Positivity of Lagrangians and vanishing of cohomology for microfunctions at the boundary

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Abstract

We compare the microlocal Levi forms of a pair of real submanifolds of class C^2 of a complex manifold, and discuss their geometric meaning under contact transformation.

This enables us to recover, and even improve, our former results in [2], [6] on vanishing of cohomology for microfunctions which are the "boundary version" of those in [1] and [9].

1 General Notations

Let X be a complex analytic manifold, denote by $\pi : T^*X \longrightarrow X$ its conormal bundle, and by $\dot{\pi} : \dot{T}^*X \longrightarrow X$ the conormal bundle with the zero-section removed. Denote by $\overline{X}$ the complex conjugate to X, and by $X^{\mathbb{R}}$ the underlying real analytic manifold. The identification of $X^{\mathbb{R}}$ to the diagonal of $X \times \overline{X}$ induces an identification $T^*(X^{\mathbb{R}}) \xrightarrow{\sim} (T^*X)^{\mathbb{R}}$ that we will often use in the following. In particular, if M is a C^2-submanifold of $X^{\mathbb{R}}$, we identify its conormal bundle to a C^1-submanifold $T^*_M X$ of $(T^*X)^{\mathbb{R}}$.

Let $M \subset X^{\mathbb{R}}$ be a C^2-submanifold, let $p \in \dot{T}^*_M X$, and set $x = \pi(p)$. Let ϕ be a C^2-function at x such that $\phi|_M = 0$, and $d\phi(x) = p$. Choose a system of local coordinates (z) on X. The Hermitian form $L_M(p)$ on the complex tangent space $T^{\mathbb{C}}_x M = T_x M \cap iT_x M$ with matrix $(\partial^2_{z_i \bar{z}_j}\phi(x))_{ij}$, neither depends on the choice of the equation $\phi = 0$ of M, nor on the system of coordinates. This is called the Levi form of M at p, and we denote by $s^{+,-,0}_M(p)$ the numbers of positive, negative, and null eigenvalues of $L_M(p)$.

We denote by $D^b(X)$ the derived category of the category of complexes of sheaves of $\mathbb{C}$-vector spaces on X with bounded cohomology. For $F \in D^b(X)$,

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we denote by $SS(F)$ the micro-support of F in the sense of [9], [10]. This is a closed conic involutive set of $T^*X^{\mathbb{R}}$ which describes the directions of "non-propagation" for the cohomology of F. We denote by $D^b(X;p)$ the localization of $D^b(X)$ by the null-system $\{F \in D^b(X); p \notin SS(F)\}$.

Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. If A is a locally closed set of X, we denote by $\mathbb{C}_A$ the sheaf which is 0 on $X \setminus A$ and the constant sheaf with stalk $\mathbb{C}$ on A. We will make use of the complex $\mu_A(\mathcal{O}_X) = \mu hom(\mathbb{C}_A, \mathcal{O}_X)$, where $\mu hom(\cdot,\cdot)$ denotes the bifunctor of microlocalization defined by [9] (cf also [15]). Recall that if M is a real analytic manifold, and X is a complexification of M, then (up to the orientation and a shift) $\mu_M(\mathcal{O}_X)$ is the sheaf of Sato's microfunctions on $T^*_M X$. Moreover, according to [15], if $\Omega \subset M$ is an open subset, the complex $\mu_\Omega(\mathcal{O}_X)$ is the natural framework for the study of microlocal boundary value problems.

2 Positivity

Let X be a complex manifold of dimension n, and let M be a C^2-submanifold of $X^{\mathbb{R}}$ of codimension l. Let $p \in \dot{T}^*_M X$, let $x = \pi(p)$, and set:

$$d^-_M(p) = l + s^-_M(p) - \gamma_M(x), \tag{2.1}$$
$$d^+_M(p) = n - s^+_M(p) + \gamma_M(x), \tag{2.2}$$

where $\gamma_M(x) = \dim((T^*_M X)_x \cap i(T^*_M X)_x)$ is the "lack of genericity" of M (recall that $M \subset X$ is generic if and only if $\gamma_M(x) \equiv 0$).

