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Poisson problems for semilinear Brinkman systems on Lipschitz domains in $\mathbb{R}^n$

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Abstract. The purpose of this paper is to combine a layer potential analysis with the Schauder fixed point theorem to show the existence of solutions of the Poisson problem for a semilinear Brinkman system on bounded Lipschitz domains in $\mathbb{R}^n$ $(n \geq 2)$ with Dirichlet or Robin boundary conditions and data in L^2-based Sobolev spaces. We also obtain an existence and uniqueness result for the Dirichlet problem for a special semilinear elliptic system, called the Darcy–Forchheimer–Brinkman system.

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Keywords. Semilinear Brinkman system · Lipschitz domain · Poisson problem · Layer potential operators · Sobolev spaces · Fixed point theorem.

1. Introduction

The layer potential methods have a well-known role in the analysis of boundary value problems for the Stokes system, but also of other elliptic boundary value problems (see, e.g., [6,17,25,31,33,40,43,50]). The Dirichlet and Neumann problems for the Laplace equation in Lipschitz domains have been investigated by Dahlberg and Kenig [7]. Fabes et al. [14] used a layer potential method to treat the Neumann problem for the Poisson equation on Lipschitz domains. Lanzani and Méndez [27] shown the existence and uniqueness of the solution to the Poisson problem for the Laplace equation with Robin boundary condition on Lipschitz domains in $\mathbb{R}^n$ $(n \geq 3)$ and with boundary data in Besov spaces, by exploiting a layer potential method. Lanzani and Shen [28] have studied the Laplace equation with Robin boundary conditions in a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ $(n \geq 3)$, by considering the boundary data in $L^p(\partial\Omega)$ spaces, $p \in (1, 2+\varepsilon)$, for some $\varepsilon > 0$. They have exploited a single-layer potential technique to obtain existence and uniqueness results with non-tangential maximal function estimate. The authors obtained similar results for the Poisson problem for the three-dimensional Lamé system with Robin boundary condition. All solutions have been expressed in terms of layer potentials. Mitrea and Mitrea [35] obtained sharp well-posedness results for the Poisson problem for the Laplace equation with mixed boundary conditions on bounded Lipschitz domains. The authors generalized previous results obtained in [14,18]. The Robin problem for the Laplace–Beltrami operator on Lipschitz domains in compact Riemannian manifolds has been studied by Mitrea and Taylor [39, Theorem 4.2]. Fabes et al. [13] developed a layer potential method in order to show the solvability of the Dirichlet problem for the Stokes system on Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, with L^2-boundary data. Dahlberg et al. [8] studied the Dirichlet and Neumann problems for the Lamé system in Lipschitz domains in $\mathbb{R}^n$ $(n \geq 3)$. Russo and Tartaglione [44] studied the Robin problem associated with the Stokes system in a bounded or exterior Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, by using a double-layer potential approach (see also [4,43,46]). Medková studied in [32, Theorems 4.3, 5.6] the Robin

The authors dedicate their work to Professor Miloslav Feistauer on the occasion of his 70th birthday.

problem for the homogeneous Stokes system in a bounded domain $G \subseteq \mathbb{R}^3$ with connected boundary ∂G of class $C^{1,\alpha}$, $\alpha \in (0,1)$, and the boundary data in $C^\alpha(\partial G, \mathbb{R}^3)$, or in $L^s(\partial G, \mathbb{R}^3)$, $s \in (1,\infty)$, in terms of a single-layer potential, whose unknown density is the solution of an integral equation of the second kind. Such a solution has been obtained explicitly in terms of a Neumann series. Mitrea and Wright [40] exploited layer potential methods to develop a powerful analysis of the main boundary value problems for the Stokes system in arbitrary Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$ (see also [29]). Mitrea et al. [36] defined the Stokes operator on Lipschitz domains in $\mathbb{R}^n$ in the case of Neumann boundary conditions. By using a single-layer potential technique, Mitrea and Taylor [38] studied the L^2-Dirichlet problem for the Stokes system in arbitrary Lipschitz domains on a smooth compact Riemannian manifold and extended the results obtained in [13] on Lipschitz domains in Euclidean setting. In addition, Dindoš and Mitrea [12] used a layer potential approach to treat the Poisson problems for the Stokes and Navier–Stokes systems on C^1 and Lipschitz domains in smooth compact Riemannian manifolds with data in Sobolev or Besov spaces. The authors in [23] constructed pseudodifferential Brinkman operators as operators with variable coefficients that extend the differential Brinkman operator from the Euclidean setting to compact Riemannian manifolds. They shown existence and uniqueness results for related transmission problems on C^1 domains of arbitrary dimension, or on Lipschitz domains of dimension ≤ 3, on a compact Riemannian manifold. In [24], these results were extended to the case of Lipschitz domains on compact Riemannian manifolds of arbitrary dimension, with data in L^2-based Sobolev spaces.

