

RESULTS OF THE APPLICATION OF AN ALGORITHM FOR L_2 MODEL REDUCTION

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An algorithm for L_2 -optimal model reduction in frequency domain is outlined. Some significant examples are illustrated to show the performance of the method and to compare the related results with those obtained using alternative techniques.

1. Introduction

The problem of constructing lower-order models approximating higher-order systems has been considered with interest in recent literature both for simulation purposes and for control system design.¹ Many approaches have in fact been adopted; some refer to the time domain and to state variables, others refer to the frequency domain and to input-output representations. From a different point of view, we might subdivide the reduction methods into: (i) empirically-based procedures, (ii) methods leading to the retention of a certain number of parameters of the original system, e.g. the coefficients of the first terms of a series expansion (MacLaurin, asymptotic, Chebyshev, etc.) and/or some appropriate performance indices (covariances, energies, time moments, singular values, etc.), and (iii) methods leading to the minimization of a suitable norm of the approximation error (L_2 norm, Hankel norm, etc.).

Clearly, it is often arbitrary to choose the reduction criterion even if the nature of the specific problem at hand or the availability of computational tools may influence this choice. However, once the criterion has been selected, it is not very meaningful to compare the related results with those obtained by referring to a different objective. Instead, the comparison does make sense with reference to the

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computational complexity. This is the reason why the search for more efficient reduction algorithms has been considered with great attention.

The approximation criterion adopted in this note is the minimization of the (squared) L_2 norm of the error, which is a classic topic both in approximation theory² and in system theory³; in fact, even the empiric or truncation methods belonging to categories (i) and (ii) above are often judged on the basis of the resulting L_2 norm (cf., e.g. Ref. 4). However, despite this interest, only few algorithms for its minimization have been conceived and further research in this field is still needed.

It has long been known^{5,6} that, in the case of single-input single-output (SISO) continuous-time systems, necessary conditions for an i th order strictly proper transfer function

$$g(s) = n_g(s) / d_g(s) \quad (1)$$

to be the L_2 -optimal rational approximant of a given high-order rational and strictly proper stable transfer function

$$f(s) = n_f(s) / d_f(s) \quad (2)$$

are

$$g^{(i)}(-p_k^*) = f^{(i)}(-p_k^*), \quad i = 0, \dots, \mu_k; \quad k = 1, \dots, r', \quad (3)$$

where p_k is a pole of multiplicity μ_k of $g(s)$, $*$ denotes complex conjugate, r' is the number of distinct poles of $g(s)$, and superscript (i) denotes the i th derivative.

It has also been shown⁷ that a solution always exists in the Hardy space H_2 (stable and strictly proper $g(s)$), that it is in general unique, even if more *local* minima may occur, and that it is of maximum order r , which in turn implies that the goodness of the approximation increases with the model order (this is therefore an intrinsic property of the criterion and not of the specific algorithm used to achieve the optimum).

Result (3) has been extended to multi-input multi-output (MIMO) discrete-time systems by Kimura,⁸ who has limited attention to the linear case in which the poles of the approximating transfer function matrix are fixed. In Ref. 9 the present authors have shown that the necessary conditions that must be satisfied by the optimal approximating transfer matrix of a MIMO continuous-time system, without restrictions on its poles, again correspond to a set of interpolation constraints. In Sec. 2, these constraints are expressed in a compact form from which an efficient algorithm is derived. The procedure turns out to be a valid alternative to the other available techniques, as shown by the representative examples of Sec. 3 that are taken from the relevant literature. Our attention here focuses on the practical aspects of this promising method (results, starting guesses and related number of iterations, comparison with the performances of alternative methods).

2. Outline of the Method

By assuming that the poles of $g(s)$ are simple and its coefficients are real, it is immediately seen that the interpolation conditions (3) are equivalent to the polynomial identity:

$$n_f(s)d_g(s) - n_g(s)d_f(s) = u(s)d_g^2(-s), \quad (4)$$

where $\deg\{u(s)\} = \deg\{d_f(s)\} - \deg\{d_g(s)\} - 1$, in that the reduced models obtained from the solution of Eqs. (3) (nonlinear in the unknown parameters of $g(s)$) coincide with those obtained by equating the coefficients of the equal powers of s on both sides of (4). In fact, relations (3) simply mean that the numerator of the error function

$$e(s) := f(s) - g(s) = \frac{n_f(s)d_g(s) - n_g(s)d_f(s)}{d_f(s)d_g(s)} \quad (5)$$

must include r' zeros of multiplicities $\mu_k + 1$ at the opposites $-p_k$ of the poles of $g(s)$, as expressed by (4) for the case of simple poles.

