Logical Characterizations of Behavioral Relations on Transition Systems of Probability Distributions

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Logical Characterizations of Behavioral Relations on Transition Systems of Probability Distributions

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Probabilistic nondeterministic processes are commonly modeled as probabilistic LTSs (PLTSs). A number of logical characterizations of the main behavioral relations on PLTSs have been studied. In particular, Parma and Segala [2007] and Hermanns et al. [2011] define a probabilistic Hennessy-Milner logic interpreted over probability distributions, whose corresponding logical equivalence/preorder when restricted to Dirac distributions coincide with standard bisimulation/simulation between the states of a PLTS. This result is here extended by studying the full logical equivalence/preorder between (possibly non-Dirac) distributions in terms of a notion of bisimulation/simulation defined on a LTS whose states are distributions (dLTS). We show that the well-known spectrum of behavioral relations on nonprobabilistic LTSs as well as their corresponding logical characterizations in terms of Hennessy-Milner logic scales to the probabilistic setting when considering dLTSs.

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1. INTRODUCTION

Formal methods for concurrent and distributed system specification and verification have been extended to encompass the behavior of probabilistic systems. In a standard nonprobabilistic setting, systems are commonly modeled as labeled transition systems (LTSs) and verification techniques are based on two major tools: temporal logics and behavioral relations. Logics are used to specify the properties that systems have to satisfy, while behavioral equivalence/preorder relations are used as appropriate abstractions that reduce the space of system states. Precise relationships have been established between these two approaches: the foundational work of van Glabbeek [2001] shows how a wide spectrum of observational equivalences for concurrent processes can be logically characterized in terms of Hennessy-Milner-like modal logics (HML).

A number of behavioral relations and temporal logics tailored for probabilistic systems have been put forward (see e.g. [Deng and van Glabbeek 2010; van Glabbeek et al. 1990; Hansson and Jonsson 1994; Hennessy 2012; Hermanns et al. 2011; Larsen

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and Skou 1991; Parma and Segala 2007; Segala and Lynch 1995]). Probabilistic LTSs (PLTSs, a.k.a. probabilistic automata) are a prominent model for formalizing probabilistic systems since they allow to model both probabilistic and nondeterministic behaviors. In PLTSs, a state s evolves through a labeled transition to a probability distribution over states that defines the probabilities of reaching the possible successor states of s. Accordingly, the standard probabilistic extension [Segala and Lynch 1995] of the simulation relation requires that if a state s progresses to a distribution d, then a simulating state s' needs to mimic such a transition by moving to a distribution d' that is related to d through a so-called weight function. This definition is a conservative extension of the simulation relation on nonprobabilistic LTSs since a LTS can be viewed as a particular PLTS where the target of transitions are the so-called Dirac distributions, i.e., state distributions δ_s such that $\delta_s(s) = 1$ and $\delta_s(t) = 0$ for any $t \neq s$.

Several modal logics have been proposed in order to provide a logical characterization of probabilistic simulation and bisimulation. Larsen and Skou [1991]'s logic as well as Hansson and Jonsson [1994]'s PCTL logic are interpreted over states of probabilistic systems that do not express nondeterminism such as reactive models and discrete-time Markov chains. On the other hand, Parma and Segala [2007] show that richer probabilistic models that encode pure nondeterminism (besides probabilistic choice), such as PLTSs, call for a richer logic. They therefore suggest a probabilistic extension of HML whose formulae are interpreted over state distributions rather than states, and they show that two states s and t are bisimilar if and only if their corresponding Dirac distributions δ_s and δ_t satisfy the same set of formulae. However, nothing is stated in [Parma and Segala 2007] about logically equivalent distributions that are not Dirac distributions. The logical characterizations in [Parma and Segala 2007] have been later extended by Hermanns et al. [2011] to simulation relations and to image-infinite PLTSs.

In this paper we study the full logical equivalence between (possibly non-Dirac) distributions that is induced by Parma and Segala [2007]'s logic. We show that this logic actually characterizes a novel and natural notion of simulation (bisimulation) between distributions of a PLTS, so that the standard state simulation (bisimulation) on PLTSs can be indeed retrieved by a suitable restriction to Dirac distributions. Furthermore, the transition relation of a PLTS is lifted to a transition relation between distributions that gives rise to a corresponding LTS on distributions, called dLTS. This allows us to lift behavioral relations on PLTSs to corresponding behavioral relations on dLTSs. Such a move from PLTSs to dLTSs provides the following advantages:

— Parma and Segala [2007]'s logic turns out to be equivalent to a logic $\mathcal{L}$ whose diamond predecessor operator is interpreted on the dLTS in accordance with the standard semantics on LTSs. In this regard, this logic best suits as probabilistic extension of Hennessy-Milner logic. In particular, $\mathcal{L}$ characterizes a (bi)simulation relation between (possibly non-Dirac) distributions which is equivalent to that characterized by Parma and Segala's logic.
— A spectrum of behavioral relations can be defined on dLTSs along the lines of the well-known approach on LTSs [van Glabbeek et al. 1990]. These preorder/equivalence relations between distributions can be then projected back to states, thus providing a spectrum of (probabilistic) preorder/equivalence relations between states of PLTSs.

This approach is studied on a number of well known probabilistic relations appearing in literature, namely simulation, probabilistic simulation, failure simulation, and their corresponding bisimulations. A discussion about related approaches is the subject of the final section, that also hints at future work.

This is an extended and revised version of the conference paper [Crafa and Ranzato 2011b].

2. SIMULATION AND BISIMULATION ON PROBABILISTIC LTSS

2.1. Basic Notions

Given a set X and a relation $R \subseteq X \times X$, we write xRy for $(x, y) \in R$. If $x \in X$ and $Y \subseteq X$ then $R(x) \triangleq \{y \in X \mid xRy\}$ and $R(Y) \triangleq \cup_{x\in Y} R(x)$. A set $U \subseteq X$ is R-closed if $R(U) \subseteq U$.

Let $R \subseteq X \times X$ be a preorder on X, namely, a reflexive and transitive relation. We denote by $\mathcal{K}_R$ the kernel of R, namely the largest equivalence relation contained in R, which is $\mathcal{K}_R = R \cap R^{-1}$. When R is a preorder, we have that: (1) a set $U \subseteq X$ is R-closed iff $R(U) = U$; (2) if U is R-closed then there exists a family of equivalence classes $\{C_i\}_{i\in I}$ of the kernel $\mathcal{K}_R$ such that $U = \cup_{i\in I} C_i = \cup_{i\in I} R(C_i)$; (3) a R-closed set U is finitely generated if there exists a finite set of equivalence classes $C_1, \ldots, C_k$ ($k \geq 1$) of the kernel $\mathcal{K}_R$ such that $U = C_1 \cup \ldots \cup C_k$.

$\mathrm{Distr}(X)$ denotes the set of (stochastic) distributions on a set X, i.e., the set of functions $d : X \to [0, 1]$ such that $\sum_{x\in X} d(x) = 1$. The support of a distribution d is defined by $\mathrm{supp}(d) \triangleq \{x \in X \mid d(x) > 0\}$. Also, if $Y \subseteq X$ then $d(Y) \triangleq \sum_{y\in Y} d(y)$. If $\mathrm{supp}(d) = \{x_1, ..., x_n\}$ then d is also denoted by $(x_1/d(x_1), ..., x_n/d(x_n))$. The Dirac distribution on $x \in X$, denoted by δ_x, is the distribution that assigns probability 1 to x (and 0 otherwise).

A probabilistic LTS (PLTS) is a tuple $\mathcal{M} = \langle \Sigma, Act, \to \rangle$ where Σ is a countable set of states, Act is a countable set of actions and $\to \subseteq \Sigma \times Act \times \mathrm{Distr}(\Sigma)$ is a countable transition relation, where $(s, a, d) \in \to$ is denoted by $s \xrightarrow{a} d$. For any action $a \in Act$, the predecessor operator $\mathrm{pre}_a : \wp(\mathrm{Distr}(\Sigma)) \to \wp(\Sigma)$ is defined by $\mathrm{pre}_a(D) \triangleq \{s \in \Sigma \mid \exists d \in D. s \xrightarrow{a} d\}$, while the successor operator $\mathrm{post}_a : \wp(\Sigma) \to \wp(\mathrm{Distr}(\Sigma))$ is defined as $\mathrm{post}_a(S) \triangleq \{d \in \mathrm{Distr}(\Sigma) \mid \exists s \in S.\, s \xrightarrow{a} d\}$. $\mathcal{M}$ is image-finite when for any state s and action a, $\mathrm{post}_a(\{s\})$ is a finite set.

2.2. Lifting Relations

The definitions of probabilistic behavioral relations often rely on so-called weight functions [Segala 1995], that are used to lift a relation between states to a relation between distributions. For our purposes, it is not needed to recall the definition of weight function as we will use the following equivalent characterizations (see [Desharnais 1999; Hermanns et al. 2011; Zhang 2008; Zhang et al. 2008]), where condition (3) is proved in [Hermanns et al. 2011, Lemma 5.2].

Definition 2.1 (*Lifting*). Let $R \subseteq X \times X$ be a relation on a set X. Then, the *lifting* of R to distributions is the relation $\sqsubseteq_R \subseteq \mathrm{Distr}(X) \times \mathrm{Distr}(X)$ defined as follows:

$$d \sqsubseteq_R e \;\text{ if }\; d(U) \leq e(R(U)) \;\text{ for any set } U \subseteq \mathrm{supp}(d). \tag{1}$$

If R is a preorder then

$$d \sqsubseteq_R e \;\Leftrightarrow\; d(U) \leq e(U) \;\text{ for any } R\text{-closed set } U \subseteq X. \tag{2}$$

If R is a preorder then

$$d \sqsubseteq_R e \;\Leftrightarrow\; d(U) \leq e(U) \;\text{ for any finitely-generated } R\text{-closed set } U \subseteq X. \tag{3}$$

□

It is easy to see that if $R \subseteq R'$ then $\sqsubseteq_R \subseteq \sqsubseteq_{R'}$. Moreover, if R is symmetric then $\sqsubseteq_R$ is also a symmetric relation, that we also denote with $\equiv_R$. When R is an equivalence relation on X, it turns out that $d \sqsubseteq_R e$ iff $d(B) = e(B)$ for any equivalence class B of R (see e.g. [Hermanns et al. 2011, Lemma 3.2]). The following easy properties of the lifting relation will be useful later on.

LEMMA 2.2. *Let* $d, e \in \mathrm{Distr}(\Sigma)$ *and* $R \subseteq \Sigma \times \Sigma$. *If* $d \sqsubseteq_R e$ *then*

(1) *for any* $x \in \mathrm{supp}(d)$ *there exists* $y \in \mathrm{supp}(e)$ *such that* $y \in R(x)$*;*
(2) *for any* $y \in \mathrm{supp}(e)$ *there exists* $x \in \mathrm{supp}(d)$ *such that* $y \in R(x)$.

PROOF. For property (1), notice that if $d \sqsubseteq_R e$ then we have that for any $x \in \mathrm{supp}(d)$, $0 < d(x) \leq e(R(x))$, which implies that there exists $y \in \mathrm{supp}(e)$ such that $y \in R(x)$. As far as property (2) is concerned, if $d \sqsubseteq_R e$ we have that $1 = d(\mathrm{supp}(d)) \leq e(R(\mathrm{supp}(d)))$, which implies $\mathrm{supp}(e) \subseteq R(\mathrm{supp}(d))$, that is, for any $y \in \mathrm{supp}(e)$ there exists $x \in \mathrm{supp}(d)$ such that $y \in R(x)$. $\Box$

We also notice that any relation on distributions $\mathcal{R} \subseteq \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma)$ embeds a corresponding relation on states that can be obtained by restricting $\mathcal{R}$ to Dirac distributions. This is formalized by a mapping $\Delta : \wp(\mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma)) \to \wp(\Sigma \times \Sigma)$ defined as follows:

$$\Delta(\mathcal{R}) \triangleq \{(s,t) \in \Sigma \times \Sigma \mid (\delta_s, \delta_t) \in \mathcal{R}\}.$$

Note that if $\mathcal{R}$ is a symmetric/preorder/equivalence relation on $\mathrm{Distr}(\Sigma)$ then $\Delta(\mathcal{R})$ is correspondingly a symmetric/preorder/equivalence relation on Σ. Also, it is easy to notice that if $\mathcal{R}' \subseteq \mathcal{R}$ then $\sqsubseteq_{\Delta(\mathcal{R}')} \subseteq \sqsubseteq_{\Delta(\mathcal{R})}$, and if $\mathcal{R}, \mathcal{R}'$ are equivalences then $\equiv_{\Delta(\mathcal{R}')} \subseteq \equiv_{\Delta(\mathcal{R})}$.

2.3. Simulation and Bisimulation

The standard notions of simulation and bisimulation on PLTSs (dating back to [Larsen and Skou 1991]) go as follows.

Definition 2.3 (*Simulation*). Given a PLTS $\mathcal{M} = \langle \Sigma, Act, \to \rangle$, a relation $R \subseteq \Sigma \times \Sigma$ is a *simulation* on $\mathcal{M}$ if for all $s, t \in \Sigma$ such that sRt,

if $s \xrightarrow{a} d$ then there exists $e \in \mathrm{Distr}(\Sigma)$ such that $t \xrightarrow{a} e$ and $d \sqsubseteq_R e$. $\Box$

We define $R_{\mathrm{sim}} \triangleq \cup\{R \subseteq \Sigma \times \Sigma \mid R$ is a simulation on $\mathcal{M}\}$. It is easily seen that R_{sim} turns out to be a preorder relation which is the greatest simulation on $\mathcal{M}$ and is called simulation preorder (or similarity) on $\mathcal{M}$. Simulation equivalence P_{sim} on $\mathcal{M}$ is defined as the kernel of the simulation preorder, i.e., $P_{\mathrm{sim}} \triangleq R_{\mathrm{sim}} \cap R_{\mathrm{sim}}^{-1}$.

Definition 2.4 (*Bisimulation*). A symmetric relation $S \subseteq \Sigma \times \Sigma$ is a *bisimulation* on $\mathcal{M}$ if for all $s, t \in \Sigma$ such that sSt,

if $s \xrightarrow{a} d$ then there exists $e \in \mathrm{Distr}(\Sigma)$ such that $t \xrightarrow{a} e$ and $d \equiv_S e$. $\Box$

Let us define $P_{\mathrm{bis}} \triangleq \cup\{S \subseteq \Sigma \times \Sigma \mid S$ is a bisimulation on $\mathcal{M}\}$. Then, P_{bis} turns out to be an equivalence relation which is the greatest bisimulation on $\mathcal{M}$ and is called bisimilarity on $\mathcal{M}$.

3. A NEW NOTION OF SIMULATION

In order to provide a modal logical characterization of the basic behavioral relations on probabilistic models that also encode pure nondeterminism, such as PLTSs, Parma and Segala [2007] put forward an extension of Hennessy-Milner logic whose formulae are interpreted over distributions on the states of a PLTS. In particular, they show that two states s_1 and s_2 are bisimilar if and only if their corresponding Dirac distributions δ_{s_1} and δ_{s_2} satisfy the same set of modal formulae. Successively, this logical characterization has been extended to the simulation relation by Hermanns et al. [2011]. However, nothing is stated in [Hermanns et al. 2011; Parma and Segala 2007] about distributions that turn out to be logically equivalent although they are not Dirac distributions.

In the following, we give a novel notion of simulation (and correspondingly bisimulation) between distributions which: (i) characterizes the full logical equivalence of [Parma and Segala 2007]'s logic between possibly non-Dirac distributions and (ii) boils down to standard simulation (and bisimulation) between the states of a PLTS when restricted to Dirac distributions.