Lemma 2.1. *(cf [3]) Let $S \subset M \subset X$ be C^2-submanifolds, and let $p \in S \times_M \dot{T}^*_M X$. Then*

$$d^\pm_M(p) \le d^\pm_S(p) \le d^\pm_M(p) + \operatorname{codim}_M S. \tag{2.3}$$

In [4] (see [7]) a notion of positivity for Lagrangian manifolds is introduced, after the works [11], [12], [14]. Restricting the attention to conormal bundles, the aim of this section is to relate this notion to the following conditions (2.4)–(2.6) (cf Remark 2.4).

Let (M_1, M_2) be a pair of C^2-submanifolds of X, and let $p \in \dot{T}^*_{M_1} X \cap \dot{T}^*_{M_2} X$. We shall consider the conditions:

$$R = M_1 \cap M_2 \text{ is a submanifold of class } C^2, \tag{2.4}$$
$$ip \notin T^*_R X, \tag{2.5}$$
$$d_{M_1}(p) = \operatorname{codim}_{M_2} R + d_{M_2}(p). \tag{2.6}$$

Notice that if $S \subset M$, $ip \notin T^*_S X$ and $\operatorname{codim}_M S = 1$, then Lemma 2.1 implies that either (M, S) or (S, M) satisfy (2.6).

Theorem 2.2. *Let (M_1, M_2) satisfy (2.4)–(2.6). Then there exists a germ of complex homogeneous contact transformation χ at $p \in \dot{T}^*X$ such that, for $q = \chi(p)$, $y = \pi(q)$:*

$$\begin{cases} \chi(T^*_{M_1}X) = T^*_{N_1}X \text{ for } N_1 \subset X \text{ with } \operatorname{codim}_X N_1 = 1,\ s^-_{N_1}(q) = 0, \\ \chi(T^*_{M_2}X) = T^*_{N_2}X \text{ for } N_2 \subset X \text{ with } \operatorname{codim}_X N_2 = 1,\ s^-_{N_2}(q) = 0, \\ {N_1}^+ \supset {N_2}^+ \text{ at } y, \end{cases} \tag{2.7}$$

where ${N_1}^+$, ${N_2}^+$ denote the closed half-spaces with boundary N_1, N_2 and interior conormal q.

Proof. (a) By Lemma 2.1 one has $d_{M_1} \leq d_R \leq d_{M_2} + \operatorname{codim}_{M_2} R$, and hence (2.6) is satisfied if and only if $d_{M_1} = d_R$ and $d_R = d_{M_2} + \operatorname{codim}_{M_2} R$.

(b) Conditions (2.4)–(2.5) ensure that there exists a germ of contact transformation χ at p such that:

$$\begin{cases} \chi(T^*_{M_1}X) = T^*_{N_1}X,\ \operatorname{codim}_X N_1 = 1,\ s^-_{N_1}(q) = 0, \\ \chi(T^*_R X) = T^*_S X,\ \operatorname{codim}_X S = 1, \\ \chi(T^*_{M_2}X) = T^*_{N_2}X,\ \operatorname{codim}_X N_2 = 1. \end{cases} \tag{2.8}$$

(c) Notice that since N_i and S are hypersurfaces, one has the isomorphisms $\mathbb{C}_{N_i} \simeq \mathbb{C}_{N_i^+}$, $\mathbb{C}_S \simeq \mathbb{C}_{S^+}$ in $D^b(X;q)$. According to [10, Chapter 7], let

$$\Phi_K : D^b(X;p) \xrightarrow{\sim} D^b(X;q)$$

be a quantization of χ. Here $K \in D^b(X \times X; (p, -q))$ is a simple sheaf of shift $-n$ at $(p, -q)$ along the Lagrangian manifold associated to χ. The restriction morphism $\mathbb{C}_{M_1} \longrightarrow \mathbb{C}_R$ is transformed by Φ_K to a non null morphism

$$\begin{aligned} \mathbb{C}_{{N_1}^+} &\longrightarrow \mathbb{C}_{S^+}[d_R(p) - d_{M_1}(p) - s^-_S(q)] \\ &= \mathbb{C}_{S^+}[-s^-_S(q)]. \end{aligned}$$

Since

$$\operatorname{Hom}_{D^b(X;q)}(\mathbb{C}_{{N_1}^+}, \mathbb{C}_{S^+})[-s^-_S(q)] \simeq H^{-s^-_S(q)}(\mathrm{R}\Gamma_{{N_1}^+}(\mathbb{C}_{S^+}))_y,$$

and the first term is non zero, it follows that $s^-_S(q) = 0$ and $S^+ \subset {N_1}^+$ at y.

