

The Group of Autoprojectivities of the Finite Irreducible Coxeter Groups

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INTRODUCTION

Given a group G , we denote by $L(G)$ the lattice of all subgroups of G . A *projectivity* of a group G onto a group \bar{G} is any lattice isomorphism from $L(G)$ onto $L(\bar{G})$, and an *autoprojectivity* of G is any projectivity of G onto itself.

An interesting problem is to know in which cases a projectivity of G onto a group \bar{G} is induced by an isomorphism. In this context, a group G is said to be *strongly lattice determined* if every projectivity of G onto a group \bar{G} is induced by an isomorphism. It is clear that G is strongly lattice determined if and only if the following two conditions are satisfied:

- (i) G projective to \bar{G} implies G isomorphic to \bar{G} ,
- (ii) every autoprojectivity of G is induced by an automorphism.

In [2, 3] we considered the second problem for simple algebraic groups G over the algebraic closure of a finite field. One step in our procedure was the following. Let T be a maximal torus of G , and let $W = N(T)/T$ be the Weyl group. We showed that every autoprojectivity φ of G fixing T also fixes $N(T)$, so that it induces in a natural way an autoprojectivity $\bar{\varphi}$ of W (cf. Proposition 3.1 in [2]). We proved in [3] that if the characteristic of the field is odd and G is not of type A_2 , then every autoprojectivity of G is induced by a unique automorphism of G . From this it follows that for every autoprojectivity φ of G fixing T , there exists an automorphism of W inducing $\bar{\varphi}$. The motivation for this paper was to study if this is a general

property of Weyl groups. Of course one cannot expect this to be true in general, since there could be autoprojectivities which are not index-preserving. This kind of problem was completely solved by Zacher for symmetric groups (over any set) in [16, 17].

It is well known that the Weyl groups are Coxeter groups (even if not all the finite Coxeter groups arise in this way). Here we are interested in the groups of autoprojectivities of the finite irreducible Coxeter groups. For the class of finite Coxeter groups problem (i) has completely been solved by Uzawa in [14].

The main result of the present paper is that *if W is a finite irreducible Coxeter group, then every index-preserving autoprojectivity of W is induced by a (unique) automorphism if and only if either W is not dihedral or W is dihedral of order $2n$, with $n = 2, 4, 3, 6, 12$ (for Weyl groups we do not assume irreducibility) (Theorem 4.6). We also determine completely the group of autoprojectivities of dihedral groups (Corollary 3.5).*

Taking into account the results of Uzawa on projective images of Coxeter groups [14], we prove that *a finite irreducible Coxeter group is strongly lattice determined if and only if either W has rank at least 3, or if it is dihedral of order $2n$, with $n = 2, 4, 6$ or 12 (Theorem 4.8).*

In a forthcoming paper we shall relax the irreducibility condition.

Notation

S_n is the symmetric group on n elements.

The symbol \cup denotes set theoretic union.

If G is a group, G' is the derived subgroup of G .

Let X be a group. $\text{Aut } L(X)$ is the group of all autoprojectivities of X and $I(X)$ is the group of index-preserving autoprojectivities of X . If α is an automorphism of X , we denote by α^* the autoprojectivity of X induced by α . We get the homomorphism $*$: $\text{Aut } X \rightarrow \text{Aut } L(X)$. Our main concern is to study surjectivity of $*$ for Coxeter groups.

1. COXETER GROUPS

It is well known that the finite Coxeter groups are precisely the finite reflection groups. For a given Coxeter group, we shall always consider its structure as a reflection group arising from the standard geometric representation. Here we just recall some definitions and facts (we use the notation from [6]). A *Coxeter system* is a pair (W, S) consisting of a group W and a set of generators S , subject only to relations of the form

$$(ss')^{m(s,s')} = 1 \quad (s, s' \text{ in } S),$$

where $m(s, s) = 1$, $m(s, s') = m(s', s) \geq 2$ for $s \neq s'$. We shall always assume S to be a finite set. Its cardinality is called the rank of (W, S) . W is called a *Coxeter group*, and S a *Coxeter generating set*. When W is a Coxeter group we assume a Coxeter generating set S fixed. To a Coxeter system (W, S) there is associated the Coxeter graph. (W, S) is *irreducible* if the Coxeter graph is connected. A finite Coxeter group is *crystallographic* if $m(s, t) \in \{2, 3, 4, 6\}$ for $s \neq t$. These are precisely the Weyl groups arising from semisimple Lie algebras over \mathbb{C} .

