

COMPLEX MANIFOLDS IN Q -CONVEX BOUNDARIES

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ABSTRACT. We consider a C^∞ boundary $b\Omega \subset \mathbb{C}^n$ which is q -convex in the sense that its Levi-form has positive trace on every complex q -plane. We prove that $b\Omega$ is tangent of infinite order to the complexification of each of its submanifolds which is complex tangential and of finite bracket type. This generalizes Diederich-Fornaess [2] from pseudoconvex to q -convex domains. We also readily prove that the rows of the Levi-form are $\frac{1}{2}$ -subelliptic multipliers for the $\bar{\partial}$ -Neumann problem on q -forms (cf. Ho [4]). This allows to run the Kohn algorithm of [5] in the chain of ideals of subelliptic multipliers for q -forms. If $b\Omega$ is real analytic and the algorithm sticks on q -forms, then it produces a variety of holomorphic dimension q , and in fact, by our result above, a complex q -manifold which is not only tangent but indeed contained in $b\Omega$. Altogether, the absence of complex q -manifolds in $b\Omega$ produces a subelliptic estimate on q -forms.

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1. COMPLEX q -MANIFOLDS IN THE BOUNDARY AND THE KOHN ALGORITHM ON q -FORMS

Let Ω be a smooth domain in \mathbb{C}^n defined by $r = 0$ with $\partial r \neq 0$, and M a smooth CR submanifold of $b\Omega$ of CR dimension q and CR codimension p . We assume that M is “complex tangential” to $b\Omega$ in the sense that

$$(1.1) \quad TM \subset T^{\mathbb{C}}b\Omega.$$

Condition (1.1) is familiar in the ambient of peak-interpolation sets. If M is minimal in the sense of Tumanov, it is endowed with a “wedge complexification” of dimension $q+p$, that is, a complex $(q+p)$ -manifold \mathcal{W} of wedge type with edge M (cf. [6]). When $b\Omega$ is pseudoconvex, then $\mathcal{W} \subset b\Omega$; this refines Bedford-Fornaess [1] which is in turn a development of Diederich-Fornaess [2]. In fact, according to [6], \mathcal{W} is made out of analytic discs attached to M . The pseudoconvexity of $b\Omega$ brings the discs inside Ω and their complex tangency to $b\Omega$, which follows from (1.1), brings them in $b\Omega$. For the last implication, we have just to apply Hopf Lemma to a plurisubharmonic Hölder exhaustion

function of Ω of type $-(-r)^\eta$, for η close to 1, restricted to each disc. We weaken the hypothesis of pseudoconvexity and assume that $b\Omega$ is q -convex, that is, for a choice of the Hermitian metric, the trace of the Levi form $L_{b\Omega} = \partial\bar{\partial}r|_{T^{\mathbb{C}}b\Omega}$ is positive on every complex q -plane of $T^{\mathbb{C}}b\Omega$, the complex tangent bundle to $b\Omega$.

We strengthen the hypothesis of minimality and assume that M is of “finite bracket type”, that is, the subsequent brackets of C^∞ vector fields with values in $T^{\mathbb{C}}M$ generate the whole tangent bundle TM . Note that when M is real analytic, finite type and minimality coincide.

Theorem 1.1. *Let $b\Omega$ be q -convex and let $M \subset b\Omega$ be complex tangential and of finite bracket type. Then \mathcal{W} is tangent to $b\Omega$ of infinite order along M .*

The proof follows in Section 2.

