

Extending the Belavin-Knizhnik “wonderful formula” by the characterization of the Jacobian

Marco Matone

*Dipartimento di Fisica “G. Galilei” and Istituto Nazionale di Fisica Nucleare,
Università di Padova,
Via Marzolo, 8, 35131 Padova, Italy*

E-mail: matone@pd.infn.it

ABSTRACT: A long-standing question in string theory is to find the explicit expression of the bosonic measure, a crucial issue also in determining the superstring measure. Such a measure was known up to genus three. Belavin and Knizhnik conjectured an expression for genus four which has been proved in the framework of the recently introduced vector-valued Teichmüller modular forms. It turns out that for $g \geq 4$ the bosonic measure is expressed in terms of such forms. In particular, the genus four Belavin-Knizhnik “wonderful formula” has a remarkable extension to arbitrary genus whose structure is deeply related to the characterization of the Jacobian locus. Furthermore, it turns out that the bosonic string measure has an elegant geometrical interpretation as generating the quadrics in \mathbb{P}^{g-1} characterizing the Riemann surface. All this leads to identify forms on the Siegel upper half-space that, if certain conditions related to the characterization of the Jacobian are satisfied, express the bosonic measure as a multiresidue in the Siegel upper half-space. We also suggest that it may exist a super analog on the super Siegel half-space.

KEYWORDS: Bosonic Strings, Superstrings and Heterotic Strings

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1 Introduction

Denote by y_1, \dots, y_{3g-3} some complex analytic coordinates on the moduli space of genus $g \geq 2$ compact Riemann surfaces \mathcal{M}_g , and by τ the Riemann period matrix. According to the Belavin-Knizhnik theorem the genus g partition function of the Polyakov bosonic string [1] is [2]

$$Z_g = \int_{\mathcal{M}_g} \frac{F \wedge \bar{F}}{(\det \operatorname{Im} \tau)^{13}},$$

where

$$F := F(y_1, \dots, y_{3g-3}) dy_1 \wedge \dots \wedge dy_{3g-3},$$

is a holomorphic $(3g-3, 0)$ form which vanishes nowhere on \mathcal{M}_g . Furthermore, F has a second-order pole, due to the tachyon, at the Deligne-Mumford boundary of \mathcal{M}_g .

Consider the Siegel upper half-space

$$\mathfrak{H}_g := \{Z \in M_g(\mathbb{C}) \mid {}^t Z = Z, \operatorname{Im} Z > 0\},$$

and the Thetanullwerte

$$\chi_k(Z) := \prod_{\delta \text{ even}} \theta[\delta](0, Z),$$

$Z \in \mathfrak{H}_g$, with $k = 2^{g-2}(2^g + 1)$. For $g=2$ it has been shown in [3–5] that F is proportional to

$$\frac{\wedge_{i \leq j}^2 d\tau_{ij}}{\chi_5^2(\tau)}.$$

For $g=3$, it has been conjectured in [3, 4] and proved in [6–8], that F is proportional to

$$\frac{\wedge_{i \leq j}^3 d\tau_{ij}}{\chi_{18}^{1/2}(\tau)}.$$

Finding the higher genus expression of F is a long-standing problem involving basic questions which are not only of a purely technical nature, rather they concern the foundations of string theories. In particular, it turns out that the problem of the effective characterization of the Jacobian locus (Schottky problem) is deeply related to string theory. This is already evident once one considers that the Fay's trisecant formula [10], which is the higher genus version of the bosonization (Cauchy determinantal) formula, characterizes, even if not effectively, the Jacobian. Furthermore, the even unimodular lattices E_8 and D_{16}^+ of the heterotic strings turn out to characterize the Jacobian at $g = 4$ (see below), and perhaps all trigonal curves or n -gonal curves, for some $n \equiv n(g)$.

In string theory the Schottky problem arises from the beginning, that is for $g = 4$. In this case, Belavin-Knizhnik [2] and Morozov [4] conjectured that

$$F = \frac{d\tau_{11} \wedge \cdots \wedge \widehat{d\tau_{ij}} \wedge \cdots \wedge d\tau_{44}}{S_{4ij}(\tau)}, \quad (1.1)$$

where

$$S_{4ij}(Z) := \frac{1 + \delta_{ij}}{2} \frac{\partial F_4(Z)}{\partial Z_{ij}},$$

with

$$F_g := 2^g \sum_{\delta \text{ even}} \theta^{16}[\delta](0, Z) - \left(\sum_{\delta \text{ even}} \theta^8[\delta](0, Z) \right)^2.$$