We now fix a finite Coxeter group W , with Coxeter generating set $S = \{s_1, \dots, s_n\}$, and consider W as a finite reflection group acting on the Euclidean space V of dimension n , with root system Φ , and a fixed simple system $\{\alpha_1, \dots, \alpha_n\}$, so that each s_i is the reflection relative to the vector α_i . The following two properties of finite reflections groups W will be crucial in our discussion.

Every involution of W can be written as a product of commuting reflections of W (cf. [6, Sect. 1.12, Ex. 3, p. 23]).

Every reflection in W is of the form s_α , for some α in Φ (cf. [6, Sect. 1.14]).

PROPOSITION 1.1. *The homomorphism $*$: $\text{Aut } W \rightarrow \text{Aut } L(W)$ is injective.*

Proof. This follows from the fact that W is generated by involutions.

We shall identify $\text{Aut } W$ with its image in $\text{Aut } L(W)$, so that we have $\text{Aut } W \leq I(W) \leq \text{Aut } L(W)$.

LEMMA 1.2. *Let φ be in $I(W)$. Then there exists an automorphism α of W such that $\langle s \rangle^\varphi = \langle s \rangle^\alpha$ for every s in S .*

Proof. For each s in S , let \bar{s} be the unique involution of W such that $\langle s \rangle^\varphi = \langle \bar{s} \rangle$. We have $|\bar{s}z| = |sz|$ for every s, z in S , since $\langle s, z \rangle$ is dihedral. Hence there exists a homomorphism: $\alpha : W \rightarrow W$ such that $s^\alpha = \bar{s}$ for every s in S . Since $W = \langle \bar{s} | s \in S \rangle$, α is surjective and is the required isomorphism.

DEFINITION 1.3. We put $\Gamma(W) = \{\varphi \in I(W) | \langle s \rangle^\varphi = \langle s \rangle \text{ for every } s \text{ in } S\}$.

We shall write $\Gamma_S(W)$ when we want to specify the Coxeter generating set.

COROLLARY 1.4. $I(W) = (\text{Aut } W)\Gamma(W)$ and $(\text{Aut } W) \cap \Gamma(W) = \{1\}$.

The next lemma is a key step in our discussion.

LEMMA 1.5. *Let φ be an autoprojectivity of the finite Coxeter group W such that $\langle s_\alpha \rangle^\varphi = \langle s_\alpha \rangle$ for every root α . Then φ is the identity.*

To prove the lemma we shall use the following results.

A group G is *strongly real* if for every g in G there exists an involution σ in G such that $\sigma g \sigma = g^{-1}$.

PROPOSITION 1.6. *W is strongly real.*

Proof. This was proved first by Carter for irreducible Weyl groups with a case by case inspection on the conjugacy classes [1, Theorem C], and then by Springer for the other irreducible Coxeter groups [12, Theorem 8.7]. The extension to the general case is immediate.

LEMMA 1.7. *Let G be a group whose elements are strongly real. If ψ is an autoprojectivity of G fixing every subgroup of order 2, then ψ is the identity.*

Proof. This is well known (see for instance [17, p. 122]).

To prove 1.5, it is enough to show that φ fixes every subgroup of order 2. Let σ be an involution of W . We can write $\sigma = \sigma_1 \cdots \sigma_r$, where σ_i 's are commuting reflections. By hypothesis, φ fixes all the subgroups $\langle \sigma_i \rangle$. It follows by induction that φ fixes $\langle \sigma \rangle$.

We conclude this paragraph by determining for which W every autoprojectivity is index-preserving. For this purpose we recall a definition. Let ψ be a projectivity of a group G onto a group \bar{G} . ψ is *2-regular* if it maps subgroups of order 2 to subgroups of order 2.

LEMMA 1.8. *Let G be a finite group generated by involutions. If ψ is a projectivity of G , then ψ is index-preserving if and only if it is 2-regular.*

Proof. If ψ is index-preserving then it is 2-regular. So assume ψ is 2-regular, and suppose ψ is not index-preserving. Then there exists an odd prime p such that ψ has a singularity of the first kind at p (Proposition 2.7, Sect. II in [13]). It follows that there exists a normal p -complement N in G such that G/N is abelian. But this is impossible, since G/G' is a 2-group.

PROPOSITION 1.9. *Let W be a finite Coxeter group. Then every autoprojectivity of W is index-preserving if and only if W is not dihedral of order $2p$, with p an odd prime.*

Proof. Suppose there are two commuting involutions in W . Then even every projectivity of W is 2-regular by Corollary 1.7 in [10]. Hence every projectivity of W is index-preserving by Lemma 1.8. This covers the cases when the rank of W is at least 3, and the dihedral groups of order $2n$ with even n . Since the group of rank 1 is cyclic of order 2, we are left to study the remaining groups of rank 2, that is, dihedral groups of order $2n$ with odd n . If n is not a prime, then every autoprojectivity of W is 2-regular,

hence index-preserving, by Lemma 1.1 in [10]. Finally, suppose n is an odd prime p . Then $\text{Aut } L(W) \cong S_{p+1}$, while $I(W) \cong S_p$.

