

SPACE-TIME TRAJECTORIES OF NONLINEAR SYSTEMS DRIVEN BY ORDINARY AND IMPULSIVE CONTROLS

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Abstract. In this paper we analyze nonlinear systems whose vector fields are driven by both an ordinary control v and the derivative \dot{u} of a second control u . It is clear that the presence of the derivative \dot{u} causes some problems in the attempt to define a robust notion of trajectory corresponding to a discontinuous control u . On one hand, the nonlinearity of the systems under consideration makes a mere extension of the notion of solution by means of a measure-theoretical approach impossible. On the other hand, the interaction occurring between the derivative \dot{u} of the *impulsive* control u and the ordinary control v prevent us from applying the results of [5], where a space-time extension of the concept of solution was already considered. Indeed in [5] the control (u, v) had equibounded variation, whereas in the present work v is just a measurable bounded map.

After giving a stable notion of (space-time) solution to our problem we investigate the topological structure of the set of trajectories: as a consequence of having suppressed the restraint on the variation of v we find that this set is no longer compact. This motivates the search for conditions under which the set of trajectories turns out to be closed again. Finally, in view of several applications we also analyze the case where the derivative \dot{u} is constrained to range over a closed cone.

1. Introduction. This paper is concerned with control systems of the form

$$\dot{x} = g_0(t, x, u, v) + \sum_{i=1}^m g_i(t, x, u, v) \dot{u}_i, \quad (1.1)$$

where the state x ranges in R^n , and the controls u and v take values on a closed, arcwise connected subset $U \subset R^m$ and a compact subset $V \subset R^q$, respectively.

The main feature of the control system (1.1) consists in the evident asymmetry between the roles of the control u and the control v . Indeed, the derivative of only the former appears on the right-hand side of (1.1). Loosely speaking this implies that a discontinuity of $u(\cdot)$ will produce a jump of the trajectory $x(\cdot)$, while a discontinuity of the ordinary control v does not affect the continuity of $x(\cdot)$.

Systems of the form (1.1) are of interest from the theoretical point of view—see e.g. [5, 6, 9, 15, 17, 23, 26]—in that they generalize the notion of ordinary control systems. Furthermore the study of these systems is motivated by several applications. For instance, they govern the dynamics of a class of mechanical systems where time-dependent constraints are utilized as controls [7, 8, 11, 21]. They are also utilized in aero-space engineering for the study of space-craft navigation [13, 16]. Moreover they arise in economics [10, 25], e.g. as models for market dynamics controlled by advertising.

The presence of the derivative \dot{u} gives an impulsive character to the system as soon as we admit discontinuous controls u . Observe that, in accord with the notion of Carathéodory solution to (1.1), we may consider only control components u in the class of *absolutely continuous* functions, while the control component v is allowed to be any *measurable* map.

Yet, in view of the asymptotic behaviour of minimizing sequences connected to optimum problems, we are led to admit control components u which merely are maps with bounded variation, so the derivative \dot{u} can be interpreted as a measure. This fact demands an extension of the classical notion of solution of the system (1.1).

In the linear case, i.e., when the vector fields g_i depend only on t ([14], [24]), the notion of solution can be extended by simply interpreting equation (1.1) as an equation in measure. On the contrary, this method cannot be pursued in the nonlinear case. Indeed, as observed by Hájek in [12], the resulting concept of solution would not satisfy elementary properties of well-posedness. Therefore, we avoid the measure-theoretical approach—which was pursued e.g. in [2, 19, 22, 27]—and perform the needed extension in the same spirit as in [4, 5, 26]. That is, by just identifying controls and trajectories with their graphs we relate (1.1) with an equivalent space-time control system. Secondly, we extend both the notion of control and the one of trajectory by allowing space-time valued maps, whose spatial components may also evolve during parameters' intervals where the time variable remains constant.

A posteriori we justify the proposed extension by proving the continuity of the corresponding input-output map with respect to appropriate parameter-free pseudometrics. In particular the continuity of the input-output map implies that the set of trajectories of the space-time system is included in the closure of the set of trajectories of the original system.

Under the hypothesis of equibounded variation for the control-pair (u, v) , systems of the form

$$\dot{x} = g_0(t, x, u, v) + \sum_{i=1}^m g_i(t, x, u, v) \dot{u}_i + \sum_{j=1}^q g_{m+j}(t, x, u, v) \dot{v}_j, \quad (1.2)$$

were already considered in [5, 6, 20]. However, the bound on the variation of (u, v) prevents us to regard (1.1) as the specialization of (1.2) to the case where $g_{m+1} = \dots = g_{m+q} = 0$. Indeed, apart from measurability, here we do not impose any other condition on v . Of course, the converse is true, i.e., (1.1) is a generalization of the system (1.2): this can be seen by simply identifying the pair (u, v) in (1.2) with the control u in (1.1).

As a consequence of the fact that we do not restrain the variation of v , we lose any compactness property of the set of (space-time) trajectories of (1.1). Indeed, the interaction between the ordinary control v and the derivative \dot{u} complicates the structure of the set of trajectories. Yet, the lost compactness of this set can be recovered by adding a convexity condition which generalizes the usual convexity condition for ordinary control systems.

In the last section of the present paper we investigate the dynamics of a control system obtained from (1.1) by constraining the derivative \dot{u} to range over a closed cone C . In particular, it is shown that if the cone C is a convex subset of the positive orthant $[0, +\infty]^m$ and the vector fields g_i satisfy a suitable independence assumption, the afore-mentioned convexity condition turns out to be considerably simplified.

The constraint on the \dot{u} 's direction appears quite natural in several applications and is then assumed in most papers on impulsive control [2, 17, 18, 22, 27]. In these papers the derivative \dot{u} is replaced by a measure μ , so the relation $\dot{u} \in C$ is expressed by saying that the range of μ must belong to C . We remark that in the quoted papers the holonomic constraint $u \in U$ is not considered at all, while here we impose it together with the nonholonomic constraint $\dot{u} \in C$.

Notations. Throughout this paper we shall adopt the following notations. $B_m[x_0, r]$ is the closed ball of \mathbb{R}^m centered at x_0 with radius r , and $B_m(x_0, r)$ is its interior. $AC([a, b], U)$ denotes the set of absolutely continuous functions from $[a, b]$ into U , and $B([a, b], V)$ is

the set of Borel measurable functions from $[a, b]$ into V . Let u be a map from $[a, b]$ to U . We recall that u is *Borel measurable* if the preimage $u^{-1}(E)$ of any Borel set E is a Borel set, while u is *Lebesgue measurable* if $u^{-1}(E)$ is just a Lebesgue measurable set. Moreover, we recall that the total variation $V_a^b(u)$ is defined as the supremum of the values

$$\sum_{i=0}^{r-1} |u(t_{i+1}) - u(t_i)|,$$

as the subset $\{t_0, \dots, t_r\}$, $a = t_0 < \dots < t_r = b$, varies among the partitions of the interval $[a, b]$. The set of functions $u : [a, b] \rightarrow U$ whose variation is bounded will be denoted by $BV([a, b], U)$. Incidentally, we recall that for every $u \in AC([a, b], U)$ one has

$$V_a^b(u) = \int_a^b |\dot{u}(s)| ds.$$

2. Space-time embedding. We consider a control system of the form

$$\dot{x} = g_0(t, x, u, v) + \sum_{i=1}^m g_i(t, x, u, v) \dot{u}_i, \quad (2.1)$$

where x ranges in \mathbb{R}^n , and u and v take values on a closed arcwise connected subset $U \subset \mathbb{R}^m$ and a compact subset $V \subset \mathbb{R}^q$, respectively. The vector fields g_0, \dots, g_m are assumed Lipschitz continuous on $\mathbb{R}^{1+n+m+q}$. Hence for any initial condition

$$x(t_1) = x_1 \in \mathbb{R}^n, \quad u(t_1) = u_1 \in U, \quad (2.2)$$

and for any control $(u, v) \in AC([t_1, t_2], U) \times B([t_1, t_2], V)$, with $0 \leq t_1 < t_2 \leq T$ and $u(t_1) = u_1$, there exists a uniquely defined (Carathéodory) solution $x[t_1, x_1, u_1; u, v]$ of (2.1), (2.2); indeed, since the derivative \dot{u} is integrable, the right-hand side of (2.1) satisfies the Carathéodory conditions for global existence (and uniqueness) of solutions. Note that the assumed Borel measurability of the control v is not restrictive, since for each Lebesgue measurable map v^* there is a Borel measurable map v which coincides with v^* almost everywhere. On the other hand, the class of Borel measurable maps has the advantage to be closed under superposition, and this property will be useful when we shall deal with reparametrizations of the time.

A control (u, v) belonging to $AC([t_1, t_2], U) \times B([t_1, t_2], V)$ will be called *regular*. It is clear that non regular controls are needed as soon as one considers minimum problems. Indeed it can happen that a minimizing sequence of regular controls (u_n, v_n) does not converge to a regular control, as is shown by the following simple example.

Example 2.1. Let $U = [0, 1]$ and consider the optimal control problem:

$$\text{minimize } x_2(1),$$

$$\dot{x}_1 = \dot{u}$$

$$\dot{x}_2 = \eta(t, x_1)$$

$$x_1(0) = x_2(0) = u(0) = 0,$$

where the map η is defined by

$$\eta(\alpha, \beta) \doteq \begin{cases} \gamma(\alpha, \beta) - \gamma(\alpha, 0), & \forall \alpha \leq 1/2 \\ \gamma(\alpha, \beta) - \gamma(\alpha, 2), & \forall \alpha \geq 1/2, \end{cases}$$

with

$$\gamma(\alpha, \beta) \doteq 3(\beta - 1)^4 + 4(\alpha - 1/2)(\beta - 1)^3 - 6(\beta - 1)^2 - 12(\alpha - 1/2)(\beta - 1).$$

Note that for $\alpha \leq 1/2$, the map $\eta(\alpha, \cdot)$ attains its global minimum at $\beta = 0$, while, for $\alpha \geq 1/2$, the global minimum is reached at $\beta = 2$. Moreover, in both cases the minimum value of the map $\eta(\alpha, \cdot)$ is zero.