It turns out that F_4 , the Schottky-Igusa form, vanishes only on the Jacobian. Furthermore, there is a nice relation between F_g and the theta series Θ_Λ corresponding to $\Lambda = E_8$ and $\Lambda = D_{16}^+$

$$F_g = 2^{-2g} (\Theta_{D_{16}^+} - \Theta_{E_8}^2).$$

Eq. (1.1) has several consequences. First, it shows that the bosonic string is strictly related to the question of characterizing the Jacobian. Furthermore, it follows that the genus four bosonic partition function can be expressed as residue formula in the Siegel upper half-space [2, 4]

$$Z_4 = \int_{\mathfrak{H}_4} \frac{1}{(\det \text{Im} Z)^{13}} \left| \frac{\wedge_{i \leq j}^4 dZ_{ij}}{F_4(Z)} \right|^2. \quad (1.2)$$

As stressed by Morozov in [9], such a “wonderful formula” has not attracted the due attention. Its elegance and simplicity suggest the intriguing possibility that there exists a formulation of the bosonic string on the Siegel upper half-space. This would be of considerable interest also in superstring theory as one may investigate its formulation on the super analog of the Siegel upper half-space.

A more rigorous derivation of eq. (1.1) has been proposed in [11]. More recently, eq. (1.1) has been proven in [12] in the context of vector-valued Teichmüller modular forms, a framework that shed light on its geometrical origin. The aim of the present paper is to further investigate its structure and then generalize it to the case of $g \geq 5$. In section 2 we shortly review the vector-valued Teichmüller modular forms, in particular it turns out that for $g \geq 4$ the bosonic measure is expressed in terms of them. In section

3 we discuss how such relation implies that the bosonic string measure characterizes the Riemann surface in \mathbb{P}^{g-1} .

In section 4 we show that the vector-valued Teichmüller modular forms are expressed as the determinant of the coefficients of the polynomials of degree n in \mathbb{P}^{g-1} , this is the key point that will lead, as a corollary, to the higher genus extension of the Belavin-Knizhnik formula (1.1), derived in section 5. We will also consider a natural candidate for the extension of (1.2) and will suggest it should be possible finding its super analog on the super Siegel half-space.

2 The bosonic measure and vector-valued Teichmüller modular form

In this section we shortly review the main results of [12]. In particular, we will see some that for $g \geq 4$ the bosonic measure is expressed in terms of the vector-valued Teichmüller modular forms.

Denote by C a compact Riemann surface of genus $g \geq 2$ and by

$$M_n := \binom{g+n-1}{n},$$

and

$$N_n := (2n-1)(g-1) + \delta_{n1},$$

the dimension of $\text{Sym}^n H^0(K_C)$ and $H^0(K_C^n)$ respectively. Set

$$K_n := M_n - N_n.$$

We will also use the notation

$$M \equiv M_2 = g(g+1)/2, \quad N \equiv N_2 = 3g-3, \quad K \equiv K_2 = (g-2)(g-3)/2.$$

Let $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ be a symplectic basis of $H_1(C, \mathbb{Z})$. Denote by $\{\omega_i\}_{1 \leq i \leq g}$ the basis of $H^0(K_C)$ with the standard normalization $\oint_{\alpha_i} \omega_j = \delta_{ij}$ and by $\tau_{ij} = \oint_{\beta_i} \omega_j$ the Riemann period matrix.

Consider the moduli space of principally polarized abelian varieties $\mathcal{A}_g = \mathfrak{H}_g / \text{Sp}(2g, \mathbb{Z})$. According to Torelli's theorem, the morphism

$$i : \mathcal{M}_g \rightarrow \mathcal{A}_g,$$

which on (geometric) points takes the algebraic curves to its Jacobian, is injective. The question of characterizing the image of i is the Schottky problem.

Denote by $\mathcal{C}_g \xrightarrow{\pi} \mathcal{M}_g$ the universal curve over \mathcal{M}_g and by $L_n = R\pi_*(K_{\mathcal{C}_g/\mathcal{M}_g}^n)$ the vector bundle on \mathcal{M}_g of rank N_n with fiber $H^0(K_C^n)$ at the point of \mathcal{M}_g representing C . Let $\lambda_n := \det L_n$ be the determinant line bundle. The Mumford isomorphism is [13]

$$\lambda_n \cong \lambda_1^{\otimes c_n},$$

where $c_n := 6n^2 - 6n + 1$. The Mumford forms $\mu_{g,n}$ are the unique, up to a constant, holomorphic section of $\lambda_n \otimes \lambda_1^{-\otimes c_n}$ nowhere vanishing on \mathcal{M}_g . Let $\{\phi_i^n\}_{1 \leq i \leq N_n}$ be a basis of $H^0(K_C^n)$, $n \geq 2$. It turns out that