3.1. Parma and Segala's Logic

Parma and Segala [2007]'s logic, here denoted by $\mathcal{L}_\forall$, is syntactically defined as follows:

$$\mathcal{L}_\forall \ni \phi ::= \top \mid \bigwedge_{i\in I} \phi_i \mid \neg\phi \mid \Diamond_a \phi \mid [\phi]_p$$

where I is a (possibly infinite) countable set of indices, $a \in Act$ and p is a rational number in $[0,1]$. Given a PLTS $\langle \Sigma, Act, \rightarrow\rangle$, the notion of a distribution satisfying a logical formula is inductively defined as follows: for any distribution $d \in \mathrm{Distr}(\Sigma)$,

$$\begin{array}{lll} d \models \top & & \\ d \models \bigwedge_I \phi_i & \text{iff} & \text{for any } i \in I,\ d \models \phi_i \\ d \models \neg\phi & \text{iff} & d \not\models \phi \\ d \models \Diamond_a\phi & \text{iff} & \forall x \in \mathrm{supp}(d).\, \exists e \in \mathrm{Distr}(\Sigma).\ x \xrightarrow{a} e \text{ and } e \models \phi \\ d \models [\phi]_p & \text{iff} & d(\{s \in \Sigma \mid \delta_s \models \phi\}) \geq p \end{array}$$

The first three clauses are standard. The modal connective $\Diamond_a$ is a probabilistic counterpart of HML's diamond operator: $\Diamond_a\phi$ is satisfied by a distribution d whenever *any state* $x \in \mathrm{supp}(d)$ reaches through an a-labeled transition a distribution e that satisfies the formula ϕ. As the formulae $\Diamond_a\phi$ only deal with transitions of the PLTS, a further modal operator $[\cdot]_p$ is needed to take into account the probabilities that distributions assign to sets of states. More precisely, a distribution d satisfies a formula $[\phi]_p$ when d assigns a probability of at least p to the set of states whose Dirac distributions satisfy the formula ϕ. This logic is here referred to as $\mathcal{L}_\forall$ in order to stress the *universal* flavor of the semantics of its diamond operator $\Diamond_a$. Given a distribution d, we will use the following notation: $\mathrm{Sat}_{\mathcal{L}_\forall}(d) \triangleq \{\phi \in \mathcal{L}_\forall \mid d \models \phi\}$.

Definition 3.1 (*Logical equivalence and preorder*). Two distributions $d, e \in \mathrm{Distr}(\Sigma)$ are logically equivalent for $\mathcal{L}_\forall$, written $d \equiv_{\mathcal{L}_\forall} e$, when $\mathrm{Sat}_{\mathcal{L}_\forall}(d) = \mathrm{Sat}_{\mathcal{L}_\forall}(e)$. Moreover, we denote by $\leq_{\mathcal{L}_\forall}$ the corresponding logical preorder, i.e., $d \leq_{\mathcal{L}_\forall} e$ when $\mathrm{Sat}_{\mathcal{L}_\forall}(d) \subseteq \mathrm{Sat}_{\mathcal{L}_\forall}(e)$. □

Let $\mathcal{L}_\forall^+$ be the positive (i.e., negation-free) and finitely disjunctive (i.e., only finite disjunctions are allowed) fragment of $\mathcal{L}_\forall$, that is:

$$\mathcal{L}_\forall^+ \ni \phi ::= \top \mid \bigwedge_{i\in I} \phi_i \mid \phi_1 \vee \phi_2 \mid \Diamond_a \phi \mid [\phi]_p$$

The following result by Parma and Segala [2007, Theorem 1] and Hermanns et al. [2011, Theorems 5.3, 6.1] shows that the logical equivalence induced by $\mathcal{L}_\forall$ and the logical preorder induced by $\mathcal{L}_\forall^+$, when restricted to Dirac distributions, correspond, respectively, to bisimulation and simulation. Notice that the simulation preorder is logically characterized by negation-free formulae, reflecting the fact that simulation, differently from bisimulation, is not a symmetric relation.

THEOREM 3.2 ([HERMANNS ET AL. 2011; PARMA AND SEGALA 2007]). *For all* $s, t \in \Sigma$*,*

(1) $s\, R_{\mathrm{sim}}\, t$ *if and only if* $\delta_s \leq_{\mathcal{L}_\forall^+} \delta_t$*;*

(2) $s\, P_{\mathrm{bis}}\, t$ *if and only if* $\delta_s \equiv_{\mathcal{L}_\forall} \delta_t$.

Let us remark that the above logical characterization for simulation R_{sim} holds for the positive restriction of the logic $\mathcal{L}_\forall$ to finite rather than infinite disjunctions of formulae. This comes as a consequence of the characterization of weight functions in Definition 2.1 (3), proved in [Hermanns et al. 2011, Lemma 5.2].

In what follows, our goal is to define a notion of simulation and bisimulation between distributions that allows us to extend Theorem 3.2 to *generic* (viz. possibly non-Dirac) *distributions* in order to encode the full operational match of the logical preorder $\le_{\mathcal{L}_\forall^+}$ and equivalence $\equiv_{\mathcal{L}_\forall}$.

3.2. $\forall$d-simulations

We observe that the semantics of the diamond operator of Parma and Segala's logic highlights a key difference with the semantics of the standard diamond operator in Hennessy-Milner logic HML. In the case of standard LTSs, the semantics of the diamond operator of HML induces the predecessor operator $\mathrm{pre}_a^{\mathrm{LTS}}$ of the LTS, meaning that the standard predecessor operator $\mathrm{pre}_a^{\mathrm{LTS}} : \wp(\Sigma) \to \wp(\Sigma)$ of the LTS satisfies the following equation: for any ϕ in HML,

$$\mathrm{pre}_a^{\mathrm{LTS}}(\llbracket\phi\rrbracket) = \{s \in \Sigma \mid s \models \Diamond_a \phi\}.$$

Analogously, the semantic definition of the diamond operator of $\mathcal{L}_\forall$ in a PLTS induces an operator $\mathrm{ppre}_a^\forall : \wp(\mathrm{Distr}(\Sigma)) \to \wp(\mathrm{Distr}(\Sigma))$ which for any $\phi \in \mathcal{L}_\forall$ is defined on the sets $\llbracket\phi\rrbracket \in \wp(\mathrm{Distr}(\Sigma))$ as follows:

$$\mathrm{ppre}_a^\forall(\llbracket\phi\rrbracket) \triangleq \{d \in \mathrm{Distr}(\Sigma) \mid d \models \Diamond_a \phi\}.$$

If $\mathrm{pre}_a : \wp(\mathrm{Distr}(\Sigma)) \to \wp(\Sigma)$ denotes the PLTS predecessor operator then we have that $d \models \Diamond_a\phi$ iff $\mathrm{supp}(d) \subseteq \mathrm{pre}_a(\llbracket\phi\rrbracket)$. These observations lead us to define the *probabilistic predecessor* operator $\mathrm{ppre}_a^\forall$ as follows: for all $D \in \wp(\mathrm{Distr}(\Sigma))$,

$$\mathrm{ppre}_a^\forall(D) \triangleq \{d \in \mathrm{Distr}(\Sigma) \mid \mathrm{supp}(d) \subseteq \mathrm{pre}_a(D)\}.$$

However, one key point to observe is that, differently from the standard predecessor operator $\mathrm{pre}_a^{\mathrm{LTS}}$ of LTSs, this probabilistic predecessor $\mathrm{ppre}_a^\forall$ *does not preserve set unions*, i.e., it is not true in general that, for any $D_1, D_2 \subseteq \mathrm{Distr}(\Sigma)$, $\mathrm{ppre}_a^\forall(D_1 \cup D_2) = \mathrm{ppre}_a^\forall(D_1) \cup \mathrm{ppre}_a^\forall(D_2)$. In fact, $\mathrm{supp}(d) \subseteq \mathrm{pre}_a(D_1 \cup D_2)$ does not imply $\mathrm{supp}(d) \subseteq \mathrm{pre}_a(D_1)$ nor $\mathrm{supp}(d) \subseteq \mathrm{pre}_a(D_2)$. For example, for a basic PLTS like $\langle\{x_1,x_2,x_3,x_4\},\{a\},\{x_1 \xrightarrow{a} \delta_{x_3}, x_2 \xrightarrow{a} \delta_{x_4}\}\rangle$ (which actually is a LTS), for the distribution $d = (x_1/0.5, x_2/0.5)$ we have that $d \in \mathrm{ppre}_a^\forall(\{\delta_{x_3}, \delta_{x_4}\})$, while $d \notin \mathrm{ppre}_a^\forall(\delta_{x_3}) \cup \mathrm{ppre}_a^\forall(\delta_{x_4})$.

It is also worth noting that, in general, an operator $f : \wp(X) \to \wp(X)$ defined on a powerset $\wp(X)$ preserves set unions if and only if there exists a relation $R \subseteq X \times X$ whose corresponding predecessor operator $\mathrm{pre}_R = \lambda Y.\{x \in X \mid \exists y \in Y.\, xRy\}$ coincides with f. As a consequence, *one cannot define* a transition relation between distributions of the PLTS whose corresponding predecessor operator coincides with $\mathrm{ppre}_a^\forall$.

In logical terms, the above remark reads as lack of distributity of the diamond connective $\Diamond_a$ w.r.t. logical disjunction in Parma and Segala's logic $\mathcal{L}_\forall$.

Example 3.3. Consider the PLTS

$$\langle\{x_1,x_2,x_3,x_4,x_5\},\{a,b,c\},\{x_1 \xrightarrow{a} \delta_{x_3}, x_2 \xrightarrow{a} \delta_{x_4}, x_3 \xrightarrow{b} \delta_{x_5}, x_4 \xrightarrow{c} \delta_{x_5}\}\rangle.$$

Let us consider a non-Dirac distribution like $d = (x_1/0.5, x_2/0.5)$. Then, it turns out that $d \models \Diamond_a(\Diamond_b\top \vee \Diamond_c\top)$ while $d \not\models \Diamond_a(\Diamond_b\top) \vee \Diamond_a(\Diamond_c\top)$. $\square$

It is worth remarking that distributivity w.r.t. logical disjunction of a diamond connective modeling the "possibly" modality holds already in the weakest modal logic **K** and therefore should be a basic desirable property for any modal logic that characterizes simulation and bisimulation in probabilistic systems.

A first definition of simulation (and bisimulation) between distributions, that we call $\forall d$*-simulation*, is directly inspired by the logic $\mathcal{L}_\forall$ and the probabilistic predecessor $\mathrm{ppre}_a^\forall$. In particular, the two distinctive modal operators of $\mathcal{L}_\forall$ are mirrored in two defining conditions of simulation between distributions. More precisely, condition (1) of the next definitions encodes a kind of universal transfer property — directly derived from the semantics of the diamond operator in $\mathcal{L}_\forall$ — that similar/bisimilar distributions should respect. On the other hand, condition (2), peculiar of the probabilistic setting, deals with the probabilities assigned by similar/bisimilar distributions to sets of related states. Let $\mathcal{M} = \langle \Sigma, Act, \to\rangle$ be a PLTS.

Definition 3.4 ($\forall d$*-simulation*). A relation $\mathcal{R} \subseteq \mathrm{Distr}(\Sigma)\times\mathrm{Distr}(\Sigma)$ is a $\forall d$*-simulation* on $\mathcal{M}$ if for all $d, e \in \mathrm{Distr}(\Sigma)$, if $d\,\mathcal{R}\,e$ then:

(1) for all $D \subseteq \mathrm{Distr}(\Sigma)$, if $\mathrm{supp}(d) \subseteq \mathrm{pre}_a(D)$ then $\mathrm{supp}(e) \subseteq \mathrm{pre}_a(\mathcal{R}(D))$;
(2) $d \sqsubseteq_{\Delta(\mathcal{R})} e$. □

Definition 3.5 ($\forall d$*-bisimulation*). A symmetric relation $\mathcal{S} \subseteq \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma)$ is a $\forall d$*-bisimulation* on $\mathcal{M}$ if for all $d, e \in \mathrm{Distr}(\Sigma)$, if $d\,\mathcal{S}\,e$ then:

(1) for all $D \subseteq \mathrm{Distr}(\Sigma)$, if $\mathrm{supp}(d) \subseteq \mathrm{pre}_a(D)$ then $\mathrm{supp}(e) \subseteq \mathrm{pre}_a(\mathcal{S}(D))$;
(2) $d \equiv_{\Delta(\mathcal{S})} e$. □

Given a PLTS $\mathcal{M}$, we define:

$$\mathcal{R}^\forall_{\mathrm{sim}} \triangleq \cup\{\mathcal{R} \subseteq \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma) \mid \mathcal{R} \text{ is a } \forall\text{d-simulation on } \mathcal{M}\},$$
$$\mathcal{P}^\forall_{\mathrm{bis}} \triangleq \cup\{\mathcal{S} \subseteq \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma) \mid \mathcal{S} \text{ is a } \forall\text{d-bisimulation on } \mathcal{M}\}.$$

PROPOSITION 3.6. *$\mathcal{R}^\forall_{\mathrm{sim}}$ is a preorder and the greatest $\forall d$-simulation on $\mathcal{M}$. Also, $\mathcal{P}^\forall_{\mathrm{bis}}$ is an equivalence relation and the greatest $\forall d$-bisimulation on $\mathcal{M}$.*

PROOF. It is straightforward to check that $\mathcal{R}^\forall_{\mathrm{sim}}$ is a preorder. We prove that it is a $\forall$d-simulation, which implies that it is the greatest one. Assume that $d\,\mathcal{R}^\forall_{\mathrm{sim}}\,e$. Hence, there exists a $\forall$d-simulation $\mathcal{R}$ such that $d\,\mathcal{R}\,e$. Therefore, we have that $d \sqsubseteq_{\Delta(\mathcal{R})} e$, which implies $d \sqsubseteq_{\Delta(\mathcal{R}^\forall_{\mathrm{sim}})} e$ because $\Delta(\mathcal{R}) \subseteq \Delta(\mathcal{R}^\forall_{\mathrm{sim}})$. Now, if $d \xrightarrow{a} f$, for some $f \in \mathrm{Distr}(\Sigma)$, then there exists $g \in \mathrm{Distr}(\Sigma)$ such that $e \xrightarrow{a} g$ and $g \in \mathcal{R}(f) \subseteq \mathcal{R}^\forall_{\mathrm{sim}}(f)$.
The proof for $\forall$d-bisimulation follows the same lines. □

Therefore, we call $\mathcal{R}^\forall_{\mathrm{sim}}$ the $\forall$d-simulation preorder on $\mathcal{M}$, while $\mathcal{P}^\forall_{\mathrm{bis}}$ is called the $\forall$d-bisimilarity on $\mathcal{M}$. It turns out that these notions allow us to fulfill our goal of extending Theorem 3.2 to generic distributions. In fact, we have that the $\forall$d-simulation preorder fully captures the logical preorder induced by $\mathcal{L}^+_\forall$ while $\forall$d-bisimilarity fully captures the logical equivalence induced by $\mathcal{L}_\forall$.

THEOREM 3.7. *For any $d, e \in \mathrm{Distr}(\Sigma)$,*

(1) $d\,\mathcal{R}^\forall_{\mathrm{sim}}\,e$ *if and only* $d \leq_{\mathcal{L}^+_\forall} e$*;*
(2) $d\,\mathcal{P}^\forall_{\mathrm{bis}}\,e$ *if and only* $d \equiv_{\mathcal{L}_\forall} e$*.*

PROOF. Let us prove (1).
($\Rightarrow$) Assume that $d\,\mathcal{R}^\forall_{\mathrm{sim}}\,e$. We prove that for all $\phi \in \mathcal{L}^+_\forall$, if $d \models \phi$ then $e \models \phi$. We proceed by structural induction on ϕ.