The holomorphic dimension of a variety $V \subset b\Omega$ at z_o is defined by

$$(1.2) \quad \text{hol dim}_{z_o} V = \sup_{U_{z_o}} \inf_{z \in U \cap V} \dim_{\mathbb{C}}(T^{\mathbb{C}}V \cap \text{Ker}L_{b\Omega}),$$

for U_{z_o} ranging through the family of neighborhoods of z_o . Remark that $TV \cap \text{Ker}L_{b\Omega}$ is involutive; moving from z_o to a nearby point where the real and the CR ranks are constant, we may apply Frobenius Theorem and produce a foliation by smooth leaves of CR-dimension q . We select a leaf M , denote by \mathcal{L} the Lie span of $T^{\mathbb{C}}M$, and observe that $\mathcal{L} \subset \text{Ker}L_{b\Omega} \subset T^{\mathbb{C}}b\Omega$. By redefining z_o , if necessary, we may assume that $\mathcal{L} = TM$; thus M is complex tangential and of finite type. Altogether, we have obtained

Corollary 1.2. *(i) Let $b\Omega$ be q -convex and let $V \subset b\Omega$ have holomorphic dimension q at z_o . Then, there is $M \subset V$ of CR-dimension $\geq q$ whose wedge complexification \mathcal{W} is tangent of infinite order to $b\Omega$.*

(ii) If, moreover, $b\Omega$ and V are real analytic, then \mathcal{W} is contained in $b\Omega$ and is a (complex) manifold not just a wedge manifold.

Our purpose is now to run the Kohn algorithm in a q -convex domain and to show that, when it goes through, it produces a subelliptic estimate for q -forms. This requires a minor effort in adapting the proof by Kohn [5] in which the domain is pseudoconvex in the usual sense.

We choose an orthonormal basis $\omega_1, \dots, \omega_n = \partial r$ of $(1, 0)$ forms, and the dual basis L_j of $(1, 0)$ vector fields. In this basis, we denote by (r_{ij}) the matrix of $\partial\bar{\partial}r$ and by $u = \sum'_{|J|=k} u_J \bar{\omega}_J$ an antiholomorphic q form

with summation being taken over ordered multiindices $|J| = q$. The form is assumed to belong to the domain $D_{\bar{\partial}^*}$ of $\bar{\partial}^*$ that is, to satisfy $u_J|_{b\Omega} \equiv 0$ when $n \in J$; we denote by $C_c^\infty(\bar{\Omega} \cap U)^q$ the space of q -forms

with support in a neighborhood U of a boundary point $z_o \in b\Omega$ with smooth coefficients up to $b\Omega$. We also denote by $||| \cdot |||_\epsilon$ the *tangential* Sobolev norm (cf. [5]).

Proposition 1.3. *Let $b\Omega$ be q -convex; then*

$$(1.3) \quad \sum_{|K|=k-1}' \sum_i ||| \sum_j r_{ij} \bar{u}_{jK} |||_{\frac{1}{2}}^2 \leq \|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 + \|u\|^2$$

for any $u \in D_{\bar{\partial}^*} \cap C_c^\infty(\bar{\Omega} \cap U)^k$, $k \geq q$.

We express (1.3) by saying that each row of $\partial\bar{\partial}r$ is a $\frac{1}{2}$ -subelliptic row-multiplier on k -form. We use the notation $Q(u, u)$ for the energy of the $\bar{\partial}$ -Neumann problem, that is, the term in the right of (1.3).

Proof. We show that for any $v \in C_c^\infty(U' \cap \bar{\Omega})^k$, for $U' \supset \supset U$, and for any derivative D , we have

$$(1.4) \quad \left| \sum_{|K|=k-1}' \sum_{ij=1, \dots, n} \int_{\Omega} r_{ij} u_{iK} D \bar{v}_{jK} dV \right|^2 \lesssim Q(u, u) + \sum_j \|\bar{L}_j(v)\|^2$$

$$+ \sum_{|K|=k-1}' \sum_{ij=1, \dots, n} \int_{b\Omega} r_{ij} v_{iK} \bar{v}_{jK} dS.$$