$$\mu_{g,n} = \frac{\kappa[\omega]^{(2n-1)^2}}{\kappa[\phi^n]} \frac{\phi_1^n \wedge \cdots \wedge \phi_{N_n}^n}{(\omega_1 \wedge \cdots \wedge \omega_g)^{c_n}},$$

where

$$\kappa[\omega] := \frac{\det \omega_i(p_j) \sigma(y) \prod_1^g E(y, p_i)}{\theta(\sum_1^g p_i - y - \Delta) \prod_1^g \sigma(p_i) \prod_{i < j}^g E(p_i, p_j)},$$

and

$$\kappa[\phi^n] := \frac{\det \phi_i^n(p_j)}{\theta(\sum_1^{N_n} p_i - (2n-1)\Delta) \prod_1^{N_n} \sigma(p_i)^{2n-1} \prod_{i < j}^{N_n} E(p_i, p_j)},$$

$n \geq 2$, where we followed the notation of [12]. Note that $\mu_{g,n}$ is independent of the choice of the basis of $H^0(K_C^n)$.

In the case of $n = 2$ the Mumford form is strictly related to the Belavin-Knizhnik form F , that they obtained from the curvature form of the determinant of Laplace operators,

$$F = \frac{\kappa[\omega]^9}{\kappa[\phi^2]} \phi_1^2 \wedge \cdots \wedge \phi_{3g-3}^2.$$

The fact that the bosonic string measure is determined by the Mumford form $\mu_{g,2}$ has been first discovered by Manin [14] who noticed that $c_2 = 13$ in Mumford's formula coincides with the half of the string critical dimension. The expression of $\mu_{g,2}$ in terms of theta functions has been given by Beilinson and Manin [15] whereas $\mu_{g,n}$ has been obtained by Verlinde and Verlinde [16] and Fay [17].

It is useful to use a single indexing, for example, instead of $\omega_i \omega_j$, with $i, j = 1, \dots, g$, we may use the shorter notation $\omega_i^{(2)}$, $i = 1, \dots, g(g+1)/2$. More generally, let us consider the basis $\tilde{\omega}_1^{(n)}, \dots, \tilde{\omega}_{M_n}^{(n)}$ of $\text{Sym}^n H^0(K_C)$ whose elements are symmetrized tensor products of n -tuples of vectors of the basis $\omega_1, \dots, \omega_g$, taken with respect to an arbitrary ordering. The image $\omega_i^{(n)}$, $i = 1, \dots, M_n$, of $\tilde{\omega}_i^{(n)}$ under $\psi : \text{Sym}^n H^0(K_C) \rightarrow H^0(K_C^n)$ is surjective for $g = 2$ and for C non-hyperelliptic of genus $g > 2$.

Let us consider $\mu_{g,2}$. For $g > 3$ one has $K > 0$, so that using N elements of $\text{Sym}^2 H^0(K_C)$ as basis for $H^0(K_C^2)$ leads to free indices. More generally for $\mu_{g,n}$ this happens when $K_n > 0$ and one uses N_n elements of $\text{Sym}^n H^0(K_C)$ as basis for $H^0(K_C^n)$. This leads to the concept of vector-valued Teichmüller modular forms [12]. For example, in the case $n = 2$, $g > 3$, one is led to consider wedge products $\wedge_{k=1}^{3g-3} \omega_{i_k} \omega_{j_k}$, or, upon taking the Kodaira-Spencer map

$$\omega_i \omega_j \mapsto \frac{1}{2\pi i} d\tau_{ij},$$

the wedge products

$$\wedge_{k=1}^{3g-3} d\tau_{i_k j_k},$$

so that there are free indices.

The vector-valued Teichmüller modular forms are [12]

$$[i_{N_n+1}, \dots, i_{M_n} | \tau] := \epsilon_{i_1, \dots, i_{M_n}} \frac{\kappa[\omega_{i_1}^{(n)}, \dots, \omega_{i_{N_n}}^{(n)}]}{\kappa[\omega]^{(2n-1)^2}}, \quad (2.1)$$

$i_1, \dots, i_{M_n} \in \{1, \dots, M_n\}$. Note that

$$[i_{N_n+1}, \dots, i_{M_n} | \tau] = \epsilon_{i_1, \dots, i_{M_n}} \frac{\omega_{i_1}^{(n)} \wedge \dots \wedge \omega_{i_{N_n}}^{(n)}}{(\omega_1 \wedge \dots \wedge \omega_g)^{c_n} \mu_{g,n}}, \quad (2.2)$$

so that, since $\mu_{g,n}$ is nowhere vanishing and holomorphic on \mathcal{M}_g , it follows that even $[i_{N_n+1}, \dots, i_{M_n} | \tau]$ is holomorphic on \mathcal{M}_g and vanishes when $\omega_{i_1}^{(n)}, \dots, \omega_{i_{N_n}}^{(n)}$ is not a basis of $H^0(K_C^n)$. In particular, for τ belonging to the closure of the locus of hyperelliptic Riemann period matrices \mathcal{H}_g in \mathfrak{H}_g , $[i_{N_n+1}, \dots, i_{M_n} | \tau]$ has zeroes of order at least $(n-1)(g-1)-1$ [12].