— the cases $\phi = \top, \bigwedge_{i\in I}\phi_i, \phi_1 \vee \phi_2$ are straightforward.

— Let $\phi = \Diamond_a\psi$ and assume that $d \models \Diamond_a\psi$. By definition we have that $\mathrm{supp}(d) \subseteq \mathrm{pre}_a(\{h \mid h \models \psi\})$. From the hypothesis $d\ \mathcal{R}^{\forall}_{\mathrm{sim}}\ e$ we have that $\mathrm{supp}(e) \subseteq \mathrm{pre}_a(\mathcal{R}^{\forall}_{\mathrm{sim}}(\{h \mid h \models \psi\}))$. By inductive hypothesis on ψ, if $g \in \mathcal{R}^{\forall}_{\mathrm{sim}}(\{h \mid h \models \psi\})$ then $g \models \psi$, so that $\mathcal{R}^{\forall}_{\mathrm{sim}}(\{h \mid h \models \psi\}) \subseteq \{h \mid h \models \psi\}$. Hence, $\mathrm{pre}_a(\mathcal{R}^{\forall}_{\mathrm{sim}}(\{h \mid h \models \psi\})) \subseteq \mathrm{pre}_a(\{h \mid h \models \psi\})$, and in turn $\mathrm{supp}(e) \subseteq \mathrm{pre}_a(\{h \mid h \models \psi\})$, i.e., $e \models \Diamond_a\psi$.

— Let $\phi = [\psi]_p$ and assume that $d \models [\psi]_p$. By definition we have that $d(\{s \mid \delta_s \models \psi\}) \geq p$. From $d\ \mathcal{R}^{\forall}_{\mathrm{sim}}\ e$ we have that $d \sqsubseteq_{\Delta(\mathcal{R}^{\forall}_{\mathrm{sim}})} e$. Hence, we know that $p \leq d(\{s \mid \delta_s \models \psi\}) \leq e(\Delta(\mathcal{R}^{\forall}_{\mathrm{sim}})\{s \mid \delta_s \models \psi\}) = e(\{t \mid \delta_t \in \mathcal{R}^{\forall}_{\mathrm{sim}}(\{\delta_s \mid \delta_s \models \psi\})\})$. By inductive hypothesis on ψ, if $\delta_t \in \mathcal{R}^{\forall}_{\mathrm{sim}}(\{\delta_s \mid \delta_s \models \psi\})$ then $\delta_t \models \psi$, so that $\{t \mid \delta_t \in \mathcal{R}^{\forall}_{\mathrm{sim}}(\{\delta_s \mid \delta_s \models \psi\})\} \subseteq \{t \mid \delta_t \models \psi\}$, and in turn $e(\{t \mid \delta_t \in \mathcal{R}^{\forall}_{\mathrm{sim}}(\{\delta_s \mid \delta_s \models \psi\})\}) \leq e(\{t \mid \delta_t \models \psi\})$. Therefore, $p \leq e(\{t \mid \delta_t \models \psi\})$, i.e. $e \models [\psi]_p$.

($\Leftarrow$) Assume that $d \leq_{\mathcal{L}^+_{\forall}} e$ and let us show that $d\ \mathcal{R}^{\forall}_{\mathrm{sim}}\ e$. It is sufficient to prove that the relation

$$Q \triangleq \{(d,e) \in \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma) \mid \mathrm{Sat}_{\mathcal{L}^+_{\forall}}(d) \subseteq \mathrm{Sat}_{\mathcal{L}^+_{\forall}}(e)\}$$

is a $\forall$d-simulation, since this implies $Q \subseteq \mathcal{R}^{\forall}_{\mathrm{sim}}$. Firstly we observe that Q is a preorder. We first prove the condition (2) of $\forall$d-simulation, namely, if $(d,e) \in Q$ then $d \sqsubseteq_{\Delta(Q)} e$. Since $\Delta(Q)$ is a preorder, by using Definition 2.1 (3), we show that for any $U = \Delta(Q)(U)$ which is finitely generated, we have that $d(U) \leq e(U)$. Notice that $\Delta(Q) = \{(s,t) \in \Sigma \times \Sigma \mid \mathrm{Sat}_{\mathcal{L}^+_{\forall}}(\delta_s) \subseteq \mathrm{Sat}_{\mathcal{L}^+_{\forall}}(\delta_t)\}$. Let $\{[s_j]\}_{j\in J}$ be an enumeration of the equivalence classes of the kernel $\mathcal{K}_{\Delta(Q)}$ of the preorder $\Delta(Q)$, where $[s_j] = \{t \in \Sigma \mid \mathrm{Sat}_{\mathcal{L}^+_{\forall}}(\delta_{s_j}) = \mathrm{Sat}_{\mathcal{L}^+_{\forall}}(\delta_t)\}$. Let us observe that for all $i,j \in J$, if $s_j \notin \Delta(Q)([s_i])$ then $\mathrm{Sat}_{\mathcal{L}^+_{\forall}}(\delta_{s_i}) \not\subseteq \mathrm{Sat}_{\mathcal{L}^+_{\forall}}(\delta_{s_j})$, so that there exists a formula $\psi_{ij} \in \mathcal{L}^+_{\forall}$ such that $\delta_{s_i} \models \phi_{ij}$ and $\delta_{s_j} \not\models \phi_{ij}$. For any $i \in J$, we define $\phi_i \triangleq \bigwedge\{\phi_{ij} \mid j \in J,\ s_j \notin \Delta(Q)([s_i])\}$. By construction, it turns out that $\{t \in \Sigma \mid \delta_t \models \phi_i\} = \Delta(Q)([s_i])$. Since U is finitely generated, we know that $U = \cup_{i\in F}[s_i]$ for some finite subset $F \subseteq I$ indices of equivalence classes in $\{[s_i]\}_{i\in I}$, and observe that since $U = \Delta(Q)(U)$, $U = \cup_{i\in F}[s_i] = \cup_{i\in F}\Delta(Q)([s_i])$. We thus define $\psi_U \triangleq \bigvee_{i\in F}\phi_i$ and we observe that since ψ_U is a finite disjunction of formulae in $\mathcal{L}^+_{\forall}$, we have that $\psi_U \in \mathcal{L}^+_{\forall}$. Therefore, $\{t \in \Sigma \mid \delta_t \models \psi_U\} = \{t \in \Sigma \mid \exists i \in F. \delta_t \models \phi_i\} = \cup_{i\in F}\Delta(Q)([s_i]) = U$. As a consequence, $d \models [\psi_U]_{d(U)}$. Hence, from $e \in Q(d)$ we obtain $e \models [\psi_U]_{d(U)}$, that is, $d(U) \leq e(\{t \in \Sigma \mid \delta_t \models \psi_U\}) = e(U)$.

Let us now prove condition (1) of $\forall$d-simulation. Let $(d,e) \in Q$ and $\mathrm{supp}(d) \subseteq \mathrm{pre}_a(D)$ for some $D \subseteq \mathrm{Distr}(\Sigma)$ and suppose, by contradiction, that $\mathrm{supp}(e) \not\subseteq \mathrm{pre}_a(Q(D))$. Hence, there exists $y \in \mathrm{supp}(e)$ such that $y \notin \mathrm{pre}_a(Q(D))$. By condition (2) of $\forall$d-simulation already shown above, we have that $d \sqsubseteq_{\Delta(Q)} e$. Thus, for $y \in \mathrm{supp}(e)$, by Lemma 2.2 (2), there exists $x \in \mathrm{supp}(d)$ such that $y \in \Delta(Q)(x)$. Hence, $\delta_y \in Q(\delta_x)$, that is, $\mathrm{Sat}_{\mathcal{L}^+_{\forall}}(\delta_x) \subseteq \mathrm{Sat}_{\mathcal{L}^+_{\forall}}(\delta_y)$. Since $\mathrm{supp}(d) \subseteq \mathrm{pre}_a(D)$, we have that there exists some $f \in D$ such that $x \xrightarrow{a} f$. Moreover, for any $g \in \mathrm{Distr}(\Sigma)$, if $y \xrightarrow{a} g$ then $g \notin Q(f)$, otherwise we would have that $g \in Q(D)$ and therefore we would get the contradiction $y \in \mathrm{pre}_a(Q(D))$. Thus, for any $g \in \mathrm{Distr}(\Sigma)$ such that $y \xrightarrow{a} g$, there exists a formula $\psi_{f,g} \in \mathcal{L}^+_{\forall}$ such that $f \models \psi_{f,g}$ and $g \not\models \psi_{f,g}$. We consider the formula

$$\psi^f \triangleq \bigwedge\{\psi_{f,g} \mid g \in \mathrm{Distr}(\Sigma),\ y \xrightarrow{a} g\}$$

which is in $\mathcal{L}^+_{\forall}$. Then, by construction, it turns out that $\delta_x \models \Diamond_a\psi^f$ whereas $\delta_y \not\models \Diamond_a\psi^f$. This is therefore a contradiction to $\mathrm{Sat}_{\mathcal{L}^+_{\forall}}(\delta_x) \subseteq \mathrm{Sat}_{\mathcal{L}^+_{\forall}}(\delta_y)$.

Let us now prove (2).
($\Rightarrow$) Assume that $d \ \mathcal{P}^{\forall}_{\text{bis}} \ e$ and let us prove that for all $\phi \in \mathcal{L}_\forall$, $d \models \phi$ iff $e \models \phi$. We proceed by structural induction on ϕ.

— The cases $\phi = \top, \neg\phi_1, \bigwedge_{i\in I} \phi_i$ are immediate.
— Let $\phi = \Diamond_a\psi$. If $d \models \Diamond_a\psi$ then, by using the same proof above for the corresponding case in simulation, $e \models \Diamond_a\psi$. Moreover, since $\mathcal{P}^{\forall}_{\text{bis}}$ is an equivalence relation, symmetrically we also have that $e \models \Diamond_a\psi$ implies $d \models \Diamond_a\psi$.
— Let $\phi = [\psi]_p$. By using the same proof above for the corresponding case in simulation and the fact that $\mathcal{P}^{\forall}_{\text{bis}}$ is an equivalence relation, we have that $d \models [\psi]_p$ iff $e \models [\psi]_p$.

($\Leftarrow$) Let us prove that $d \equiv_{\mathcal{L}_\forall} e$ implies $d \ \mathcal{P}^{\forall}_{\text{bis}} \ e$. Similarly to the case of simulation, it is enough to show that the relation

$$R \triangleq \{(d,e) \in \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma) \mid \mathrm{Sat}_{\mathcal{L}_\forall}(d) = \mathrm{Sat}_{\mathcal{L}_\forall}(e)\}$$

is a $\forall$d-bisimulation. We first observe that R is an equivalence relation. Analogously to the case of simulation, the first condition of $\forall$d-bisimulation is shown by contradiction. Let $(d,e) \in R$ and $\mathrm{supp}(d) \subseteq \mathrm{pre}_a(D)$ for some $D \subseteq \mathrm{Distr}(\Sigma)$ and suppose that $\mathrm{supp}(e) \not\subseteq \mathrm{pre}_a(R(D))$, i.e., there exists $y \in \mathrm{supp}(e)$ such that $y \notin \mathrm{pre}_a(R(D))$. If some $h \in \mathrm{post}_a(y)$ then h cannot belong to $R(D)$, so that for any $f \in D$, $\mathrm{Sat}_{\mathcal{L}_\forall}(f) \neq \mathrm{Sat}_{\mathcal{L}_\forall}(h)$. Hence, for any $f \in D$, there exists $\psi_{h,f} \in (\mathrm{Sat}_{\mathcal{L}_\forall}(f) \smallsetminus \mathrm{Sat}_{\mathcal{L}_\forall}(h)) \cup (\mathrm{Sat}_{\mathcal{L}_\forall}(h) \smallsetminus \mathrm{Sat}_{\mathcal{L}_\forall}(f))$. Since $\mathcal{L}_\forall$ is closed under negation, this implies that there exists $\phi_{h,f} \in \mathcal{L}_\forall$ such that $f \models \phi_{h,f}$ and $h \not\models \phi_{h,f}$. We thus consider the formula

$$\phi_y \triangleq \bigwedge_{h\in\mathrm{post}_a(y)} \bigvee_{f\in D} \phi_{h,f}$$

(note that $\mathrm{post}_a(y) \neq \varnothing$) which is in $\mathcal{L}_\forall$ and, by construction, is such that $d \models \Diamond_a\phi_y$ and $e \not\models \Diamond_a\phi_y$, namely, which is a contradiction to $\mathrm{Sat}_{\mathcal{L}_\forall}(d) = \mathrm{Sat}_{\mathcal{L}_\forall}(e)$.
It is left to prove that if $(d,e) \in R$ then $d \equiv_{\Delta(R)} e$. As R is an equivalence relation, $\Delta(R)$ is an equivalence relation. Hence, given a block B of $\Delta(R)$, we have to show that $d(B) = e(B)$. Let $\{[s_j]\}_{j\in J}$ be an enumeration of the equivalence classes of the equivalence relation $\Delta(R)$, where $[s_j] = \{t \in \Sigma \mid \mathrm{Sat}_{\mathcal{L}_\forall}(\delta_{s_j}) = \mathrm{Sat}_{\mathcal{L}_\forall}(\delta_t)\}$. Let us observe that for all $i,j \in J$, if $s_j \notin [s_i]$, that is $j \neq i$, then $\mathrm{Sat}_{\mathcal{L}_\forall}(\delta_{s_i}) \neq \mathrm{Sat}_{\mathcal{L}_\forall}(\delta_{s_j})$, thus, since $\mathcal{L}_\forall$ is closed under negation, there exists a formula $\psi_{ij} \in \mathcal{L}_\forall$ such that $\delta_{s_i} \models \phi_{ij}$ and $\delta_{s_j} \not\models \phi_{ij}$. For any $i \in J$, we define $\phi_i \triangleq \bigwedge_{j\in J, j\neq i} \psi_{ij}$, so that, by construction, we have that $\{t \in \Sigma \mid \delta_t \models \phi_i\} = [s_i]$. Hence, for some $i \in J$, we have that $B = [s_i]$ and in turn we define $\phi_B \triangleq \phi_i$. Since $\{t \in \Sigma \mid \delta_t \models \psi_B\} = B$, we have that $d \models [\psi_B]_{d(B)}$. Hence, from $(d,e) \in R$ we obtain $e \models [\psi_B]_{d(B)}$, that is, $d(B) \leq e(\{t \in \Sigma \mid \delta_t \models \psi_B\}) = e(B)$. $\square$

3.3. d-simulations

Consider the transfer property of a $\forall$d-simulation $\mathcal{R}$, namely condition (1) of Definition 3.4: using the definition of $\mathrm{ppre}^{\forall}_a$, this can be equivalently stated as

$$\text{for all } D \subseteq \mathrm{Distr}(\Sigma), \text{ if } d \in \mathrm{ppre}^{\forall}_a(D) \text{ then } e \in \mathrm{ppre}^{\forall}_a(\mathcal{R}(D)) \qquad (\dagger)$$

Since $\mathrm{ppre}^{\forall}_a$ does not preserve set unions, the statement $d \in \mathrm{ppre}^{\forall}_a(D)$ is not equivalent to the existential quantification $\exists f \in D.\, d \in \mathrm{ppre}^{\forall}_a(f)$, so that the above condition ($\dagger$) does not scale to the standard transfer property of simulations on LTSs. As a consequence, while the transfer property that characterizes simulations on LTSs admits a natural game characterization [Nielsen and Clausen 1994] (since predecessor operators in LTSs are additive), it is not clear whether a game-based characterization can be also given for simulations on PLTSs. It is therefore interesting to ask whether a suitable definition of an additive (i.e., preserving arbitrary set unions) probabilistic predecessor operator between distributions can be given.