(d) The restriction morphism $\mathbb{C}_{M_2} \longrightarrow \mathbb{C}_R$ induces by duality a non zero morphism $\mathbb{C}_R \longrightarrow \mathbb{C}_{M_2}[\operatorname{codim}_{M_2} R]$. Applying Φ_K, we get a non zero morphism

$$\begin{aligned} \mathbb{C}_{S^+} &\longrightarrow \mathbb{C}_{{N_2}^+}[\operatorname{codim}_{M_2} R + d_{M_2}(p) - d_R(p) - s^-_{N_2}(q)] \\ &= \mathbb{C}_{{N_2}^+}[-s^-_{N_2}(q)]. \end{aligned}$$

Thus again, $s^-_{N_2}(q) = 0$ and ${N_2}^+ \supset S^+$ at y. This proves the statement. □

Recall that an object $F \in D^b(X)$ which verifies $SS(F) \subset T^*_M X$ at p, is microlocally isomorphic (i.e. isomorphic in the category $D^b(X;p)$) to $C^\cdot_M$ for a complex of $\mathbb{C}$-vector spaces $C^\cdot$. This criterion, stated in [9, Proposition 6.2.1] for C^2-submanifolds M, easily extends to C^1-submanifolds (cf [2]).

Recall from [4] that a complex $L_M \in D^b(X;p)$ is called a Levi simple sheaf along $T^*_M X$ at p, when it is simple along $T^*_M X$ with shift $-d_M(p) + \frac{1}{2}\operatorname{codim}_X M$ at p. Levi simple sheaves are unique up to isomorphism, and we recall from [9, Chapter 7] that if χ is a germ of a contact transformation which interchanges $T^*_M X$ with $T^*_N X$, and if Φ_K is a quantization of χ as above, then $\Phi_K(L_M) = L_N$ for a Levi simple sheaf L_N along $T^*_N X$ at q. In particular, if χ is as in (2.7), then

$$\begin{aligned} \chi_* \mu hom(L_{M_1}, L_{M_2})_q &\simeq \mu hom(L_{N_1}, L_{N_2})_q \simeq \mu hom(\mathbb{C}_{N_1}, \mathbb{C}_{N_2})_q \\ &\simeq \mu hom(\mathbb{C}_{N_1{}^+}, \mathbb{C}_{N_2{}^+})_q \simeq \mathrm{R}\Gamma_{N_1{}^+}(\mathbb{C}_{N_2{}^+})_y \simeq \mathbb{C}, \end{aligned} \tag{2.9}$$

i.e.

$$\mu hom(L_{M_1}, L_{M_2})_p \simeq \mathbb{C}. \tag{2.10}$$

On the other hand, if (2.10) holds for a pair (M_1, M_2) satisfying (2.4)–(2.5), then a transformation χ with the above properties may be constructed as in the proof of Theorem 2.2.

Proposition 2.3. *Let $S \subset M \subset X^{\mathbb{R}}$ be C^2-submanifolds with $p \in S \times_M \dot{T}^*_M X$ and $ip \notin T^*_S X$. Then (M, S) (resp. (S, M)) satisfies (2.6) if and only if it satisfies (2.10).*

Proof. One has $\mu hom(\mathbb{C}_M, \mathbb{C}_S) \simeq \mathbb{C}_{S \times_M T^*_M X}$, and hence:

$$\begin{aligned} \mu hom(L_M, L_S)_p &\simeq \mu hom(\mathbb{C}_M, \mathbb{C}_S)_p[-d_s(p) + d_M(p)] \\ &= \mathbb{C}[-d_S(p) + d_M(p)]. \end{aligned}$$

Moreover

$$\begin{aligned} \mu hom(L_S, L_M)_p &\simeq \mu hom(\mathbb{C}_M, \mathbb{C}_S)_{-p}[-d_S(p) + d_M(p) + \operatorname{codim}_M S] \\ &= \mathbb{C}[-d_S(p) + d_M(p) + \operatorname{codim}_M S]. \end{aligned}$$

□

Remark 2.4. In view of the above discussion, we conclude that for a pair (M_1, M_2) satisfying (2.4)–(2.5), condition (2.6) is equivalent to $T^*_{M_1} X > T^*_R X > T^*_{M_2} X$, where $>$ means that the conormal bundles are relatively positive in the sense of [5].