Since $\mu_{g,n}$ is invariant under the modular transformations

$$\tau \mapsto \gamma \cdot \tau = (A\tau + B)(C\tau + D)^{-1}, \quad (2.3)$$

$\gamma \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g := \mathrm{Sp}(2g, \mathbb{Z})$, it follows by (2.2) that [12]

$$\begin{aligned} \sum_{j_1, \dots, j_{K_n}=1}^{M_n} \rho^{(n)}(\gamma)_{k_1 j_1} \dots \rho^{(n)}(\gamma)_{k_{K_n} j_{K_n}} [j_1, \dots, j_{K_n} | \gamma \cdot \tau] \\ = \det(C\tau + D)^{d_n} [k_1, \dots, k_{K_n} | \tau], \end{aligned} \quad (2.4)$$

where

$$\rho^{(n)}(\gamma) \cdot (\omega_{k_1} \dots \omega_{k_n}) = \sum_{j_1, \dots, j_n=1}^g \omega_{j_1} \dots \omega_{j_n} (C\tau + D)_{j_1 k_1}^{-1} \dots (C\tau + D)_{j_n k_n}^{-1},$$

$k_1, \dots, k_n = 1, \dots, g$, and

$$d_n := 6n^2 - 6n + 1 - \binom{g+n-1}{n-1}. \quad (2.5)$$

Note that $d_2 = g - 12$, so that $g = 12$ is a critical point.¹

3 Bosonic measure and the quadrics in \mathbb{P}^{g-1}

Interestingly, vector-valued modular forms are the natural generalization of the classical elliptic modular forms, and therefore basic even for number theory, as seen by studying the cohomology of the universal abelian variety. The theory of vector-valued modular forms is not particularly developed, only the case of low genus has been investigated. The fact that the Mumford forms define an infinite set of vector-valued Teichmüller modular forms is of interest even in such a context.

¹It is worth noticing that $g = 12$ is a critical value for the Ikeda lift and for the slope conjecture.

A nice geometrical feature of the vector-valued Teichmüller modular forms derived from the Mumford forms is that these describe the Riemann surface (or canonical curve) in terms of quadrics in \mathbb{P}^{g-1} . Recently, such quadrics have been expressed as determinantal relations among quadratic differentials [18–20].

An embedding of C in \mathbb{P}^{g-1} is given by $p \mapsto (\omega_1(p), \dots, \omega_g(p))$. More generally, any element of $H^0(K_C^n)$ defines a homogeneous n -degree polynomial in \mathbb{P}^{g-1} by

$$\phi^n := \sum_{i_1, \dots, i_n} B_{i_1, \dots, i_n} \omega_{i_1} \cdots \omega_{i_n} \mapsto \sum_{i_1, \dots, i_n} B_{i_1, \dots, i_n} X_{i_1} \cdots X_{i_n},$$

where X_1, \dots, X_g are homogeneous coordinates on \mathbb{P}^{g-1} . A basis of $H^0(K_C^n)$ corresponds to a basis of the homogeneous polynomials of degree n in \mathbb{P}^{g-1} that are not zero when restricted to C . One may identify C with the ideal of all the polynomials in \mathbb{P}^{g-1} vanishing at C . Such an ideal is generated by the quadrics

$$\sum_{j=1}^M C_j^i X_j^{(2)} = 0, \quad N+1 \leq i \leq M,$$

where we used the single index notation $X_j^{(2)}$ instead of $X_i X_j$. The canonical curves that are not cut out by such quadrics are trigonal or isomorphic to a smooth plane quintic. In these cases the ideal is generated by the quadrics above together with a suitable set of cubics corresponding to linear relations among holomorphic 3-differentials.

A property of the vector-valued Teichmüller modular forms is that they provide the coefficients of such polynomials. Namely, for each integer $n \geq 2$ and for all $i_2, \dots, i_{K_n} \in \{1, \dots, M_n\}$ we have [12]

$$\sum_{i=1}^{M_n} [i, i_2, \dots, i_{K_n} | \tau] \omega_i^{(n)}(x) = 0. \quad (3.1)$$

In particular, for $n = 2$ these are all the quadrics characterizing the canonical curve in projective space. The explicit form of the coefficients of the quadrics has been derived in [18–20]. However, finding their expression in terms of theta-constants is still an open question. The vector-valued Teichmüller modular forms seem the natural candidates to find such expressions.