For $D = L_k$, (1.4) follows from Schwartz inequality. For $D = \bar{L}_k$, $k < n$, it follows from integration by parts, Schwartz inequality, and basic estimate for u . Finally, for $D = \bar{L}_n$, we write

$$(1.5) \quad \sum_{|K|=k-1}' \sum_{ij=1, \dots, n} \int_{\Omega} r_{ij} u_{iK} \bar{L}_n \bar{v}_{jK} dV = \sum_{|K|=k-1}' \sum_{i,j < n} \int_{b\Omega} r_{ij} u_{iK} \bar{v}_{jK} dS$$

$$+ O\left(\sum_j \|\bar{L}_j u\| \|v\|\right).$$

Using again Schwartz inequality on $b\Omega$ for the positive 2-form $\sum_{|K|=k-1}' \sum_{ij=1, \dots, n} r_{ij} u_{iK} \bar{u}_{jK}$ over k -vectors u , we get

$$\sum_{|K|=k-1}' \left| \sum_{ij} r_{ij} u_{iK} \bar{v}_{jK} dV \right| \leq \left(\sum_{|K|=k-1}' \sum_{ij} r_{ij} u_{iK} \bar{u}_{jK} \right)^{\frac{1}{2}} \left(\sum_{|K|=k-1}' \sum_{ij=1, \dots, n} r_{ij} v_{iK} \bar{v}_{jK} \right)^{\frac{1}{2}},$$

and this yields (1.4) from (1.5) and the basic estimate. We use now (1.4) for $v_{jK} = \sum_i r_{ij} u_{iK}$. Reasoning as in [5] p. 97, we get

$$(1.6) \quad \left| \sum_{|K|=k-1}' \sum_{ijk} \int_{\Omega} r_{ij} u_{iK} D(r_{kj} \bar{u}_{kK}) dV \right| \lesssim Q(u, u).$$

Using the microlocal factorization $\Lambda^1 = \Lambda^{\frac{1}{2}}\Lambda^{\frac{1}{2}}$ for the tangential standard elliptic pseudodifferential operator of order 1 (together with the fact that the different derivatives D 's represent the full Λ^1), we get (1.3) from (1.6). \square

We recall briefly the Kohn's algorithm. We define, in a neighborhood of z_o , the chain of ideals $I_1^q \subset I_2^q \subset \dots \subset I_h^q$ and of modules $M_1^q \subset M_2^q \subset \dots \subset M_h^q$, starting from

$$\begin{cases} M_1^q = \{\partial r, \partial_i \bar{\partial} r\}_{i=1, \dots, n} \\ I_1^q = \sqrt{r, \det_{n-q+1} M_1^q}^{\mathbb{R}} \end{cases} \quad \text{where } \sqrt{\cdot}^{\mathbb{R}} \text{ denotes the real radical,}$$

and, inductively,

$$\begin{cases} M_h^q = \{M_{h-1}^q, \partial I_{h-1}^q\}, \\ I_h^q = \sqrt{I_{h-1}^q, \det_{n-q+1} M_h^q}^{\mathbb{R}}. \end{cases}$$

By Proposition 1.3, and by Garding inequality, M_1^q is made out of $\frac{1}{2}$ -subelliptic row multipliers (that is, (1.3) holds) and I_1^q is an ideal of $\frac{1}{2}$ -subelliptic multipliers over q -forms. By [5] Proposition 4.7, the full chain of M_h^q 's (resp. I_h^q 's) is made out of subelliptic row multipliers (resp. function multipliers). The proof of this point remains unchanged from pseudoconvex to q -convex domains.

We take our conclusions. If $1 \in I_h^q$ for some h , then we have a subelliptic estimate (for some ϵ depending on the number h of steps and on the operation of radical) on q -forms and, in fact, on k -forms for any $k \geq q$. If, instead, $I_{h+1}^q = I_h^q$ (and I_h^q does not capture 1), this reveals under the extra assumption $b\Omega \in C^\omega$, that $V = V(I_h^q)$, the zero-set of I_h^q , has holomorphic dimension $\geq q$. By Corollary 1.2 this implies the existence of a complex q -manifold in $b\Omega$. Putting altogether, we get the proof of