Since for each $u \in AC([0, 1], U)$ with $u(0) = 0$ we have $u(t) = x_1(t)$, it is easy to verify that

$$u_n(t) = \begin{cases} 0, & \forall t \in [0, \frac{1}{2} - \frac{1}{n}], \\ n(t - (\frac{1}{2} - \frac{1}{n})), & \forall t \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}], \\ 2, & \forall t \in [\frac{1}{2} + \frac{1}{n}, 1] \end{cases}$$

defines a minimizing sequence which converges to a piecewise-constant map exhibiting a discontinuity at $t = 1/2$.

When the control component u merely belongs to $BV([t_1, t_2], U)$, the derivative \dot{u} can be interpreted as a Radon measure, and one could regard (2.1) as an equation *in measure*—see e.g. [27]. Yet we do not adopt this approach, for it leads to an unstable notion of solution.

Instead, like in [5] we embed the original control system into an extended control system, where the control (u, v) may evolve during intervals (of the new parameter) where the time is constant. To accomplish this program, let us first look at the regular case: in place of a regular control (u, v) one can consider its graph $(t, u(t), v(t))$ and replace (2.1), (2.2) with the Cauchy problem corresponding to the state $y = (t, x, u)$,

$$\begin{cases} \dot{y}_0 = 1 \\ \dot{y}_j = g_{0j}(y, v) + \sum_{i=1}^m g_{ij}(y, v) \dot{u}_i & j = 1, \dots, n \\ \dot{y}_{n+k} = \dot{u}_k & k = 1, \dots, m, \end{cases}$$

$$y(t_1) = (t_1, x_1, u_1),$$

where g_{ij} denotes the j -th component of the vector field g_i . To get rid of the particular parameter t one can also consider a reparametrization of the graph $(\varphi_0(s), \varphi(s), \psi(s)) \doteq (t(s), u(t(s)), v(t(s)))$, where $t(s) (= \varphi_0(s))$ is a Lipschitz continuous nondecreasing map from $[0, 1]$ onto $[t_1, t_2]$. Thus one obtains the control system

$$\dot{y} = \sum_{i=0}^m \hat{g}_i(y, \psi) \dot{\varphi}_i, \quad (2.3)$$

with the initial condition

$$y(0) = (t_1, x_1, u_1), \quad (2.4)$$

where differentiation, still denoted with a dot, is now referred to the new parameter s and \hat{g}_i denotes the i -th column of the $(1+n+m) \times (1+m)$ matrix

$$\hat{G}(y, v) \doteq \begin{pmatrix} 1 & 0 & \dots & 0 \\ g_{01} & g_{11} & \dots & g_{m1} \\ \vdots & \vdots & \ddots & \vdots \\ g_{0n} & g_{1n} & \dots & g_{mn} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Observe that the extended system (2.3) has a *parameter-free* character. Precisely, if $(\tilde{\varphi}_0, \tilde{\varphi}, \tilde{\psi})(s) \doteq (\varphi_0, \varphi, \psi)(\tilde{s}(s))$, with $\tilde{s}(\cdot)$ a reparametrization of $[0, 1]$, one can check that $\tilde{y}(s) = y(\tilde{s}(s))$, where \tilde{y} and y denote the solutions to (2.3), (2.4) corresponding to the space-time controls $(\tilde{\varphi}_0, \tilde{\varphi}, \tilde{\psi})$ and $(\varphi_0, \varphi, \psi)$, respectively. By allowing the components (φ, ψ) to evolve even on the intervals where φ_0 is constant, in Definition 2.1 we introduce a notion of (space-time) control which is more general than the graph of a regular control.

Definition 2.1. The control system (2.3) is called the *space-time control system* relative to (2.1), and a map

$$(\varphi_0, \varphi, \psi) : [0, 1] \rightarrow [t_1, t_2] \times U \times V,$$

$0 \leq t_1 < T, t_1 \leq t_2 \leq T$, is called a *space-time control* for (2.3), (2.4) if

- (i) $(\varphi_0, \varphi)(0) = (t_1, u_1)$;
- (ii) (φ_0, φ) is Lipschitz continuous;
- (iii) φ_0 is surjective and monotone nondecreasing;
- (iv) ψ is Borel measurable.

The set of space-time controls will be denoted by $\Gamma(t_1, u_1)$. Finally, a solution of the space-time control system (2.3) will be called a *space-time trajectory*.

We observe that, because of the form of equation (2.3), one is allowed to replace a given control component ψ with any other element of its L^∞ -equivalence class. The (unique) space-time trajectory of (2.3), (2.4) corresponding to a space-time control $(\varphi_0, \varphi, \psi)$ will be denoted by $y[t_1, x_1, u_1; \varphi_0, \varphi, \psi]$. In the situations where the initial conditions are kept fixed we also adopt the simpler notations Γ and $y[\varphi_0, \varphi, \psi]$ instead of $\Gamma(t_1, u_1)$ and $y[t_1, x_1, u_1; \varphi_0, \varphi, \psi]$, respectively.

Remark 2.1. It is clear that a space-time control is not necessarily the graph of a regular control. In fact, this happens if and only if φ is constant on every subinterval $[\gamma_1, \gamma_2] \subset [0, 1]$ where φ_0 is constant. On the contrary, in general there exist subintervals of $[0, 1]$ where φ evolves while φ_0 remains constant. The latter will be called *intervals of instantaneous evolution*.

A (non regular) control $(u, v) \in BV([t_1, t_2], U) \times B([t_1, t_2], V)$ can be naturally associated to a class of space-time controls, called the *graph-completions* of (u, v) , as specified by the following definition.

Definition 2.2. A *graph completion* of $(u, v) \in BV([t_1, t_2], U) \times B([t_1, t_2], V)$ is a space-time control $(\varphi_0, \varphi, \psi)$ satisfying

- (i) $\varphi_0([0, 1]) = [t_1, t_2]$;
- (ii) $\forall t \in [t_1, t_2], \exists s \in [0, 1]$ such that $(\varphi_0, \varphi, \psi)(s) = (t, u(t), v(t))$.

Hence, while there does not exist a well-defined notion of solution to (2.1), (2.2) corresponding to a control $(u, v) \in BV([t_1, t_2], U) \times B([t_1, t_2], V)$, there are no problems in considering the (space-time) solution $y[\varphi_0, \varphi, \psi]$ corresponding to a graph-completion $(\varphi_0, \varphi, \psi)$ of (u, v) . Obviously, this solution depends on the chosen graph-completion; loosely speaking it depends not only on u and v , but also on the restrictions of φ and ψ to the intervals of instantaneous evolution. Notice that this latter set of intervals is at most countable and contains those intervals on which the constant value of the time represents a discontinuity instant of u . If the subset U is convex, an example of graph-completion is easily obtained by filling the gaps of u by means of rectilinear arcs and by assigning arbitrary values to $\psi (\equiv v)$ on the intervals parameterizing these arcs; the Lipschitz continuity is easily achieved by reparameterizing the so obtained space-time curve by the arc-length abscissa. In a simplified situation where the fields do not depend on v , this is what is done in [5].

We remark that in general there is no reason to privilege the rectilinear bridging of gaps. For example, in minimum problems one can find an optimal control which contains intervals of instantaneous evolution; in such a case it is the functional to minimize which imposes the form of the instantaneous arc.

3. Topologies for space-time controls and space-time trajectories. In order to study the input-output map associated to (2.3), (2.4), in this section we introduce topologies on the set of space-time controls and the set of the corresponding trajectories. In fact, the resulting continuity of the input-output map will be utilized in order to investigate the structure of the set of solutions. Both the topologies we are going to consider keep track of the parameter free character of the extended system (2.3).

We shall associate to each space-time control $(\varphi_0, \varphi, \psi)$ a new space-time control which will be called the *canonical parameterization* of $(\varphi_0, \varphi, \psi)$. To this end, if (φ_0, φ) is not identically constant, let us consider the map σ from $[0, 1]$ onto itself defined by

$$\sigma(s) \doteq \frac{V_0^s(\varphi_0, \varphi)}{V_0^1(\varphi_0, \varphi)} = \frac{\int_1^s |(\dot{\varphi}_0, \dot{\varphi})| ds}{\int_1^1 |(\dot{\varphi}_0, \dot{\varphi})| ds}. \quad (3.1)$$

For every $s \in [0, 1]$, $\sigma(s)$ is the *normalized variation* of the curve (φ_0, φ) on the interval $[0, s]$. It is a monotone nondecreasing Lipschitz continuous map from $[0, 1]$ onto $[0, 1]$; moreover it is constant on every interval where (φ_0, φ) is constant.

Now, if (φ_0, φ) is constant on the whole interval $[0, 1]$, set

$$(\varphi_0^c, \varphi^c, \psi^c) \doteq (\varphi_0, \varphi, \psi),$$

while, if (φ_0, φ) is not constant, set

$$(\varphi_0^c, \varphi^c, \psi^c)(\sigma) \doteq (\varphi_0, \varphi, \psi)(s), \quad \sigma = \sigma(s). \quad (3.2)$$

In principle (3.2) only defines a multivalued map. Yet (φ_0^c, φ^c) is univalued and ψ^c is uniquely determined almost everywhere. More precisely we have

Proposition 3.1. *The relation (3.2) defines a Lipschitz continuous map (φ_0^c, φ^c) on $[0, 1]$, and the derivative $(\dot{\varphi}_0^c, \dot{\varphi}^c)$, which exists almost everywhere, has constant modulus equal to $V_0^1(\varphi_0, \varphi)$. Moreover, (3.2) defines a univalued Borel-measurable map almost everywhere in $[0, 1]$.*

To prove Proposition 3.1 we need Lemma 3.1 below on *pseudoinverses* of monotone nondecreasing maps. If $\alpha : [0, 1] \rightarrow [0, 1]$ is a monotone nondecreasing surjective (continuous) map, we set

$$\hat{\alpha}^{-1}(\xi) \doteq \min \alpha^{-1}(\{\xi\}),$$

where $\alpha^{-1}(\{\xi\})$ denotes the preimage of the singleton $\{\xi\}$. Since $\alpha^{-1}(\{\xi\})$ is a compact interval of $[0, 1]$, the map $\hat{\alpha}^{-1}$ is well defined. It will be called the *pseudoinverse* of the map α .

