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ONE-DIMENSIONAL SYMMETRY FOR SEMILINEAR EQUATIONS WITH UNBOUNDED DRIFT

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Abstract. We consider semilinear equations with unbounded drift in the whole of $\mathbb{R}^n$ and we show that monotone solutions with finite energy are one-dimensional.

1. **Introduction.** In the paper [9] E. De Giorgi formulated the celebrated conjecture that bounded monotone solutions to the Allen-Cahn equation

$$\Delta u = u^3 - u \tag{1}$$

are necessarily one-dimensional (in the sense that the level sets are hyperplanes) at least if $n \leq 8$. This conjecture has been proved by Ghoussoub and Gui in [17] (see also [3]) in dimension $n = 2$, and by Ambrosio and Cabré [2] in dimension $n = 3$ (see also [1]), and a counterexample has been given by del Pino, Kowalczyk and Wei in [10] for $n \geq 9$. Under the additional assumption that u connects -1 to 1, a proof has been presented by Savin [22] in dimension $n \leq 8$.

In this paper we consider the semilinear elliptic equation

$$\Delta u + c(z)u_z + \langle \nabla_y g(y), \nabla_y u\rangle + f(u) = 0, \tag{2}$$

where we write $x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$. A solution u of (2) of the form

$$u(x) = u_0(\langle \omega, x\rangle) \quad \forall x \in \mathbb{R}^n,$$

where $u_0 : \mathbb{R} \longrightarrow \mathbb{R}$ and $\omega \in \mathbb{R}^n$ with $|\omega| = 1$ will be called *one-dimensional*.

We are interested in symmetry results for solutions u which are monotone in the z-variable, i.e. satisfy

$$u_z(x) > 0 \qquad \forall x \in \mathbb{R}^n. \tag{3}$$

In particular, we will show that, under suitable assumptions, monotone solutions to (2) are necessarily one-dimensional (see Theorem 1.1).

Our methods rely on the geometric approach developed in [13] (see also [6, 7, 11, 12, 16, 23]), and our computations follow those in [14, 15], where the authors prove Liouville type results for stable solutions to elliptic equations in complete Riemmanian manifolds with nonnegative Ricci curvature.

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1.1. **Main result.** Let us state the main result of this paper:

Theorem 1.1. *Assume that* $f:\mathbb{R}\to\mathbb{R}$ *is locally Lipschitz,* $g\in C^2(\mathbb{R}^{n-1})$, $c\in C^1(\mathbb{R})$ *and that*

$$c'(z)\,\mathrm{I}_{n-1}\geq \nabla_y^2 g(y) \qquad \textit{for every } (y,z)\in\mathbb{R}^{n-1}\times\mathbb{R}, \tag{4}$$

where I_{n-1} *denotes the identity matrix on* $\mathbb{R}^{n-1}$*. Let* $C\in C^2(\mathbb{R})$ *be a primitive of* c*, and let* u *be a solution to* (2) *satisfying* (3) *and one of the following conditions:*

a)

$$\int_{\mathbb{R}^n}|\nabla u|^2 e^{g(y)+C(z)}dzdy<+\infty\,; \tag{5}$$

b) For all $z\in\mathbb{R}$

$$\int_{\mathbb{R}^{n-1}}|\nabla u|^2 e^{g(y)+C(z)}dy\leq K \qquad \text{for some } K>0\,; \tag{6}$$

c) $n=2$ *and for all* $(y,z)\in\mathbb{R}^n$

$$|\nabla u|^2 e^{g(y)+C(z)}\leq K \qquad \text{for some } K>0\,; \tag{7}$$

then u *is one-dimensional, and*

$$\langle\big(c'(z)\,\mathrm{I}_{n-1}-\nabla_y^2 g(y)\big)\,\nabla_y u,\nabla_y u\rangle=0\,. \tag{8}$$

In particular, if the strict inequality holds in (4) *for some* (y,z) *then* u *depends only on* z*.*

From Theorem 1.1 we get the following corollaries which extend a result in [6], valid for the Ornstein-Uhlenbeck case $C(z)=-z^2/2$, $g(y)=-|y|^2/2$.