Let us therefore consider the following alternative definition of probabilistic predecessor operator:

$$\begin{aligned}&\mathrm{ppre}_a : \wp(\mathrm{Distr}(\Sigma)) \to \wp(\mathrm{Distr}(\Sigma))\\ &\mathrm{ppre}_a(D) \triangleq \{d \in \mathrm{Distr}(\Sigma) \mid \mathrm{supp}(d) \cap \mathrm{pre}_a(D) \neq \varnothing\}\end{aligned}$$

Hence, this definition corresponds to the existential version of the $\mathrm{ppre}_a^{\forall}$ operator: in order for a distribution d to be a probabilistic predecessor of a distribution e it is now sufficient that the support of d contains some state that reaches e. In this sense, ppre_a has an *existential* flavor as opposed to the *universal* flavor of $\mathrm{ppre}_a^{\forall}$. We will make this observation precise in Section 6 through a formalization in the standard abstract interpretation framework [Cousot and Cousot 1977; Cousot and Cousot 1979].

The above ppre_a operator clearly preserves arbitrary set unions, so that a corresponding transition relation between distributions can be defined as follows: $d \xrightarrow{a} e$ iff $d \in \mathrm{ppre}_a(\{e\})$, namely,

$$d \xrightarrow{a} e \;\text{ iff }\; \exists x \in \mathrm{supp}(d).\, x \xrightarrow{a} e \qquad (*)$$

This allows us to lift a PLTS to a LTS of distributions, that we call dLTS (distribution-based LTS). Accordingly, the following notions of simulation/bisimulation based on the standard transfer property in LTSs naturally arise.

Definition 3.8 (*d-simulation*). A relation $\mathcal{R} \subseteq \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma)$ is a *d-simulation* on $\mathcal{M}$ if for all $d, e \in \mathrm{Distr}(\Sigma)$, if $d\,\mathcal{R}\,e$ then:

(1) if $d \xrightarrow{a} f$ then there exists $g \in \mathrm{Distr}(\Sigma)$ such that $e \xrightarrow{a} g$ and $f\,\mathcal{R}\,g$;
(2) $d \sqsubseteq_{\Delta(\mathcal{R})} e$. $\square$

Definition 3.9 (*d-bisimulation*). A symmetric relation $\mathcal{S} \subseteq \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma)$ is a *d-bisimulation* on $\mathcal{M}$ if for all $d, e \in \mathrm{Distr}(\Sigma)$, if $d\,\mathcal{S}\,e$ then:

(1) if $d \xrightarrow{a} f$ then there exists $g \in \mathrm{Distr}(\Sigma)$ such that $e \xrightarrow{a} g$ and $f\,\mathcal{S}\,g$;
(2) $d \equiv_{\Delta(\mathcal{S})} e$. $\square$

It turns out that ∀d-(bi)simulations and d-(bi)simulations are equivalent notions. In spite of the fact they rely on rather different transfer properties, their second defining condition, peculiar to the probabilistic setting, is powerful enough to bridge this gap. More precisely, the following proof shows that this depends on the easy properties of the lifting relation $\sqsubseteq_R$ stated in Lemma 2.2.

LEMMA 3.10. *If $\mathcal{R} \subseteq \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma)$ then $\mathcal{R}$ is a ∀d-simulation on $\mathcal{M}$ iff $\mathcal{R}$ is a d-simulation on $\mathcal{M}$. If $\mathcal{R}$ is symmetric then $\mathcal{R}$ is a ∀d-bisimulation on $\mathcal{M}$ iff $\mathcal{R}$ is a d-bisimulation on $\mathcal{M}$.*

PROOF. ($\Rightarrow$) Let $\mathcal{R}$ be a ∀d-simulation. To prove that $\mathcal{R}$ is a d-simulation it is enough to show that if $d\,\mathcal{R}\,e$ and $d \xrightarrow{a} f$ then $e \xrightarrow{a} g$ for some $g \in \mathcal{R}(f)$. From $d \xrightarrow{a} f$ we know that there exists $x \in \mathrm{supp}(d)$ such that $x \xrightarrow{a} f$. Also, from $d\,\mathcal{R}\,e$ we have that $d \sqsubseteq_{\Delta(\mathcal{R})} e$, so that, by Lemma 2.2 (1), there exists $y \in \mathrm{supp}(e)$ such that $y \in \Delta(\mathcal{R})(x)$, i.e., $\delta_y \in \mathcal{R}(\delta_x)$. Hence, from $x \xrightarrow{a} f$, i.e., $\mathrm{supp}(\delta_x) \subseteq \mathrm{pre}_a(\{f\})$, by definition of ∀d-simulation, we have that $\mathrm{supp}(\delta_y) \subseteq \mathrm{pre}_a(\mathcal{R}(\{f\}))$, i.e., $y \xrightarrow{a} g$ for some $g \in \mathcal{R}(f)$. Therefore, $e \xrightarrow{a} g$ as desired.
($\Leftarrow$) Let $\mathcal{R}$ be a d-simulation. Let us prove that if $d\,\mathcal{R}\,e$, $\mathrm{supp}(d) \subseteq \mathrm{pre}_a(D)$, for some $D \subseteq \mathrm{Distr}(\Sigma)$, and $y \in \mathrm{supp}(e)$ then $y \in \mathrm{pre}_a(\mathcal{R}(D))$. From $d\,\mathcal{R}\,e$, we have that $d \sqsubseteq_{\Delta(\mathcal{R})} e$ so that for $y \in \mathrm{supp}(e)$, by Lemma 2.2 (2), there exists $x \in \mathrm{supp}(d)$ such that $y \in \Delta(\mathcal{R})(x)$,

i.e., $\delta_y \in \mathcal{R}(\delta_x)$. Since $\mathrm{supp}(d) \subseteq \mathrm{pre}_a(D)$, $x \xrightarrow{a} f$ for some $f \in D$, and in turn $\delta_x \xrightarrow{a} f$. Therefore, $\delta_y \in \mathcal{R}(\delta_x)$ implies that $\delta_y \xrightarrow{a} g$ for some $g \in \mathcal{R}(f)$, so that $y \xrightarrow{a} g$ for some $g \in \mathcal{R}(D)$, that is, $y \in \mathrm{pre}_a(\mathcal{R}(D))$.

The proof for bisimulation is similar. □

We define

$$\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}} \triangleq \cup\{\mathcal{R} \subseteq \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma) \mid \mathcal{R} \text{ is a d-simulation on } \mathcal{M}\},$$
$$\mathcal{P}^{\mathrm{d}}_{\mathrm{bis}} \triangleq \cup\{\mathcal{S} \subseteq \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma) \mid \mathcal{S} \text{ is a d-bisimulation on } \mathcal{M}\},$$

so that, by Lemma 3.10, $\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}} = \mathcal{R}^{\forall}_{\mathrm{sim}}$ and $\mathcal{P}^{\mathrm{d}}_{\mathrm{bis}} = \mathcal{P}^{\forall}_{\mathrm{bis}}$. In turn, $\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}$ is a preorder and the greatest d-simulation on $\mathcal{M}$, called d-simulation preorder, while $\mathcal{P}^{\mathrm{d}}_{\mathrm{bis}}$ is an equivalence and the greatest d-bisimulation on $\mathcal{M}$, called d-bisimulation equivalence or d-bisimilarity.

The following remark shows how (bi)simulations generate d-(bi)simulations and vice versa.

PROPOSITION 3.11.

(1) *If $\mathcal{R}$ is a d-simulation then $\Delta(\mathcal{R})$ is a simulation.*
(2) *If R is a simulation then $\sqsubseteq_R$ is a d-simulation and $\Delta(\sqsubseteq_R) = R$.*
(3) *If $\mathcal{S}$ is a d-bisimulation then $\Delta(\mathcal{S})$ is a bisimulation.*
(4) *If S is a bisimulation then $\equiv_S$ is a d-bisimulation and $\Delta(\equiv_S) = S$.*

PROOF. Let us prove (1). Assume that $s\ \Delta(\mathcal{R})\ t$ and $s \xrightarrow{a} d$. We thus have that $\delta_s\ \mathcal{R}\ \delta_t$ and $\delta_s \xrightarrow{a} d$. Since $\mathcal{R}$ is a d-simulation, $\delta_t \xrightarrow{a} e$ for some $e \in \mathcal{R}(d)$, so that $t \xrightarrow{a} e$, and $d \sqsubseteq_{\Delta(\mathcal{R})} e$. Thus, $\Delta(\mathcal{R})$ is a simulation.

Let us turn to (2). We first show that $\Delta(\sqsubseteq_R) = R$: $s\,\Delta(\sqsubseteq_R)\,t$ iff $\delta_s \sqsubseteq_R \delta_t$ iff $\delta_s(U) \leq \delta_t(R(U))$ for any $U \subseteq \mathrm{supp}(\delta_s) = \{s\}$ iff $t \in R(s)$. Let us now check that $\sqsubseteq_R$ is a d-simulation. Suppose that $d \sqsubseteq_R e$ and $d \xrightarrow{a} f$. Hence, there exists $x \in \mathrm{supp}(d)$ such that $x \xrightarrow{a} f$. From $d \sqsubseteq_R e$, by Lemma 2.2 (1), we have that there exists $y \in \mathrm{supp}(e)$ such that $y \in R(x)$. Since R is a simulation, there exists a distribution g such that $y \xrightarrow{a} g$ and $f \sqsubseteq_R g$. Hence, $e \xrightarrow{a} g$ and, since $\Delta(\sqsubseteq_R) = R$, $f \sqsubseteq_{\Delta(\sqsubseteq_R)} g$.

The proofs for (3) and (4) involving bisimulations are similar. □

As a consequence, it turns out that the simulation preorder R_{sim} can be retrieved from the d-simulation preorder $\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}$ by restricting $\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}$ to Dirac distributions and, conversely, $\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}$ can be characterized as the lifting to distributions of R_{sim}.

THEOREM 3.12.

(1) $\Delta(\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}) = R_{\mathrm{sim}}$ *and* $\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}} = \sqsubseteq_{R_{\mathrm{sim}}}$.
(2) $\Delta(\mathcal{P}^{\mathrm{d}}_{\mathrm{bis}}) = P_{\mathrm{bis}}$ *and* $\mathcal{P}^{\mathrm{d}}_{\mathrm{bis}} = \equiv_{P_{\mathrm{bis}}}$.

PROOF. Let us prove that $\Delta(\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}) = R_{\mathrm{sim}}$ On the one hand, $\Delta(\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}) \subseteq R_{\mathrm{sim}}$ because, by Proposition 3.11 (1), $\Delta(\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}})$ is a simulation. On the other hand, observe that for any simulation R, by Proposition 3.11 (2), we know that $\sqsubseteq_R$ is a d-simulation such that $\Delta(\sqsubseteq_R) = R$. Hence $\sqsubseteq_R \subseteq \mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}$, so that $R = \Delta(\sqsubseteq_R) \subseteq \Delta(\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}})$ for any simulation R, so that $R_{\mathrm{sim}} \subseteq \Delta(\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}})$.

Let us now prove that $\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}} = \sqsubseteq_{R_{\mathrm{sim}}}$. On the one hand, $\sqsubseteq_{R_{\mathrm{sim}}} \subseteq \mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}$ because, by Proposition 3.11 (2), $\sqsubseteq_{R_{\mathrm{sim}}}$ is a d-simulation. On the other hand, notice that if $d\,\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}\,e$ then, by definition of d-simulation, $d \sqsubseteq_{\Delta(\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}})} e$. By Proposition 3.11 (1), $\Delta(\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}})$ is

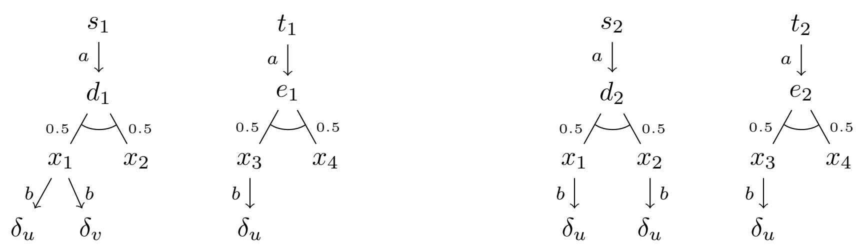

Fig. 1. A pair of PLTSs.

a simulation so that $\Delta(\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}) \subseteq R_{\mathrm{sim}}$, which in turn implies $\sqsubseteq_{\Delta(\mathcal{R}^{\mathrm{d}}_{\mathrm{sim}})} \subseteq \sqsubseteq_{R_{\mathrm{sim}}}$. Hence, $d \sqsubseteq_{R_{\mathrm{sim}}} e$.

The proofs for (2) involving bisimulations are similar. $\square$

Example 3.13. Consider the leftmost PLTS depicted in Figure 1. Observe that the relation $\mathcal{R}_1 = \{(\delta_{s_1}, \delta_{t_1}),\ (d_1, e_1)\} \cup \{(d, d) \mid d \in \mathrm{Distr}(\Sigma)\}$ is not a d-simulation since $d_1 \xrightarrow{b} \delta_v$ while the only b-transition departing from e_1 is $e_1 \xrightarrow{b} \delta_u$ and $\delta_u \notin \mathcal{R}_1(\delta_v)$. On the other hand, consider the equivalence relation $\mathcal{R}_2$ on distributions whose corresponding partition includes the following blocks of equivalent distributions:

$$\{\delta_{s_1}, \delta_{t_1}\},\ \{d_1, e_1\},\ \{\delta_{x_1}, \delta_{x_3}\},\ \{\delta_{x_2}, \delta_{x_4}, \delta_u, \delta_v\}$$

so that $\Delta(\mathcal{R}_2) = \{\{s_1, t_1\}, \{x_1, x_3\},\ \{x_2, x_4, u, v\}\}$. It is easy to check that $\mathcal{R}_2$ is a d-bisimulation: we have that every pair of distributions in $\mathcal{R}_2$ respects the transfer property of d-bisimulation and is $\equiv_{\Delta(\mathcal{R}_2)}$-equivalent. Hence, δ_{s_1} and δ_{t_1} are d-bisimilar, and therefore, by Theorem 3.12 (1), s_1 and t_1 are similar states, as well as the non-Dirac distributions d_1 and e_1 result to be d-bisimilar.