Lemma 3.1. *The pseudoinverse $\hat{\alpha}^{-1} : [0, 1] \rightarrow [0, 1]$ is strictly increasing and is a right inverse of α , i.e., it satisfies*

$$\alpha \circ \hat{\alpha}^{-1}(\xi) = \xi \quad (3.3)$$

for every $\xi \in [0, 1]$. Furthermore, if $\gamma = \alpha \circ \beta$, where α, β, γ are nondecreasing maps of $[0, 1]$ onto itself, one has

$$\hat{\gamma}^{-1}(\xi) = \hat{\beta}^{-1} \circ \hat{\alpha}^{-1}(\xi) \quad (3.4)$$

for every $\xi \in [0, 1]$.

Proof. Equality (3.3) is straightforward, since $\hat{\alpha}^{-1}(\xi) \in \alpha^{-1}(\{\xi\})$. As for (3.4), one has

$$\hat{\gamma}^{-1}(\xi) = \min(\alpha \circ \beta)^{-1}(\{\xi\}) = \min \beta^{-1}(\alpha^{-1}(\{\xi\})) \leq \min \beta^{-1}(\{\min \alpha^{-1}(\{\xi\})\}),$$

for every $\xi \in [0, 1]$. Since the monotonicity of β implies $r \geq s$ whenever $r \in \beta^{-1}(\alpha^{-1}(\{\xi\}))$ and $s \in \beta^{-1}(\{\min \alpha^{-1}(\{\xi\})\})$, one has also

$$\min \beta^{-1}(\alpha^{-1}(\{\xi\})) \geq \min \beta^{-1}(\{\min \alpha^{-1}(\{\xi\})\}),$$

so

$$\hat{\gamma}^{-1}(\xi) = \min \beta^{-1}(\{\min \alpha^{-1}(\{\xi\})\}) = \hat{\beta}^{-1} \circ \hat{\alpha}^{-1}(\xi) \quad \text{for every } \xi \in [0, 1].$$

Proof of Proposition 3.1. The facts concerning (φ_0^c, φ^c) are well known since σ is the arc-length parameter divided by $V_0^1(\varphi_0, \varphi)$. Let us prove the statement concerning ψ^c . Observe that, since σ is a nondecreasing map from $[0, 1]$ onto $[0, 1]$, the set \mathcal{N} of points ξ such that $\sigma^{-1}(\{\xi\})$ is a nondegenerate interval is at most countable. Hence the map ψ^c is defined almost everywhere by (3.2), as it coincides with the map $\psi \circ \hat{\sigma}^{-1}$ on the subset $[0, 1] \setminus \mathcal{N}$. Since ψ^c needs only to be defined almost everywhere on $[0, 1]$, it is not restrictive to set $\psi^c(\xi) = \psi \circ \hat{\sigma}^{-1}(\xi)$ for every $\xi \in [0, 1]$. Thus the Borel measurability of ψ^c follows from the monotonicity of $\hat{\sigma}^{-1}$. \square

Thanks to Proposition 3.1 we can give the notion of *canonical parameterization*.

Definition 3.1. The space-time control $(\varphi_0^c, \varphi^c, \psi^c)$ defined by relation (3.2) is called the *canonical parameterization* of $(\varphi_0, \varphi, \psi)$.

Definition 3.2. Let $(\varphi_{01}, \varphi_1, \psi_1)$, $(\varphi_{02}, \varphi_2, \psi_2)$ be two space-time controls and let $(\varphi_{01}^c, \varphi_1^c, \psi_1^c)$, $(\varphi_{02}^c, \varphi_2^c, \psi_2^c)$ be their canonical parameterizations. Then $(\varphi_{01}, \varphi_1, \psi_1)$ will be called *equivalent* to $(\varphi_{02}, \varphi_2, \psi_2)$ if $(\varphi_{01}^c, \varphi_1^c)(s) = (\varphi_{02}^c, \varphi_2^c)(s)$, $\forall s \in [0, 1]$ and $\psi_1^c(s) = \psi_2^c(s)$ for almost every s in $[0, 1]$.

An important example of equivalence is provided by the case of two space-time controls where the latter is obtained from the former by reparametrization. Precisely, we have

Proposition 3.2. *Consider two space-time controls $(\varphi_{01}, \varphi_1, \psi_1)$, $(\varphi_{02}, \varphi_2, \psi_2)$ such that, for a Lipschitz continuous monotone nondecreasing map α from $[0, 1]$ onto itself, one has*

$$(\varphi_{02}, \varphi_2, \psi_2)(s) = (\varphi_{01}, \varphi_1, \psi_1) \circ \alpha(s)$$

for every $s \in [0, 1]$. Then the space-time controls $(\varphi_0, \varphi, \psi)$, $(\varphi_0, \varphi_2, \psi_2)$ are equivalent.

Proof. Setting

$$\sigma_i = \frac{\int_0^s |(\varphi_{0i}, \dot{\varphi}_i)|(s') ds'}{\int_0^1 |(\varphi_{0i}, \dot{\varphi}_i)|(s') ds'},$$

with $i = 1, 2$, we have

$$\sigma_2 = \sigma_1 \circ \alpha. \quad (3.5)$$

Indeed, if

$$L = \int_0^1 |(\varphi_{01}, \dot{\varphi}_1)|(s) ds = \int_0^1 |(\varphi_{02}, \dot{\varphi}_2)|(s) ds,$$

then for $i = 1, 2$ $L \cdot \sigma_i(s)$ represents the length of the restriction of the curve $(\varphi_{0i}, \dot{\varphi}_i)$ to the interval $[0, s]$. Hence (3.5) follows from the same definition of $(\varphi_{02}, \varphi_2)$ and by the fact that the length of a curve is independent of its parameterization.

Denoting by $\hat{\sigma}_1^{-1}, \hat{\sigma}_2^{-1}, \hat{\alpha}^{-1}$ the pseudoinverses of $\sigma_1, \sigma_2, \alpha$, respectively, by (3.5) and Lemma 3.1 we obtain

$$\hat{\sigma}_2^{-1}(\xi) = \hat{\alpha}^{-1} \circ \hat{\sigma}_1^{-1}(\xi)$$

for every $\xi \in [0, 1]$. Then Lemma 3.1 again implies

$$\alpha \circ \hat{\sigma}_2^{-1} = \hat{\sigma}_1^{-1},$$

which finally yields

$$\psi_2^c = \psi_1 \circ \alpha \circ \hat{\sigma}_2^{-1} = \psi_1 \circ \hat{\sigma}_1^{-1} = \psi_1^c. \quad \square$$

Proposition 3.3 below illustrates the relationship between the trajectory $y[\varphi_0, \varphi, \psi]$ corresponding to a space-time control $(\varphi_0, \varphi, \psi)$ and the trajectory $y[\varphi_0^c, \varphi^c, \psi^c]$ corresponding to the canonical parameterization $(\varphi_0^c, \varphi^c, \psi^c)$.

Proposition 3.3. Fix the initial condition (2.4). Then the trajectories

$$y[\varphi_0, \varphi, \psi], \quad y[\varphi_0^c, \varphi^c, \psi^c]$$

satisfy the relation

$$y[\varphi_0^c, \varphi^c, \psi^c](\xi) = y[\varphi_0, \varphi, \psi](\sigma^{-1}(\{\xi\})) \quad (3.6)$$

for every $\xi \in [0, 1]$, where $\sigma^{-1}(\{\xi\})$ denotes the preimage of the singleton $\{\xi\}$.

Proof. Like in the proof of the previous proposition, let $\mathcal{N} \subset [0, 1]$ denote the subset of points ξ whose preimages $\sigma^{-1}(\xi)$ are nondegenerate intervals. If $\xi \in \mathcal{N}$, then $|(\dot{\varphi}_0, \dot{\varphi})|(s) = 0$ for almost every s belonging to the interval $[a, b] \doteq \sigma^{-1}(\xi)$. Since for every $s \in [a, b]$ one has

$$y[\varphi_0, \varphi, \psi](s) - y[\varphi_0, \varphi, \psi](a) = \int_a^s \sum_{i=0}^m \hat{g}_i(y(s'), \psi(s')) \dot{\varphi}_i(s') ds',$$

it follows that $y[\varphi_0, \varphi, \psi]$ is constant on the whole $[a, b]$. Thus in order to prove (3.6) it is sufficient to show that

$$y[\varphi_0^c, \varphi^c, \psi^c] \circ \sigma(s) = y[\varphi_0, \varphi, \psi](s) \quad (3.7)$$

for every $s \in [0, 1]$. Indeed, setting $y^c \doteq y[\varphi_0^c, \varphi^c, \psi^c]$ and recalling that $\sigma = \frac{1(\varphi_0, \dot{\varphi})}{V_0(\varphi_0, \dot{\varphi})} = 0$ almost everywhere in $\sigma^{-1}(\mathcal{N})$, for every $\bar{s} \in [0, 1]$ we have

$$\begin{aligned} y^c \circ \sigma(\bar{s}) - (t_1, x_1, u_1) &= \int_0^{\sigma(\bar{s})} \sum_{i=0}^m \hat{g}_i(y^c(s'), \psi^c(s')) \frac{d\varphi_i^c}{ds'} ds' \\ &= \int_0^{\bar{s}} \sum_{i=0}^m \hat{g}_i(y^c(\sigma(s)), \psi^c(\sigma(s))) \frac{d\varphi_i^c}{ds'}(\sigma(s)) \frac{d\sigma}{ds}(s) ds \\ &= \int_0^{\bar{s}} \left[\sum_{i=0}^m \hat{g}_i(y^c \circ \sigma(s), \psi(s)) \frac{d\varphi_i}{ds}(s) \right] \cdot \chi_{[0,1] \setminus \sigma^{-1}(\mathcal{N})}(s) ds \\ &= \int_0^{\bar{s}} \left[\sum_{i=0}^m \hat{g}_i(y^c \circ \sigma(s), \psi(s)) \frac{d\varphi_i}{ds}(s) \right] \cdot \chi_{\sigma^{-1}(\mathcal{N})}(s) ds \\ &\quad + \int_0^{\bar{s}} \left[\sum_{i=0}^m \hat{g}_i(y^c \circ \sigma(s), \psi(s)) \frac{d\varphi_i}{ds}(s) \right] \cdot \chi_{[0,1] \setminus \sigma^{-1}(\mathcal{N})}(s) ds \\ &= \int_0^{\bar{s}} \sum_{i=0}^m \hat{g}_i(y^c \circ \sigma(s), \psi(s)) \frac{d\varphi_i}{ds}(s) ds, \end{aligned}$$

where for any subset $E \subset [0, 1]$, χ_E denotes the characteristic function of E . Hence both $y^c \circ \sigma$ and y solve the Cauchy problem (2.3), (2.4); since the solution of the latter is unique, (3.7) holds true for every $s \in [0, 1]$. \square

Let us introduce a pseudometric δ^c on the space of space-time controls. For every pair $(\varphi_0, \varphi_1, \psi_1), (\varphi_0, \varphi_2, \psi_2)$ let us set

$$\delta^c((\varphi_0, \varphi_1, \psi_1), (\varphi_0, \varphi_2, \psi_2)) \doteq \|(\varphi_0^c, \varphi_1^c) - (\varphi_0^c, \varphi_2^c)\| + \|\psi_1^c - \psi_2^c\|_1,$$

where $\|\cdot\|$ and $\|\cdot\|_1$ denote the C^0 norm and the L^1 norm, respectively. Hence two space-time controls have δ^c pseudodistance equal to zero if and only if they are equivalent. Incidentally this implies that δ^c induces a metric on the quotient space.