Theorem 1.4. *Assume that in a neighborhood of z_o , $b\Omega$ is real analytic, q -convex, and contains no germ of holomorphic manifold of dimension $\geq q$. Then a subelliptic estimate in degree $k \geq q$ for the $\bar{\partial}$ -Neumann problem holds in a neighborhood U of z_o , that is, for some ϵ we have*

$$\|u\|_\epsilon^2 \lesssim Q(u, u) \quad \text{for any } u \in D_{\bar{\partial}^*} \cap C_c^\infty(\bar{\Omega} \cap U)^k.$$

Example 1.5. In \mathbb{C}^3 , consider the domain Ω defined by

$$x_3 > -|z_1|^2 |z_2|^2 + \left(\frac{1}{4}|z_1|^4 + \frac{3}{4}|z_2|^4\right).$$

Here $b\Omega$ is real analytic, there are no complex 2-manifolds at 0 but just the complex curve defined by $z_1 = z_2$. Also, if we compute the

Levi form of $b\Omega$ in the metric in which $\pi_z^{-1}(1, 0, 0)$ and $\pi_z^{-1}(0, 1, 0)$ (for $\pi_z : T_z b\Omega \rightarrow \mathbb{C}^2 \times \mathbb{R}$ being the projection along the x_3 -axis) is an orthonormal system for $T_z^{\mathbb{C}} b\Omega$, we have

$$L_{b\Omega} = \begin{bmatrix} -|z_2|^2 + |z_1|^2 & -\bar{z}_1 z_2 \\ -z_1 \bar{z}_2 & -|z_1|^2 + 3|z_2|^2 \end{bmatrix}.$$

It follows

$$\text{trace } L_{b\Omega} = 2|z_2|^2 \geq 0.$$

Thus we have a subelliptic estimate in degree 2 according to Theorem 1.4. Note that this example could not be explained neither by usual pseudoconvexity nor by *strong 2-pseudoconvexity*. In fact

$$\begin{aligned} \det L_{b\Omega} &= (|z_1|^2 - |z_2|^2)(3|z_2|^2 - |z_1|^2) - |z_1|^2 |z_2|^2 \\ &= -|z_1|^4 - 3|z_2|^4 + 3|z_1|^2 |z_2|^2 \\ &\leq 0. \end{aligned}$$

Thus,

- $b\Omega$ is not pseudoconvex (because $\det L_{b\Omega} \leq 0$ implies that there are eigenvalues of opposite sign),
- $b\Omega$ does not satisfy $Z(2)$ (in the sense of [3]) because there are no positive eigenvalues at 0.

2. PROOF OF THEOREM 1.1

We adapt the proof of [2] Proposition 3 to the new situation in which $b\Omega$ is no more pseudoconvex but just q -convex. We move to a nearby point that we still denote by z_o at which the “multitype” in the sense of (i)–(v) below is minimal (in the lessicographic order). We observe that the wedge complexification \mathcal{W} can be (non-uniquely) continued to a smooth manifold without boundary W of real dimension $2(q+p)$. Since \mathcal{W} is holomorphic, then W is “approximately holomorphic” at M . By a linear unitary coordinate change we can assume that $z_o = 0$, $T_{z_o} M = \mathbb{C}^q \times \mathbb{R}^p \times \{0\}$ and $T_{z_o} W = \mathbb{C}^q \times \mathbb{C}^p \times \{0\}$ and $T_{z_o} b\Omega = \mathbb{C}^{n-1} \times i\mathbb{R}$. We observe that the projection π along the z_n -axis is transversal to W ; thus $\pi(W)$ and $\pi^{-1}\pi(W)$ are real manifolds of dimension $2(q+p)$ and $2(q+p+1)$ respectively. We use the notation $t := n - (p+q+1)$. We suppose that $\pi^{-1}\pi(W)$ is defined by real equations $\mu_j(z') = 0$, $j = 1, \dots, 2t$ such that, putting $f_j =: \mu_j + i\mu_{t+j}$, $j \leq t$, we have $\bar{\partial} f_j = O_M^\infty$, and W is graphed over $\pi(W)$ by $z_n = h + ig$ with $\bar{\partial}(h + ig) = O_M^\infty$; here O_M^∞ denotes a zero of infinite order at M . Clearly M is defined by $x_n - h = 0$, $y_n - g = 0$, $\rho = 0$, $\mu = 0$ (where by ρ and μ we denote the full set of the ρ_j 's and μ_j 's). We consider the Hermitian metric on \mathbb{C}^n in which Ω is q -convex and the induced Euclidian metric on \mathbb{R}^{2n} .