A related issue is the problem of the effective characterization of the Jacobian. This has been explicitly solved only for $g = 4$: there is a weight 8 Siegel modular form vanishing only on the Jacobian, this is the Schottky-Igusa form F_4 . We have seen that F_4 defines $[i|\tau]$. More generally, a similar relation between vector-valued Teichmüller modular forms and the characterization of the Jacobian should hold even for $g > 4$. To see this, let us first explicitly show the case $g = 4$. Let \mathcal{I}_g be the closure of the locus of Riemann period matrices in \mathfrak{H}_g . Since F_4 vanishes identically on \mathcal{I}_4 , we have [12]

$$0 = \partial_t F_4(\tau) = \sum_{i \leq j} \frac{\partial F_4}{\partial Z_{ij}} \Big|_{Z=\tau} \partial_t \tau_{ij} = \sum_{i \leq j} \frac{\partial F_4}{\partial Z_{ij}} \Big|_{Z=\tau} d\tau_{ij}(\partial_t). \quad (3.2)$$

It follows by (3.1) that

$$[ij|\tau] = cS_{4ij}(Z),$$

$i, j = 1, \dots, g$, where c is a constant. Note that here the double index notation $[ij|\tau]$ instead of $[i|\tau]$, $i = 1, \dots, M$, has been used. It turns out that there is a simple relation relating the discriminant of the quadric and χ_{68} [12]

$$\det S_4(\tau) = d\chi_{68}(\tau)^{1/2},$$

where d is a constant. Note that $\det S_4$ and $\chi_{68}^{1/2}$ are modular forms (of weight 34) only when restricted to \mathcal{I}_4 . The above results also provide a straightforward proof that for $g = 4$, $n = 2$, the Mumford form is

$$\mu_{4,2} = \pm \frac{1}{cS_{4ij}} \frac{\omega_1 \omega_1 \wedge \dots \wedge \widehat{\omega_i \omega_j} \wedge \dots \wedge \omega_4 \omega_4}{(\omega_1 \wedge \dots \wedge \omega_4)^{13}}, \quad (3.3)$$

as conjectured by Belavin and Knizhnik [2] and by Morozov [4].

4 Vector-valued Teichmüller modular forms as determinants

A basic question is to understand what is the analog structure of $\mu_{4,2}$ in the case of $\mu_{g,n}$, for arbitrary g and n . Are there forms that, like F_4 generate the vector-valued Teichmüller modular forms for any n and g ? This may shed light on algebraic-geometrical structures underlying the bosonic and supersymmetric string measures.

There is a natural answer to the above question.

Theorem 4.1.

$$[i_{N_n+1}, \dots, i_{M_n}|\tau] = \det_j \quad C_{i_k}^j(\tau), \quad (4.1)$$

$$k \in \{N_n+1, \dots, M_n\}$$

where $C^j(\tau) \equiv (C_1^j(\tau), \dots, C_{M_n}^j(\tau))$ are $K_n = M_n - N_n$ linearly independent vectors, non-trivial solutions of the degree n polynomial equations

$$\sum_{i=1}^{M_n} C_i^j(\tau) \omega_i^{(n)} = 0, \quad (4.2)$$

$j = 1, \dots, K_n$. Furthermore,

$$\sum_{i=1}^{M_n} \rho^{(n)}(\gamma)_{ki} C_i^j(\gamma \cdot \tau) = \sum_{l=1}^{K_n} A_l^j \det(C\tau + D)^{n_l} C_k^l(\tau), \quad (4.3)$$

$k = 1, \dots, M_n$, where $\det A = 1$ and

$$\sum_{k=1}^{K_n} n_k = d_n. \quad (4.4)$$

Proof. To prove eq. (4.1) let use the fact that there are K_n linearly independent vectors $D^j(\tau) \equiv (D_1^j(\tau), \dots, D_{M_n}^j(\tau))$ non-trivial solutions of

$$\sum_{i=1}^{M_n} D_i^j(\tau) \omega_i^{(n)} = 0, \quad (4.5)$$

$j = 1, \dots, K_n$. Then we rewrite this in the form

$$\sum_{k \in M_n \setminus \{1, \dots, N_n-1, N_n+1\}} D_{i_k}^j(\tau) \omega_{i_k}^{(n)} = - \sum_{k \in \{1, \dots, N_n-1, N_n+1\}} D_{i_k}^j(\tau) \omega_{i_k}^{(n)},$$