The choice of this topology will be justified in the next section, where we show the continuity of the input-output map. A simpler choice would have been represented by the $C^0 \times L^1$ topology; unfortunately, with respect to this topology the input-output map is not continuous—see Remark 4.1.

Let us conclude this section by introducing a pseudometric on the set of trajectories of the control system (2.3), (2.4). Since our main goal is the continuity of the input-output map, we must choose a pseudometric such that the pseudodistance between a path and a reparametrization is zero. This prerequisite is satisfied by the pseudodistance δ defined as follows. If y_1, y_2 are two continuous paths from $[0, 1]$ into \mathbb{R}^{1+n+m} we set

$$\delta(y_1, y_2) = \min_{(\alpha_1, \alpha_2)} \|y_1 \circ \alpha_1 - y_2 \circ \alpha_2\|,$$

where the minimum is taken over all pairs (α_1, α_2) of nondecreasing reparametrization of the interval $[0, 1]$. The operator δ was already introduced in [5], where it was also shown that δ is actually a pseudometric which vanishes on pairs formed by a path y and a reparametrization $y \circ \alpha$. We remark that an adaptation of the δ^c pseudometric to continuous maps would satisfy this latter prerequisite; however such a choice would not guarantee the continuity of the input-output map.

4. Continuity of the input-output map. In this section we prove a continuity property of the input-output map corresponding to the extended system (2.3), (2.4). Actually, some kind of continuity of the trajectories of (2.3), (2.4) on controls (and on initial conditions) is essential in order to justify the space-time extension proposed in the previous section. We remark that the noncommutativity of the vector fields \hat{g}_i plays a crucial role. Roughly, this noncommutativity is the reason why the jump of a trajectory at an instant \bar{t} depends not only on the jump of the control but also on the *instantaneous evolution* at \bar{t} . In fact in the conservative case [6]—where all Lie brackets $[g_i, g_j]$ vanish—the jump of the state is a function of the jump of the control and there is no need of considering a space-time extension of the original system.

In connection with the original problem (2.1), (2.2) we consider the class of admissible controls

$$W_K(t_1, u_1) \doteq \left\{ (u, v) \in AC([t_1, t_2], U) \times B([t_1, t_2], V) : \begin{array}{l} 0 \leq t_1 < t_2 \leq T, \quad u(t_1) = u_1, \quad V_{t_1}^{t_2}(u) \leq K \end{array} \right\}. \quad (4.1)$$

The bound on the variation $V_{t_1}^{t_2}(u)$ of u is a natural restraint which can be encountered in several applications. In fact, it prevents phenomena of discontinuity of the input-output map which are connected to the noncommutativity of the vector fields \hat{g}_i . In the setting of the extended problem (2.3), (2.4), where intervals of instantaneous evolution are allowed, the (extended) set $\Gamma_K(t_1, u_1)$ of admissible space-time controls is defined by

$$\Gamma_K(t_1, u_1) \doteq \{(\varphi_0, \varphi, \psi) \in \Gamma(t_1, u_1) : V_0^1(\varphi) \leq K\}. \quad (4.2)$$

It is clear that we can identify the original set $W_K(t_1, u_1)$ of admissible controls with the subset $\Gamma_K^+(t_1, u_1) \subset \Gamma_K(t_1, u_1)$ defined by

$$\Gamma_K^+(t_1, u_1) \doteq \left\{ \begin{array}{l} (\varphi_0, \varphi, \psi) \in \Gamma_K(t_1, u_1) : \\ (\varphi_0, \varphi, \psi) \text{ is the graph of a regular control } (u, v) \end{array} \right\}. \quad (4.3)$$

Whenever there is no danger of confusion, we shall write $W_K, \Gamma_K, \Gamma_K^+$ instead of $W_K(t_1, u_1), \Gamma_K(t_1, u_1), \Gamma_K^+(t_1, u_1)$, respectively.

The following proposition states that the family of space-time controls is nothing but the closure of the original family of controls.

Proposition 4.1. Γ_K^+ is dense in Γ_K with respect to the δ^c pseudometric.

Proof. Fix $(\varphi_0, \varphi, \psi) \in \Gamma_K$ and denote by $(\varphi_0^c, \varphi^c, \psi^c)$ its canonical parameterization. Let φ_{0_n} be defined by

$$\varphi_{0_n}(s) \doteq \varphi_0^c(s) + \frac{s}{n}, \quad \forall s \in [0, 1],$$

whenever $\varphi_0(1) = t_2 < T$. Moreover, set

$$\varphi_{0_n}(s) \doteq t_1 + (T - t_1) \frac{(\varphi_0^c(s) - t_1) + \frac{s}{n}}{(T - t_1) + \frac{1}{n}}, \quad \forall s \in [0, 1],$$

in the case where $\varphi_0(1) = t_2 = T$, and consider the sequence $((\varphi_{0_n}, \varphi^c, \psi^c))_n \subset \Gamma_K^+$. For any n let σ_n denote the increasing Lipschitz continuous map

$$\sigma_n(s) \doteq \frac{\int_0^s |(\dot{\varphi}_{0_n}, \dot{\varphi}^c)|(s') ds'}{\int_0^1 |(\dot{\varphi}_{0_n}, \dot{\varphi}^c)|(s') ds'}, \quad \forall s \in [0, 1].$$

By the definition of φ_{0_n} it follows that if (φ_0, φ) is constant then σ_n coincides with the identity map. Otherwise one has

$$|(\dot{\varphi}_{0_n}, \dot{\varphi}^c)(s) - |(\dot{\varphi}_0^c, \dot{\varphi}^c)(s)| \leq |\dot{\varphi}_{0_n}(s) - \dot{\varphi}_0^c(s)| \leq \frac{1}{n} \quad \text{for a.e. } s \in [0, 1].$$

Moreover,

$$|(\dot{\varphi}_{0_n}, \dot{\varphi}^c)(s) = \int_0^1 |(\dot{\varphi}_0^c, \dot{\varphi}^c)|(s') ds' \quad \text{for a.e. } s \in [0, 1].$$

Hence one has

$$|\dot{\sigma}_n(s) - 1| \leq \frac{\frac{1}{n} + \int_0^1 |(\dot{\varphi}_0^c, \dot{\varphi}^c)(s) - |(\dot{\varphi}_{0_n}, \dot{\varphi}^c)(s)| ds}{V_0^1(\varphi_0, \varphi) - \frac{1}{n}} \leq \frac{2}{n(V_0^1(\varphi_0, \varphi) - \frac{1}{n})}$$

for almost every $s \in [0, 1]$ and for any n large enough. Therefore both the σ_n 's and their inverses σ_n^{-1} 's are equi-Lipschitzian and tend uniformly to the identity map. In particular this implies that $(\varphi_{0_n}, \varphi^c) \circ \sigma_n^{-1}$ converges uniformly to (φ_0^c, φ^c) and $\psi^c \circ \sigma_n^{-1}$ converges to ψ in L^1 . Since $(\varphi_{0_n}, \varphi^c, \psi^c) \circ \sigma_n^{-1}$ coincides with the canonical parameterization of $(\varphi_{0_n}, \varphi^c, \psi^c)$ we have

$$\delta^c((\varphi_{0_n}, \varphi^c, \psi^c), (\varphi_0, \varphi, \psi)) = \|(\varphi_{0_n}, \varphi^c) \circ \sigma_n^{-1} - (\varphi_0^c, \varphi^c)\| + \|\psi^c \circ \sigma_n^{-1} - \psi\|_1.$$

Hence we conclude that the sequence $((\varphi_{0_n}, \varphi^c, \psi^c))_n$ converges to $(\varphi_0, \varphi, \psi)$ in the δ^c pseudometric. \square

We now prove the continuity of the input-output functional. By *input* we mean here a pair formed by an initial condition (t_1, x_1, u_1) and a space-time control $(\varphi_0, \varphi, \psi) \in \Gamma_K(t_1, u_1)$. So the input-output functional, say \mathcal{I} , maps a pair $(t_1, x_1, u_1; \varphi_0, \varphi, \psi)$ into the corresponding trajectory $y[t_1, x_1, u_1; \varphi_0, \varphi, \psi]$ of (2.3), (2.4). We endow the set of initial conditions with the Euclidean topology. Moreover we shall consider the δ^c pseudometric on the set of space-time controls and the δ pseudometric on the set of space-time trajectories.

Theorem 4.1. The input-output map \mathcal{I} is continuous.

Proof. Let Λ^* be the set of inputs $(t_1, x_1, u_1; \varphi_0, \varphi, \psi)$ such that the map (φ_0, φ) is Lipschitzian with rank $L \doteq T + K$. We begin by proving the continuity of the restriction $\mathcal{I}|_{\Lambda^*}$ of the input-output map to Λ^* when the latter is endowed with the topology induced by the product topology of $\mathbb{R}^{1+n+m} \times C^0([0, 1], [0, T] \times U) \times L^1([0, 1], V)$ and the range is endowed with the C^0 topology.