Corollary 1. *Let* C,g *bounded above and satisfying* (4)*. Assume also that* $n=2$ *or*

$$\int_{\mathbb{R}^{n-1}}e^{g(y)}dy<+\infty\,. \tag{9}$$

Let $u\in W^{1,\infty}(\mathbb{R}^n)$ *be a solution to* (2) *satisfying* (3)*, then* u *is one-dimensional.*

Proof. If C,g are bounded above and $u\in W^{1,\infty}(\mathbb{R}^n)$, then (7) holds, and moreover condition (9) implies (6). The thesis then follows directly from Theorem 1.1. ☐

Remark 1. From [19, Th. 2.4 and Rem. 2.5] it follows that, if $\nabla^2 g(y)$ and $c'(z)$ are uniformly bounded below, every bounded solution to (2) belongs to $W^{1,\infty}(\mathbb{R}^n)$.

Corollary 2. *Let* C,g *be concave, satisfying* (4) *and* $-C,-g$ *coercive. Let* $u\in L^\infty(\mathbb{R}^n)$ *be a solution to* (2) *satisfying* (3)*, then* u *is one-dimensional.*

Proof. In [8, Thm 2.5, Cor. 4.3] it is proved that if $-C,-g$ are convex and coercive then any (weak) solution to (2) such that

$$\int_{\mathbb{R}^n}u^2 e^{g(y)+C(z)}dzdy<+\infty$$

also satisfies (5) (see Remark 3). In particular, any bounded solution to (2) satisfies (5), and we can conclude by Theorem 1.1. ☐

When $c(z) \equiv c \in \mathbb{R}$, solutions to (2) correspond to traveling (or standing if $c = 0$) wave solutions to the reaction-diffusion equation:

$$v_t = \Delta v + \langle \nabla_y g, \nabla_y v \rangle + f(v) \qquad \text{in } \mathbb{R}^n \times (0, +\infty). \tag{10}$$

A traveling wave solution is a particular solution v to (10), uniformly translating in the z-direction at constant speed c, of the form

$$v(t, x) = u(y, z - ct).$$

We refer to [24, 27] and references therein for classical results about existence and uniqueness of traveling waves in infinite cylinders.

Corollary 3. *Let g be concave and let $v(t, x) = u(y, z - ct)$ be a traveling or a standing wave solution to (10). If u satisfies one of the three conditions of Theorem 1.1, then u is one-dimensional. Moreover u depends only on z unless $n = 2$, g is constant and (7) holds.*

Conditions (5), (6), (7) are quite restrictive. However, traveling wave solutions satisfying these conditions are relevant to propagation and are sometimes called *variational traveling waves*. We refer to [20, 21] for a general analysis of such solutions, including necessary and sufficient conditions for existence.

If these conditions are not satisfied, equation (10) admits in general traveling waves which are not one-dimensional even in the case $n = 2$ and $g = 0$ (see [4, 5, 18]).

2. **A Ornstein-Uhlenbeck type equations.**

More generally, we shall consider the following equation of Ornstein-Uhlenbeck type:

$$\Delta u + \langle \nabla G(x), \nabla u \rangle + f(u) = 0 \qquad x \in \mathbb{R}^n, \tag{11}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function and $G \in C^2(\mathbb{R}^n)$.

Notice that solutions to (11) are critical point of the functional

$$I(u) := \int_{\mathbb{R}^n} \left(\frac{|\nabla u|^2}{2} + F(u) \right) e^{G(x)} dx, \tag{12}$$

where $F'(t) = -f(t)$. We define the function $\lambda_G \in C^0(\mathbb{R}^n)$ as

$$\lambda_G(x) := \text{ maximal eigenvalue of } \nabla^2 G(x). \tag{13}$$

Observe that, if $G(x) := g(y) + C(z)$, then (11) reduces to (2), and $\lambda_G(x) \geq C''(z)$ for every $x \in \mathbb{R}^n$.