Existence results for boundary value problems with nonlinear boundary conditions are known, and we mention the work of Klingelhöfer [20,21], the contributions of Begehr and Hsiao [2], and Begehr and Hile [1]. Nonlinear boundary value problems for elliptic systems have been also studied in [9,26]. The authors in [22] combined a layer potential analysis with a fixed point theorem to show the existence result for a nonlinear Neumann-transmission problem for the Stokes and Brinkman systems on Euclidean Lipschitz domains with boundary data in L^p spaces, Sobolev spaces, and also in Besov spaces. A nonlinear Neumann condition has been imposed on an external Lipschitz boundary together with transmission conditions on the interface between two adjacent Lipschitz domains. Dindoš [10] obtained existence and uniqueness results for semilinear elliptic problems on Lipschitz domains in Riemannian manifolds. The author extended results for L^p Dirichlet and Neumann boundary value problems associated with linear second-order elliptic equations on Lipschitz domains to a class of semilinear elliptic problems. Dindoš and Mitrea [11] combined various results from the linear theory for the Poisson problem associated with the Laplace operator in the framework of Sobolev–Besov spaces on Lipschitz domains, which have been obtained in [14,18,37], with a fixed point theorem, and developed a sharp theory for semilinear Poisson problems of the type $\triangle u - N(x,u) = F(x)$ on Lipschitz domains in compact Riemannian manifolds, equipped with Dirichlet and Neumann boundary conditions. Fitzpatrick and Pejsachowicz [15] developed an additive, integer-valued degree theory for a class of quasilinear Fredholm mappings between real Banach spaces of the form $f(x) = L(x)x + C(x)$, where C is a compact operator and, for each x, $L(x)$ is a Fredholm operator of index zero. Such a class does not possess a homotopy-invariant degree. The authors introduced a homotopy invariant of paths of linear Fredholm operators with invertible end- points, called the parity, which provides a complete description of the possible changes in sign of the degree. Then the authors proved existence, multiplicity and bifurcation results. Applications have been given for fully nonlinear elliptic operators with general nonlinear elliptic boundary conditions when the coefficients are sufficiently smooth.

The purpose of this paper was to use a layer potential analysis and the Schauder fixed point theorem in order to show the existence of solutions of a Poisson problem for a semilinear Brinkman system on a bounded Lipschitz domain $\mathfrak{D} \subseteq \mathbb{R}^n$ $(n \geq 2)$ with Dirichlet or Robin boundary condition and data in Sobolev spaces. The nonlinear term in the semilinear Brinkman system is written in terms of an essentially bounded Carathéodory function $\mathcal{P}$ from $\mathfrak{D} \times \mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}^n \otimes \mathbb{R}^n$, which satisfies a nonnegativity condition [see (4.36)]. First, we show the well-posedness of the corresponding linear Poisson problem, i.e., the existence and uniqueness of the solution in the aforementioned spaces (see Theorems 4.1, 5.2), together

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with some useful regularity estimates (see Lemmas 4.2, 5.3). Then, by using the well-posedness result from the linear case and the Schauder fixed point theorem, we show the desired existence result for the semilinear Poisson problem (see Theorems 4.4 and 5.4). Theorem 6.1 provides an existence and uniqueness result for the Dirichlet problem associated with the semilinear Darcy–Forchheimer–Brinkman system (6.1) with small boundary data.