In the case of MIMO systems, if we denote by

$$F(s) = N_F(s)d_F^{-1}(s) \quad (6)$$

the original $m_o \times m_i$ transfer matrix, where $d_F(s)$ is the n th degree (Hurwitz) least common denominator (l.c.d.) of the entries of $F(s)$,

$$G(s) = N_G(s)d_G^{-1}(s) \quad (7)$$

the approximating (real) transfer matrix, where $d_G(s)$ is the r th degree l.c.d. of the entries of $G(s)$,

$$E(s) = F(s) - G(s) = [N_F(s)d_G(s) - N_G(s)d_F(s)]d_F^{-1}(s)d_G^{-1}(s) \quad (8)$$

the approximation error matrix,

$$J = \|E(s)\|^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{tr} \{ E(j\omega) E^T(-j\omega) \} d\omega \quad (9)$$

the squared L_2 norm to be minimized, then the necessary conditions for optimality generalizing (3) are, in the case in which the r poles of $G(s)$ are distinct,

$$F(-p_k) = G(-p_k), \quad k = 1, \dots, r, \quad (10)$$

$$\text{tr} \{ [F^{(1)}(-p_k) - G^{(1)}(-p_k)]^T R_k \} = 0, \quad k = 1, \dots, r, \quad (11)$$

where R_k is the residue matrix at pole p_k . Similar conditions hold in the case of multiple poles.

Again, the interpolation conditions (10) and (11) can be expressed in the form of the following polynomial identities

$$N_F(s)d_G(s) - N_G(s)d_F(s) = Q(s)d_G(-s), \quad (12)$$

$$\text{tr} \{Q(s)^T N_G(-s)\} = v(s)d_G(-s), \quad (13)$$

where $Q(s)$ is an $m_o \times m_i$ polynomial matrix whose elements have at most the degree $n - 1$, and $v(s)$ is a polynomial of degree at most $n - 2$.

By equating the coefficients of the equal powers of s on both sides of (12) and (13), we form a set of $(m_o m_i + 1)(n + r) - 1$ equations in the same number of unknowns represented by: (i) the $(m_o m_i + 1)r$ coefficients of $G(s)$, and (ii) the $(m_o m_i + 1)n - 1$ coefficients of $Q(s)$ and $v(s)$. All these equations have degree 2 in the considered unknowns and, therefore, the solution of the above set is not an easy task unless a suitable "expedient" is used.

To this purpose an iterative procedure has been conceived which entails, at each step, the solution of a set of linear equations corresponding to the polynomial identities

$$N_F(s)d_{G,(m+1)}(s) - N_{G,(m+1)}(s)d_F(s) = Q_{(m+1)}(s)d_{G,(m)}(-s), \quad (14)$$

$$\text{tr} \{Q_{(m+1)}(s)^T N_{G,(m)}(-s)\} = v_{(m+1)}(s)d_{G,(m)}(-s), \quad (15)$$

where subscript (m) denotes the quantities already computed in the previous m th iteration and subscript $(m + 1)$ denotes those to be determined in the current $(m + 1)$ th iteration. In (14) and (15) the unknowns are therefore the coefficients of the polynomials with subscript $(m + 1)$, which appear linearly.

On the basis of Eqs. (14) and (15) a MATLAB program has been written. By exploiting their peculiar structure, it only requires the inversion of matrices of order r despite the rather large number of auxiliary variables (coefficients of $Q(s)$ and $v(s)$). Clearly, the procedure must be started from suitably guessed values of the coefficients of $d_{G,(0)}(s)$ and $N_{G,(0)}(s)$: experience suggests^{10,11} that a good choice corresponds to the retention of the dominant poles because the *global* optimum is often located in their neighborhood. The stopping criterion is based on the distance of the points in the parameter space corresponding to two consecutive iterations. All examples considered show that the algorithm rapidly converges to local minima and escapes from the other critical points, thus behaving like a "good" gradient algorithm; a theoretical analysis is now being performed.

Concerning the choice of the order of the reduced model, since its McMillan degree is included between r and $r \cdot \min(m_o, m_i)$, no problem arises for SISO, SIMO and MISO systems. For MIMO systems, instead, one must set $r < \nu / \min(m_o, m_i)$, where ν is the (given) McMillan degree of the original system, to ensure that the order of the minimal realization of the reduced model (and not only the number of its poles) is less than that of the original system; usually, such a choice is not too restrictive.

In the next section, we report the results obtained by applying the proposed algorithm to some not trivial examples already considered by other authors in connection with the same and/or a different reduction criterion.