Let $(t_1^*, x_1^*, u_1^*; \varphi_0^*, \varphi^*, \psi^*) \in \Lambda^*$ be fixed, and let $y^* \doteq y[t_1^*, x_1^*, u_1^*; \varphi_0^*, \varphi^*, \psi^*]$ be the corresponding solution of (2.3), (2.4). Since $y^*([0, 1])$ is a compact subset of $[0, T] \times \mathbb{R}^n \times U$, for any $\eta > 0$ the set

$$\Omega \doteq \bigcup_{s \in [0, 1]} B_{1+n+m}(y^*(s), \eta)$$

is bounded in \mathbb{R}^{1+n+m} . For $i = 0, \dots, m$ replace the vector field \hat{g}_i with a Lipschitz continuous vector field $h_i : \mathbb{R}^{1+n+m+q} \rightarrow \mathbb{R}^{1+n+m}$ which has compact support, and coincides with \hat{g}_i on $\Omega \times V$. Let N be an upper bound for both the $|h_i|$ and their Lipschitz constants, i.e., let

$$|h_i(y, v)| \leq N, \quad |h_i(y', v') - h_i(y, v)| \leq N(|y' - y| + |v' - v|), \\ \forall (y, v), (y', v') \in \mathbb{R}^{1+n+m+q}, \quad \forall i = 0, \dots, m.$$

Consider the space $E = \mathbb{R}^{1+n+m} \times C^0([0, 1], \mathbb{R}^{1+m}) \times L^1([0, 1], \mathbb{R}^q)$ with the norm

$$\|(t_1, x_1, u_1; \varphi_0, \varphi, \psi)\|_E \doteq |(t_1, x_1, u_1)| + \|(\varphi_0, \varphi)\| + \|\psi\|_1.$$

Moreover, endow the space $F = C^0([0, 1], \mathbb{R}^{1+n+m})$ with the norm

$$\|y\|_F \doteq \sup\{e^{-\lambda s}|y(s)| : s \in [0, 1]\}, \quad \lambda = 2N(1+m)L,$$

which is equivalent to the supnorm.

Let $(t_1, x_1, u_1; \varphi_0, \varphi, \psi) \in \Lambda^*$. The solution $z[t_1, x_1, u_1; \varphi_0, \varphi, \psi]$ of the Cauchy problem

$$\dot{z}(s) = \sum_{i=0}^m h_i(z, \psi) \dot{\varphi}_i(s), \quad z(0) = (t_1, x_1, u_1),$$

coincides with the solution of the implicit equation

$$z = \Psi(t_1, x_1, u_1; \varphi_0, \varphi, \psi; z),$$

where

$$\Psi(t_1, x_1, u_1; \varphi_0, \varphi, \psi; z)(s) \doteq (t_1, x_1, u_1) + \int_0^s \sum_{i=0}^m h_i(z(\sigma), \psi(\sigma)) \dot{\varphi}_i(\sigma) d\sigma. \quad (4.4)$$

Now we are in the position of utilizing the following corollary of the Contraction Mapping Theorem—see e.g. ([3], Corollary 3.1.4).

Lemma 4.1. *Let A, B be closed subsets of the Banach spaces E, F , respectively. Let $\Psi : A \times B \rightarrow B$ be a map such that*

- i) $\|\Psi(a, b_1) - \Psi(a, b_2)\|_F \leq \frac{1}{2} \|b_1 - b_2\|_F, \quad \forall a \in A, \forall b_1, b_2 \in B;$
- ii) $\forall b \in B, a \mapsto \Psi(a, b)$ is continuous from A to B .

Then for each $a \in A$ there exists a unique $b = b(a)$ such that $b(a) = \Psi(a, b(a))$ and $a \mapsto b(a)$ is continuous from A to B .

Let us identify the subsets A and B in Lemma 4.1 with Λ^* and the subset formed by the maps in F having Lipschitz constant $N(1+m)L$, respectively. It is straightforward to check that Ψ maps $A \times B$ into B . Moreover, the condition i) of Lemma 4.1 is easily verified. Indeed let $(t_1, x_1, u_1; \varphi_0, \varphi, \psi)$ be an input in Λ^* and let z, \hat{z} be maps in F having Lipschitz constant $N(1+m)L$. Set $\mu = \|z - \hat{z}\|_F$. By $|z(s) - \hat{z}(s)| \leq \mu e^{\lambda s}$, we have

$$\begin{aligned} & e^{-\lambda s} |\Psi(t_1, x_1, u_1; \varphi_0, \varphi, \psi; z)(s) - \Psi(t_1, x_1, u_1; \varphi_0, \varphi, \psi; \hat{z})(s)| \\ & \leq e^{-\lambda s} \int_0^s \sum_{i=0}^m |h_i(z(\sigma), \psi(\sigma)) - h_i(\hat{z}(\sigma), \psi(\sigma))| \cdot |\dot{\varphi}_i(\sigma)| d\sigma \\ & \leq e^{-\lambda s} \int_0^s N(1+m) |z(\sigma) - \hat{z}(\sigma)| \cdot L d\sigma \leq e^{-\lambda s} \int_0^s N(1+m) \mu e^{\lambda \sigma} L d\sigma \leq \frac{1}{2} \|z - \hat{z}\|_F. \end{aligned}$$

Let us see that condition ii) as well is verified by the map defined in (4.4). Fix $z(\cdot) \in B$ and consider a sequence $((t_{1n}, x_{1n}, u_{1n}; \varphi_{0n}, \varphi_n, \psi_n))_{n \in \mathbb{N}}$ in Λ^* converging to

$(t_1, x_1, u_1; \varphi_0, \varphi, \psi)$. Then

$$\begin{aligned} & |\Psi(t_{1n}, x_{1n}, u_{1n}; \varphi_{0n}, \varphi_n, \psi_n; z)(s) - \Psi(t_1, x_1, u_1; \varphi_0, \varphi, \psi; z)(s)| \\ & \leq |(t_{1n}, x_{1n}, u_{1n}) - (t_1, x_1, u_1)| + \int_0^s \sum_{i=0}^m |h_i[z(\sigma), \psi_n(\sigma)] - h_i[z(\sigma), \psi(\sigma)]| |\dot{\varphi}_n(\sigma)| d\sigma \\ & \quad + \left| \int_0^s \sum_{i=0}^m h_i[z(\sigma), \psi(\sigma)] [\dot{\varphi}(\sigma) - \dot{\varphi}_n(\sigma)] d\sigma \right| \\ & \leq |(t_{1n}, x_{1n}, u_{1n}) - (t_1, x_1, u_1)| + N(1+m)L \|\psi_n - \psi\|_1 \\ & \quad + \left| \int_0^s \sum_{i=0}^m h_i[z(\sigma), \psi(\sigma)] [\dot{\varphi}(\sigma) - \dot{\varphi}_n(\sigma)] d\sigma \right|. \end{aligned} \quad (4.5)$$

By hypothesis, $|(t_{1n}, x_{1n}, u_{1n}) - (t_1, x_1, u_1)|$ and $\|\psi_n - \psi\|_1$ converge to zero. In order to prove that the right-hand side of (4.5) converges uniformly to zero on $[0, 1]$, it remains to show that the sequence

$$f_n(s) \doteq \left| \int_0^s \sum_{i=0}^m h_i[z(\sigma), \psi(\sigma)] [\dot{\varphi}(\sigma) - \dot{\varphi}_n(\sigma)] d\sigma \right|$$

converges uniformly to zero. Since $(\varphi_{0n}, \varphi_n)$ converges uniformly to (φ_0, φ) on $[0, s]$, $\forall s \in [0, 1]$, the maps $(\dot{\varphi}_{0n}, \dot{\varphi}_n)$'s tend to $(\dot{\varphi}_0, \dot{\varphi})$ in the weak* topology of $L^\infty([0, s], \mathbb{R}^{1+m})$; hence the f_n 's converge pointwise to 0. The uniform convergence to zero of the f_n now follows from the Ascoli-Arzelà Theorem, for the f_n 's are equibounded and equi-Lipschitzian.

Therefore all the hypotheses of Lemma 4.1 are satisfied; the map

$$(t_1, x_1, u_1; \varphi_0, \varphi, \psi) \mapsto z[t_1, x_1, u_1; \varphi_0, \varphi, \psi]$$

is thus continuous on Λ^* . Hence \mathcal{I}_{Λ^*} as well is continuous, because whenever the trajectory $z[t_1, x_1, u_1; \varphi_0, \varphi, \psi]$ is entirely contained inside Ω it coincides with the solution $y[t_1, x_1, u_1; \varphi_0, \varphi, \psi]$ of (2.3), (2.4).

To conclude the proof, observe that if we fix the initial condition (2.4), by Proposition 3.3 it follows that two space-time controls having the same canonical parameterization are mapped into trajectories whose δ pseudodistance is zero.

Given an input pair $(t_1, x_1, u_1; \varphi_0, \varphi, \psi)$ let $(\varphi_0^c, \varphi^c, \psi^c)$ denote the canonical parameterization of $(\varphi_0, \varphi, \psi)$. Then $(t_1, x_1, u_1; \varphi_0^c, \varphi^c, \psi^c) \in \Lambda^*$, and by the first part of the proof for every $\epsilon > 0$ there exists an $\eta > 0$ such that for every input $(\hat{t}_1, \hat{x}_1, \hat{u}_1; \varphi_0^*, \varphi^*, \psi^*) \in \Lambda^*$ satisfying

$$|(t_1, x_1, u_1) - (\hat{t}_1, \hat{x}_1, \hat{u}_1)| + \|(\varphi_0^c, \varphi^c) - (\varphi_0^*, \varphi^*)\| + \|\psi^c - \psi^*\|_1 \leq \eta, \quad (4.6)$$

one has

$$\|y[t_1, x_1, u_1; \varphi_0^c, \varphi^c, \psi^c] - y[\hat{t}_1, \hat{x}_1, \hat{u}_1; \varphi_0^*, \varphi^*, \psi^*]\| \leq \epsilon.$$

Now let $(\hat{t}_1, \hat{x}_1, \hat{u}_1; \hat{\varphi}_0, \hat{\varphi}, \hat{\psi})$ be an input pair such that

$$|(t_1, x_1, u_1) - (\hat{t}_1, \hat{x}_1, \hat{u}_1)| + \delta^c(\varphi_0, \varphi, \psi), (\hat{\varphi}_0, \hat{\varphi}, \hat{\psi}) \leq \eta, \quad (4.7)$$

and let $(\hat{\varphi}_0^c, \hat{\varphi}^c, \hat{\psi}^c)$ denote the canonical parameterization of $(\hat{\varphi}_0, \hat{\varphi}, \hat{\psi})$. By identifying $(\hat{\varphi}_0^c, \hat{\varphi}^c, \hat{\psi}^c)$ with $(\varphi_0^*, \varphi^*, \psi^*)$ in (4.6) we obtain

$$\delta(y, \hat{y}) \leq \delta(y, y^c) + \delta(y^c, \hat{y}^c) + \delta(\hat{y}^c, \hat{y}) \leq \|y^c - \hat{y}^c\| \leq \epsilon,$$

where we have written y , \hat{y} , y^c , and \hat{y}^c in place of $y[t_1, x_1, u_1; \varphi_0, \varphi, \psi]$, $y[\hat{t}_1, \hat{x}_1, \hat{u}_1; \hat{\varphi}_0, \hat{\varphi}, \hat{\psi}]$, $y[t_1, x_1, u_1; \varphi_0^c, \varphi^c, \psi^c]$, and $y[\hat{t}_1, \hat{x}_1, \hat{u}_1; \varphi_0^*, \varphi^*, \psi^*]$, respectively. Hence the theorem is proved.