In the next section we shall deal with Weyl groups.

2. AUTOPROJECTIVITIES OF WEYL GROUPS

In this section we assume that W is a Weyl group. We fix an element φ in $\Gamma(W)$. We prove that φ fixes every subgroup of W generated by a reflection. We shall make use of the following result of Yacovlev.

LEMMA 2.1. *Let $G = \langle \sigma, \tau \rangle$, σ, τ involutions, and let ψ be an index-preserving projectivity of G , $\langle \sigma \rangle^\psi = \langle \bar{\sigma} \rangle$, and $\langle \tau \rangle^\psi = \langle \bar{\tau} \rangle$. If $|\sigma\tau| \in \{2, 3, 4, 6, 12, \infty\}$, then $\langle \sigma\tau\sigma \rangle^\psi = \langle \bar{\sigma}\bar{\tau}\bar{\sigma} \rangle$.*

Proof. See Lemma 6.1 in [15].

PROPOSITION 2.2. *We have $\langle s_\alpha \rangle^\varphi = \langle s_\alpha \rangle$ for every root α .*

Proof. Since each reflection is conjugate under W to a simple reflection, we are left to prove the following. Let w be in W , and let s_i be in S . Then $\langle ws_iw^{-1} \rangle^\varphi = \langle ws_iw^{-1} \rangle$. We prove this by induction on the length $\ell(w)$ of w . If $\ell(w) = 0$, then there is nothing to prove. So assume $\ell(w) \geq 1$. We can write $w = s_jw'$, for some w' with $\ell(w') < \ell(w)$. Let $\beta = w'(\alpha_i)$. By induction we have $\langle s_\beta \rangle^\varphi = \langle s_\beta \rangle$. Moreover, $ws_iw^{-1} = s_js_\beta s_j$. If $s_\beta = s_j$, then there is nothing to prove. Otherwise we have $|s_js_\beta| \in \{2, 3, 4, 6\}$ since W is a Weyl group. Then we conclude by applying 2.1 to $\langle s_j, s_\beta \rangle$.

THEOREM 2.3. *Let W be a Weyl group. Then every index-preserving autoprojectivity of W is induced by a unique automorphism of W .*

Proof. We have to prove that $\Gamma(W) = \{1\}$. This follows from 2.2 and 1.5.

In the next two sections we shall deal with finite Coxeter groups which are not crystallographic. We shall only consider irreducible groups, that is, dihedral groups, and the groups H_3 and H_4 .

3. DIHEDRAL GROUPS

Our aim is to determine the group of autoprojectivities of dihedral groups. Let $W = \langle s_1, s_2 \rangle$. We put $\sigma = s_1$ and $\rho = s_1s_2$. Then, if the order of ρ is n , we have $W = \langle \rho \rangle \cup \{\sigma\rho^k | k = 0, \dots, n-1\}$, the group usually denoted by D_{2n} , $n \geq 2$.

If H is a subgroup of D_{2n} , then either $H \leq \langle \rho \rangle$, or $H = \langle \rho^b, \sigma \rho^a \rangle$, where b is a divisor of n . We put $\mathcal{A}(H) = \{\sigma \rho^k, k = 0, \dots, n-1\} \cap H$. We shall give a complete description of the groups $\text{Aut } D_{2n}$, $I(D_{2n})$, and $\text{Aut } L(D_{2n})$, depending on n . For this purpose we write $n = p_1^{m_1} \cdots p_r^{m_r}$, where p_1, \dots, p_r are increasing primes.

Among the groups D_{2n} , there is only one which is abelian, namely the Klein group V_4 . We have $\text{Aut } V_4 \cong S_3$ and $\text{Aut } D_{2n} \cong (\mathbb{Z}/n\mathbb{Z}) \times \triangleleft (\mathbb{Z}/n\mathbb{Z})^\times$ if $n \neq 2$.