to express $\omega_{i_{N_n}}$ in

$$\omega_{i_1}^{(n)} \wedge \dots \wedge \omega_{i_{N_n}}^{(n)},$$

in terms of $\omega_{i_1}^{(n)}, \dots, \omega_{i_{N_n}-1}^{(n)}, \omega_{i_{N_n}+1}^{(n)}$, to find that, up to a sign,

$$\det_j \sum_{k \in \{N_n, N_n+2, \dots, M_n\}} D_{i_k}^j(\tau) \omega_{i_1}^{(n)} \wedge \dots \wedge \omega_{i_{N_n}}^{(n)},$$

is equal to

$$\det_j \sum_{k \in \{N_n+1, \dots, M_n\}} D_{i_k}^j(\tau) \omega_{i_1}^{(n)} \wedge \dots \wedge \omega_{i_{N_n}-1}^{(n)} \wedge \omega_{i_{N_n}+1}^{(n)}.$$

Since

$$\det_j \sum_{k \in \{N_n+1, \dots, M_n\}} D_{i_k}^j(\tau),$$

satisfies eq. (3.1) and is completely antisymmetric as $[i_{N_n+1}, \dots, i_{M_n} | \tau]$, it follows that they differ by a global term given by the determinant of the non degenerate matrix X such that $C^j = \sum_{l=1}^{K_n} X_l^j D^l$. Eqs. (4.3) and (4.4) follow by (2.4) and (2.5) respectively. \square

5 The Mumford-Polyakov form at any genus

A consequence of Theorem 4.1 is that the bosonic measure, corresponding to the case $n = 2$, is built in terms of the determinant of the coefficients of the quadrics describing the Riemann surface in \mathbb{P}^{g-1} . Also, note that the range of the indices of the determinant is from 1 to $K = M - N = (g-2)(g-3)/2$, which is the codimension of the locus of Riemann period matrices in the Siegel upper half-space. This is another piece of evidence that the Schottky problem is strictly related to string theory.

Let us choose an arbitrary point $p_0 \in C$ and consider the Abel-Jacobi map $I(p) := (I_1(p), \dots, I_g(p))$

$$I_i(p) := \int_{p_0}^p \omega_i,$$

$p \in C$. It embeds C into the Jacobian $J(C) := \mathbb{C}^g / L_\tau$, $L_\tau := \mathbb{Z}^g + \tau \mathbb{Z}^g$.

Denote by Θ_s the locus of the $e \in \mathbb{Z}^g$ where $\theta(e, Z)$ and its gradient vanish. It turns out that the map $P \mapsto \theta(I(P) \pm e)$, $e \in J(C)$ vanishes identically on C for all $P_0 \in C$ if

and only if $e \in \Theta_s$. As a consequence $\frac{\partial^2 \theta}{\partial u_j \partial u_k}(\pm e)$ and $\frac{\partial^3 \theta}{\partial u_j \partial u_k \partial u_l}(\pm e)$ are the coefficients of the quadrics and cubics describing C in \mathbb{P}^{g-1} , that is

$$\sum_{j,k=1}^g \frac{\partial^2 \theta}{\partial u_j \partial u_k}(\pm e) \omega_j \omega_k = 0, \quad (5.1)$$

$$\sum_{j,k,l=1}^g \frac{\partial^3 \theta}{\partial u_j \partial u_k \partial u_l}(\pm e) \omega_j \omega_k \omega_l = 0. \quad (5.2)$$

By means of the heat equation

$$\frac{\partial^2 \theta(u, Z)}{\partial u_j \partial u_k} = 2\pi i(1 + \delta_{jk}) \frac{\partial \theta(u, Z)}{\partial Z_{jk}},$$

and using the single indexing, one may rewrite eq. (5.1) in the form

$$\sum_{j=1}^K \frac{\partial \theta}{\partial Z_j}(\pm e)|_{Z=\tau} \omega_j^{(2)} = 0. \quad (5.3)$$

Since as e varies in Θ_s eq. (5.1) generates the ideal of quadrics passing through the curve C , it follows by (4.2) that there are $e^1, \dots, e^K \in \Theta_s$, such that the C_i^j in Theorem 4.1 are

$$C_i^j(\tau) = \sum_{l=1}^K \Lambda_l^j \frac{\partial \theta}{\partial Z_i}(e^l)|_{Z=\tau},$$

$j = 1, \dots, K$, for some matrix Λ_k^j , so that

$$[i_{N+1}, \dots, i_M | \tau] = \det_j \sum_{k \in \{N+1, \dots, M\}} \Lambda_l^j \frac{\partial \theta}{\partial Z_{i_k}}(e^l)|_{Z=\tau}.$$