Let $S \subset M$ with $ip \notin T^*_S X$, assume $\operatorname{codim}_M S = 1$, and let Ω be an open component of $M \setminus S$. We denote by ρ the projection $M \times_X T^*X \longrightarrow T^*M$, and by $N^*(\Omega)$ the conormal cone to Ω in M. Let χ satisfy (2.7), and let Φ_K be a quantization of χ as above.

Proposition 2.5. *([16]) In the above situation, the pair (M, S) (resp. (S, M)) satisfies (2.6) if and only if $\Phi_K(\mathbb{C}_{\overline{\Omega}}) = \mathbb{C}_Y[d_M(p) - 1]$ (resp. $\Phi_K(\mathbb{C}_\Omega) = \mathbb{C}_Y[d_M(p) - 1]$), where $Y \subset X$ is a C^1-hypersurface containing R.*

Proof. We put $\Lambda_1^+ = \overline{\Omega} \times_M T^*_M X$, $\Lambda_2^{\pm} = \rho^{-1}(\pm N^*(\Omega)|_S)$, $\Sigma = S \times_M T^*_M X$, $\Lambda_1^{+\circ} = \Lambda_1^+ \backslash \Sigma$, $\Lambda_2^{\pm\circ} = \Lambda_2^{\pm} \backslash \Sigma$. We have $SS(\mathbb{C}_{\overline{\Omega}}) = \Lambda_1^+ \cup \Lambda_2^+$, $SS(\mathbb{C}_\Omega) = \Lambda_1^+ \cup \Lambda_2^-$.

Let χ satisfy (2.7). Then either $\chi(\Lambda_1^+ \cup \Lambda_2^+)$ or $\chi(\Lambda_1^+ \cup \Lambda_2^-)$ is the conormal bundle to a C^1-hypersurface $Y \subset X$. This implies that either $\Phi_K(\mathbb{C}_{\overline{\Omega}})$ or $\Phi_K(\mathbb{C}_\Omega)$ is isomorphic to $\mathbb{C}_Y$ in $D^b(X; q)$. Note that $\Phi_K(\mathbb{C}_{\overline{\Omega}})$ (resp. $\Phi_K(\mathbb{C}_\Omega)$) has constant shift $d_M(p) - \frac{1}{2}$ in $\Lambda_1^{+\circ} \cup \Lambda_2^{+\circ}$ (resp. $\Lambda_1^{+\circ} \cup \Lambda_2^{-\circ}$) if and only if $d_M(p) = d_S(p)$ (resp. $d_M(p) = d_S(p) - 1$). The conclusion follows. □

3 Application to vanishing of cohomology for microfunctions at the boundary

Using the results of the previous section, we may recover and even improve our former results in [5]. Let X be a complex manifold of dimension n, M a real C^2-submanifold of codimension l, Ω an open set of M with C^2-boundary $S = \partial\Omega$ of $\operatorname{codim}_M S = r$ (Ω locally on one side of S when $r = 1$).

Theorem 3.1. *(i) Let $p \in S \times_M \dot{T}^*_M X$ and assume $ip \notin T^*_S X$. Then:*

$$H^j(\mu_\Omega(\mathcal{O}_X))_p = 0 \quad \forall j \notin [d_M^-(p), d_M^+(p) + r - 1]. \tag{3.1}$$

(ii) Assume in addition that $d_M^-(p')$ and $d_S^-(p'')$ are constant for $p' \in \overline{\Omega} \times_M T^*_M X$, $p'' \in S \times_X \rho^{-1}(-N^*(\Omega))$ near p, and that $d_S^-(p) = d_M^-(p) + 1$. Then

$$H^j(\mu_\Omega(\mathcal{O}_X))_p = 0 \quad \forall j \neq d_M^-(p). \tag{3.2}$$

Proof. For the completeness of our exposition we begin by repeating some arguments of [9, Theorems 11.3.1, 11.3.5].