To study $I(D_{2n})$, we introduce some notation. Let k be a natural number. We define a certain subgroup of the permutation group S_k on $\mathbb{Z}/k\mathbb{Z}$ in the following way. We consider the set $S(\mathbb{Z}/k\mathbb{Z})$ of all cosets in $\mathbb{Z}/k\mathbb{Z}$, partially ordered by inclusion. We put

$$T_k = \{\delta \in S_k \mid L \in S(\mathbb{Z}/k\mathbb{Z}) \Leftrightarrow L^\delta \in S(\mathbb{Z}/k\mathbb{Z})\}.$$

T_k is the subgroup of S_k , whose elements induce an automorphism of $S(\mathbb{Z}/k\mathbb{Z})$. For every γ in T_k , and every coset $L = a + b\mathbb{Z}/k\mathbb{Z}$ of $\mathbb{Z}/k\mathbb{Z}$, we have $L^\gamma = a^\gamma + b\mathbb{Z}/k\mathbb{Z}$.

To describe the structure of T_n , we first consider the case when k is a prime power. $S(\mathbb{Z}/p^\alpha\mathbb{Z})$ is then a tree of length α . By Lemma 5 in [5], T_{p^α} is the permutational wreath product denoted by $(S_p)^\alpha$. In particular, T_{p^α} has order $(p!)^{1+p+\dots+p^{\alpha-1}}$.

PROPOSITION 3.1. T_n is isomorphic to $T_{p_1^{m_1}} \times \cdots \times T_{p_r^{m_r}}$.

Proof. One can define an isomorphism of T_n onto $T_{p_1^{m_1}} \times \cdots \times T_{p_r^{m_r}}$ using the canonical isomorphism between $\mathbb{Z}/n\mathbb{Z}$ and $(\mathbb{Z}/p_1^{m_1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_r^{m_r}\mathbb{Z})$. We omit the details.

We shall prove that if n is not 2, then $I(D_{2n})$ is isomorphic to T_n .

Let φ be in $I(D_{2n})$, $n \neq 2$. We define the map $\theta_\varphi : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ by $\langle \sigma \rho^{a\theta_\varphi} \rangle = \langle \sigma \rho^a \rangle^\varphi$, for a in $\mathbb{Z}/n\mathbb{Z}$.

PROPOSITION 3.2. θ_φ lies in T_n .

Proof. We just write θ for θ_φ . It is enough to show that if $m|n$, then $a \equiv b \pmod{m}$ implies $a\theta \equiv b\theta \pmod{m}$. Let us consider the subgroup $\langle \rho^m, \sigma \rho^a \rangle$. We get $\langle \sigma \rho^a \rangle, \langle \sigma \rho^b \rangle \leq \langle \rho^m, \sigma \rho^a \rangle$, so that $\langle \sigma \rho^a \rangle^\varphi, \langle \sigma \rho^b \rangle^\varphi \leq \langle \rho^m, \sigma \rho^a \rangle^\varphi$. Hence $\sigma \rho^{a\theta_\varphi}$ and $\sigma \rho^{b\theta_\varphi}$ both lie in $\langle \rho^m, \sigma \rho^a \rangle^\varphi$, which is a dihedral subgroup of the same order of $\langle \rho^m, \sigma \rho^a \rangle$. Therefore $a\theta \equiv b\theta \pmod{m}$.

We denote by Θ the homomorphism from $I(W)$ to T_n , given by $\varphi \mapsto \theta_\varphi$. Let now δ be in T_n . We define $\varphi_\delta : L(W) \rightarrow L(W)$ in the following way. We first define a permutation, that we still denote by δ , of the set $\mathcal{A}(W)$ by $(\sigma \rho^a)^\delta = \sigma \rho^{a\delta}$. Let H be a subgroup of W . Then $H = \langle \rho^b \rangle \cup \mathcal{A}(H)$, $b|n$. We put $H^{\varphi_\delta} = \langle \rho^b \rangle \cup \mathcal{A}(H)^\delta$.

PROPOSITION 3.3. φ_δ is an index-preserving autoprojectivity of W .

Proof. We just write φ for φ_δ . Let $H = \langle \rho^b \rangle \cup \mathcal{A}(H)$ be a subgroup of W . We show that H^φ is a subgroup. This is obvious if $H \leq \langle \rho \rangle$. So assume $\mathcal{A}(H)$ is nonempty, and let $\sigma\rho^a$ be an element of $\mathcal{A}(H)$, so that $\mathcal{A}(H) = \{\sigma\rho^{a+kb} | k \text{ in } \mathbb{Z}\}$. Since δ is in T_n , we get $\mathcal{A}(H)^\delta = \{\sigma\rho^{a\delta+kb\delta} | k \text{ in } \mathbb{Z}\}$, so that $H^\varphi = \langle \rho^b, \sigma\rho^{a\delta} \rangle$. Therefore φ is a bijection of $L(W)$. It is enough to show that it is inclusion preserving. Let $H \leq K \leq W$. Then $H = \langle \rho^b \rangle \cup \mathcal{A}(H)$, $K = \langle \rho^c \rangle \cup \mathcal{A}(K)$. $H \leq K$ implies that $c|b$ and $\mathcal{A}(H) \subseteq \mathcal{A}(K)$. Hence $H^\varphi = \langle \rho^b \rangle \cup \mathcal{A}(H)^\delta \subseteq \langle \rho^c \rangle \cup \mathcal{A}(K)^\delta = K^\varphi$. φ is clearly index-preserving.