3. Examples

3.1. Example 1

Let us first consider a simple, yet informative, example which was also treated by Wilson,¹² Moore,¹³ and Hyland and Bernstein.¹⁴ It refers to an SISO system whose original transfer function is:

$$f(s) = \frac{4 + s}{150 + 245s + 113s^2 + 19s^3 + s^4}, \quad (16)$$

whose poles are in $-1, -3, -5$ and -10 , and whose impulse response energy is $\|f(s)\|^2 = 2.693765 \cdot 10^{-4}$. Since, according to (1), in the case of SISO systems Eqs. (14) and (15) reduce to the single equation

$$n_f(s)d_{g,(m+1)}(s) - n_{g,(m+1)}(s)d_f(s) = u_{(m+1)}(s)d_{g,(m)}^2(-s), \quad (17)$$

the initial guess only concerns the denominator $d_{g,(0)}(s)$. By starting the suggested algorithm from an initial denominator $d_{g,(0)}(s)$ with all coefficients equal to 1 (very far from the optimal denominator obtained and even "unstable" for $r = 3$), and by adopting the following stringent stopping criterion:

$$\frac{\max_i |a_{i,(m+1)} - a_{i,(m)}|}{\min_i |a_{i,(m+1)}|} < 0.001 \quad (18)$$

where $a_{i,(k)}, k = m, m+1$, denote the coefficients of polynomial $d_{g,(k)}(s)$, we have obtained in only 5 iterations the following reduced models of order 3 and 2 and related index values:

$$g_{iii}(s) = \frac{1.070399 - 0.002929s + 5.715404 \cdot 10^{-5}s^2}{40.116974 + 55.486442s + 16.127431s^2 + s^3}, \quad (19)$$

$$J_{iii} = 4.585602 \cdot 10^{-10}; \quad (20)$$

$$g_{ii}(s) = \frac{0.073021 - 0.003223s}{2.760151 + 3.610528s + s^2}, \quad (21)$$

$$J_{ii} = 4.158469 \cdot 10^{-7}; \quad (22)$$

and the following first-order model in 11 iterations:

$$g_i(s) = \frac{0.014772}{0.495281 + s}, \quad (23)$$

$$J_i = 4.907489 \cdot 10^{-5}. \quad (24)$$

The above models correspond to the *global* optima and, in fact, their relative errors:

$$\delta = J^{1/2} / \|f(s)\| \quad (25)$$

practically coincide with those obtained in Ref. 14, as shown in Table 1 where the values of δ for the models found by Moore¹³ using balanced coordinates and by Wilson¹² using the L_2 criterion are also reported (as pointed out in Ref. 13, the result of Wilson seems to imply a lack of final convergence).

Table 1. Relative error δ .

Model order	Ref. 12	Ref. 13	Ref. 14	Present
3	—	0.001311	0.001306	0.001305
2	0.04097	0.03938	0.03929	0.03929
1	—	0.4321	0.4268	0.4268

3.2. Example 2

As a second example, we consider the 6th order three-spring-dashpot system studied by Gawronski and Juang¹⁵ in their example 14. Its original transfer function is:

$$f(s) = -\frac{45.36405 + 0.906008s + 24.831974s^2 + 0.248135s^3 + 2.1182s^4}{119.0845 + 3.56619s + 180.579348s^2 + 3.609306s^3 + 32.972538s^4 + 0.3295s^5 + s^6}, \quad (26)$$

whose poles are in complex conjugate pairs with very small damping factors at $-0.003798 \mp j0.873796$, $-0.029698 \mp j2.437375$, $-0.131253 \mp j5.121724$, and whose impulse response energy is $\|f(s)\|^2 = 4.076344$. By adopting the same stopping criterion as in Ex. 1 and starting again from $d_{g,(0)} = \sum_{i=0}^r s^i$, we have found the following optimal reduced models of order from 5 to 2 in less than 10 iterations for r even (as is expected, the cases of r odd require many more iterations and improve only a little the optimum index value upon the value for the immediately smaller even value of r):

$$g_v(s) = -\frac{11.539689 + 6.799404s + 5.374538s^2 + 1.773668s^3 + 0.085233s^4}{28.405190 + 20.656818s + 38.674047s^2 + 27.461745s^3 + 1.670691s^4 + s^5}, \quad (27)$$

$$J_v = 0.092439; \quad (28)$$

$$g_{iv}(s) = -\frac{6.778126 + 0.097789s + 1.871034s^2 + 0.011927s^3}{20.083737 + 0.402648s + 27.070606s^2 + 0.273231s^3 + s^4}, \quad (29)$$

