = 1$, $\operatorname{codim}_M S = r$.

Theorem 3.4. *Let $T^*_M X$ be I-symplectic, let $S \times_M T^*_M X$ be regular involutive, let (3.1) hold, and assume:*

$$s^-(M,p) - s^-(S,p) = \delta(S,p) - \gamma(S,p). \tag{3.18}$$

Then

$$H^j \mu_{\Lambda\setminus\widetilde{\Sigma}}(\mathcal{O}_X) = 0 \quad \textit{for } j \neq 1 + s^-(M,p). \tag{3.19}$$

Proof. We have $1 + \delta(S) - \gamma(S) + s^-(S) = 1 + s^-(M)$. Hence by Theorem 3.1, $\mu_W(\mathcal{O}_X)$ and $\mu_M(\mathcal{O}_X)$ are both concentrated in degree $1+s^-(M)$. Moreover owing to Proposition 3.3 we have a morphism

$$H^{1+s^-(M)}\mu_W(\mathcal{O}_X) \to H^{1+s^-(M)}\mu_M(\mathcal{O}_X),$$

which is injective due to [S2].

Remark 3.5. We will give in [D'A-M-Z] the following extension of the previous results. Let $\mu(p) := \lambda_M(p) \cap i\lambda_M(p)$ and $\delta(M,p) := \dim_{\mathbf{C}}(\mu(p))$. The preceding results can be extended to the following situation:

$$\begin{cases} M \text{ is generic} \\ \delta(M,p) \text{ is constant for } p \in T^*_M X \\ T_p\Sigma \supset \mu(p) \\ T_p\Sigma^{\mu(p)} \text{ is regular involutive in } \lambda_M^{\mu(p)}(p). \end{cases} \tag{3.20}$$

In this frame, the results on involutivity and "genericity" of §1 still hold, and we also have

$$\dim_{\mathbf{C}}(T\widetilde{\Sigma} \cap iT\widetilde{\Sigma}) = r + \delta(M),$$
$$\dim_{\mathbf{C}}(T\widetilde{\Sigma} \cap iT\widetilde{\Sigma} \cap \lambda_0) = \dim_{\mathbf{C}}(\lambda_S \cap i\lambda_S \cap \lambda_0),$$
$$\dim_{\mathbf{R}}(T\widetilde{\Sigma} \cap \lambda_0) = 1 + \delta(S) - \delta(M).$$

In particular if $\delta(S)$ is constant in p, then there exists $W \subset X$ with $\widetilde{\Sigma} = T^*_W X$. We still have

$$\begin{aligned} s^{\pm}(W) &= s^{\pm}(S), \\ s^0(W) &= r + \delta(M) - \gamma(S) \\ &= s^0(M) + \operatorname{codim}_W S. \end{aligned}$$

Thus repeating step by step the proof of Theorem 3.1 we get the following statement:

Assume that (3.20) is fulfilled and that $\delta(S)$ is constant in p (and hence $\widetilde{\Sigma} = T^*_W X$). Then

(i) $H^j \mu_W(\mathcal{O}_X) = 0$ *for* $j \neq 1 + s^-(S) + \delta(S) - \delta(M) - \gamma(S)$,

(ii) *If moreover* $s^-(M) - s^-(S) = \delta(S) - \delta(M) - \gamma(S)$ *then there is a natural morphism*

$$\mu_W(\mathcal{O}_X) \to \mu_M(\mathcal{O}_X)$$

and if one defines $\mu_{\Lambda \setminus \widetilde{\Sigma}}(\mathcal{O}_X)$ *as in (3.17), one has:*

$$H^j \mu_{\Lambda \setminus \widetilde{\Sigma}}(\mathcal{O}_X) = 0 \quad \text{for } j \neq 1 + s^-(M).$$

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