We denote by Δ the homomorphism from T_n to $I(W)$, given by $\delta \mapsto \varphi_\delta$.

PROPOSITION 3.4. We have

$$I(V_4) \cong S_3.$$

$$I(D_{2n}) \cong T_n, \quad \text{if } n \neq 2.$$

Proof. The first part is clear. The second part follows from the fact that the homomorphisms Θ and Δ previously defined are the inverses of each other.

COROLLARY 3.5.

$$\begin{aligned} \text{Aut } L(V_4) &\cong S_3. \\ \text{Aut } L(D_{2n}) &\cong T_n \text{ if } n \text{ is not a prime,} \\ \text{Aut } L(D_{2p}) &\cong S_{p+1} \text{ if } p \text{ is an odd prime.} \end{aligned}$$

Proof. This follows from 3.4 and 1.9.

We can now compare the groups $\text{Aut } D_{2n}$ and $I(D_{2n})$.

THEOREM 3.6. Let D_{2n} be the dihedral group of order $2n$, $n \geq 2$. Then every index-preserving autoprojectivity of D_{2n} is induced by an automorphism if and only if $n = 2, 4, 3, 6$, or 12 .

Proof. Since $I(V_4) \cong S_3$, we get $\text{Aut } V_4 = I(V_4)$. So assume $n \neq 2$. From 3.4 and 3.1, $I(W)$ has order $(p_1!)^{1+p_1+\dots+p_1^{m_1-1}} \dots (p_r!)^{1+p_r+\dots+p_r^{m_r-1}}$. On the other hand, $|\text{Aut } W| = p_1^{2m_1-1}(p_1-1) \dots p_r^{2m_r-1}(p_r-1)$. Hence $\text{Aut } W = I(W)$ if and only if $n = 4, 3, 6$, or 12 .

Remark. From Theorem 3.6, we can improve the result of Yacovlev we mentioned in the previous paragraph. In fact, in the situation of Lemma 6.1 in [15], let $G = \langle \sigma, \tau \rangle$, σ, τ involutions, and let $\psi : G \rightarrow \overline{G}$ be an index-preserving projectivity, $\langle \sigma \rangle^\psi = \langle \bar{\sigma} \rangle$ and $\langle \tau \rangle^\psi = \langle \bar{\tau} \rangle$. If $|\sigma\tau| = \infty$, then it is known that ψ is induced by an isomorphism (and this is used in

Yacovlev's proof). But now we can say that this also holds if $|\sigma\tau| \in \{2, 3, 4, 6, 12\}$, since then \overline{G} must be isomorphic to G .

4. THE GROUPS H_3 AND H_4

In this paragraph we deal with the cases left out so far, that is, with the groups of type H_3 and H_4 . We conclude by giving the list of the finite irreducible Coxeter groups which are strongly lattice determined.

We prove that for the groups H_3 and H_4 every autoprojectivity is induced by an automorphism. We begin with $W = H_3$. For this group the situation is very easy, since H_3 is the direct product of its center $\langle -1 \rangle$ and its rotation subgroup H_3^+ which is isomorphic to the alternating group $\text{Alt}(5)$.

PROPOSITION 4.1. $\text{Aut } L(H_3) = \text{Aut } H_3$.

Proof. Let φ be an autoprojectivity of H_3 . Then φ induces an autoprojectivity of H_3^+ which is induced by a unique automorphism β of H_3^+ (cf. [11]). If we denote by α the unique automorphism of H_3 inducing β on H_3^+ , we get that φ is induced by α .

We now deal with the group of type H_4 , the group with graph

$$\begin{array}{ccccccc} \circ & \equiv & \circ & - & \circ & - & \circ \\ 1 & & 2 & & 3 & & 4 \end{array} \quad (*)$$

We shall find an appropriate Coxeter generating set for W . We first make some observations. The finite simple group $\text{Alt}(5)$ (which is isomorphic to $\text{PSL}_2(5)$) can be presented in the following way:

$$\text{Alt}(5) = \langle x, y | x^5 = y^2 = (xy)^3 = 1 \rangle \quad [7, \text{I}, 19.9].$$

Suppose we are given such a presentation. From a direct calculation it follows that there are exactly two elements t in $\text{Alt}(5)$ such that $t^2 = 1$, $txt = x^{-1}$, $(ty)^3 = 1$. These two elements are $t_1 = yx^2yx^{-2}y$ and $t_2 = yx^{-2}yx^2y$ (one can take for instance $x = (12345)$ and $y = (15)(34)$, then t must be either $(15)(24)$ or $(25)(34)$). We also observe that if a, b are nontrivial elements of $\text{Alt}(5)$ such that $a^5 = b^2 = (ab)^3 = 1$, then $\langle a, b \rangle = \text{Alt}(5)$.