Consider now the rightmost PLTS in Figure 1. Here, we have that s_2 simulates t_2 but t_2 does not simulate s_2, while the distribution d_2 d-simulates e_2. In fact, consider the following relation between distributions:

$$\mathcal{R}_3 = \{(\delta_{t_2}, \delta_{s_2}), (e_2, d_2), (\delta_u, \delta_u)\} \cup \{(\delta_{x_4}, \delta_{x_i})\}_{i=1,\dots,4} \cup \{(\delta_{x_3}, \delta_{x_i})\}_{i=1,2,3}.$$

Then, $\mathcal{R}_3$ is a d-simulation, since every pair of distributions in $\mathcal{R}_3$ respects the transfer property of d-simulation and belongs to $\sqsubseteq_{\Delta(\mathcal{R}_3)}$. For instance, let us check that $e_2 \sqsubseteq_{\Delta(\mathcal{R}_3)} d_2$: by Definition 2.1, it is enough to check that for all $U \subseteq \mathrm{supp}(e_2)$, $e_2(U) \leq d_2(\Delta(\mathcal{R}_3)(U))$. The nonempty subsets of $\mathrm{supp}(e_2)$ are: $U_1 = \{x_3\}$, $U_2 = \{x_4\}$ and $U_3 = \{x_3, x_4\}$, so that we have

$$\begin{aligned}
0.5 = e_2(\{x_3\}) \leq d_2(\Delta(\mathcal{R}_3)(\{x_3\})) = d_2(\{x_1, x_2, x_3\}) = 1 \\
0.5 = e_2(\{x_4\}) \leq d_2(\Delta(\mathcal{R}_3)(\{x_4\})) = d_2(\{x_1, x_2, x_3, x_4\}) = 1 \\
1 = e_2(\{x_3, x_4\}) \leq d_2(\Delta(\mathcal{R}_3)(\{x_3, x_4\})) = d_2(\{x_1, x_2, x_3, x_4\}) = 1
\end{aligned}$$

The fact that t_2 does not simulate s_2 can be retrieved by the fact that e_2 does not d-simulate d_2. In fact, by Theorem 3.12 (1), if t_2 simulates s_2 then δ_{t_2} d-simulates δ_{s_2} and this would imply that there exists a d-simulation $\mathcal{R}$ such that $d_2 \sqsubseteq_{\Delta(\mathcal{R})} e_2$. However, this latter condition implies that for $\{x_1, x_2\} = \mathrm{supp}(d_2)$, $1 = d_2(\{x_1, x_2\}) \leq e_2(\Delta(\mathcal{R})(\{x_1, x_2\}))$, which can be true only if $\mathrm{supp}(e_2) = \{x_3, x_4\} \subseteq \Delta(\mathcal{R})(\{x_1, x_2\})$. Hence, in particular, we would obtain $\delta_{x_4} \in \mathcal{R}(\{\delta_{x_1}, \delta_{x_2}\})$, which is a contradiction since δ_{x_4} cannot d-simulate a b-transition. $\square$

4. A NEW LOGIC FOR SIMULATION

Besides the above notions of d-simulation/d-bisimulation, the additive operator ppre_a allows us to provide a corresponding new interpretation for the diamond connective. Let us denote simply by $\mathcal{L}$ the logic whose syntax coincides with $\mathcal{L}_\forall$ and whose semantics is identical to that of $\mathcal{L}_\forall$ but for the diamond connective $\Diamond_a$, which is interpreted as follows:

$$d \models \Diamond_a \phi \;\text{ iff }\; \exists e \in \mathrm{Distr}(\Sigma).\ d \xrightarrow{a} e \text{ and } e \models \phi$$

This is therefore the *standard modal interpretation* of the diamond connective on a dLTS, namely a LTS whose "states" are distributions and whose transitions $\xrightarrow{a}$ between distributions are defined by condition $(*)$ in Section 3.2. Observe that this interpretation of $\Diamond_a$ inhibits the simple counterexample in Example 3.3.

We denote by $\mathcal{L}^+$ the positive and finitely disjunctive fragment of $\mathcal{L}$. It turns out that the preorder $\leq_{\mathcal{L}^+}$ and the equivalence $\equiv_{\mathcal{L}}$ logically characterize, respectively, the d-simulation preorder and d-bisimulation equivalence.

THEOREM 4.1. *For any* $d, e \in \mathrm{Distr}(\Sigma)$*,*

(1) $d\, \mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}\, e$ *if and only* $d \leq_{\mathcal{L}^+} e$*;*
(2) $d\, \mathcal{P}^{\mathrm{d}}_{\mathrm{bis}}\, e$ *if and only* $d \equiv_{\mathcal{L}} e$*.*

PROOF. Let us show (1).
$(\Rightarrow)$ Assume that $d\, \mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}\, e$. We prove that for all $\phi \in \mathcal{L}^+$ if $d \models \phi$ then $e \models \phi$. We proceed by structural induction on $\phi \in \mathcal{L}^+$.

— The cases $\phi = \top, \bigwedge_{i \in I} \phi_i, \phi_1 \vee \phi_2$ are straightforward.
— Let $\phi = \Diamond_a \psi$ and assume that $d \models \Diamond_a \psi$. Hence, there exists f such that $d \xrightarrow{a} f$ and $f \models \psi$. From $d\, \mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}\, e$, there exists $g \in \mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}(f)$ such that $e \xrightarrow{a} g$. From $f \models \psi$, by induction on ψ, $g \models \psi$, so that $e \models \Diamond_a \psi$.
— The proof for the case $\phi = [\psi]_p$ is analogous to the proof of the corresponding case of Theorem 3.7 (1).

$(\Leftarrow)$ Suppose that $d \leq_{\mathcal{L}^+} e$ and let us prove that $d\, \mathcal{R}^{\mathrm{d}}_{\mathrm{sim}}\, e$. It is sufficient to prove that the relation

$$Q = \{(d, e) \in \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma) \mid \mathrm{Sat}_{\mathcal{L}^+}(d) \subseteq \mathrm{Sat}_{\mathcal{L}^+}(e)\}$$

is a d-simulation. Let us observe that Q is trivially a preorder. Let us prove by contradiction condition (1) of d-simulation. Let $(d, e) \in Q$ so that $\mathrm{Sat}_{\mathcal{L}^+}(d) \subseteq \mathrm{Sat}_{\mathcal{L}^+}(e)$. Assume that there exists $f \in \mathrm{Distr}(\Sigma)$ such that $d \xrightarrow{a} f$ and, by contradiction, that for any g such that $e \xrightarrow{a} g$ it holds $g \notin Q(f)$, i.e., there exists there exists a formula $\psi_{f,g} \in \mathcal{L}^+$ such that $f \models \psi_{f,g}$ and $g \not\models \psi_{f,g}$. Consider the formula $\psi^f \triangleq \bigwedge\{\psi_{f,g} \mid g \in \mathrm{Distr}(\Sigma),\ e \xrightarrow{a} g\}$ in $\mathcal{L}^+$. Then, by construction, it turns out that $d \models \Diamond_a \psi^f$ whereas $e \not\models \Diamond_a \psi^f$, which is a contradiction to $\mathrm{Sat}_{\mathcal{L}^+}(d) \subseteq \mathrm{Sat}_{\mathcal{L}^+}(e)$.
It remains then to prove that if $(d, e) \in Q$ then $d \sqsubseteq_{\Delta(Q)} e$. Since $\Delta(Q)$ is a preorder, by using Definition 2.1 (3), we show that for any $U = \Delta(Q)(U)$ which is finitely generated, we have that $d(U) \leq e(U)$. This part of the proof follows the same lines of the analogous part of the proof of Theorem 3.7 (1), and is therefore omitted.

Let us now show (2).
$(\Rightarrow)$ Suppose that $d\, \mathcal{P}^{\mathrm{d}}_{\mathrm{bis}}\, e$ and let us prove that for all $\phi \in \mathcal{L}$, $d \models \phi$ iff $e \models \phi$ by structural induction on ϕ.

— The cases $\phi = \top, \neg\phi_1, \bigwedge_{i \in I} \phi_i$ are immediate.

— The cases $\phi = \Diamond_a \psi$ and $\phi = [\psi]_p$ are shown by resorting to the same proof above for the corresponding cases in simulation and to the fact that $\mathcal{P}^{\mathrm{d}}_{\mathrm{bis}}$ is an equivalence and therefore is symmetric.

($\Leftarrow$) Let us prove that $d \equiv_{\mathcal{L}} e$ implies $d\, \mathcal{P}^{\mathrm{d}}_{\mathrm{bis}}\, e$. It is sufficient to prove that the relation

$$\mathcal{R} = \{(d,e) \in \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma) \mid \mathrm{Sat}_{\mathcal{L}}(d) = \mathrm{Sat}_{\mathcal{L}}(e)\}$$

is a d-bisimulation. We first note that $\mathcal{R}$ is an equivalence relation. Let us prove the first condition of d-bisimulation by contradiction. Let $(d,e) \in \mathcal{R}$, $d \xrightarrow{a} f$ and for all $g \in \mathrm{Distr}(\Sigma)$ such that $e \xrightarrow{a} g$, by contradiction assume that $g \notin \mathcal{R}(f)$, i.e., there exists a formula $\phi_{f,g} \in \mathcal{L}$ such that $\phi_{f,g} \in (\mathrm{Sat}_{\mathcal{L}}(f) \smallsetminus \mathrm{Sat}_{\mathcal{L}}(g)) \cup (\mathrm{Sat}_{\mathcal{L}}(g) \smallsetminus \mathrm{Sat}_{\mathcal{L}}(f))$. Since $\mathcal{L}$ is closed under negation, we have that there exists a formula $\psi_{f,g} \in \mathcal{L}$ such that $f \models \psi_{f,g}$ and $g \not\models \psi_{f,g}$. We thus consider the formula $\psi^f \triangleq \bigwedge\{\psi_{f,g} \mid g \in \mathrm{Distr}(\Sigma),\ e \xrightarrow{a} g\}$ which is in $\mathcal{L}$. Then, by construction, it turns out that $d \models \Diamond_a \psi$ whereas $e \not\models \Diamond_a \psi$, which is a contradiction to $\mathrm{Sat}_{\mathcal{L}}(d) = \mathrm{Sat}_{\mathcal{L}}(e)$.
It is left to prove that if $(d,e) \in \mathcal{R}$ then $d \equiv_{\Delta(\mathcal{R})} e$. This part of the proof follows the same lines of the analogous part of the proof of Theorem 3.7 (2), and is therefore omitted. □

5. COMPARING LOGICS

As a straight consequence of Theorem 3.7, Lemma 3.10 and Theorem 4.1, it turns out that the logical preorders and equivalences induced, respectively, by the positive logics $\mathcal{L}^+_\forall$ and $\mathcal{L}^+$ and by the full logics $\mathcal{L}_\forall$ and $\mathcal{L}$ coincide.

COROLLARY 5.1. $\leq_{\mathcal{L}^+_\forall} = \leq_{\mathcal{L}^+}$ *and* $\equiv_{\mathcal{L}_\forall} = \equiv_{\mathcal{L}}$.

Let us now compare these logics w.r.t. their expressive powers. Given a generic logic L, which is interpreted over a PLTS $\mathcal{M} = \langle \Sigma, Act, \rightarrow \rangle$, the semantics of L on $\mathcal{M}$ is defined as $\mathrm{Sem}_{\mathcal{M}}(L) \triangleq \{[\![\varphi]\!]^{\mathcal{M}} \mid \varphi \in L\}$, where $[\![\varphi]\!]^{\mathcal{M}} \triangleq \{d \in \mathrm{Distr}(\Sigma) \mid d \models \varphi\}$. Hence, two logics (L_1, L_2) have the same expressive power (or expressiveness) when for any $\mathcal{M}$, $\mathrm{Sem}_{\mathcal{M}}(L_1) = \mathrm{Sem}_{\mathcal{M}}(L_2)$, while L_1 is more expressive than L_2 when for any $\mathcal{M}$, $\mathrm{Sem}_{\mathcal{M}}(L_2) \subseteq \mathrm{Sem}_{\mathcal{M}}(L_1)$.

In the following, we show that the full logics $\mathcal{L}$ and $\mathcal{L}_\forall$ have the same expressive power, while, for their positive and finitely disjunctive fragments, we show that $\mathcal{L}^+$ is strictly more expressive than $\mathcal{L}^+_\forall$. To this purpose, we consider two encoding maps $\bar{\cdot} : \mathcal{L}_\forall \to \mathcal{L}$ and $\tilde{\cdot} : \mathcal{L} \to \mathcal{L}_\forall$ which are inductively defined as follows:

$$\bar{\phi} \triangleq \begin{cases} \top, \neg\bar{\psi}, \bigwedge_{i\in I} \bar{\psi}_i & \text{if } \phi = \top, \neg\psi, \bigwedge_{i\in I} \psi_i \\ [\Diamond_a \bar{\psi}]_1 & \text{if } \phi = \Diamond_a \psi \\ [\bar{\psi}]_p & \text{if } \phi = [\psi]_p \end{cases}$$

$$\tilde{\phi} \triangleq \begin{cases} \top, \neg\tilde{\psi}, \bigwedge_{i\in I} \tilde{\psi}_i & \text{if } \phi = \top, \neg\psi, \bigwedge_{i\in I} \psi_i \\ \bigvee_{p>0} [\Diamond_a \tilde{\psi}]_p & \text{if } \phi = \Diamond_a \psi \\ [\tilde{\psi}]_p & \text{if } \phi = [\psi]_p \end{cases}$$

LEMMA 5.2. *Let* $\mathcal{M} = \langle \Sigma, Act, \rightarrow \rangle$ *be a PLTS.*

(1) *For any* $\phi \in \mathcal{L}_\forall$, $[\![\phi]\!]^{\mathcal{M}}_{\mathcal{L}_\forall} = [\![\bar{\phi}]\!]^{\mathcal{M}}_{\mathcal{L}}$.
(2) *For any* $\phi \in \mathcal{L}$, $[\![\phi]\!]^{\mathcal{M}}_{\mathcal{L}} = [\![\tilde{\phi}]\!]^{\mathcal{M}}_{\mathcal{L}_\forall}$.

PROOF. Let us prove (1). We show that $d \models_{\mathcal{L}_\forall} \phi$ iff $d \models_{\mathcal{L}} \bar{\phi}$ by structural induction on the formula $\phi \in \mathcal{L}_\forall$. The cases $\phi = \top, \neg\psi, \bigwedge_{i \in I} \psi_i$ are immediate. The case $\phi = [\psi]_p$ is a simple application of induction: $d \models_{\mathcal{L}_\forall} \phi$ iff $d(\{s \mid \delta_s \models_{\mathcal{L}_\forall} \psi\}) \geq p$ iff (by inductive hypothesis) $d(\{s \mid \delta_s \models_{\mathcal{L}} \bar{\psi}\}) \geq p$ iff $d \models_{\mathcal{L}} [\bar{\psi}]_p$. For the case $\phi = \Diamond_a \psi$, we have that $d \models_{\mathcal{L}_\forall} \phi$ iff $\mathrm{supp}(d) \subseteq \mathrm{pre}_a(\{h \mid h \models_{\mathcal{L}_\forall} \psi\})$. By induction on ψ, $\{h \mid h \models_{\mathcal{L}_\forall} \psi\} = \{h \mid h \models_{\mathcal{L}} \bar{\psi}\}$, so that $d \models_{\mathcal{L}_\forall} \phi$ iff $\mathrm{supp}(d) \subseteq \mathrm{pre}_a(\{h \mid h \models_{\mathcal{L}} \bar{\psi}\})$ iff $d(\mathrm{pre}_a(\{h \mid h \models_{\mathcal{L}} \bar{\psi}\})) = 1$ iff $d(\{x \mid x \xrightarrow{a} h,\ h \models_{\mathcal{L}} \bar{\psi}\}) = 1$ iff $d(\{x \mid \delta_x \models_{\mathcal{L}} \Diamond_a \bar{\psi}\}) = 1$ iff $d \models_{\mathcal{L}} [\Diamond_a \bar{\psi}]_1$.