$$J_{iv} = 0.095748; \quad (30)$$

$$g_{iii}(s) = -\frac{0.693990 + 0.209460s + 0.329663s^2}{1.621741 + 0.779574s + 2.131501s^2 + s^3}, \quad (31)$$

$$J_{iii} = 0.268407; \quad (32)$$

$$g_{ii}(s) = -\frac{0.209478 + 0.003489s}{0.763438 + 0.007599s + s^2}, \quad (33)$$

$$J_{ii} = 0.293443. \quad (34)$$

By limiting attention to the model of order 4, which was studied in Ref. 15, it turns out that the poles of the globally L_2 -optimal model, which are at $-0.003798 \mp j0.873772$ and $-0.132817 \mp j5.127129$, practically coincide with four poles of the original system. This is also the case in the improved balancing methods in limited time adopted in Ref. 15, and confirms that the poles of the L_2 -optimal models tend to occur, in general, near those of the full-order system.¹⁰ Therefore, it is reasonable to start the algorithm from a denominator polynomial that retains the dominant poles¹¹ of the original system. In fact, with such an initial guess the globally optimal model (27) is found in 4 iterations only.

3.3. Example 3

Finally, we consider the 26th order 2-input 2-output model of a truss structure.¹⁶ This is schematically represented in Fig. 1 where u_1 and u_2 denote the inputs (forces) and y_1 and y_2 denote the outputs (displacements).

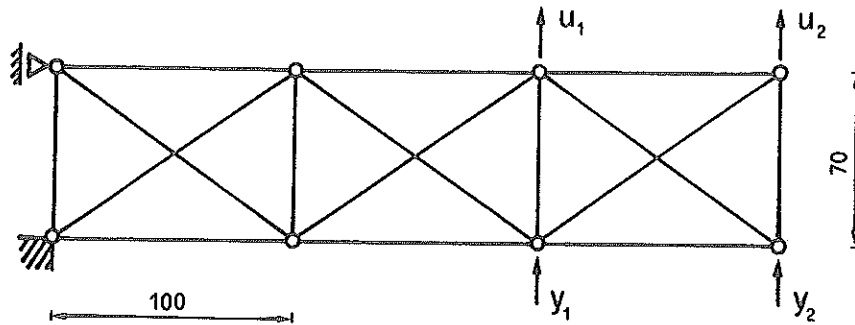


Fig. 1. Truss structure.

The detailed description of the system is not given here but is available to the interested reader. We only report the results concerning the reduced models with 6 and 4 poles, respectively.

The best model with l.c.d. degree equal to 6 has been found in 163 iterations starting from an initial model with all its numerators and denominator coefficients equal to 1 (very far from the final solution). The comparatively large number of iterations is certainly due both to the intrinsic complexity of the problem and to the wide range within which the polynomial coefficients are distributed (this makes

the adopted stopping criterion (18) very stringent; to overcome this difficulty, one could resort to a suitable rescaling of the polynomial coefficients). Again, it turns out that the L_2 -optimal model practically retains six poles of the original system (belonging to the set of eight poles closest to the origin). The corresponding index value and relative error are

$$J = 2.610572 \cdot 10^{-6}, \quad \delta = 0.162892. \quad (35)$$

Note that in this case the model with the same number of poles obtained through balancing is characterized by a slightly greater value of J , i.e., $J = 2.610952 \cdot 10^{-6}$.

The L_2 -optimal model with l.c.d. degree equal to 4 has been found in 12 iterations starting from an initial model with all its coefficients equal to 1. It turns out that the global optimum practically retains four poles of the original system, which in turn are very close to the four poles with the smallest magnitude retained by the above model of order 6. The index value and relative error for the model with 4 poles are

$$J = 4.564383 \cdot 10^{-6}, \quad \delta = 0.215389. \quad (36)$$

The value of J for the model with the same number of poles obtained through balancing is $J = 4.564400 \cdot 10^{-6}$.

As shown by the two cases above, the balanced models for the considered system are very close to the L_2 -optimal models even if this is not always true.⁴

4. Conclusions

The purpose of this paper has been to present a new algorithm for the approximation of rational transfer functions by minimizing the L_2 -norm of the error. It is characterized by remarkable computational simplicity if compared to alternative techniques. This is essentially achieved by suitably manipulating the necessary interpolation conditions for optimality (12) and (13), which allows us to convert the considered nonlinear problem into a sequence of linear problems. The results obtainable have been evaluated with the aid of practical examples.

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