In this metric, we choose an orthonormal basis $\{X_{0,i}\}_{i=1}^{p_0}$ of $T^{\mathbb{C}}M$ and a completion to a full basis of TM

$$(2.1) \quad \{X_{0,i}\}_{i=1}^{p_0}, \{X_{1,i}\}_{i=1}^{p_1}, \dots, \{X_{s,i}\}_{i=1}^{p_s} \quad \text{with } p_0 = 2q \text{ and } \sum_{j=1}^s p_j = p.$$

We may assume that

- (i) any j -iterated bracket of the $X_{0,i}$'s is in the span of the $X_{h,i}$'s for $h \leq j$,
- (ii) $X_{j,i} = [X_{0,\nu}, X_{j-1,\mu}]$ modulo $\text{Span}\{X_{j',i}\}_{j' \leq j-1}$ for suitable $X_{0,\nu} \in \text{Span}\{X_{0,i}\}$ and $X_{j-1,\mu} \in \text{Span}\{X_{h,i}\}_{h \leq j-1}$ when $j \geq 1$.

This is an immediate consequence of Jacobi identity. We put $\mathcal{L}^0 = T^{\mathbb{C}}M$, write, inductively, $\mathcal{L}^j = \text{Span}\{\mathcal{L}^{j-1}, [X_{0,\nu}, X_{j-1,\mu}]\}_{\nu,\mu}$ and decompose

$$TM = \mathcal{L}^0 \oplus \frac{\mathcal{L}^1}{\mathcal{L}^0} \oplus \dots \oplus \frac{\mathcal{L}^s}{\mathcal{L}^{s-1}}.$$

We can assume that our linear unitary transformation gives $\frac{\mathcal{L}^j}{\mathcal{L}^{j-1}} \Big|_{z_0} = \{0\} \times \mathbb{R}^{p_j} \times \{0\}$. Also, we can choose our basis so that, in addition to (i)–(ii) we also have

- (iii) each group $\{X_{j,i}\}_{i=1,\dots,p_j}$ is orthogonal one to another for different j .
- (iv) in a basis $z_{0,1}, \dots, z_{0,q}, z_{1,1}, \dots, z_{1,p_1}, \dots$ of \mathbb{C}^{q+p} we have $X_{0,i}|_{z_0} = \partial_{x_i}$, $X_{0,i+q}|_{z_0} = \partial_{y_i}$, $i \leq q$, and $X_{j,i}|_{z_0} = \partial_{x_{j,i}}$ for $j \geq 1$,
- (v) M is the intersection of W with the set defined by $\rho_{j,i} = 0$, $j \geq 1$, where the $\rho_{j,i}$'s are functions on $\pi(W)$ with $\text{Span}\{\text{Re } \partial \rho_{j,i}\} = \text{Span}\{\text{Re } \partial y_{j,i}\}$ and with $\langle \partial \bar{\rho}_{h,l}, L_{j,i} \rangle = 0$ for any $h \geq j+1$.

Note that, in particular, (v) implies that $\partial \bar{\partial} \rho_h(L_{j,i}, \bar{L}_{j',i'}) = 0$ for any $j, j' \leq h-2$.