2.1. *h*-stable solutions.

We denote by μ the measure on $\mathbb{R}^n$ with density $e^{G(x)}$ w.r.t. the Lebesgue measure, and we let $W^{k,p}_\mu(\mathbb{R}^n) \subset W^{k,p}_{\mathrm{loc}}(\mathbb{R}^n)$, for $k, p \in \mathbb{N}$, be the corresponding Sobolev spaces.

Notice that, if G is concave, then μ is a finite measure iff

$$\lim_{|x| \to +\infty} G(x) = -\infty\,.$$

We introduce now the notion of h-stability for solutions to (11).

Definition 2.1. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. A solution u to (11) is h-stable if

$$\int_{\mathbb{R}^n} \Big(|\nabla \varphi|^2 - f'(u)\varphi^2 \Big) d\mu \geq \int_{\mathbb{R}^n} h(x)\varphi^2 d\mu \qquad \forall\ \varphi \in C^1_c(\mathbb{R}^n)\,.$$

If $h \equiv 0$, then u is said to be stable.

We recall that a function $u \in W^{1,2}_{\rm loc}(\mathbb{R}^n)$ is a weak solution to (11) if

$$\int_{\mathbb{R}^n} \big(\langle \nabla u, \nabla\varphi\rangle - f(u)\varphi\big) d\mu = 0 \qquad \forall\, \varphi \in C^1_c(\mathbb{R}^n)\,.$$

Note that every critical point of the functional I in (12) is a weak solution to (11). By classical elliptic regularity theory, if u is a weak solution then $u \in C^{2,\alpha}(\mathbb{R}^n)$ for all $\alpha < 1$, in particular it is also a classical solution to (11).

Remark 2. The function $u \in W^{1,2}_{\rm loc}(\mathbb{R}^n)$ is a local minimizer of the functional I in (12) if I does not decrease under compactly supported perturbations, i.e.

$$I(u) \le I(v) \qquad \text{whenever } v \in W^{1,2}_{\rm loc}(\mathbb{R}^n) \text{ and } \{u \neq v\} \subset K \subset\subset \mathbb{R}^n.$$

Every local minimizer of I is a stable weak solution to (11).
In [6] the authors show that, when $G(x) = -|x|^2/2$, monotone solutions to (11) are -1-stable (i.e. stable with respect to the constant function $h \equiv -1$).

In the following we will consider h-stable solutions to (11) which have *finite energy*, in the sense that

$$|\nabla u| \in L^2_\mu(\mathbb{R}^n). \tag{14}$$

Note that if $G(y,z) = g(y) + C(z)$, this condition reduces to (5). When $n = 2$, we can substitute this condition with

$$|\nabla u|^2 e^G \in L^\infty(\mathbb{R}^n). \tag{15}$$

Remark 3. If the measure μ is finite then $L^\infty(\mathbb{R}^n) \subset L^2_\mu(\mathbb{R}^n)$. If the function G is concave, by [8, Thm 2.5, Cor. 4.3] this implies that every bounded solution to (11) belongs to $W^{2,2}_\mu(\mathbb{R}^n)$ and hence satisfies (14).

On the other hand, assumption (14) can be satisfied also when μ is not finite: for instance, if $G(x) = g(y)$ is such that (9) holds and $f(s) = s - s^3$, the function

$$u(z) = \tanh\left(\frac{z}{\sqrt{2}}\right)$$

is a monotone stable solution to (11) with finite energy.

3. λ_G-stability and finite energy imply one-dimensional symmetry.

We now show that λ_G-stable solutions to (11), where λ_G is defined in (13), which satisfy (14) or (15) are one-dimensional. Similar results for stable solutions have been obtained in the setting of Riemannian manifolds with nonnegative Ricci curvature in [14, 15].

Given a differentiable function $v : \mathbb{R}^n \to \mathbb{R}$, we set $v_i := \partial_i v$ for all $i = 1, \dots, n$.