Since $\theta(e^l)$ vanishes for $Z = \tau$, we have the basic fact that the vector-valued Teichmüller modular forms are given by a Jacobian. Set

$$F_g^j(\tau) := \sum_{l=1}^K \Lambda_l^j \theta(e^l)|_{Z=\tau}. \quad (5.4)$$

We have proved the following

Theorem 5.1.

$$[i_{N+1}, \dots, i_M | \tau] = \det_j \frac{\partial F_g^j(Z)}{\partial Z_{i_k}}|_{Z=\tau}. \quad (5.5)$$

As a corollary, it follows by (2.2) that for any g the Mumford-Polyakov form is

$$\mu_{g,2} = \frac{\epsilon_{i_1, \dots, i_M} \omega_{i_1}^{(2)} \wedge \dots \wedge \omega_{i_N}^{(2)}}{(\omega_1 \wedge \dots \wedge \omega_g)^{13} \det_j \frac{\partial F_g^j(Z)}{\partial Z_{i_k}}|_{Z=\tau}}. \quad (5.6)$$

Such an expression for $\mu_{g,2}$ provides strong evidence that the Mumford-Polyakov form can be expressed as a multiresidue. More precisely, if the unimodular matrix A in Theorem 4.1 is diagonal and under suitable conditions on the divisors of $F_g^j(Z)$, the bosonic partition function could be then expressed as a multiresidue on the Siegel upper half-space

$$Z_g = \int_{\mathfrak{H}_g} \frac{1}{\det(\text{Im}Z)^{13}} \left| \frac{\prod_{i \leq j}^g dZ_{ij}}{\prod_{j=1}^{(g-2)(g-3)/2} F_g^j(Z)} \right|^2. \quad (5.7)$$

If the above properties for $F_g^j(Z)$ could be satisfied, then finding them would provide an effective solution of the Schottky problem. In the case of $g = 5$, it is known that the Jacobian variety $J(C)$ is not characterized by the vanishing of three globally defined Siegel modular forms. Nevertheless, by (4.3) it follows that the weights n_j of $F_g^j(Z)$ must satisfy the condition

$$\sum_{j=1}^K n_j = 12 - g,$$

which implies that in general $F_g^j(Z)$ are not holomorphic on \mathfrak{H}_g . In particular, note that the n_j may be not integers. If the unimodular matrix A in (4.3) is not diagonal, then one should investigate a different expression of the integrand in (5.7) and construct invariants from F_g^j and \bar{F}_g^j . In this respect, it would be interesting to check possible non homogeneous terms in their transformation properties and/or a dependence of F_g^j on \bar{Z} that either vanishes on $J(C)$ or that disappears once one considers its Jacobian on $J(C)$. It may be that some higher genus generalization of the Mock theta functions play a rôle in such a context.

The construction in such a section can be extended to Mumford forms with arbitrary n . This is related to the generalized Beltrami differentials associated to $H^0(K_C^n)$ [21]. In particular, one should extend the map $\omega_i^{(2)} \mapsto \frac{1}{2\pi i} d\tau_i^{(2)}$ to arbitrary n

$$\omega_i^{(n)} \mapsto \frac{1}{2\pi i} d\tau_i^{(n)},$$

that defines the tangent space to the moduli space associated to the holomorphic n -differentials, that is the moduli space of vector bundles on Riemann surfaces. This is related to the chiral split for the higher order diffeomorphism anomalies. The Wess-Zumino conditions correspond to the cocycle identities (see section 3.4 of [21]).

We observe that recently the problem of superstring perturbation theory attracted renewed interest [22–26]–[52]. Interestingly, the vector-valued Teichmüller modular forms also appear in the superstring measure, in particular the cosmological constant for $g = 4$ turns out to be given in terms of $\mu_{4,2}$ and F_4 , namely

$$\frac{F_4}{\partial_{\tau_j} F_4} \frac{d\tau_1 \wedge \cdots \wedge \widehat{d\tau_j} \wedge \cdots \wedge d\tau_{44}}{(\omega_1 \wedge \cdots \wedge \omega_4)^5} = 0, \quad (5.8)$$

and vanishes just because F_4 characterizes the Jacobian locus. F_g gives a vanishing cosmological constant also for $g \leq 3$. The fact that F_g is the difference between of the two theta series associated to the even unimodular lattices E_8 and D_{16}^+ of the two heterotic

strings may be due to foundational aspects. It would be natural to guess that F_g characterizes the cosmological constant for any g . However, it has been shown in [48] that getting the amplitudes by degenerating the Riemann surfaces, one sees that after GSO projection, the two-point function does not vanish at genus four, as expected from space-time supersymmetry arguments. This is a strong hint that the Grushevsky ansatz [37] the superstring measure should be corrected for $g > 4$. Very recently in an interesting paper, Dunin-Barkowski, Sleptsov and Stern [52] have shown that there is a minor modification of Grushevsky ansatz [37] leading to a vanishing cosmological constant even for $g = 5$ that reproduces, by degeneration, a vanishing two-point function at $g = 4$. However, they also proved that a similar mechanism fails starting from $g = 6$.