We identify the $X_{j,i} \in TM$ to the real or imaginary parts of vector fields $L_{j,i} \in \mathbb{C}(TM + JTM) \cap T^{1,0}\mathbb{C}^n$ defined by

$$(2.2) \quad \{L_{0,i} := X_{0,i} + iX_{0,q+i}\}_{i=1}^q, \{L_{1,i} := X_{1,i} + iJX_{1,i}\}_{i=1}^{p_1}, \dots, \{L_{s,i} := X_{s,i} + iJX_{s,i}\}_{i=1}^{p_s}.$$

Since $\mathbb{C}(TM + JTM) \cap T^{1,0}\mathbb{C}^n \subset T^{1,0}b\Omega|_M$, we extend the $L = L_{j,i}$ from M to the whole $b\Omega$ as sections of $T^{1,0}b\Omega$ keeping unchanged their notation. We can also arrange that the $L_{j,i}$ are extended from M to W so that $\langle \partial \mu_j, L \rangle = O_M^\infty$, $j = 1, \dots, 2t$. For that, we extend them with the request $\langle \partial f_j, L \rangle \equiv 0$, $j = 1, \dots, t$; since $\langle \bar{\partial} f_j, L \rangle = O_M^\infty$, the conclusion follows remembering that the μ 's are the real and imaginary parts of

the f 's. By (iii) above, and by the fact that \mathcal{L}^0 is invariant under J , we have that the $L_{j,i}$, $j \geq 1$, are orthogonal to $\mathbb{C}\mathcal{L}^0$; this stays true also outside M for the extended vector fields. Moreover, possibly after renormalization, the $L_{0,i}$ can be chosen so that they form an orthonormal system.

Recall that for the equation $z_n = h + ig$ of W , we have supposed $\bar{\partial}(h + ig) = O_M^\infty$ and thus, in particular, $\partial\bar{\partial}h = O_M^\infty$. Thus, if $b\Omega$ is graphed by $x_n = h + \sigma$ (which serves as a definition of σ), we have $L_{b\Omega} = \partial\bar{\partial}\sigma|_{T^{1,0}b\Omega} + O_M^\infty$. We also denote by $r := x_n - (h + \sigma)$ a defining function for $b\Omega$. Note that $\sigma = 0$ on M ; we want to prove that

$$\sigma = O(\rho^\infty) \quad \text{when } y_n - g = 0 \text{ and } \mu = 0,$$

and hence W is tangent of infinite order to $b\Omega$ along M . We expand

$$(2.3) \quad \sigma = \sum_{|I|=k} a_I \rho^I + O(\rho^{k+1}) + \mathcal{E} + \mathcal{E}_1,$$

where I is a multi bi-index in the (j, i) 's and where $\mathcal{E} = O(y_n - g)$ and $\mathcal{E}_1 = O(\mu)$. We observe that

$$(2.4) \quad \partial\bar{\partial}\mathcal{E}(L, \bar{L}) = O(\langle dz_n, L \rangle \langle \partial\rho, L \rangle) + |\langle dz_n, L \rangle|^2 + O(y_n - g).$$

In particular, recalling that $\langle \partial r, L_{j,i} \rangle = 0$ and $\langle \partial\rho, L_{0,i} \rangle = O(\rho)$, we have for $y_n - g = 0$

$$(2.5) \quad \partial\bar{\partial}\mathcal{E}(L_{j,i}, \bar{L}_{j',i'}) = O(\rho^{k-1}), \quad \partial\bar{\partial}\mathcal{E}(L_{j,i}, \bar{L}_{0,i'}) = O(\rho^k), \quad \partial\bar{\partial}\mathcal{E}(L_{0,i}, \bar{L}_{0,i'}) = O(\rho^{k+1}).$$

As for $\partial\bar{\partial}\mathcal{E}_1$, recalling also $\langle \partial\mu, L \rangle = O_M^\infty$, we have

$$(2.6) \quad \begin{aligned} \partial\bar{\partial}\mathcal{E}_1(L, \bar{L}) &\sim \partial\bar{\partial}\mu(L, \bar{L}) + |\langle \partial\mu, L \rangle| (|\langle \partial\rho, L \rangle| + |\langle dz_n, L \rangle|) + O(\mu) \\ &= O_M^\infty + O(\rho^\infty) + O(\mu). \end{aligned}$$