Lemma 3.1. *Let $u \in W^{1,2}_{\rm loc}(\mathbb{R}^n)$ be a weak solution to (11). Then for any $i = 1, \dots, n$ and $\varphi \in C^1_c(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} \left(\langle \nabla u_i, \nabla\varphi\rangle - \langle \nabla u, \nabla G_i\rangle\, \varphi - f'(u) u_i \varphi\right) d\mu(x) = 0.$$

Proof. It suffices to prove (3.1) for $\varphi \in C_c^\infty(\mathbb{R}^n)$. From (2.1), applied with φ replaced by φ_i, we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \langle \nabla u, \nabla \varphi_i \rangle - f(u)\varphi_i \, d\mu(x) \\ &= \int_{\mathbb{R}^n} -\langle \nabla u_i, \nabla \varphi \rangle - \langle \nabla u, \nabla \varphi \rangle G_i + f'(u) u_i \varphi + f(u)\varphi G_i \, d\mu(x) \\ &= \int_{\mathbb{R}^n} -\langle \nabla u_i, \nabla \varphi \rangle + f'(u) u_i \varphi + f(u)\varphi G_i \, d\mu(x) + \\ &\quad + \int_{\mathbb{R}^n} -\langle \nabla u, \nabla(\varphi G_i) \rangle + \langle \nabla u, \nabla G_i \rangle \varphi \, d\mu(x). \end{aligned}$$

Recalling (2.1), applied with φ replaced by φG_i, we obtain the thesis. □

Proposition 1. *Let $h \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and u be a h-stable solution to (11). Then for every $\varphi \in C_c^1(\mathbb{R}^n)$ we have*

$$\begin{aligned} \int_{\mathbb{R}^n} \Big(|\nabla^2 u|^2 - |\nabla |\nabla u||^2 + \big\langle \big(h(x)\mathrm{I}_n - \nabla^2 G(x)\big) \nabla u, \nabla u \big\rangle \Big) \varphi^2 d\mu(x) \quad (16) \\ \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 d\mu(x) \, . \end{aligned}$$

Proof. Let $\varphi \in C_c^1(\mathbb{R}^n)$. Using (3.1) with test function $u_i \varphi^2$ we obtain

$$\int_{\mathbb{R}^n} \big\langle \nabla u_i, \nabla(u_i \varphi^2) \big\rangle - f'(u) u_i^2 \varphi^2 d\mu(x) = \int_{\mathbb{R}^n} \langle \nabla u, \nabla G_i \rangle u_i \varphi^2 d\mu(x). \quad (17)$$

Summing over i, (17) gives

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla^2 u|^2 \varphi^2 + \frac{1}{2} \big\langle \nabla |\nabla u|^2, \nabla \varphi^2 \big\rangle - f'(u) |\nabla u|^2 \varphi^2 d\mu(x) \quad (18) \\ = \int_{\mathbb{R}^n} \big\langle \nabla^2 G(x) \, \nabla u, \nabla u \big\rangle \varphi^2 \, d\mu(x). \end{aligned}$$

Using (2.1) with test function $|\nabla u|\varphi$ we then get

$$\begin{aligned} \int_{\mathbb{R}^n} h(x) |\nabla u|^2 \varphi^2 d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla(|\nabla u|\varphi)|^2 - f'(u)|\nabla u|^2 \varphi^2 d\mu(x) \quad (19) \\ = \int_{\mathbb{R}^n} \varphi^2 |\nabla |\nabla u||^2 + |\nabla u|^2 |\nabla \varphi|^2 + \frac{1}{2} \big\langle \nabla |\nabla u|^2, \nabla \varphi^2 \big\rangle - f'(u) |\nabla u|^2 \varphi^2 d\mu(x). \end{aligned}$$