Let us prove (2). We show that $d \models_{\mathcal{L}} \phi$ iff $d \models_{\mathcal{L}_\forall} \tilde{\phi}$ by induction on the structure of the formula $\phi \in \mathcal{L}$. The cases $\phi = \top, \neg\psi, \bigwedge_{i \in I} \psi_i, [\psi]_p$ are like in point (1). For the case $\phi = \Diamond_a \psi$, we have that $d \models_{\mathcal{L}} \phi$ iff $\exists x \in \mathrm{supp}(d)$ such that $x \in \mathrm{pre}_a(\{h \mid h \models_{\mathcal{L}} \psi\})$. By induction on ψ, we have that $\{h \mid h \models_{\mathcal{L}} \psi\} = \{h \mid h \models_{\mathcal{L}_\forall} \tilde{\psi}\}$. Hence, $d \models_{\mathcal{L}} \phi$ iff $d(\mathrm{pre}_a(\{h \mid h \models_{\mathcal{L}} \psi\})) > 0$ iff $d(\{x \mid x \xrightarrow{a} h,\ h \models_{\mathcal{L}} \bar{\psi}\}) > 0$ iff $d(\{x \mid \delta_x \models_{\mathcal{L}_\forall} \Diamond_a \tilde{\psi}\}) > 0$ iff $\exists p > 0.\ d \models_{\mathcal{L}_\forall} [\Diamond_a \tilde{\psi}]_p$ iff $d \models_{\mathcal{L}_\forall} \bigvee_{p>0} [\Diamond_a \tilde{\psi}]_p$. $\square$

COROLLARY 5.3. *$(\mathcal{L}_\forall, \mathcal{L})$ have the same expressive power.*

It is worth observing that this equivalence of expressiveness between $\mathcal{L}_\forall$ and $\mathcal{L}$ depends on the fact that the semantics of the "universal" diamond connective of $\mathcal{L}_\forall$ can be encoded in $\mathcal{L}$ that instead features an "existential" diamond connective and vice versa. In particular, the $\mathcal{L}_\forall$ semantics of a diamond formula $\Diamond_a \phi$, i.e. $[\![\Diamond_a \phi]\!]_{\mathcal{L}_\forall} = \{d \mid \mathrm{supp}(d) \subseteq \mathrm{pre}_a(\{e \mid e \models_{\mathcal{L}_\forall} \phi\})\}$, can be obtained in $\mathcal{L}$ through the formula $[\Diamond_a \phi]_1$, whose semantics is indeed $[\![[\Diamond_a \phi]_1]\!]_{\mathcal{L}} = \{d \mid d(\{x \mid \delta_x \models_{\mathcal{L}} \Diamond_a \phi\}) = 1\}$. On the other hand, the encoding of a diamond formula in $\mathcal{L}$ as a formula in $\mathcal{L}_\forall$ is more tricky. The $\mathcal{L}$-semantics of $\Diamond_a \phi$ is given by all the distributions whose support contains at least a state that may move to a distribution that satisfies ϕ, i.e., $[\![\Diamond_a \phi]\!]_{\mathcal{L}} = \{d \mid d(\{x \mid \exists e.\ x \xrightarrow{a} e,\ e \models_{\mathcal{L}} \phi\}) > 0\}$. This semantics can be therefore expressed in $\mathcal{L}_\forall$ by requiring that $d \models_{\mathcal{L}_\forall} [\Diamond_a \phi]_p$ for some $p > 0$. The existential quantification on a rational number $p > 0$ can be therefore expressed as a logical formula by means of an infinite countable disjunction and it is therefore expressible in $\mathcal{L}_\forall$.

Let us now focus on the positive and finitely disjunctive logics $\mathcal{L}_\forall^+$ and $\mathcal{L}^+$.

THEOREM 5.4. *$\mathcal{L}^+$ is strictly more expressive than $\mathcal{L}_\forall^+$.*

PROOF. Lemma 5.2 (1) also shows that $\mathcal{L}^+$ is more expressive than $\mathcal{L}_\forall^+$, because the encoding $\bar{\cdot}$ can be restricted to an encoding from $\mathcal{L}_\forall^+$ to $\mathcal{L}^+$. It can be observed that this does not hold for the encoding $\tilde{\cdot}$ of Lemma 5.2 (2), because the encoding $\tilde{\cdot}$ relies on infinite disjunctions that are not allowed in $\mathcal{L}_\forall^+$.

Let us therefore describe an example showing that the logic $\mathcal{L}^+$ is strictly more expressive than $\mathcal{L}_\forall^+$. Consider a PLTS $\mathcal{M} = \langle \{x_1, x_2\}, \{a\}, \{x_1 \xrightarrow{a} d = (x_1/0.5, x_2/0.5)\} \rangle$ that contains two states x_1, x_2 and a single transition from x_1 to the distribution $d = (x_1/0.5, x_2/0.5)$. In the logic $\mathcal{L}^+$, we have that $[\![\Diamond_a \top]\!]_{\mathcal{L}}^{\mathcal{M}} = \mathrm{Distr}(\Sigma) \setminus \{\delta_{x_2}\}$, since any distribution different from δ_{x_2} contains x_1 in its support, and therefore has an outgoing a-transition. Let us show that there is no formula in $\mathcal{L}_\forall^+$ whose semantics is $\mathrm{Distr}(\Sigma) \setminus \{\delta_{x_2}\}$. Consider the $\mathcal{L}_\forall^+$-formulae $\top$, $\Diamond_a \top$ and $[\Diamond_a \top]_p$, with $p > 0$, whose semantics are as follows: $[\![\top]\!]_{\mathcal{L}_\forall}^{\mathcal{M}} = \mathrm{Distr}(\Sigma)$, $[\![\Diamond_a \top]\!]_{\mathcal{L}_\forall}^{\mathcal{M}} = \{\delta_{x_1}\}$, $[\![[\Diamond_a \top]_p]\!]_{\mathcal{L}_\forall}^{\mathcal{M}} = \{d \mid d(\{x_1\}) \geq p\}$. We observe that $\mathrm{Sem}_{\mathcal{M}}(\mathcal{L}_\forall^+) = \{[\![\top]\!]_{\mathcal{L}_\forall}^{\mathcal{M}}, [\![\Diamond_a \top]\!]_{\mathcal{L}_\forall}^{\mathcal{M}}\} \cup \{[\![[\Diamond_a \top]_p]\!]_{\mathcal{L}_\forall}^{\mathcal{M}} \mid p > 0\}$, because this set of semantics is closed under applications of infinite intersections, finite unions, probabilistic predecessor and the semantics of the operator $[\cdot]_p$. It is thus enough to observe that $\mathrm{Distr}(\Sigma) \setminus \{\delta_{x_2}\} \notin \mathrm{Sem}_{\mathcal{M}}(\mathcal{L}_\forall^+)$. Actually, we also note that $\mathrm{Distr}(\Sigma) \setminus \{\delta_{x_2}\}$ can only be expressed as the infinite union $\cup_{p>0} [\![[\Diamond_a \top]_p]\!]_{\mathcal{L}_\forall^+}^{\mathcal{M}}$. $\square$

Example 5.5. Consider again the rightmost PLTS in Figure 1. We have already observed in Example 3.13 that s_2 simulates t_2 whilst t_2 does not simulate s_2. The fact that t_2 does not simulate s_2 can be easily proved by exhibiting a formula that is satisfied by δ_{s_2} but not by δ_{t_2}. We provide both a formula in $\mathcal{L}^+_\forall$ and an equivalent formula in $\mathcal{L}^+$ obtained through the encoding $\bar{\cdot}$ of Lemma 5.2:

$$\begin{array}{lll} \text{(1) let } \phi \triangleq \Diamond_a \Diamond_b \top \in \mathcal{L}^+_\forall; & \text{then } \delta_{s_2} \models_{\mathcal{L}^+_\forall} \phi & \text{and } \delta_{t_2} \not\models_{\mathcal{L}^+_\forall} \phi \\ \text{(2) let } \phi' \triangleq \Diamond_a [\Diamond_b \top]_1 \in \mathcal{L}^+; & \text{then } \delta_{s_2} \models_{\mathcal{L}^+} \phi' & \text{and } \delta_{t_2} \not\models_{\mathcal{L}^+} \phi' \end{array}$$

To see (1), observe that $\delta_{s_2} \models_{\mathcal{L}^+_\forall} \phi$ since $\mathrm{supp}(\delta_{s_2}) \subseteq \mathrm{pre}_a(\{d_2\})$ and $\mathrm{supp}(d_2) \subseteq \mathrm{pre}_b(\{\delta_u\})$ with $\delta_u \models_{\mathcal{L}^+_\forall} \top$. On the other hand, $\mathrm{supp}(\delta_{t_2}) \subseteq \mathrm{pre}_a(\{e_2\})$ but there is no distribution f such that ($f \models_{\mathcal{L}^+_\forall} \top$ and) $\mathrm{supp}(e_2) \subseteq \mathrm{pre}_b(\{f\})$, because $x_4 \in \mathrm{supp}(e_2)$ has no outgoing transition.
In order to check (2), notice that $\delta_{s_2} \models_{\mathcal{L}^+} \phi'$ since $\delta_{s_2} \xrightarrow{a} d_2$, and $d_2(\{x \mid \delta_x \models_{\mathcal{L}^+} \Diamond_b \top\}) = 1$ since for any $x \in \mathrm{supp}(d_2)$ it holds $\delta_x \xrightarrow{b} \delta_u$ with $\delta_u \models_{\mathcal{L}^+} \top$. On the other hand, $\delta_{t_2} \not\models_{\mathcal{L}^+} \phi'$ since $\delta_{t_2} \xrightarrow{a} e_2$ and $e_2(\{x \mid \delta_x \models_{\mathcal{L}^+} \Diamond_b \top\}) = 0.5 \not\geq 1$ because for the state $x_4 \in \mathrm{supp}(e_2)$ it holds $\delta_{x_4} \not\models_{\mathcal{L}^+} \Diamond_b \top$. □

6. STATES AS ABSTRACT INTERPRETATION OF DISTRIBUTIONS

Unlike LTSs and their standard behavioral relations, whose definitions rely on a single notion of system state, PLTSs as well as the corresponding notions of simulation/bisimulation involve two notions of system state, namely a bare state and a probabilistic state modeled as a state distribution. We have shown above how PLTSs can be embedded into dLTSs, that is, LTSs of probabilistic states that involve a single (but richer) notion of system state, i.e. state distributions. We show in this section how to formalize a systematic embedding of states into distributions by viewing states as abstractions of distributions. The intuition is that Dirac distributions allow us to view states as an abstraction of distributions, namely the map $\delta : \Sigma \to \mathrm{Distr}(\Sigma)$ such that $\delta(x) \triangleq \delta_x$ can be viewed as a function that embeds states into distributions. The other way round, the support map $\mathrm{supp} : \mathrm{Distr}(\Sigma) \to \wp(\Sigma)$ can be viewed as a function that abstracts a distribution d to the set of states in the support of d.

Let us recall that in the standard abstract interpretation framework for specifying sound approximations of system models [Cousot and Cousot 1977; Cousot and Cousot 1979; Cousot and Cousot 1992], approximations of a concrete semantic domain are encoded by abstract domains that are specified by Galois insertions (GIs for short) or, equivalently, by adjunctions. The notion of approximation on a concrete/abstract domain is encoded by a partial order $\leq$ where, traditionally, $x \leq y$ means that y is a sound approximation of x. Concrete and abstract approximation orders, denoted by $\leq_C$ and $\leq_A$, must be related by a GI. Let us recall that a GI of an abstract domain $\langle A, \leq_A \rangle$ into a concrete domain $\langle C, \leq_C \rangle$ is determined by a surjective abstraction map $\alpha : C \to A$ and a 1-1 concretization map $\gamma : A \to C$ such that $\alpha(c) \leq_A a \Leftrightarrow c \leq_C \gamma(a)$ and is here denoted by (α, C, A, γ). In a GI, $\alpha(c)$ intuitively provides the best approximation in A of a concrete value c while $\gamma(a)$ is the concrete value that a abstractly represents.

In our case, in order to cast δ as a concretization map in abstract interpretation, we need to lift its definition to powersets, namely we need to provide its so-called "collecting" version [Cousot and Cousot 1977; Cousot and Cousot 1979; Cousot and Cousot 1992]. Firstly, we observe that $\{\delta(x)\} = \{d \in \mathrm{Distr}(\Sigma) \mid \mathrm{supp}(d) \subseteq \{x\}\}$. Hence, this leads us to define the following concretization function $\gamma^\forall : \wp(\Sigma) \to \wp(\mathrm{Distr}(\Sigma))$:

$$\gamma^\forall(S) \triangleq \{d \in \mathrm{Distr}(\Sigma) \mid \mathrm{supp}(d) \subseteq S\}.$$

This is a *universal* concretization function, meaning that $d \in \gamma^{\forall}(S)$ if and only if all the states in $\mathrm{supp}(d)$ are contained in S. One can dually define an *existential* concretization map $\gamma^{\exists} : \wp(\Sigma) \to \wp(\mathrm{Distr}(\Sigma))$ as

$$\gamma^{\exists}(S) \triangleq \{d \in \mathrm{Distr}(\Sigma) \mid \mathrm{supp}(d) \cap S \neq \varnothing\},$$

where $d \in \gamma^{\exists}(S)$ if there exists some state in the support of d which is contained in S. Actually, these two mappings give rise to a pair of GIs (i.e., sound approximations as formalized in abstract interpretation) where $\wp(\mathrm{Distr}(\Sigma))$ and $\wp(\Sigma)$ play, respectively, the role of concrete and abstract domains. The approximation order is encoded by the subset relation (i.e., logical implication) in the case of $\gamma^{\forall}$ and by the superset relation (i.e., logical co-implication) in the case of $\gamma^{\exists}$. The dual maps, systematically obtained by adjunction from $\gamma^{\forall}$ and $\gamma^{\exists}$, are $\alpha^{\forall}, \alpha^{\exists} : \wp(\mathrm{Distr}(\Sigma)) \to \wp(\Sigma)$ defined as follows:

$$\begin{aligned}
\alpha^{\forall}(X) &\triangleq \{s \in \Sigma \mid \exists d \in X.\ s \in \mathrm{supp}(d)\},\\
\alpha^{\exists}(X) &\triangleq \{s \in \Sigma \mid \forall d \in \mathrm{Distr}(\Sigma).\ s \in \mathrm{supp}(d) \Rightarrow d \in X\}.
\end{aligned}$$

LEMMA 6.1. *$(\alpha^{\forall}, \wp(\mathrm{Distr}(\Sigma))_{\subseteq}, \wp(\Sigma)_{\subseteq}, \gamma^{\forall})$ and $(\alpha^{\exists}, \wp(\mathrm{Distr}(\Sigma))_{\supseteq}, \wp(\Sigma)_{\supseteq}, \gamma^{\exists})$ are GIs.*

PROOF. Given $X \in \wp(\mathrm{Distr}(\Sigma))$ and $S \in \wp(\Sigma)$, we have that

$$\begin{aligned}
\alpha^{\forall}(X) \subseteq S &\quad\Leftrightarrow\\
\{s \in \Sigma \mid \exists d \in X.\ s \in \mathrm{supp}(d)\} \subseteq S &\quad\Leftrightarrow\\
\cup_{d \in X} \mathrm{supp}(d) \subseteq S &\quad\Leftrightarrow\\
\forall d \in X.\, \mathrm{supp}(d) \subseteq S &\quad\Leftrightarrow\\
X \subseteq \gamma^{\forall}(S).&
\end{aligned}$$

Also, $\alpha^{\forall}$ is surjective because, for any $S \in \wp(\Sigma)$, $\alpha^{\forall}(\gamma^{\forall}(S)) = S$. Therefore, $(\alpha^{\forall}, \wp(\mathrm{Distr}(\Sigma))_{\subseteq}, \wp(\Sigma)_{\subseteq}, \gamma^{\forall})$ is a GI. Moreover, $(\alpha^{\exists}, \wp(\mathrm{Distr}(\Sigma))_{\supseteq}, \wp(\Sigma)_{\supseteq}, \gamma^{\exists})$ is a dual GI because $\alpha^{\exists}(X) = \Sigma \smallsetminus (\alpha^{\forall}(\mathrm{Distr}(\Sigma) \smallsetminus X))$ and $\gamma^{\exists}(S) = \mathrm{Distr}(\Sigma) \smallsetminus \gamma^{\forall}(\Sigma \smallsetminus S)$. □