Remark 4.1. Note that Theorem 4.1 fails to hold if the δ^c topology is replaced with the $C^0 \times L^1$ topology, as shown by the following example.

Consider the control system on $[0, 1]$,

$$\dot{x} = v\dot{u}, \quad x(0) = 0,$$

where $x \in \mathbb{R}$ and the control variable (u, v) can range over $[-1, 1] \times \{-1, 1\}$. Let us restrain the variation $V_0^1(u)$ to be less or equal than 1. For any $n \geq 1$ consider the map

$$(w_n, v_n)(s) \doteq \begin{cases} (0, 1), & 0 \leq t \leq 1 - \frac{1}{2^n} \\ (2^n, 1), & 1 - \frac{1}{2^n} + \frac{2i}{2^{n+1}} < t \leq 1 - \frac{1}{2^n} + \frac{2i+1}{2^{n+1}} \\ (-2^n, -1), & 1 - \frac{1}{2^n} + \frac{2i+1}{2^{n+1}} < t \leq 1 - \frac{1}{2^n} + \frac{2i+2}{2^{n+1}}, \end{cases}$$

$i = 0, \dots, 2^n - 1$, and define the control (u_n, v_n) by setting

$$u_n(s) = \int_0^s w_n(s') ds'$$

for every $s \in [0, 1]$. It is straightforward to check that $V_0^1(u_n) = 1$, $\forall n \in \mathbb{N}$ and that (u_n, v_n) converges to the constant control $(u, v) \equiv (0, 1)$ in the product topology of $C^0 \times L^1$. However the solutions x_n corresponding to the controls (u_n, v_n) do not converge in the C^0 norm to the constant trajectory $x \equiv 0$. Indeed for every $n \geq 1$ one has $x_n(1) = 1$.

5. Closure of the set of trajectories. By Proposition 4.1 and Theorem 4.1 it follows that each trajectory of the space-time system (2.3), (2.4) can be approximated in the δ metric by the graphs of a suitable sequence of trajectories of the original system (2.1), (2.2). In other words the set of space-time trajectories is included in the δ closure of the set of trajectories' graphs of the original system. In general this inclusion is a proper one, i.e., there exist sequences of space-time trajectories which converge in the δ metric to a map which is not a solution of (2.3), (2.4). Actually this is not very much surprising, since the same phenomena occurs in the non impulsive case unless the sets of velocities is convex.

Let us denote by S_K the set of trajectories corresponding to regular controls (u, v) , i.e., let us set

$$S_K \doteq \{x \in AC([t_1, t_2], \mathbb{R}^n) : x = x[u, v], \quad (u, v) \in W_K(t_1, u_1)\}, \quad (5.1)$$

where $x[u, v]$ is a shorter notation for the trajectory $x[t_1, x_1, u_1; u, v]$. Analogously, we define the set of admissible space-time trajectories

$$\Sigma_K \doteq \{y \in AC([0, 1], \mathbb{R}^{1+n} \times U) : y = y[\varphi_0, \varphi, \psi], \quad (\varphi_0, \varphi, \psi) \in \Gamma_K(t_1, u_1)\}, \quad (5.2)$$

where $y[\varphi_0, \varphi, \psi]$ still denotes the solution to (2.3), (2.4). Since the original set W_K of admissible controls is identified with the subset $\Gamma_K^+ \subset \Gamma_K$, the set of the graphs of the functions belonging to S_K coincides with the subset Σ_K^+ of the space-time trajectories corresponding to space-time controls in Γ_K^+ .

Theorem 5.1. Assume the following.

i) For some $C > 0$ one has

$$|g_i(y, v)| \leq C(1 + |y|), \quad \forall (y, v) \in [0, T] \times \mathbb{R}^n \times U \times V, \quad \forall i = 0, \dots, m; \quad (5.3)$$

ii) the set

$$\hat{G}(y) \doteq \left\{ \begin{array}{l} z \in \mathbb{R}^{1+n+m} : \quad z = \sum_{i=0}^m \hat{g}_i(y, v) w_i, \\ v \in V, \quad w \in [0, +\infty[\times \mathbb{R}^m \end{array} \right\}$$

is convex for all $y \in [t_1, T] \times \mathbb{R}^n \times U$.

Then the subset $\Sigma_K \subset C^0([0, 1], \mathbb{R}^{1+n+m})$ is compact with respect to the δ topology. Moreover, it coincides with the closure of Σ_K^+ .

Proof. We begin by proving that Σ_K is compact under the hypothesis that U coincides with the whole \mathbb{R}^m . Let $(y_v)_v$ be a sequence of trajectories of Σ_K . Hence for every $v \in \mathbb{N}$ there is a space-time control $(\varphi_{0_v}, \varphi_v, \psi_v) \in \Gamma_K$ such that $y_v = y[\varphi_{0_v}, \varphi_v, \psi_v]$. Let y_v^c denote the space-time trajectory corresponding to the canonical parameterization $(\varphi_{0_v}^c, \varphi_v^c, \psi_v^c)$ of $(\varphi_{0_v}, \varphi_v, \psi_v)$. Since every $(\varphi_{0_v}^c, \varphi_v^c)$ is Lipschitz continuous with rank $K + T$, by using hypothesis (5.3) and the Gronwall's inequality one has that $\{y_v^c : v \in \mathbb{N}\}$ is a set of equibounded and equi-Lipschitz maps. By Ascoli-Arzelà theorem it follows that there exists a subsequence, still denoted by $(y_v^c)_v$, which converges uniformly to a Lipschitz continuous map y^* . In particular, since

$$y_{0_v}^c = \varphi_{0_v}^c, \quad y_{(n+\alpha)_v}^c = \varphi_{\alpha_v}^c, \quad \forall \alpha = 1, \dots, m,$$

by setting $(\varphi_0^*, \varphi_1^*, \dots, \varphi_m^*) \doteq (y_0^*, y_{n+1}^*, \dots, y_{n+m}^*)$, we have that

$$\|(\varphi_{0_v}^c, \varphi_{1_v}^c, \dots, \varphi_{m_v}^c) - (\varphi_0^*, \varphi_1^*, \dots, \varphi_m^*)\| \rightarrow 0, \quad (5.4)$$

as v tends to ∞ . We claim that

$$y^* \in \Sigma_K. \quad (5.5)$$

To prove (5.5), let us replace $\hat{G}(y)$ with the multivalued map $\hat{G}^*(y)$ defined

$$\hat{G}^*(y) \doteq \left\{ \begin{array}{l} z \in \mathbb{R}^{1+n+m} : \quad z = \sum_{i=0}^m \hat{g}_i(y, v) w_i, \\ v \in V, \quad w \in B_{1+m}[0, K+T] \cap ([0, +\infty[\times \mathbb{R}^m) \end{array} \right\}.$$

Hypothesis ii) and the regularity of the vector fields \hat{g}_i 's ($i = 0, \dots, m$) imply that \hat{G}^* is a Hausdorff continuous multifunction with convex, compact values. Furthermore, each y_v^c satisfies the differential inclusion

$$\dot{y} \in \hat{G}^*(y), \quad (5.6)$$

and the boundary condition $y(0) = (t_1, x_1, u_1)$. Hence Filippov's theorem (see e.g. [1]), yields the existence of measurable controls ψ^* and w^* with values in V and in $B_{1+m}[0, K+T] \cap ([0, +\infty[\times \mathbb{R}^m)$, respectively, such that

$$\dot{y}^* = \sum_{i=0}^m \hat{g}_i(y^*, \psi^*) w_i^*. \quad (5.7)$$

In particular one has

$$\dot{\varphi}_i^*(s) = w_i^*(s) \quad \text{for a.e. } s \in [0, 1] \quad (i = 0, \dots, m);$$

hence (5.7) can be rewritten in the form

$$\dot{y}^* = \sum_{i=0}^m \hat{g}_i(y^*, \psi^*) \dot{\varphi}_i^*.$$

Hence y^* belongs to Σ_K , as (5.4) implies that the control $(\varphi_0^*, \varphi^*, \psi^*)$ is an element of Γ_K . Since $\delta(y_v, y_v^c) = 0$ for every $v \in \mathbb{N}$ —see Proposition 3.3—the sequence y_v tends to y^* and the theorem is proved in the case where $U = \mathbb{R}^m$.

To prove that Σ_K is compact in the general case where U does not coincide with the whole \mathbb{R}^m , observe that since $\varphi_v^c(s) \in U$ for every $v \in \mathbb{N}$ and $s \in [0, 1]$, it follows by (5.4) that $\varphi^*(s) \in U$ for every $s \in [0, 1]$. It remains to prove that Σ_K^+ is dense in Σ_K . Since Σ_K is closed in $C^0([0, 1], \mathbb{R}^{1+n+m})$ with respect to the δ pseudometric, it is sufficient to prove that the δ closure of Σ_K^+ contains Σ_K . In fact, this is a consequence of the density of Γ_K^+ in Γ_K in the δ^c pseudometric and of the continuity of the input-output map \mathcal{I} .