For this reason, when evaluating $\partial\bar{\partial}\sigma$ on L as above, we can assume without loss of generality that $\mathcal{E}_1 = 0$ in (2.3). We call k the first integer for which there is in (2.3) a non-trivial occurrence a_I for $|I| = k$; we wish to show that k cannot exist finite. First, the inclusion $TW|_M \subset T^{\mathbb{C}}b\Omega|_M$ implies $k \geq 2$. We first show that k cannot be odd. In fact, by a choice of $L = X + iJX$, $X \in \mathcal{L}^j$, $j \geq 1$ such that $\partial\bar{\partial}\sigma(L, \bar{L})$ is obtained by differentiating two factors once, we get

$$(2.7) \quad \partial\bar{\partial}\sigma(L, \bar{L}) = \sum_{|I|=k-2} a_I \rho^I + O(\rho^{k-1}) + \partial\bar{\partial}\mathcal{E}(L, \bar{L}),$$

with $a_{I'} \neq 0$ for at least one I' . By the first of (2.5), the last term in (2.7) can be neglected. Thus the form in the right of (2.7), having odd

order, it changes sign. On the other hand

$$(2.8) \quad \partial\bar{\partial}\sigma(L_{0,i}, \bar{L}_{0,i'}) = O(\rho^{k-1}).$$

Define a q -plane by $Q_q := \text{Span}\{L, L_{0,i}\}_i$ (for any choice of $q-1$ between the indices i); we can conclude that $\text{trace}_{Q_q}\partial\bar{\partial}\sigma$ changes sign, which violates the q -convexity of $b\Omega$. Thus k cannot be odd.

We show that k cannot be even, either. We first remove any possible term with a factor of $\rho_{1,i}$ in the homogeneous expression of degree k of σ , that is, $\sum a_{(1,i)I'}\rho^{(1,i)I'}$. We have

$$(2.9) \quad \partial\bar{\partial}\sigma|_{\mathbb{C}\mathcal{L}^0} = \sum_{|I'|=k-1} \left(\sum_i a_{(1,i)I'}\partial\bar{\partial}\rho_{1,i}|_{\mathbb{C}\mathcal{L}^0} \right) \rho^{I'} + O(\rho^k) + \partial\bar{\partial}\mathcal{E}|_{\mathbb{C}\mathcal{L}^0}.$$

By the third of (2.5), the last term in (2.9) can be neglected. If, for some $|I'| = k-1$, we have $\text{trace}_{\mathbb{C}\mathcal{L}^0}(\sum_i a_{(1,i)I'_o}\partial\bar{\partial}\rho_{1,i}) \neq 0$, then $\text{trace}_{\mathbb{C}\mathcal{L}^0}(\partial\bar{\partial}\sigma)$ changes sign since the $\rho^{I'}$ vary independently.

Otherwise, assume

$$(2.10) \quad \text{trace}_{\mathbb{C}\mathcal{L}^0} \left(\sum_i a_{(1,i)I'}\partial\bar{\partial}\rho_{1,i} \right) = 0 \quad \text{for any } I'.$$

Recall that the commutators of the $L'_{0,i}$ s span a space of dimension p_1 ; by Cartan formula, this is equivalent as to saying that the Levi matrices $\partial\bar{\partial}\rho_{1,i}|_{\mathbb{C}\mathcal{L}^0}$, $i = 1, \dots, p_1$ are independent. Thus, from $\sum_i a_{(1,i)I'_o}\rho_{1,i} \neq 0$ for some I'_o , we get for some vector of $\mathbb{C}\mathcal{L}^0$, say $L_{0,1}$,

$$(2.11) \quad \sum_i a_{(1,i)I'_o}\partial\bar{\partial}\rho_{1,i}(L_{0,1}, \bar{L}_{0,1}) \neq 0.$$