As observed in the above proof, $\alpha^{\forall}/\gamma^{\forall}$ and $\alpha^{\exists}/\gamma^{\exists}$ are dual abstractions, i.e.,

$$\alpha^{\exists} = \neg\, \alpha^{\forall} \neg \quad \text{and} \quad \gamma^{\exists} = \neg\, \gamma^{\forall} \neg$$

where $\neg\, \alpha^{\forall} \neg(X) = \Sigma \smallsetminus \alpha^{\forall}(\mathrm{Distr}(\Sigma) \smallsetminus X)$ and $\neg\, \gamma^{\forall} \neg(S) = \mathrm{Distr}(\Sigma) \smallsetminus \gamma^{\forall}(\Sigma \smallsetminus S)$. Moreover, it is not hard to see that $\alpha^{\forall}$ is the additive extension of the supp function, while $\alpha^{\exists}$ is its co-additive extension, i.e.,

$$\alpha^{\forall}(X) = \cup_{d \in X} \mathrm{supp}(d) \quad \text{and} \quad \alpha^{\exists}(\mathrm{Distr} \smallsetminus X) = \cap_{d \in X} \Sigma \smallsetminus \mathrm{supp}(d).$$

These two abstract domains thus provide dual universal/existential ways for logically approximating sets of distributions into sets of states. One interesting point in these formal abstractions lies in the fact that they allow us to systematically obtain the above probabilistic predecessor operators $\mathrm{ppre}_a^{\forall}$ and ppre_a in a dLTS from the predecessor operator pre_a of the corresponding PLTS. Recall that in a PLTS the predecessor operator $\mathrm{pre}_a : \wp(\mathrm{Distr}(\Sigma)) \to \wp(\Sigma)$ maps a set of distributions into a set of states. Here, $\wp(\Sigma)$ can be therefore viewed as a universal/existential abstraction of $\wp(\mathrm{Distr}(\Sigma))$, so that, correspondingly, pre_a can be viewed as an *abstract predecessor function*, since its co-domain actually is an abstract domain. Consequently, the output of this abstract function can be projected back to distributions using the corresponding concretization map. Interestingly, it turns out that the corresponding *concrete predecessor functions*, obtained by composing the operator pre_a with either $\gamma^{\forall}$ or $\gamma^{\exists}$, exactly coincide with the two probabilistic predecessors $\mathrm{ppre}_a^{\forall}$ and ppre_a.

LEMMA 6.2. $\mathrm{ppre}_a^{\forall} = \gamma^{\forall} \circ \mathrm{pre}_a$ *and* $\mathrm{ppre}_a = \gamma^{\exists} \circ \mathrm{pre}_a$.

PROOF. Clear by definition. $\square$

Thus, in equivalent terms, the predecessor operator pre_a is the best correct universal/existential approximation of the operators $\mathrm{ppre}_a^{\forall}/\mathrm{ppre}_a$, for the universal/existential abstractions $\alpha^{\forall}/\gamma^{\forall}$ and $\alpha^{\exists}/\gamma^{\exists}$.

7. A SPECTRUM OF PROBABILISTIC RELATIONS OVER DLTSS

The approach developed above advocates for a general methodology for defining behavioral relations between states of a PLTS: first define a "lifted" behavioral relation between distributions of the corresponding dLTS and then restrict this definition to Dirac distributions. As discussed above, this approach works satisfactorily for simulation and bisimulation on PLTSs. In what follows, we show that this technique is indeed more general since it can be applied to a number of known behavioral relations on PLTSs. We focus only on the "simulation" version of these behavioral relations since the approach easily scales to the corresponding "bisimulation" counterparts.

7.1. Probabilistic Simulation

Segala [1995] and Segala and Lynch [1995] put forward a weaker variant of simulation where a state transition $s \xrightarrow{a} d$ can be matched by a so-called combined transition from a state t, namely a convex combination of distributions reachable from t. Probabilistic simulation originates from the intuition of interpreting nondeterministic choice in PLTSs as being based on schedulers. We show that the underlying idea of probabilistic simulation can be easily lifted to transitions in dLTSs.

Let $\mathcal{M} = \langle \Sigma, Act, \to \rangle$ be a PLTS, $\{s \xrightarrow{a} d_i\}_{i \in I}$ a (countable) family of transitions in $\mathcal{M}$ and $\{p_i\}_{i \in I}$ a corresponding family of probabilities in $[0, 1]$ such that $\sum_{i \in I} p_i = 1$. Let $d \in \mathrm{Distr}(\Sigma)$ be the convex combination $d = \sum_{i \in I} \; p_i d_i$. Then, $\langle s, a, \sum_{i \in I} p_i d_i \rangle$, denoted by $s \overset{a}{\rightsquigarrow} d$, is called a *combined transition* in $\mathcal{M}$.

Definition 7.1 (*Probabilistic simulation [Segala and Lynch 1995]*). A relation $R \subseteq \Sigma \times \Sigma$ is a *probabilistic simulation* on $\mathcal{M}$ if for all $s, t \in \Sigma$ such that sRt,

if $s \xrightarrow{a} d$ then there exists $e \in \mathrm{Distr}(\Sigma)$ such that $t \overset{a}{\rightsquigarrow} e$ and $d \sqsubseteq_R e$.

The greatest probabilistic simulation on $\mathcal{M}$ exists, it is a preorder relation called *probabilistic simulation preorder* (on $\mathcal{M}$) and denoted by R_{psim}. $\square$

Let us apply our approach in order to lift the above notion of combined transition to distributions.

Definition 7.2 (*Combined and hyper d-transitions*).

— Let $d, e \in \mathrm{Distr}(\Sigma)$. Then, $d \overset{a}{\rightsquigarrow} e$ if there exists $s \in \mathrm{supp}(d)$ such that $s \overset{a}{\rightsquigarrow} e$. $d \overset{a}{\rightsquigarrow} e$ is called a *combined d-transition*.

— Let $\{d \xrightarrow{a} d_i\}_{i \in I}$ be a family of transitions in a dLTS and $\{p_i\}_{i \in I}$ a corresponding family of probabilities such that $\sum_{i \in I} p_i = 1$. Then the triple $\langle d, a, \sum_{i \in I} p_i d_i \rangle$, compactly denoted by $d \overset{a}{\rightrightarrows} e$ where $e = \sum_{i \in I} p_i d_i$, is called a *hyper d-transition*. $\square$

Let us remark that the notion of hyper d-transition is stronger than that of combined d-transition, because $d \overset{a}{\rightsquigarrow} e$ implies $d \overset{a}{\rightrightarrows} e$ but not vice versa. Moreover, our definition of hyper d-transition can be compared with analogous notions of hyper transition defined by Deng et al. [2008], Hennessy [2012], and Stoelinga [2002]. In particular, we observed that a hyper transition in the sense of both Stoelinga [2002], Deng et al. [2008] and Hennessy [2012] is a hyper d-transition, but not vice versa.

Definition 7.3 (*Hyper transitions [Deng et al. 2008; Hennessy 2012; Stoelinga 2002]*). Let $\mathcal{M} = \langle \Sigma, Act, \rightarrow \rangle$ be a PLTS.

— The *hyper transition* relation $\rightarrow_h \subseteq \mathrm{Distr}(\Sigma) \times Act \times \mathrm{Distr}(\Sigma)$ by Deng et al. [2008, Section 4] and Hennessy [2012, Definition 2.2, Lemma 2.3] is defined as follows: $d \xrightarrow{a}_h d'$ when there exist $\{s_i\}_{i \in I}$, $\{d_i\}_{i \in I}$, $\{p_i\}_{i \in I}$, with $\sum_{i \in I} \, p_i = 1$, such that:

$$(1)\;\; d = \sum_{i \in I} \, p_i \, \delta_{s_i}; \quad (2)\;\; d' = \sum_{i \in I} \, p_i \, d_i; \quad (3)\;\; \forall i \in I.\; s_i \xrightarrow{a} d_i.$$

— The *hyper transition* relation $\overset{a}{\underset{h}{\rightrightarrows}} \subseteq \mathrm{Distr}(\Sigma) \times Act \times \mathrm{Distr}(\Sigma)$ by Stoelinga [2002] is defined as follows: $d \overset{a}{\underset{h}{\rightrightarrows}} d'$ when for all $s_i \in \mathrm{supp}(d)$ there exists a combined transition $s_i \overset{a}{\rightsquigarrow} d_i$ such that $d' = \sum_{s_i \in \mathrm{supp}(d)} \, d(s_i)\, d_i$. $\square$

PROPOSITION 7.4.

(1) $d \overset{a}{\rightsquigarrow} d'$ *implies* $d \overset{a}{\rightrightarrows} d'$ *but not vice versa;*
(2) $d \xrightarrow{a}_h d'$ *implies* $d \overset{a}{\rightrightarrows} d'$ *but not vice versa;*
(3) $d \overset{a}{\underset{h}{\rightrightarrows}} d'$ *implies* $d \xrightarrow{a}_h d'$ *but not vice versa.*

PROOF. To prove (1) we observe that $d \overset{a}{\rightsquigarrow} d'$ means that there exists $s \in \mathrm{supp}(d)$ such that $s \overset{a}{\rightsquigarrow} d'$, i.e., there exist $\{s \xrightarrow{a} d_i\}_{i \in I}$ and $\{p_i\}_{i \in I}$ such that $\sum_{i \in I} p_i = 1$ and $d' = \sum_{i \in I} \, p_i d_i$. Then, there also exists a set $\{d \xrightarrow{a} d_i\}_{i \in I}$ such that $d \overset{a}{\rightrightarrows} d'$. To show that the opposite implication does not hold, consider the PLTS with $\Sigma = \{x_1, x_2, u, v\}$ and two transitions $x_1 \xrightarrow{a} \delta_u$ and $x_2 \xrightarrow{a} \delta_v$. Consider $d = (x_1/0.5, x_2/0.5, u/0, v/0)$ and $d' = (x_1/0, x_2/0, u/0.5, v/0.5) = 0.5\, \delta_u + 0.5\, \delta_v$ with $\mathrm{supp}(d) = \{x_1, x_2\}$ and $\mathrm{supp}(d') = \{u, v\}$. Then, $d \overset{a}{\rightrightarrows} d'$ but $d \overset{a}{\not\rightsquigarrow} d'$.

Let us consider (2). Observe that if $d \xrightarrow{a}_h d'$ then there exists $\{s_i \xrightarrow{a} d_i\}_{i \in I}$, with $\{s_i\}_{i \in I} = \mathrm{supp}(d)$, $\{p_i\}_{i \in I}$, with $\sum_{i \in I} \, p_i{=}1$, such that $d' = \sum_{i \in I} \, p_i \, d_i$. Then there also exists a set $\{d \xrightarrow{a} d_i\}_{i \in I}$ such that $d \overset{a}{\rightrightarrows} d'$. To prove that the converse implication does not hold, consider the PLTS with $\Sigma = \{x_1, x_2, u\}$ and a single transition $x_1 \xrightarrow{a} \delta_u$. Hence, with $d = (x_1/0.5, x_2/0.5, u/0)$, we have that $d \overset{a}{\rightrightarrows} \delta_u$ but $d \not\xrightarrow{a}_h \delta_u$.

Finally, let us consider (3). By definition of $d \overset{a}{\underset{h}{\rightrightarrows}} d'$, if we consider $\{p_i\}_{i \in I} = \{d(s_i)\}_{s_i \in \mathrm{supp}(d)}$ then $d \xrightarrow{a}_h d'$. To show that the opposite implication does not hold, we consider the PLTS with $\Sigma = \{x_1, x_2, u, v\}$ and three transitions: $x_1 \xrightarrow{a} \delta_u$, $x_1 \xrightarrow{a} \delta_v$ and $x_2 \xrightarrow{a} \delta_v$. Let $d = (x_1/0.5, x_2/0.5, u/0, v/0) = 0.25\, x_1 + 0.25\, x_1 + 0.5 x_2$ and $d' = (x_1/0, x_2/0, u/0.25, v/0.75) = 0.25\, \delta_u + 0.25\, \delta_v + 0.5\, \delta_v$. Then, we have that $d \xrightarrow{a}_h d'$ but $d \overset{a}{\underset{h}{\not\rightrightarrows}} d'$ since there are no d_1, d_2 such that $x_1 \overset{a}{\rightsquigarrow} d_1$, $x_2 \overset{a}{\rightsquigarrow} d_2$ and $d' = 0.5\, d_1 + 0.5\, d_2$. $\square$

Following the idea of probabilistic simulation in Definition 7.1, probabilistic d-simulation is defined following the pattern in Definition 3.8 for d-simulation, but using combined d-transitions rather than transitions in a dLTS.

Definition 7.5 (*Probabilistic d-simulation*). A relation $Q \subseteq \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma)$ is a *probabilistic d-simulation* on $\mathcal{M}$ if for all $d, e \in \mathrm{Distr}(\Sigma)$, if $d \, Q \, e$ then:

(1) if $d \overset{a}{\rightsquigarrow} f$ then there exists $g \in \mathrm{Distr}(\Sigma)$ such that $e \overset{a}{\rightsquigarrow} g$ and $f \, Q \, g$;
(2) $d \sqsubseteq_{\Delta(Q)} e$. $\square$

The greatest probabilistic d-simulation on $\mathcal{M}$ exists, it is a preorder relation called *probabilistic d-simulation preorder* (on $\mathcal{M}$) and denoted by $\mathcal{R}^{\mathrm{d}}_{\mathrm{psim}}$. □

PROPOSITION 7.6. *Let $\mathcal{M}$ be a PLTS. If Q is a probabilistic d-simulation on $\mathcal{M}$ then $\Delta(Q)$ is a probabilistic simulation on $\mathcal{M}$. Also, if R is a probabilistic simulation on $\mathcal{M}$ then $\sqsubseteq_R$ is a probabilistic d-simulation on $\mathcal{M}$ and $\Delta(\sqsubseteq_R) = R$.*

PROOF. The proof is completely analogous to that of Proposition 3.11 apart from using combined transitions instead of standard transitions. □

It turns out that all the results obtained in Sections 3 and 4 for simulation also hold for probabilistic simulation and are collected in the following theorem. In particular, as before, the probabilistic simulation preorder between states can be retrieved from the probabilistic d-simulation preorder by restricting it to Dirac distributions. Dually, the probabilistic d-simulation preorder coincides with the lifting of the probabilistic simulation preorder. As far as the logical characterization is concerned, Hermanns et al. [2011] (Parma and Segala [2007] considers only the case of bisimulation) show that the probabilistic simulation between states of a PLTS is logically characterized by the logical preorder — restricted to Dirac distributions — of a modal logic that has the same syntax of $\mathcal{L}^+_\forall$ but whose diamond operator is defined in terms of combined transitions on the PLTS. Let us denote by $\mathcal{L}^+_{\mathrm{p}}$ the logic $\mathcal{L}^+$ where the semantics of the diamond operator is defined in terms of combined d-transitions. Then, analogously to what has been done in Sections 3 and 4, the results in [Hermanns et al. 2011; Parma and Segala 2007] can be extended by showing that the full logical preorder between distributions induced by $\mathcal{L}^+_{\mathrm{p}}$ coincides with the probabilistic d-simulation preorder.

THEOREM 7.7.