Substituting (18) in (19) we get the result. □

Corollary 4. *Recalling that $|\nabla^2 u|^2 - |\nabla |\nabla u||^2 \geq 0$ (see Remark 4), from (16) it follows*

$$\int_{\mathbb{R}^n} \big\langle \big(h(x)\mathrm{I}_n - \nabla^2 G(x)\big) \; \nabla u, \nabla u \big\rangle \varphi^2 d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 d\mu(x). \quad (20)$$

If $h \geq \lambda_G$, from (16) and the definition of λ_G in (13) it follows

$$\int_{\mathbb{R}^n} \Big(|\nabla^2 u|^2 - |\nabla |\nabla u||^2 \Big) \varphi^2 d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 d\mu(x). \quad (21)$$

Remark 4. The Poincaré type formula (21) was first obtained by Sternberg and Zumbrun [26]. Notice that the quantity $|\nabla^2 u|^2 - |\nabla |\nabla u||^2$ has a geometric interpretation, in the sense that it can be expressed in terms of the principal curvatures of level sets of u.

More precisely, letting

$$L_{u,x} := \{y \in \mathbb{R}^n \mid u(y) = u(x)\}\,,$$

we denote by $\nabla_T u$ the tangential gradient of u along $L_{u,x} \cap \{\nabla u \neq 0\}$, and by $k_1, \ldots, k_{n-1}$ the principal curvatures of $L_{u,x} \cap \{\nabla u \neq 0\}$. Then the following formula holds (as proved in Lemma 2.1 in [25])

$$|\nabla^2 u|^2 - |\nabla|\nabla u||^2 = |\nabla_T|\nabla u||^2 + |\nabla u|^2 \sum_{j=1}^{n-1} k_j^2 \qquad \text{on } L_{u,x} \cap \{\nabla u \neq 0\}\,,$$

so that (21) becomes

$$\int_{\{\nabla u \neq 0\}} (|\nabla u|^2 \mathcal{K}^2 + |\nabla_T|\nabla u||^2)\varphi^2 \, d\mu(x) + \int_{\{\nabla u = 0\}} (|\nabla^2 u|^2 - |\nabla|\nabla u||^2)\varphi^2 \, d\mu(x)$$
$$\leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 \, d\mu(x).$$

where $\mathcal{K} := \sum_{j=1}^{n-1} k_j^2$. By Stampacchia's Theorem, since $\mu << \mathcal{L}^n$, we get

$$\nabla|\nabla u|(x) = 0 \quad \mu\text{-a.e } x \in \{|\nabla u| = 0\}$$
$$\nabla u_j(x) = 0 \quad \mu\text{-a.e } x \in \{|\nabla u| = 0\} \subseteq \{u_j = 0\}$$

for any $j = 1, \ldots, n$. Hence the previous inequality becomes

$$\int_{\{\nabla u \neq 0\}} (|\nabla u|^2 \mathcal{K}^2 + |\nabla_T|\nabla u||^2)\varphi^2 \, d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 \, d\mu(x)\,.$$

We refer to [26] and [13] for more details.

We now state the main result of this section.

Theorem 3.2. *Assume that $G \in C^2(\mathbb{R}^n)$ and $h \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $h \geq \lambda_G$. Let u be a h-stable solution to (11) such that one of the following conditions hold:*

i) u satisfies (14);
ii) $n = 2$ and u satisfies (15).

Then u is one-dimensional, i.e. there exists $\omega \in \mathbb{S}^{n-1}$ and $u_0 : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$u(x) = u_0(\langle \omega, x \rangle) \quad \forall x \in \mathbb{R}^n.$$

Moreover,

$$\big\langle \big(h(x)\mathrm{I}_n - \nabla^2 G(x)\big)\, \nabla u, \nabla u \big\rangle = 0 \qquad \forall\; x \in \mathbb{R}^n\,. \tag{22}$$

In particular, if u_0 is not constant, there are C and g of class C^2 such that

$$G(x) = C(\langle x, \omega \rangle) + g(x'), \tag{23}$$

where $x' := x - \langle x, \omega \rangle\, \omega$, and $\lambda_G(x) = h(x) = C''(\langle x, \omega \rangle)$ for all $x \in \mathbb{R}^n$.