Let us mention some possible developments in investigating the superstring measure.

1. *Focusing on the geometry of the Jacobian locus.* Starting from $g = 5$ there are more forms, namely K forms, vanishing on the Jacobian and that define the bosonic measure. If the structure of the cosmological constant at any genus should reproduce the one of $g = 4$, factorization arguments suggest that the F_g^j , $j = 1, \dots, K$, should appear in the cosmological constant too, not only in the bosonic part of the measure. This is related to the fact that there are not stable Schottky forms, as recently shown by Codogni and Shepherd-Barron [49].
2. *Trigonal curves.* Another possibility is to focus on the curious ubiquity of F_g in the expression for the cosmological constant up to $g = 4$. It is known that for $g = 5$ it vanishes only on the locus of \mathcal{M}_g corresponding to trigonal curves. This means that if one restricts the path-integral formulation to trigonal curves, then the Grushevsky ansatz would be effective at least up to $g = 5$. In this respect, it is interesting to note that trigonal curves, related to triality, have, with their three \mathbb{P}^1 , a structure reminiscent of the geometrical object defining the interactions in string theory, the pant. In this context it may be useful to recall that at the beginning of the superstring era several works were dedicated to the check, in perturbation theory, of the unitarity constraint.
3. *$n(g)$ -gonal curve.* A property of F_4 is that it identifies a subvariety, the Jacobian, which is of codimension 1. For $g > 4$ the codimension of $J(C)$ is K , so that there is not a divisor vanishing only on $J(C)$. This may be a problem in focusing on the trigonal curves. A possibility is that the case of $g = 4$ is the first one of a more subtle mechanism.² Actually, one should consider not just trigonal curves but $n(g)$ -gonal curves. In particular, one should consider $n(g)$ -gonal curves with $n(g)$ fixed by requiring that the codimension of their moduli spaces in \mathcal{M}_g be the lowest one. A $n(g)$ -gonal curve of genus g is a n -sheet covering of \mathbb{P}^1 , so that, by Hurwitz theorem, it should have $2g + 2n - 2$ ramification points. Fixing three points by the $\text{PSL}(2, \mathbb{C})$ symmetry of \mathbb{P}^1 , one sees that the locus of the moduli space of $n(g)$ -gonal curves in \mathcal{M}_g is $2g + 2n - 5$. Requiring that the $n(g)$ -gonal locus be of codimension one in \mathcal{M}_g fixes

$$2n(g) = g + 1 .$$

²I thank A. Sagnotti and R. Volpato for enlighting suggestions on the following analysis.

It follows that for g odd the space of $(g+1)/2$ -gonal curves is a divisor in \mathcal{M}_g . For g even, all the curves are $(g+2)/2$ -gonals, with the moduli space of $g/2$ -gonal curves being of codimension 2 in \mathcal{M}_g . It follows that a possible generalization of F_4 is to consider a form such that its zero divisor in \mathcal{M}_g be the locus of the $(g+1)/2$ -gonal curves for g odd, and properly containing the moduli of the $g/2$ -gonal curves for g even. Another possibility is that such a form always contains the trigonal locus, but for $g > 5$ only as sublocus. In this case, it would be interesting to understand if there exists a modular form which is a Schottky form for $g = 6$ and defines, by its zeroes, the divisor of the 4-gonals at $g = 7$.

The fact that $g = 5$ is a critical value for the superstring measure also follows by a recent result by Donagi and Witten [50, 51]. They proved that at least for $g \geq 5$ the supermoduli space is not a split supermanifold, that is the moduli space of super Riemann surfaces does not map to the moduli space of Riemann surfaces with a spin structure. In this context, it should be observed that the Berkovits pure spinor formulation of superstring theory is defined on Riemann surfaces [53, 54]. Understanding the relation between GSO projection in the RNS formalism and the Berkovits pure spinor formulation, where the GSO projection is absent, may lead to understand whether the Donagi-Witten result forbids a priori the reduction to \mathcal{M}_g .

Finally, we note that eq. (5.7) suggests that the bosonic string may be formulated on the Siegel upper half-space. Perhaps one may use the properties of the period matrices associated to the Prym varieties. An investigation in this direction may also lead to find a super analog of (5.7) on the super Siegel half-space.

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