2. Preliminaries

Consider a bounded Lipschitz domain[1] $\mathfrak{D} := \mathfrak{D}_- \subseteq \mathbb{R}^n$ $(n \geq 2)$ with boundary Γ, and let $\mathfrak{D}_+ := \mathbb{R}^n \setminus \overline{\mathfrak{D}}$. Also, let ν be the outward unit normal to Γ. For fixed $\kappa = \kappa(\Gamma) > 1$, sufficiently large, define the non-tangential maximal operator (see, e.g., [40])

$$\mathcal{N}(u)(\mathbf{x}) := \mathcal{N}_\kappa(u)(\mathbf{x}) := \sup\left\{|u(\mathbf{y})| : \mathbf{y} \in \gamma_\pm(\mathbf{x})\right\}, \quad \mathbf{x} \in \Gamma, \tag{2.1}$$

for arbitrary $u : \mathfrak{D}_\pm \to \mathbb{R}$, where $\gamma_\pm(\mathbf{x}) := \{\mathbf{y} \in \mathfrak{D}_\pm : \operatorname{dist}(\mathbf{x}, \mathbf{y}) < \kappa \operatorname{dist}(\mathbf{y}, \Gamma)\}$, $\mathbf{x} \in \Gamma$, are non-tangential approach regions lying in $\mathfrak{D}_+$ and $\mathfrak{D}_-$, respectively. Also, consider the non-tangential boundary trace operators $\mathrm{Tr}^\pm$ on Γ, as[2]

$$(\mathrm{Tr}^\pm u)(\mathbf{x}) := \lim_{\gamma_\pm(\mathbf{x}) \ni \mathbf{y} \to \mathbf{x}} u(\mathbf{y}), \quad \text{a.e. } \mathbf{x} \in \Gamma, \tag{2.2}$$

$$\mathrm{Tr}^\pm : C^\infty(\overline{\mathfrak{D}}_\pm) \to C^0(\Gamma), \quad \mathrm{Tr}^\pm u = u|_\Gamma. \tag{2.3}$$

For $p \in [1, \infty)$, $L^p(\mathbb{R}^n)$ denotes the Lebesgue space of (equivalence classes of) measurable, p-th power integrable functions on $\mathbb{R}^n$, and $L^\infty(\mathbb{R}^n)$ consists of (equivalence classes of) essentially bounded measurable functions on $\mathbb{R}^n$. For $p \in (1, \infty)$ and $s \in \mathbb{R}$, the Bessel potential space $L^p_s(\mathbb{R}^n)$ is defined by

$$L^p_s(\mathbb{R}^n) := \left\{(I - \triangle)^{-\frac{s}{2}} f : f \in L^p(\mathbb{R}^n)\right\} = \left\{\mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F} f : f \in L^p(\mathbb{R}^n)\right\}, \tag{2.4}$$

with the norm $\|f\|_{L^p_s(\mathbb{R}^n)} := \|(I - \triangle)^{-\frac{s}{2}} f\|_{L^p(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F} f\|_{L^p(\mathbb{R}^n)}$, where $\mathcal{F}$ is the Fourier transform defined on the space of tempered distributions to itself, and $\mathcal{F}^{-1}$ is its inverse. Also, $L^p_s(\mathbb{R}^n, \mathbb{R}^n) := \{f = (f_1, \ldots, f_n) : f_j \in L^p_s(\mathbb{R}^n),\ j = 1, \ldots, n\}$. In addition, $L^p_s(\mathfrak{D})$ denotes the Sobolev (or Bessel potential) space in $\mathfrak{D}$, defined by

$$L^p_s(\mathfrak{D}) := \{f \in \mathcal{D}'(\mathfrak{D}) : \exists\, g \in L^p_s(\mathbb{R}^n) \text{ such that } g|_{\mathfrak{D}} = f\}, \tag{2.5}$$

with the norm $\|f\|_{L^p_s(\mathfrak{D})} := \inf\left\{\|g\|_{L^p_s(\mathbb{R}^n)} : g \in L^p_s(\mathbb{R}^n),\ g|_{\mathfrak{D}} = f\right\}$, where $\mathcal{D}'(\mathfrak{D})$ is the space of distributions, i.e., the dual of $C^\infty_{\mathrm{comp}}(\mathfrak{D})$ equipped with the inductive limit topology.