Define $L_t = \frac{(1-t)L_{0,1} + t^2L_{1,i}}{c_t}$ (any i) where c_t is a factor which normalizes $|U_t| = 1$. We deform $\mathbb{C}\mathcal{L}^0$ to

$$Q_q = \text{Span}\{L_t, L_{0,2}, \dots, L_{0,q}\}.$$

Combination of (2.10) and (2.11) yields

$$\text{trace}_{Q_q} \left(\sum_i a_{(1,i)I'_o}\partial\bar{\partial}\rho_{1,i} \right) = tc_{I'_o} \quad \text{for } c_{I'_o} \neq 0.$$

Then, using (2.4), we have for the trace of the full $\sigma = \sum_{|I|\geq k} a_I\rho^I$

$$(2.12) \quad \text{trace}_{Q_q}\sigma = tc_{I'_o}\rho^{I'_o} + \sum_{\substack{I' \neq I'_o \\ |I'|=k-1}} c_{I'}\rho^{I'} + t^4O(\rho^{k-2}) + t^2O(\rho^{k-1}) + O(\rho^k) + \partial\bar{\partial}\mathcal{E}|_{Q_q};$$

observing that by (2.5) we have $\partial\bar{\partial}\mathcal{E}|_{Q_q} = t^4O(\rho^{k-1}) + t^2O(\rho^k) + O(\rho^{k+1})$, we see that this term can be neglected. By taking restriction to a suitable region of the plane $\mathbb{R}^r \times \mathbb{R}$ of $(\rho_{j,i}, t)$, all terms in

the right of (2.12) are negligible comparing to the first: thus, again, $\text{trace}_{Q_q}(\partial\bar{\partial}\sigma)$ changes sign.

At last, we have to consider the case when $\sum_{|I|=k} a_I \rho^I$ contains factors $\rho_{j_o,i}$ which start from $j_o > 1$. For fixed h , each group of matrices $\partial\bar{\partial}\rho_{h,i}$, $i = 1, \dots, p_h$, are independent. Thus, for a pair of vectors, say $L_{0,1} \in \mathbb{C}\mathcal{L}^0$ and $L_{j_o-1,1} \in \mathbb{C}\mathcal{L}^{j_o-1}$, and for some $|I'_o| = k - 1$, we have $(\sum_i a_{(j_o,i)I'_o} \partial\bar{\partial}\rho_{j_o,i})(L_{0,1}, \bar{L}_{j_o-1,1}) \neq 0$. But then, under the choice $L := \frac{t^{-1}L_{0,1} + L_{j_o-1,1}}{c_t}$, $t \ll 1$, (for a normalization factor c_t) we have

$$(2.13) \quad \left(\sum_i a_{(j_o,i)I'_o} \partial\bar{\partial}\rho_{j_o,i} \right)(L, \bar{L}) = c_{I'_o} \neq 0.$$

We then complete L by $q - 1$ vectors in $\mathbb{C}\mathcal{L}^0$ to an orthonormal basis of a q -space Q_q thus obtaining

$$(2.14) \quad \text{trace}_{Q_q}(\partial\bar{\partial}\sigma) = t c_{I'_o} \rho^{I'_o} + t \sum_{\substack{I' \neq I'_o \\ |I'|=k-1}} c_{I'} \rho^{I'} + t^2 O(\rho^{k-1}) + O(\rho^k),$$

where $tO(\rho^{k-1})$ comes from differentiation once with respect to L different terms $\rho_{j_o,i}$ in $(k + 1)$ -powers and where we have controlled the term $\partial\bar{\partial}\mathcal{E}$ by $t^2O(\rho^{k-1}) + O(\rho^k) + O(\rho^{k+1})$. Again, we can make negligible in the right of (2.14) the terms which follow the first and conclude that the trace changes sign, a contradiction.

In conclusion, k cannot exist neither odd nor even and therefore σ vanishes of infinite order along M .

□

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