We give the description of H_4 following [8]. Let G be the group $\text{SL}_2(5)$. Let H be the semidirect product $(G \times G) \rtimes \langle \eta \rangle$, where $\langle \eta \rangle$ is cyclic of order 2 and η acts on $G \times G$ mapping (g_1, g_2) to (g_2, g_1) for every g_1, g_2 in G . There exists an irreducible representation ρ of H on an Euclidean space of dimension 4 over \mathbb{R} with kernel $\langle (-1, -1) \rangle$. It follows that the

group $W = H / \langle (-1, -1) \rangle$ is a Coxeter group of type H_4 . For every element x of H we shall denote by $[x]$ the corresponding element of W under the natural projection. Let Φ be the set of roots of the subgroup H^ρ of the group of isometries of E . The reflections of W (that is, the elements of W which are sent to reflections in H^ρ) are those of the form $[(g, g^{-1})\eta]$.

The rotation group W^+ coincides with the derived subgroup of W [4, p. 101, Ex. 6.8], so that $[G \times G] = W^+$ is the unique subgroup of index 2 in W . The center of W is cyclic of order 2, generated by the longest element $w_0 = [(1, -1)]$. Let $\{w_1, \dots, w_4\}$ be the Coxeter generating set of W corresponding to a certain simple system Δ of Φ , with graph $(*)$. We define another Coxeter generating set $S' = \{s_1, \dots, s_4\}$ for W .

For every x in G we denote by \bar{x} the corresponding element of $\text{PSL}_2(5)$ under the natural projection. We fix two elements a, b of G such that $a^5 = 1$, $b^2 = -1$ and $(ab)^3 = 1$, for instance

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 \\ 4 & 4 \end{pmatrix}.$$

From the previous discussion we have $\langle \bar{a}, \bar{b} \rangle = \text{PSL}_2(5)$. It also follows that $G = \langle a, b \rangle$. Let $c = ba^2ba^{-2}b$, $e = b^{-1}a^{-2}b^{-1}a^2b^{-1}$. We get $c^2 = e^2 = -1$ and $ce = -a^{-1}$.

Let $\{s_1, \dots, s_4\}$ be the set of involutions $s_1 = [\eta]$, $s_2 = [(a, a^{-1})\eta]$, $s_3 = [(b^{-1}, b)\eta]$, $s_4 = [(c, c^{-1})\eta]$. We have $|s_i s_j| = |w_i w_j|$ for every i, j . We show that $\langle s_1, \dots, s_4 \rangle = W$. It is enough to show that the subgroup $M = \langle s_2 s_1, s_2 s_3, s_4 s_3 \rangle$ coincides with $[G \times G]$. We have $M = \langle [(a, a^{-1})], [(b, b^{-1})], [(c, c^{-1})] \rangle$. We define projections

$$\pi_1, \pi_2 : [G \times G] \rightarrow \text{PSL}_2(5),$$

by

$$\pi_1([(x, y)]) = \bar{x}, \quad \pi_2([(x, y)]) = \bar{y}.$$

Let $M_i = M \wedge \ker \pi_i$. Suppose $M \neq M_1 M_2$. We get $M_1 = M_1 M_2 = M_2$. Since $[(c, e)]$ lies in M , $[(1, ce)]$ is in M_1 . But \bar{c} and \bar{e} are distinct involutions of $\text{PSL}_2(5)$, so that $[(1, ce)]$ does not lie in $\ker \pi_2$. This is a contradiction. Hence $M = M_1 M_2$. It is now enough to show that $M_1 \wedge M_2$ has order at least 2. Since $ce = -a^{-1}$, we get that $w_0 = [(1, ce)]^5$ lies in M_1 . Similarly $w_0 = [(ce, 1)]^5$ lies in M_2 , so that w_0 lies in $M_1 \wedge M_2$. Therefore $\langle s_1, \dots, s_4 \rangle = W$, and $S' = \{s_1, \dots, s_4\}$ is a Coxeter generating set. We also note that besides s_4 there exists a unique involution t in $W \setminus W^+$ such that $|s_i s_4| = |s_i t|$ for every $i = 1, 2, 3$. Namely $t = [(e, e^{-1})\eta]$.