Remark 5.1. It is not difficult to verify that one obtains a condition equivalent to hypothesis ii) in Theorem 5.1 by replacing the set $[0, +\infty[\times \mathbb{R}^m$ in the definition of $\hat{\mathcal{G}}(y)$ with a subset H satisfying

$$H \supset ([0, +\infty[\times \mathbb{R}^m) \cap B_{1+m}[0, 1].$$

Remark 5.2. The convexity of the subsets

$$g_i(t, x, u, V) \doteq \{g_i(t, x, u, v) : v \in V\} \quad (i = 0, \dots, m)$$

is a necessary condition in order that hypothesis ii) is verified. Yet, it is not sufficient to ensure the closure of the set of space-time trajectories. Indeed, consider the simple control system, defined on $[0, 1]$,

$$\begin{aligned} \dot{x}_1 &= v_1 \dot{u} \\ \dot{x}_2 &= v_2 \dot{u}, \end{aligned} \quad (\text{E})$$

with the initial conditions

$$x(0) = (0, 0), \quad u(0) = 0. \quad (\text{I})$$

The control (v_1, v_2) is allowed to take values on the ball $V = B_2([0, 2], 1]$ and the control u can range over $U = \mathbb{R}$. Moreover we consider only controls u whose total variations are not larger than 1. Notice that the vector field g_0 is identically equal to zero, and $g_1(V) = V$ is a convex subset of \mathbb{R}^2 . For every $n \in \mathbb{N}$ define the (regular) control (v_{1n}, v_{2n}, u_n) by setting

$$(v_{1n}, v_{2n}, u_n)(t) = \begin{cases} (1, 2, t - \frac{i}{n}), & \forall t \in [\frac{2i}{2n}, \frac{2i+1}{2n}] \\ (-1, 2, -t + \frac{i+1}{n}), & \forall t \in [\frac{2i+1}{2n}, \frac{2i+2}{2n}] \end{cases} \quad (i = 0, \dots, n-1)$$

and extending u_n continuously at the instants $t = \frac{j}{2n}$, $j = 0, \dots, 2n$. It is straightforward

which is neither a trajectory of (I)–(E) nor a trajectory of the corresponding space-time control system.

6. The cone constraint. The convexity condition of Theorem 5.1, which yields the closure of the set of trajectories, cannot be checked by the mere convexity of the $g_i(y, V)$'s values—see Remark 5.2. Essentially this is due to the fact that the holonomic constraint $u \in U$ forces the derivative \dot{u} to range over the “too large” set $T_{u(t)}U$, where $T_{u(t)}U$ denotes the (Bouligand's) tangent cone of U at $u(t)$ —see e.g. [1].

An attempt to simplify the condition of Theorem 5.1 can be made by suitably restricting the set where \dot{u} can take its values. Actually, it happens that in many applications—see [11, 13, 17, 19]—and in most literature on impulsive control a constraint is imposed on the direction of \dot{u} [2, 17, 22, 27]. For instance, in the scalar case one could assume that \dot{u} is non-negative. More generally, when u is vector-valued the derivative \dot{u} may be constrained to range over a closed cone C . By adding this kind of constraint we obtain a wider class of problems which differ from the one considered in the previous sections in that the directional constraint $\dot{u} \in T_{u(t)}U$ is replaced with $\dot{u} \in T_{u(t)}U \cap C$.

However, even the addition of this new constraint does not succeed in simplifying the condition of Theorem 5.1 unless further assumptions are made on the cone C —see Corollary 6.1. In fact the situation improves as soon as the closed cone C is a convex subset of the positive orthant of \mathbb{R}^m —see Theorem 6.1. Indeed in this case the closure of the set of trajectories is achieved via an assumption which involves just the convexity of the sets $g_i(y, V)$.

Let us introduce formally the cone constraint problem. We recall that a cone of \mathbb{R}^m is a nonempty subset $C \subset \mathbb{R}^m$ such that $\alpha w \in C$ whenever $w \in C$ and α is a positive scalar.

Let C be a closed cone of \mathbb{R}^m . The set W_K of admissible (regular) control pairs, its space-time counterpart Γ_K^+ , and the set Γ_K of admissible space-time controls are now replaced with the subsets

$$W_K(C) \doteq \{(u, v) \in W_K : \dot{u}(t) \in C \text{ for a.e. } t \in [t_1, t_2]\}, \quad (6.1)$$

$$\Gamma_K^+(C) \doteq \{(\varphi_0, \varphi, \psi) \in \Gamma_K^+ : \dot{\varphi}(s) \in C \text{ for a.e. } s \in [0, 1]\}, \quad (6.2)$$

and

$$\Gamma_K(C) \doteq \{(\varphi_0, \varphi, \psi) \in \Gamma_K : \dot{\varphi}(s) \in C \text{ for a.e. } s \in [0, 1]\}, \quad (6.3)$$

respectively.

Observe that a space-time control $(\varphi_0, \varphi, \psi)$ belongs to $\Gamma_K(C)$ if and only if the same fact holds for its canonical parameterization $(\varphi_0^c, \varphi^c, \psi^c)$.

Like in the unconstrained case, $\Gamma_K^+(C)$ is dense in $\Gamma_K(C)$ with respect to the δ^c pseudometric, and, moreover, by Theorem 4.1 the input-output map \mathcal{I} is continuous. Yet, the subset $\Gamma_K(C)$ fails in general to be closed in Γ_K with respect to the δ^c pseudometric. The situation improves when the cone C is convex, as stated by the following proposition.

Proposition 6.1. *Let C be a closed convex cone. Then $\Gamma_K(C)$ is a closed subset of Γ_K in the δ^c pseudometric.*

Proof. Let $((\varphi_0, \varphi, \psi)_v) \subset \Gamma_K(C)$ be a sequence converging to some $(\varphi_0, \varphi, \psi) \in \Gamma_K$ in the δ^c pseudometric. If $(\varphi_0^c, \varphi_v^c, \psi_v^c)$ and $(\varphi_0^c, \varphi^c, \psi^c)$ denote the canonical parameterizations of $(\varphi_0, \varphi, \psi)_v$ and $(\varphi_0, \varphi, \psi)$, respectively, one has in particular

$$\lim \|\varphi_v^c - \varphi^c\| = 0. \quad (6.4)$$

For any $v \in \mathbb{N}$ and for any $s \in [0, 1]$ where the derivative $\dot{\varphi}^c(s)$ exists, consider the identity

$$\frac{\varphi_v^c(s+h) - \varphi_v^c(s)}{h} = \frac{1}{h} \int_s^{s+h} \dot{\varphi}_v^c(\sigma) d\sigma \quad (h \neq 0). \quad (6.5)$$

Since $\dot{\varphi}_v^c(\sigma) \in C$ for almost every $\sigma \in [0, 1]$, the right-hand side of (6.5) belongs to the closure of the convex hull of C , which under our hypotheses coincides with C itself. Taking first the limit as v tends to ∞ and then as h tends to 0, we obtain $\dot{\varphi}^c(s) \in C$, whence $(\varphi_0^c, \varphi^c, \psi^c)$ as well as $(\varphi_0, \varphi, \psi)$ belong to $\Gamma_K(C)$. \square

Let us now investigate the closure of the set of trajectories. We denote by $S_K(C)$ the subset of S_K formed by the trajectories of the original system corresponding to controls in $W_K(C)$. Similarly, $\Sigma_K(C)$ and $\Sigma_K^+(C)$ will represent the sets of admissible space-time trajectories corresponding to space-time controls in $\Gamma_K(C)$ and $\Gamma_K^+(C)$, respectively. Like in the unconstrained problem, $\Sigma_K^+(C)$ is nothing but the set of graphs of the maps belonging to $S_K(C)$. Thanks to Proposition 6.1, Theorem 5.1 yields the following result concerning the cone-constrained problem.

Corollary 6.1. Assume the sublinear growth condition i) of Theorem 5.1, and replace condition ii) with the following one:

ii)' the set

$$\hat{G}(y; C) \doteq \left\{ z \in \mathbb{R}^{1+n+m} : z = \sum_{i=0}^m \hat{g}_i(y, v) w_i, \right. \\ \left. v \in V, w \in [0, +\infty[\times C \right\}, \quad (6.6)$$

is convex for all $y \in [t_1, T] \times \mathbb{R}^n \times U$.

Then the set $\Sigma_K(C)$ is compact in the δ pseudometric, and coincides with the closure of $\Sigma_K^+(C)$.

Remark 6.1. In view of Remark 5.1, we can replace the set $[0, +\infty[\times C$ in the definition of $\hat{G}(y; C)$ with a set H such that $H \supset B_{1+m}[0, 1] \cap ([0, +\infty[\times C)$.

Remark 6.2. The fundamental hypothesis ii)' concerns the $1+n+m$ -dimensional sets $\hat{G}(y; C)$. This underlines that, though the variable x belongs to \mathbb{R}^n , the natural state space for system (2.1) is the $(1+n+m)$ -dimensional set $[0, +\infty[\times \mathbb{R}^n \times U$. In fact, the convexity of the n -dimensional sets

$$\mathcal{G}(t, x; C) \doteq \left\{ z \in \mathbb{R}^n : z = \sum_{i=0}^m g_i(t, x, u, v) w_i, \right. \\ \left. u \in U, v \in V, w \in [0, +\infty[\times C \right\}$$

does not guarantee that the set of trajectories is closed. Indeed consider the system

$$\begin{aligned} \dot{x}_1 &= v_1 \dot{u}, \\ \dot{x}_2 &= v_2 \dot{u}, \end{aligned} \quad (E)$$

with the initial conditions

$$x(0) = (0, 0), \quad u(0) = 0, \quad (0)$$

and $[t_1, t_2] \equiv [0, 1]$. Let us allow the controls (v_1, v_2) and u to take values in the sets

$$V \doteq \{(v_1, v_2) : -1 \leq v_1 \leq 1, \quad v_2 = v_1^2 + 1\}$$

and \mathbb{R} , respectively, and let us restrain the derivative \dot{u} to take values in the cone $C \doteq [0, +\infty[$. Moreover let the total variation of u be less or equal than 1. Then all sets $\mathcal{G}(t, x; C)$ coincide with the convex cone $\{(x_1, x_2) : |x_1| \leq x_2/2\}$. However the set of trajectories is not closed, for the solutions corresponding to the controls

$$(v_{1n}, v_{2n}, u_n)(t) = \begin{cases} (1, 2, t), & \forall t \in \left] \frac{2i}{2n}, \frac{2i+1}{2n} \right[\\ (-1, 2, t), & \forall t \in \left] \frac{2i+1}{2n}, \frac{2i+2}{2n} \right[\end{cases}, \quad (i = 0, \dots, n-1)$$

converge uniformly to the map $(x_1, x_2)(t) = (0, 2t)$, which is not a trajectory of (E)–(I), since it would require the implementation of a control $(0, 1, u(\cdot))$ such that u has variation equal to 2.