Proof. Let us fix $R > 1$ and let us define $\varphi(x) := \Phi(|x|)$ where $\Phi \in C^\infty(\mathbb{R})$, $|\Phi'(t)| \leq 3$ for any $t \in [R, R+1]$

$$\Phi(t) := \begin{cases} 1 & \text{if} \quad t \leq R \\ 0 & \text{if} \quad t \geq R+1. \end{cases} \tag{24}$$

Obviously $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $|\nabla \varphi(x)| \leq |\Phi'(|x|)| \leq 3$. Hence for every $R > 1$ (4) yields

$$\int_{\{\nabla u \neq 0\} \cap \bar{B}_R} \Big(|\nabla u|^2 \mathcal{K}^2 + |\nabla_T|\nabla u||^2\Big) \, d\mu(x) \leq 9 \int_{\bar{B}_{R+1} \setminus B_R} |\nabla u|^2 \, d\mu(x)$$

where $B_R := \{y \in \mathbb{R}^n \mid |y| < R\}$.

If $\nabla u \in L^2_\mu(\mathbb{R}^n)$, then

$$\lim_{R\to\infty} \int_{\bar{B}_{R+1}\setminus B_R} |\nabla u|^2 \, d\mu(x) = 0.$$

Hence (3) and (3) yield

$$k_j(x) = 0 \quad \text{and} \quad |\nabla_T |\nabla u||(x) = 0$$

for every $j = 1, \ldots, n-1$ and every $x \in \{\nabla u \neq 0\}$. From this and Lemma 2.11 in [13] we get the one-dimensional symmetry of u.

If $n = 2$ and $|\nabla u|^2 e^G \in L^\infty(\mathbb{R}^n)$, we take in (4) the following test function

$$\varphi(x) = \max\left[0, \min\left(1, \frac{\ln R^2 - \ln|x|}{\ln R}\right)\right], \tag{25}$$

Reasoning as in [13, Cor. 2.6], we then obtain

$$\int_{\{\nabla u\neq 0\}\cap \bar{B}_R} \left(|\nabla u|^2\mathcal{K}^2 + |\nabla_T|\nabla u||^2\right) d\mu(x) \leq \int_{B_{R^2}\setminus B_R} \frac{1}{|x|^2\,(\ln R)^2}|\nabla u|^2 e^{G(x)}dx.$$

When $R \to +\infty$, since $|\nabla u|^2 e^{G(x)}$ is bounded, the r.h.s. term of the previous inequality vanishes, and we conclude agian that u is one-dimensional.

Assume now that u is not constant. If we take in (20) the same test functions as above, we get

$$\int_{\mathbb{R}^n} \big\langle (h(x)\mathrm{I}_n - \nabla^2 G(x))\nabla u, \nabla u\big\rangle \, d\mu(x) = 0\,.$$

Using the fact that $u(x) = u_0(\langle \omega, x\rangle)$, we obtain that $\big\langle (h(x)\mathrm{I}_n - \nabla^2 G(x))\omega, \omega\big\rangle = 0$ for all x such that $u_0'(\langle\omega, x\rangle) \neq 0$. Since u is not constant and is a solution to the elliptic equation (11), the set of points such that $u_0'(\langle\omega, x\rangle) = 0$ has zero measure, so, by the regularity of G we conclude that

$$\big\langle (h(x)\mathrm{I}_n - \nabla^2 G(x))\omega, \omega\big\rangle = 0 \qquad \forall\, x \in \mathbb{R}^n\,,$$

which gives (22) and (23). □

Theorem 3.2 direclty implies the following Liouville type result (cfr. [14]).