To prove that $\text{Aut } L(W) = \text{Aut } W$, we are left to prove that $\Gamma_{S'}(W) = \{1\}$. By 1.5 it will be enough to show that for every φ in $\Gamma_{S'}(W)$, φ fixes all the subgroups of the form $\langle [(g, g^{-1})\eta] \rangle$. So let φ be in $\Gamma_{S'}(W)$. We have

$(W^+)^{\varphi} = W^+$ and $\langle w_0 \rangle^{\varphi} = \langle w_0 \rangle$. Moreover, we must have $\langle [(e, e^{-1})\eta] \rangle^{\varphi} = \langle [(e, e^{-1})\eta] \rangle$. If we consider the dihedral subgroups $\langle s_1, s_2 \rangle$, $\langle s_1, s_3 \rangle$, $\langle s_1, s_4 \rangle$, and $\langle s_1, [(e, e^{-1})\eta] \rangle$, it follows that φ also fixes the subgroups $\langle [(a, a^{-1})] \rangle$, $\langle [(b, b^{-1})] \rangle$, $\langle [(c, c^{-1})] \rangle$, and $\langle [(e, e^{-1})] \rangle$. Let $K = \{[(g, g)] | g \in G\}$. K is isomorphic to $\text{PSL}_2(5)$.

PROPOSITION 4.2. φ fixes every subgroup of K .

Proof. φ fixes the subgroups $\langle [(b, b^{-1})], w_0 \rangle$, $\langle [(c, c^{-1})], w_0 \rangle$, and $\langle [(e, e^{-1})], w_0 \rangle$, hence φ also fixes $\langle [(b, b)] \rangle$, $\langle [(c, c)] \rangle$, and $\langle [(e, e)] \rangle$. But $ce = -a^{-1}$ implies $G = \langle b, c, e \rangle$, so that $K = \langle [(b, b)], [(c, c)], [(e, e)] \rangle$ is fixed by φ . By [9], there exists a unique automorphism α of K inducing φ on K . Since $[(b, b)]$, $[(c, c)]$, and $[(e, e)]$ are involutions, α is the identity, and we are done.

PROPOSITION 4.3. For every x in G , φ fixes $\langle [(x, x^{-1})] \rangle$.

Proof. We may assume x of order 5, 3, or 4. If $|x| = 4$, then $x^{-1} = -x$, so $\langle [(x, x^{-1})] \rangle$ is fixed by φ , since $[(x, -x)] = [(x, x)]w_0$. Suppose x is of order 5 or 3. Let $s = [(x, x^{-1})]$. Then $\eta s \eta = s^{-1}$. Since φ is index preserving, there exists y in G of the same order of x such that $\langle s \rangle^{\varphi} = \langle [(y, y^{-1})] \rangle$ or $\langle [(y, -y^{-1})] \rangle$. But $|(y, -y^{-1})| = 2|x|$, so that $\langle s \rangle^{\varphi} = \langle [(y, y^{-1})] \rangle$. By 4.2 we have $\langle [(x, x)] \rangle^{\varphi} = \langle [(x, x)] \rangle$, so $[(y, y^{-1})]$ lies in the centralizer of $[(x, x)]$. It follows that $\bar{y} \in C_{\text{PSL}_2(5)}(\bar{x}) = \langle \bar{x} \rangle$. Hence $\langle [(x, x^{-1})] \rangle^{\varphi} = \langle [(x, x^{-1})] \rangle$, and we are done.

We can now prove the key step.

PROPOSITION 4.4. φ fixes every subgroup generated by a reflection.