We conclude by showing that if C is a closed convex cone contained in the positive orthant $[0, +\infty[^m$, then the mere convexity of the sets $g_i(y, V)$ —together with a further independence condition—yields the compactness of the set of trajectories.

We shall suppose the following independence condition:

a) there exist $m+1$ compact subsets $V_0 \subset \mathbb{R}^{q_0}, \dots, V_m \subset \mathbb{R}^{q_m}$ for some $q_0, \dots, q_m \in \mathbb{N}$, such that $V = V_0 \times \dots \times V_m$ and for each $i = 0, \dots, m$ the vector field $g_i(y, v)$ depends only on the i -th component v_i of v .

Theorem 6.1. Let C be a closed convex cone contained in the positive orthant $[0, +\infty[^m$, and assume the independence hypothesis a). Moreover, let the sublinear growth condition i) of Theorem 5.1 be satisfied. Finally, write $g_i(y, v_i)$ instead of $g_i(y, v)$ and assume the following convexity condition:

ii)'' for every $y \in [t_1, T] \times \mathbb{R}^n \times U$ and for every $i = 0, \dots, m$, the set

$$g_i(y, V^i) \doteq \{z \in \mathbb{R}^n : z = g_i(y, v_i), \quad v_i \in V_i\}$$

is convex.

Then the set of space-time trajectories $\Sigma_K(C)$ is compact in the δ pseudometric and coincides with the closure of $\Sigma_K^+(C)$.

Proof. In view of Proposition 6.1 it is sufficient to prove the theorem when the cone C coincides with the positive orthant $[0, +\infty[^m$. Moreover, on the basis of Corollary 6.1 and Remark 6.1—where we take $H \doteq [0, K+T]^{1+m}$ —it suffices to show that for every $y \in [t_1, T] \times \mathbb{R}^n \times U$ the set

$$\hat{G}^*(y; [0, +\infty[^m) \doteq \left\{ z \in \mathbb{R}^{1+n+m} : z = \sum_{i=0}^m \hat{g}_i(y, v_i) w_i, \right. \\ \left. v_i \in V_i, w \in [0, K+T]^{1+m} \quad (i = 0, \dots, m) \right\}$$

is convex. In fact, one has

$$\hat{G}^*(y; [0, +\infty[^m) = \sum_{i=0}^m \hat{g}_i(y, V_i)[0, T+K], \quad (6.7)$$

where

$$\hat{g}_i(y, V_i)[0, T + K] \doteq \{\alpha z : \alpha \in [0, T + K], z \in \hat{g}_i(y, V_i)\},$$

and the sum in (6.7) denotes the set formed by the vectors $z = z_0 + \dots + z_m$, with $z_i \in \hat{g}_i(y, V_i)[0, T + K]$. The convexity of the $\hat{g}_i(y, V_i)[0, T + K]$ follows from the convexity of the $\hat{g}_i(y, V_i)$, which in turn is a trivial consequence of the assumed convexity of $g_i(y, V_i)$. Finally $\hat{G}^*(y, [0, +\infty]^m)$ is convex, as it is the sum of $m + 1$ convex sets. \square

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QUADRATIC LYAPUNOV FUNCTIONS FOR LINEAR SKEW-PRODUCT FLOWS AND WEIGHTED COMPOSITION OPERATORS

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Abstract. The Daleckij-Krein method for constructing a quadratic Lyapunov function for the equation $f' = Df(t)$ in Hilbert space is extended to include the case of an unbounded operator D that generates a C_0 -group. The extension is applied to obtain a quadratic Lyapunov function for the case of a group of weighted composition operators generated by a flow on a compact metric space together with a cocycle over this flow. These results are used to characterize the hyperbolicity of linear skew-product flows in terms of the existence of such a Lyapunov function. Also, the "trajectorial" method for constructing the Lyapunov function is discussed. Interrelations with Schrödinger, Riccati and Hamiltonian equations are discussed and an application to geodesic flows on two-dimensional Riemannian manifolds is given.

1. Introduction. A bounded operator $T \in L(H)$ in a Hilbert space H , with scalar product (\cdot, \cdot) is called hyperbolic (HP, for brevity) provided its spectrum $\sigma(T)$ does not intersect the unit circle. An operator $D \in L(H)$ is called infinitesimally hyperbolic (IHP, for brevity) provided $\sigma(D) \cap i\mathbb{R} = \emptyset$, and uniformly W -dissipative with respect to a Hermitian operator $W \in L(H)$ provided $\operatorname{Re}(WD) < 0$, i.e., there is some $\delta > 0$ such that $((WD + D^*W)f, f) \leq -\delta \|f\|^2$ for each $f \in H$.

The version of the second Lyapunov method for the differential equation $f' = Df(t)$ in H with bounded operator D , given by Y.L. Daleckij and M.G. Krein in [6], is well known and can be stated as follows. Let $f(t) = e^{tD} f(0)$ be a solution of the differential equation. The operator e^{tD} , $t \neq 0$, is HP if and only if D is IHP. The operator D is IHP if and only if D is uniformly W -dissipative for some W . If such a W exists, a quadratic Lyapunov function $\mathcal{L} : H \times H \rightarrow \mathbb{R}$ is given by $\mathcal{L}(f, g) = (f, g)_W$ by the (perhaps indefinite) scalar product $(f, g)_W := (Wf, g)$ in H . This means $t \mapsto (f(t), f(t))_W$ decreases monotonically for the solution $f(t) = e^{tD} f(0)$ of the differential equation. Moreover, the invariant subspace H_- (resp., H_+) corresponding to the portion of the spectrum of D lying in the left (resp., right) half plane is uniformly W -positive (resp., uniformly W -negative). Here, a subspace is called uniformly W -positive provided there is some $\delta > 0$ such that for each f in the subspace, $(f, f)_W \geq \delta \|f\|^2$. Also, W can be chosen so that H_+ and H_- are W -orthogonal. Finally, W is a solution of the operator equation $WD + D^*W = -H_1$ for some $H_1 > 0$.

The method of proof used in [6] to obtain the above results can not be applied directly in case D is unbounded. In fact, starting with an infinitesimally hyperbolic operator D , the idea employed in [6] is to associate a hyperbolic operator T in order to use the integral formula for the Riesz projection P of T to obtain $H_- := \operatorname{Im} P$ and $H_+ := \operatorname{Im}(I - P)$. For a bounded IHP operator D the HP operator T is simply defined by $T = (i\xi + D)(i\xi - D)^{-1}$, $\xi \in \mathbb{R}$. However, in general, T defined in this manner fails to be HP for unbounded D .

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In particular one has

$$\dot{\varphi}_i^*(s) = w_i^*(s) \quad \text{for a.e. } s \in [0, 1] \quad (i = 0, \dots, m);$$

hence (5.7) can be rewritten in the form

$$\dot{y}^* = \sum_{i=0}^m \hat{g}_i(y^*, \psi^*) \dot{\varphi}_i^*.$$

Hence y^* belongs to Σ_K , as (5.4) implies that the control $(\varphi_0^*, \varphi^*, \psi^*)$ is an element of Γ_K . Since $\delta(y_\nu, y_\nu^c) = 0$ for every $\nu \in \mathbb{N}$ —see Proposition 3.3—the sequence y_ν tends to y^* and the theorem is proved in the case where $U = \mathbb{R}^m$.

To prove that Σ_K is compact in the general case where U does not coincide with the whole \mathbb{R}^m , observe that since $\varphi_\nu^c(s) \in U$ for every $\nu \in \mathbb{N}$ and $s \in [0, 1]$, it follows by (5.4) that $\varphi^*(s) \in U$ for every $s \in [0, 1]$. It remains to prove that Σ_K^+ is dense in Σ_K . Since Σ_K is closed in $C^0([0, 1], \mathbb{R}^{1+n+m})$ with respect to the δ pseudometric, it is sufficient to prove that the δ closure of Σ_K^+ contains Σ_K . In fact, this is a consequence of the density of Γ_K^+ in Γ_K in the δ^c pseudometric and of the continuity of the input-output map \mathcal{I} .

Remark 5.1. It is not difficult to verify that one obtains a condition equivalent to hypothesis ii) in Theorem 5.1 by replacing the set $[0, +\infty[\times \mathbb{R}^m$ in the definition of $\hat{\mathcal{G}}(y)$ with a subset H satisfying

$$H \supset ([0, +\infty[\times \mathbb{R}^m) \cap B_{1+m}[0, 1].$$

Remark 5.2. The convexity of the subsets

$$g_i(t, x, u, V) \doteq \{g_i(t, x, u, v) : v \in V\} \quad (i = 0, \dots, m)$$

is a necessary condition in order that hypothesis ii) is verified. Yet, it is not sufficient to ensure the closure of the set of space-time trajectories. Indeed, consider the simple control system, defined on $[0, 1]$,

$$\begin{aligned} \dot{x}_1 &= v_1 \dot{u} \\ \dot{x}_2 &= v_2 \dot{u}, \end{aligned} \tag{E}$$

with the initial conditions

$$x(0) = (0, 0), \quad u(0) = 0. \tag{I}$$

The control (v_1, v_2) is allowed to take values on the ball $V = B_2[(0, 2), 1]$ and the control u can range over $U = \mathbb{R}$. Moreover we consider only controls u whose total variations are not larger than 1. Notice that the vector field g_0 is identically equal to zero, and $g_1(V) = V$ is a convex subset of \mathbb{R}^2 . For every $n \in \mathbb{N}$ define the (regular) control (v_{1n}, v_{2n}, u_n) by setting

$$(v_{1n}, v_{2n}, u_n)(t) = \begin{cases} (1, 2, t - \frac{i}{n}), & \forall t \in]\frac{2i}{2n}, \frac{2i+1}{2n}[\\ (-1, 2, -t + \frac{i+1}{n}), & \forall t \in]\frac{2i+1}{2n}, \frac{2i+2}{2n}[\end{cases}, \quad (i = 0, \dots, n-1)$$

and extending u_n continuously at the instants $t = \frac{j}{2n}$, $j = 0, \dots, 2n$. It is straightforward to check that the corresponding solutions x_n converge uniformly to the map $x^*(t) = (t, 0)$,