Corollary 5. *Let $h \in C^0(\mathbb{R}^n)$ with $h \geq \lambda_G$, and u be a h-stable solution solution to (11) with finite energy. If $\lambda_G(x) < h(x)$ for some $x \in \mathbb{R}^n$, then u is constant. In particular, if u is a stable solution and $\lambda_G(x) < 0$ for some $x \in \mathbb{R}^n$, then u is constant.*

Remark 5. Recalling Remark 3, when the measure μ is finite and G is concave, Theorem 3.2 implies that bounded solutions to (11) which are λ_G-stable are one-dimensional.

4. Monotonicity implies λ_G-stability.

In this section we assume that, for every $x \in \mathbb{R}^n$, e_n is the eigenvector associated to the maximal eigenvalue $\lambda_G(x)$ of $\nabla^2 G(x)$. This implies that there exist two functions g and C such that

$$G(x) = g(y) + C(z) \quad \text{and} \quad \lambda_G(x) = C''(z). \tag{26}$$

We prove that solutions to (11) which are monotone along the z-axis are stable.

Theorem 4.1. *Assume that G satisfies (26) and u is a solution to (11) satisfying (3). Then u is λ_G-stable.*

Proof. Equation (3.1) with $i = n$ reads

$$\int_{\mathbb{R}^n} \langle \nabla u_z, \nabla\varphi\rangle - C''(z)u_z\varphi - f'(u)u_z\varphi \; d\mu(x) = 0. \tag{27}$$

Let $\varphi \in C^1_c(\mathbb{R}^n)$. Taking as test function $\frac{\varphi^2}{u_z}$ in (27), we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \left\langle \nabla u_z, \nabla\left(\frac{\varphi^2}{u_z}\right)\right\rangle - C''(z)\varphi^2 - f'(u)\varphi^2 \; d\mu(x) \\ &= \int_{\mathbb{R}^n} |\nabla\varphi|^2 - \left|\frac{\varphi}{u_z}\nabla u - \nabla\varphi\right|^2 - C''(z)\varphi^2 - f'(u)\varphi^2 \; d\mu(x) \\ &\le \int_{\mathbb{R}^n} |\nabla\varphi|^2 - C''(z)\varphi^2 - f'(u)\varphi^2 \; d\mu(x), \end{aligned}$$

which is the stability condition (2.1). □

5. **Proof of Theorem 1.1.** Observe that in (2), $G(x) = g(y) + C(z)$, and by assumption $C''(z) \ge \nabla^2 g(y)$. So (26) holds, and by Theorem 4.1 every solution to (2) satisfying (3) is λ_G-stable.

If either a) or c) holds, the thesis follows from Theorem 3.2.

Let us assume that u satisfies b). We define $\psi_R(y) := \Phi(|y|)$ where Φ is as in (24) and $\varphi_S(z)$ as follows. We fix $S > 1$ and let

$$\varphi_S(z) := \begin{cases} 3 & \text{if } |z| \le S \\ 4 - \frac{z^2}{S^2} & \text{if } S \le |z| \le 2S \\ 0 & \text{if } |z| \ge 2S. \end{cases}$$

We compute (4) with test function $\psi_R(y)\varphi_S(z)$ and obtain, recalling (6),

$$\begin{aligned} \int_{\{\nabla u \neq 0\}} \left(|\nabla u|^2\mathcal{K}^2 + |\nabla_T|\nabla u||^2\right)\psi_R^2\varphi_S^2 \; d\mu(x) &\le \int_{\mathbb{R}^n} |\nabla u|^2 {\varphi'}_S^2(z)\nabla^2\psi_R(y)d\mu(x) \\ \le \frac{4}{S^2}\int_{\mathbb{R}^n} |\nabla u|^2\nabla^2\psi_R(y)d\mu(x) &\le \frac{36K}{S^2}. \end{aligned}$$

If we let $R \to +\infty$ we obtain

$$\int_{\{\nabla u\neq 0\}\cap\{|z|\le S\}} \left(|\nabla u|^2\mathcal{K}^2 + |\nabla_T|\nabla u||^2\right)d\mu(x) \le \frac{4K}{S^2}.$$

Letting $S \to +\infty$ we then obtain (3) and we conclude as in the proof of Theorem 3.2. □

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