Proof. We have to show that φ fixes $\langle [(x, x^{-1})\eta] \rangle$ for every x in G . We may assume x of order 5, 3, or 4. By 4.3, φ fixes the dihedral subgroup $\langle \eta, [(x, x^{-1})] \rangle$. If $|x| = 4$, we are done. So assume $|x| = 3$. Then $\langle [(x, x^{-1})\eta] \rangle^{\varphi} = \langle [(x, x^{-1})\eta] \rangle$ or $\langle [(x^{-1}, x)\eta] \rangle$. Suppose that $\langle [(x, x^{-1})\eta] \rangle^{\varphi} = \langle [(x^{-1}, x)\eta] \rangle$. Then $\langle [(x, x^{-1})\eta], s_2 \rangle^{\varphi} = \langle [(x^{-1}, x)\eta], s_2 \rangle$ implies $|\bar{x}\bar{a}| = |\bar{x}^{-1}\bar{a}|$. We consider the trace map $T : G \rightarrow \mathbb{Z}/5\mathbb{Z}$. We have $T(x) = -1$, $T(xa) \neq -1$ (since the elements of order 3 in G are not diagonalizable in G) and $T(x^{-1}a) = 3 - T(xa)$. But $|\bar{x}\bar{a}| = |\bar{x}^{-1}\bar{a}|$ implies $T(x^{-1}a) = \pm T(xa)$, so that $T(xa) = -1$, which is a contradiction. Hence $\langle [(x, x^{-1})\eta] \rangle^{\varphi} = \langle [(x, x^{-1})\eta] \rangle$. We are left with the case $|x| = 5$. Let $\langle [(x, x^{-1})\eta] \rangle^{\varphi} = \langle [(x^i, x^{-i})\eta] \rangle$. There exists w in $\text{GL}_2(5)$ such that $x = waw^{-1}$. Let $y = wbw^{-1}$. We have already proved that $\langle [(y^{-1}, y)\eta] \rangle^{\varphi} = \langle [(y^{-1}, y)\eta] \rangle$, so that $\langle [(y^{-1}, y)\eta], [(x, x^{-1})\eta] \rangle^{\varphi} = \langle [(y^{-1}, y)\eta], [(x^i, x^{-i})\eta] \rangle$. As before we get $|\bar{x}\bar{y}| = |\bar{x}^i\bar{y}|$, and the same trace argument leaves us with $i = \pm 1$. To conclude, we have to exclude $i = -1$. We consider separately the cases when $\langle x \rangle = \langle a \rangle$ and $\langle x \rangle \neq \langle a \rangle$. Suppose

we have $\langle x \rangle \neq \langle a \rangle$, and let $\langle [(x, x^{-1})\eta] \rangle^\varphi = \langle [(x^{-1}, x)\eta] \rangle$. We proceed as in the case when $|x| = 3$, considering the subgroup $\langle [(x, x^{-1})\eta], s_2 \rangle$. We get $T(x) = 2$, $T(xa) \neq 2$ (since $\langle x \rangle \neq \langle a \rangle$) and $T(x^{-1}a) = 4 - T(xa)$. $|\bar{x}\bar{a}| = |\bar{x}^{-1}\bar{a}|$ implies $T(x^{-1}a) = \pm T(xa)$, so that $T(xa) = 2$, which is a contradiction. Finally assume $x = a^k$. Since $\langle s_2 \rangle^\varphi = \langle s_2 \rangle$, we get $\langle [(a^{-1}, a)\eta] \rangle^\varphi = \langle [(a^{-1}, a)\eta] \rangle$. Suppose that $\langle [(a^2, a^{-2})\eta] \rangle^\varphi = \langle [(a^{-2}, a^2)\eta] \rangle$, that is $\langle s_2 s_1 s_2 \rangle^\varphi = \langle s_2 s_1 s_2 s_1 s_2 \rangle$, and let $\tau = s_2 s_3 s_2$. Since $|s_2 s_3| = 3$, we have $\langle \tau \rangle^\varphi = \langle \tau \rangle$. Hence $\langle s_2 s_1 s_2, \tau \rangle^\varphi = \langle s_2 s_1 s_2 s_1 s_2, \tau \rangle$. But $|s_2 s_1 s_2 \tau| = 2$, while $|s_2 s_1 s_2 s_1 s_2 \tau| = 3$. This is the final contradiction and we are done.

COROLLARY 4.5. $\text{Aut } L(H_4) = \text{Aut } H_4$.

We have therefore determined the group of autoprojectivities of every finite irreducible Coxeter group. We summarize the results obtained in

THEOREM 4.6. *Let W be a finite irreducible Coxeter group. Then every index preserving autoprojectivity of W is induced by a (unique) automorphism if and only if either W is not dihedral or W is dihedral of order $2n$, with $n = 2, 4, 3, 6, 12$.*

Proof. This follows from 1.1, 2.3, 3.6, 4.1, 4.5.

COROLLARY 4.7. *Let W be a finite irreducible Coxeter group. Then $\text{Aut } L(W) = \text{Aut } W$ if and only if either W is not dihedral or W is dihedral of order $2n$, with $n = 2, 4, 6, 12$.*

Proof. This follows from 4.6 and 1.9.

Taking into account the results of Uzawa ([14]), we can determine which irreducible Coxeter groups are strongly lattice determined.

THEOREM 4.8. *Let W be a finite irreducible Coxeter group. Then W is strongly lattice determined if and only if either it has rank at least 3, or it is dihedral of order $2n$, with $n = 2, 4, 6, 12$.*

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