(1) $\Delta(\mathcal{R}^{\mathrm{d}}_{\mathrm{psim}}) = R_{\mathrm{psim}}$ *and* $\mathcal{R}^{\mathrm{d}}_{\mathrm{psim}} = \sqsubseteq_{R_{\mathrm{psim}}}$.
(2) $\mathcal{R}^{\mathrm{d}}_{\mathrm{psim}} = \leq_{\mathcal{L}^+_{\mathrm{p}}}$.

PROOF. The proof of (1) follows the lines of that of Theorem 3.12 by exploiting Proposition 7.6.
The proof of (2) is along the lines of the proof of Theorem 4.1 (1) so that we just highlight some steps.
($\Rightarrow$) Let $d\, \mathcal{R}^{\mathrm{d}}_{\mathrm{psim}}\, e$. We prove that for all $\phi \in \mathcal{L}^+_{\mathrm{p}}$ if $d \models \phi$ then $e \models \phi$, by structural induction on ϕ. The cases $\phi = \top, \bigwedge_{i\in I}\phi_i, \bigvee_{i\in I}\phi_i, [\psi]_p$ are identical to the case of simulation in Theorem 4.1 (1). Let be $\phi = \Diamond_a\psi$ and assume that $d \models \Diamond_a\psi$ so that there exists h such that $d \overset{a}{\rightsquigarrow} h$ and $h \models \psi$. From $d\, \mathcal{R}^{\mathrm{d}}_{\mathrm{psim}}\, e$ we have that there exists f such that $e \overset{a}{\rightsquigarrow} f$ with $f \in \mathcal{R}^{\mathrm{d}}_{\mathrm{psim}}(h)$. Hence, by induction on ψ, $h \models \psi$ implies $f \models \psi$, so that $e \models \Diamond_a\psi$ as desired.
($\Leftarrow$) Let us prove that $d \leq_{\mathcal{L}^+_{\mathrm{p}}} e$ implies $d\, \mathcal{R}^{\mathrm{d}}_{\mathrm{psim}}\, e$. It is sufficient to prove that the preorder

$$Q = \{(d,e) \mid \mathrm{Sat}_{\mathcal{L}^+_{\mathrm{p}}}(d) \subseteq \mathrm{Sat}_{\mathcal{L}^+_{\mathrm{p}}}(e)\}$$

is a probabilistic d-simulation. The first condition of probabilistic d-simulation is proved by contradiction. Assume that $(d,e) \in Q$, $d \overset{a}{\rightsquigarrow} f$ for some f and for any g such that $e \overset{a}{\rightsquigarrow} g$, suppose by contradiction that $g \notin Q(f)$, i.e., there exists a formula $\psi_{f,g} \in \mathcal{L}^+_{\mathrm{p}}$ such that $f \models \psi_{f,g}$ and $g \not\models \psi_{f,g}$. Then, the formula $\psi^f = \bigwedge\{\psi_{f,g} \mid g \in \mathrm{Distr}(\Sigma, e \overset{a}{\rightsquigarrow} g\} \in \mathcal{L}^+_{\mathrm{p}}$ is such that $d \models \Diamond_a\psi$ and $e \not\models \Diamond_a\psi$ which is a contradiction to $\mathrm{Sat}_{\mathcal{L}^+_{\mathrm{p}}}(d) \subseteq \mathrm{Sat}_{\mathcal{L}^+_{\mathrm{p}}}(e)$. The proof that $(d,e) \in Q$ implies $d \sqsubseteq_{\Delta(Q)} e$ is analogous to the case of simulation in Theorem 4.1 (1). □

It is worth mentioning that after Crafa and Ranzato [2011b], also Hennessy [2012] put forward a notion of "probabilistic simulation" between distributions rather than states. This is called *strong probabilistic d-simulation* [Hennessy 2012, Definition 3.16] and it is shown to capture the probabilistic extension of a standard notion of behavioural contextual equivalence. Let denote here by $\mathcal{R}^{\mathrm{d}}_{\mathrm{spsim}} \subseteq \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma)$ the corresponding preorder relation (while Hennessy [2012] deals with bisimulations we focus here on simulations). It is not necessary to recall the full definition here, since we observe that Hennessy [2012, Proposition 3.13, Theorem 3.17] shows that $\mathcal{R}^{\mathrm{d}}_{\mathrm{spsim}} = \sqsubseteq_{R_{\mathrm{psim}}}$. As a consequence of the above Theorem 7.7 (1), we have that $\mathcal{R}^{\mathrm{d}}_{\mathrm{spsim}} = \mathcal{R}^{\mathrm{d}}_{\mathrm{psim}}$, that is, the strong probabilistic d-simulation preorder is indeed an equivalent characterization of our probabilistic d-simulation preorder.

7.2. Failure Simulation

One interesting consequence of defining dLTSs as LTSs of distributions lies in the fact that the well-known van Glabbeek [2001]'s spectrum of behavioral relations on LTSs can be reformulated in terms of transitions between distributions of a dLTS. This leads to a spectrum of d-relations between distributions of a dLTS, that can be projected back into a spectrum of relations between states of a PLTS by restricting the d-relations to Dirac distributions. As an example we show how this approach works on failure simulation [van Glabbeek 2001]. A formalization and generalization of such a "lifting schema" in a suitable framework like abstract interpretation or coalgebras is left as future work.

Following a standard notation, if $A \subseteq Act$ then $s \xrightarrow{A}$ means that there exists some $a \in A$ such that $\mathrm{post}_a(s) \neq \varnothing$, while $s \stackrel{A}{\not\rightarrow}$ means that there is no outgoing transition from s which is labeled with some $a \in A$. Similarly, we use the notations $d \xrightarrow{A}$ and $d \stackrel{A}{\not\rightarrow}$ for a distribution d in a DLTS.

Definition 7.8 (*Failure simulation*). A relation $R \subseteq \Sigma \times \Sigma$ is a *failure simulation* on a PLTS when for any $s, t \in \Sigma$, if sRt then:

(1) if $s \xrightarrow{a} d$ then there exists $e \in \mathrm{Distr}(\Sigma)$ such that $t \xrightarrow{a} e$ and $d \sqsubseteq_R e$;
(2) if $s \stackrel{A}{\not\rightarrow}$ then $t \stackrel{A}{\not\rightarrow}$ for any $A \subseteq Act$. □

Definition 7.9 (*Failure d-simulation*). A relation $\mathcal{R} \subseteq \mathrm{Distr}(\Sigma) \times \mathrm{Distr}(\Sigma)$ is a *failure d-simulation* on a PLTS when for all $d, e \in \mathrm{Distr}(\Sigma)$, if $d \,\mathcal{R}\, e$ then:

(1) if $d \xrightarrow{a} f$ then there exists $g \in \mathrm{Distr}(\Sigma)$ such that $e \xrightarrow{a} g$ and $f \,\mathcal{R}\, g$;
(2) if $d \stackrel{A}{\not\rightarrow}$ then $e \stackrel{A}{\not\rightarrow}$ for any $A \subseteq Act$;
(3) $d \sqsubseteq_{\Delta(\mathcal{R})} e$. □

The lifting of a relation between states of a PLTS to a relation between distributions of the corresponding dLTS is obtained by resorting to the standard transfer property and by adding the condition (i.e., condition (3) in Definition 7.9) that deals with probabilities assigned to sets of related states.

PROPOSITION 7.10.

(1) *If Q is a failure d-simulation on $\mathcal{M}$ then $\Delta(Q)$ is a failure simulation on $\mathcal{M}$.*
(2) *If R is a failure simulation on $\mathcal{M}$ then $\sqsubseteq_R$ is a failure d-simulation on $\mathcal{M}$ and $\Delta(\sqsubseteq_R) = R$.*

PROOF. Let us prove (1). Let Q be a failure d-siumulation. By Proposition 3.11 (1), we have that $\Delta(Q)$ is a simulation, hence it is sufficient to show that if $t \in \Delta(Q)(s)$ and $t \stackrel{A}{\not\rightarrow}$ for some $A \subseteq Act$, then $s \stackrel{A}{\not\rightarrow}$. From $t \in \Delta(Q)(s)$ we have that $\delta_t \in Q(\delta_s)$ and from

$t \xrightarrow{A}$ we have that $\delta_t \xrightarrow{A}$. Since Q is a failure d-simulation, we have that $\delta_s \xrightarrow{A}$, so that $s \xrightarrow{A}$.

Let us turn to (2). Let R be a failure simulation. $\Delta(\sqsubseteq_R) = R$ comes by Proposition 3.11 (2). By Proposition 3.11 (2), $\sqsubseteq_R$ is a d-simulation, hence it is sufficient to show that if $d \sqsubseteq_R e$ and $e \xrightarrow{A}$ for some $A \subseteq Act$ then $d \xrightarrow{A}$. If $e \xrightarrow{a}$, for some $a \in A$, then there exists $y \in \mathrm{supp}(e)$ such that $y \xrightarrow{a}$. By Lemma 2.2 (2), there exists $x \in \mathrm{supp}(d)$ such that $y \in R(x)$, hence $x \xrightarrow{a}$ because R is a failure simulation. But $x \xrightarrow{a}$ implies $d \xrightarrow{a}$, that is $d \xrightarrow{A}$. □

Let R_{fsim} and $\mathcal{R}^{\mathrm{d}}_{\mathrm{fsim}}$ be, respectively, the failure simulation and d-simulation preorders on a PLTS $\mathcal{M}$. According to the LTS spectrum [van Glabbeek 2001], failure simulation can be logically characterized through a modality that characterizes which transitions cannot be fired. We follow this same approch and we denote by $\mathcal{L}^+_{\mathrm{f}}$ the logic obtained from $\mathcal{L}^+$ by adding a modality $\mathsf{ref}\langle A\rangle$, where $A \subseteq Act$, and whose semantics is defined as follows: for any $d \in \mathrm{Distr}(\Sigma)$, $d \models_{\mathcal{L}^+_{\mathrm{f}}} \mathsf{ref}\langle A\rangle$ iff $d \not\xrightarrow{A}$.

THEOREM 7.11.

(1) $\Delta(\mathcal{R}^{\mathrm{d}}_{\mathrm{fsim}}) = R_{\mathrm{fsim}}$ *and* $\mathcal{R}^{\mathrm{d}}_{\mathrm{fsim}} = \sqsubseteq_{R_{\mathrm{fsim}}}$.
(2) $\mathcal{R}^{\mathrm{d}}_{\mathrm{fsim}} = \leq_{\mathcal{L}^+_{\mathrm{f}}}$.

PROOF. The proof of (1) comes as that of Theorem 3.12, using Proposition 7.10.
Let us prove (2).

($\Rightarrow$) Let us prove that $d\ \mathcal{R}^{\mathrm{d}}_{\mathrm{fsim}}\ e$ implies that for all $\phi \in \mathcal{L}^+_{\mathrm{f}}$, if $d \models \phi$ then $e \models \phi$. As usual, we proceed by structural induction on ϕ. The cases $\phi = \top$, $\bigvee_{i\in I}\phi_i$, $\bigwedge_{i\in I}\phi_i$, $\diamond_a\psi$, $[\psi]_p$ are identical to the case of simulation (in Theorem 4.1 (1)). If $\phi = \mathsf{ref}\langle A\rangle$ and $d \models \mathsf{ref}\langle A\rangle$, then we have that $d \not\xrightarrow{A}$, so that from $d\ \mathcal{R}^{\mathrm{d}}_{\mathrm{fsim}}\ e$ we obtain, by definition of failure d-simulation, $e \not\xrightarrow{A}$, and therefore $e \models \mathsf{ref}\langle A\rangle$.

($\Leftarrow$) In order to prove that $d \leq_{\mathcal{L}^+_{\mathrm{f}}} e$ implies $d\ \mathcal{R}^{\mathrm{d}}_{\mathrm{fsim}}\ e$, we prove that the preorder $Q = \{(d,e) \mid \mathrm{Sat}_{\mathcal{L}^+_{\mathrm{f}}}(d) \subseteq \mathrm{Sat}_{\mathcal{L}^+_{\mathrm{f}}}(e)\}$ is a failure d-simulation. The fact that Q is a d-simulation comes by Theorem 4.1 (1), hence it is enough to prove that if $d\ Q\ e$ and $d \not\xrightarrow{A}$, for some $A \subseteq Act$, then $e \not\xrightarrow{A}$. From $d \not\xrightarrow{A}$ we have that $d \models \mathsf{ref}\langle A\rangle$, while from $d\ Q\ e$ we have that $e \models \mathsf{ref}\langle A\rangle$, so that $e \not\xrightarrow{A}$ as desired. □

8. RELATED AND FUTURE WORK

Simulation and bisimulation on PLTSs have been introduced by Segala [1995] and Segala and Lynch [1995] as two behavioral relations that preserve significant classes of temporal properties in the probabilistic logic PCTL [Hansson and Jonsson 1994]. Since then, a number of works put forward probabilistic extensions of Hennessy-Milner logic in order to logically characterize these relations. Larsen and Skou [1991] and Desharnais et al. [2002] investigated a probabilistic diamond connective that enhances the diamond operator of HML with the probability bounds of transitions. However, these logics are adequate just for reactive and alternating systems, which are probabilistic models that are strictly less expressive than PLTSs. Two further probabilistic variants of HML are available [Deng et al. 2008; Parma and Segala 2007]. The first one has been defined by Parma and Segala [2007] (see also [Hermanns et al. 2011] for simulation), whose formulae are interpreted on sets of probability distributions over the states of a PLTS. One distinctive connective of this logic is a modal operator $[\phi]_p$, whose semantics is the set of distributions that assigns at least probability p to the set of states whose Dirac distributions satisfy ϕ. In this paper we have shown that such a logic admits an equivalent formulation that retains the probabilistic operator $[\phi]_p$ and retrieves the diamond operator of HML by lifting it to distributions. Deng et al. [2008] and Hennessy [2012] follow a different approach. They propose a probabilistic variant of HML that is interpreted on sets of processes of the pCSP process calculus. In their logic the semantics of the diamond operator is defined in terms of hyper transitions between distributions: this notion of hyper transition is more complex than ours and has been compared with our notion of hyper transition in Section 7. Moreover, the logic by Deng et al. [2008] and Hennessy [2012] features a probabilistic operator $\bigoplus_{i\in I} p_i\phi_i$ that is satisfied by processes that correspond to distributions that can be decomposed into convex combinations of distributions that satisfy ϕ_i. Besides (bi)simulation and probabilistic (bi)simulation, this logic is able to characterize two notions of failure and forward simulation that have been proved to agree with the testing preorders on pCSP processes (see [Deng et al. 2008; Hennessy 2012]).

Deng et al. [2008]'s definition of failure simulation is quite different from ours, that we directly derived from the standard LTS spectrum [van Glabbeek 2001]. One major difference is that we define a relation between states of a PLTS which is then lifted to a relation between distributions whereas Deng et al. [2008] consider a relation between states and distributions. A precise comparison between the spectrum of behavioral relations on dLTSs and the behavioral relations defined by Deng et al. [2008] is left as subject for future work. We also plan to investigate weak transitions in dLTSs that abstract from internal, invisible, actions. Weak variants of simulation, probabilistic simulation, forward and failure simulation have been studied both by Deng et al. [2008] and Parma and Segala [2007].

As a further avenue of future work we plan to study whether and how behavioral relations on PLTSs can be computed by resorting to standard algorithms for LTSs that compute the corresponding lifted relations on a dLTS. A first step in this direction has been taken in [Crafa and Ranzato 2011a; Crafa and Ranzato 2012], where efficient algorithms to compute simulation and bisimulation on PLTSs have been derived by resorting to abstract interpretation techniques.

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