Consider two germs of contact transformations χ, $\widetilde{\chi}$ at p, such that, setting $q = \chi(p)$, $\widetilde{q} = \widetilde{\chi}(p)$, $y = \pi(q)$, $\widetilde{y} = \pi(\widetilde{q})$:

$$\begin{cases} \chi(T^*_M X) = T^*_N X \text{ for } N \subset X \text{ with } \operatorname{codim}_X N = 1,\ s_N^-(q) = 0, \\ \widetilde{\chi}(T^*_N X) = T^*_{\widetilde{N}} X \text{ for } \widetilde{N} \subset X \text{ with } \operatorname{codim}_X \widetilde{N} = 1,\ s_{\widetilde{N}}^+(\widetilde{q}) = 0. \end{cases}$$

Quantizing χ and $\widetilde{\chi}$ with kernels K and $\widetilde{K}$ as in the proof of Theorem 2.2, we get:

$$\Phi_K(\mathbb{C}_M) = \mathbb{C}_{M_1}[d^+_M(p) - n], \quad \Phi_{\widetilde{K}} = (\mathbb{C}_M) = \mathbb{C}_{M_2}[d^-_M(p) - 1].$$

This implies (as in (2.9)):

$$\chi_* \mu_M(\mathcal{O}_X)_q \simeq \mathrm{R}\Gamma_{N^+}(\mathcal{O}_X)_y[n - d^+_M(p)], \quad \widetilde{\chi}_* \mu_M(\mathcal{O}_X)_{\widetilde{q}} \simeq \mathrm{R}\Gamma_{\widetilde{N}^+}(\mathcal{O}_X)_{\widetilde{y}}[-d^-_M(p) + 1].$$

Hence:

$$H^j(\mu_M(\mathcal{O}_X))_p = 0 \quad \forall j \notin [d^-_M(p), d^+_M(p)]. \tag{3.3}$$

The same arguments hold for S.

We set now $\Omega^- = M \setminus \overline{\Omega}$ and consider the distinguished triangle

$$\mu_S(\mathcal{O}_X) \longrightarrow \mu_M(\mathcal{O}_X) \longrightarrow \mu_\Omega(\mathcal{O}_X) \oplus \mu_{\Omega^-}(\mathcal{O}_X) \xrightarrow{+1} . \tag{3.4}$$

Formulas (2.3), (3.3), and the analogous formula for S, imply the vanishing of (3.1) for $j > d^+_M(p) + r - 1$ and for $j < d^-_M(p)$, when $d^-_S(p) > d^-_M(p)$.

In order to treat the case $d^-_S(p) = d^-_M(p)$, it remains to prove the injectivity of the morphism

$$H^{d^-_M(p)}(\mu_S(\mathcal{O}_X)_p) \longrightarrow H^{d^-_M(p)}(\mu_M(\mathcal{O}_X)_p). \tag{3.5}$$

Applying Theorem 2.2 for $(M_1, M_2) = (M, S)$, we reduce (3.5) to the morphism

$$H^1_{N_2^+}(\mathcal{O}_X)_y \longrightarrow H^1_{N_1^+}(\mathcal{O}_X)_y, \tag{3.6}$$

which is clearly injective since $N_2^+ \subset N_1^+$.

(i) If $r > 1$, then $\overline{\Omega} = M$, $N^*(\Omega) = T^*M$, and therefore $d^-_M(p')$, $d^-_S(p'')$ are constant in a full neighborhood of p in $T^*_M X$ and $T^*_S X$ respectively. We recall that $(s^-_M(p') - \gamma_M(x')) - s^-_{N_1}(q')$ is constant for $q' = \chi(p')$. Since $s^-_{N_1}(q) = 0$, then $X \setminus N_1^+$ is pseudoconvex (cf. [8]), whence the complex $\mu_M(\mathcal{O}_X)_p \simeq \mathrm{R}\Gamma_{N_1^+}(\mathcal{O}_X)_y[-d^-_M(p) + 1]$ is concentrated in degree $d^-_M(p)$. The same is true for S and thus we get the conclusion by the aid of (3.4).

(ii) Let now $r = 1$, and assume $d^-_S(p) = d^-_M(p) + 1$. According to Proposition 2.5 we have

$$\chi_* \mu_\Omega(\mathcal{O}_X)_p \xrightarrow{\sim} \mathrm{R}\Gamma_{Y^+}(\mathcal{O}_X)_y[-d^-_M(p) + 1]. \tag{3.7}$$

We observe that Y is C^1 and even C^2 outside R. Moreover,

$$s^-_Y(q') \equiv 0 \quad \forall q' \in T^*_Y X,\ \pi(q') \notin R,\ q' \text{ near } q.$$

It follows that $X \setminus Y^+$ is pseudoconvex whence (3.7) is concentrated in degree $d^-_M(p)$. $\square$

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