For $s \in \mathbb{R}$ and $p \in (1, \infty)$, define $L^p_{s;0}(\mathfrak{D})$ as the space of all distributions $f \in L^p_s(\mathbb{R}^n)$ with support in $\overline{\mathfrak{D}}$ and the norm inherited from $L^p_s(\mathbb{R}^n)$ (see [18, Definition 2.6]). Note that the space $C^\infty_{\mathrm{comp}}(\mathfrak{D})$ is dense in $L^p_{s;0}(\mathfrak{D})$ for all $s \in \mathbb{R}$ and $p \in (1, \infty)$ (see [18, Remark 2.7], [37, p. 23]). For $p, p' \in (1, \infty)$, with $\frac{1}{p} + \frac{1}{p'} = 1$, and for $s > 0$, $L^p_{-s}(\mathfrak{D})$ can be defined as the space of linear functionals on $C^\infty_{\mathrm{comp}}(\mathfrak{D})$ with finite norm

$$\|f\|_{L^p_{-s}(\mathfrak{D})} := \sup\left\{|\langle f, \varphi\rangle| : \varphi \in C^\infty_{\mathrm{comp}}(\mathfrak{D}) \ \text{ with } \ \|\tilde{\varphi}\|_{L^{p'}_s(\mathbb{R}^n)} \leq 1\right\}, \tag{2.6}$$

where tilde denotes the extension by zero outside $\mathfrak{D}$ (see [18, Definition 2.8], [37, (4.13)]). For $s \in \mathbb{R}$ and $p \in (1, \infty)$, $C^\infty(\overline{\mathfrak{D}})$ is dense in $L^p_s(\mathfrak{D})$, and (see [18, Proposition 2.9], [37, (4.14)], [14, (1.9)])

$$(L^p_s(\mathfrak{D}))' = L^{p'}_{-s;0}(\mathfrak{D}), \quad L^p_{-s}(\mathfrak{D}) = \left(L^{p'}_{s;0}(\mathfrak{D})\right)', \tag{2.7}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. The spaces $L^p_s(\mathfrak{D}, \mathbb{R}^n)$, $L^p_{s;0}(\mathfrak{D}, \mathbb{R}^n)$ can be defined similarly (for a more detailed presentation of these spaces, we refer the reader to [18,19,37,40,49]).

[1] The connected open subset $\mathfrak{D} \subseteq \mathbb{R}^n$ is a *Lipschitz domain* if its boundary is locally the graph of a Lipschitz function.
[2] The superscripts $-$ and $+$ apply to non-tangential limits evaluated from $\mathfrak{D}_-$ and $\mathfrak{D}_+$, respectively.

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For $p \in (1,\infty)$ and $s \in [0,1]$, the boundary Sobolev space $L^p_s(\Gamma)$ can be defined by using the space $L^p_s(\mathbb{R}^{n-1})$, a partition of unity and pullback, and $L^p_{-s}(\Gamma)$ is the dual of $L^p_s(\Gamma)$.

Next, the notation $\langle\cdot,\cdot\rangle$ is used for the inner product in $\mathbb{R}^n$. For a subset $X \subseteq \mathbb{R}^n$, the notation $\langle\cdot,\cdot\rangle_X := {}_{(L^p_s(X))'}\langle\cdot,\cdot\rangle_{L^p_s(X)}$ stands for the pairing between the space $L^p_s(X)$ and its dual $(L^p_s(X))'$.

We now refer to the case $p=2$. Then, for $n \geq 2$ and $s \in (0,1)$, we define the space

$$L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathcal{L}_0) := \left\{(\mathbf{u},\pi) \in L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D}) : \mathcal{L}_0(\mathbf{u},\pi) = \mathbf{0}, \operatorname{div}\mathbf{u} = 0 \text{ in } \mathfrak{D}\right\}, \tag{2.8}$$

where $\mathcal{L}_0(\mathbf{u},\pi) := \Delta\mathbf{u} - \nabla\pi$, and $\|(\mathbf{u},\pi)\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathcal{L}_0)} := \|\mathbf{u}\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)} + \|\pi\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})}$.

Let us mention the following trace lemma for bounded Lipschitz domains (see [18, Proposition 3.1], [40, Theorem 2.5.2], [6], [30, Theorem 3.38], [34, Lemma 2.6]):

Lemma 2.1. *Let $\mathfrak{D} \subseteq \mathbb{R}^n (n \geq 2)$ be a bounded Lipschitz domain with boundary Γ. Let $s \in (0,1)$. Then there exists a linear and bounded operator $\operatorname{Tr}^- : L^2_{s+\frac{1}{2}}(\mathfrak{D}) \to L^2_s(\Gamma)$ whose action is compatible to that of the restriction to the boundary in (2.3). This operator is onto and has a linear and bounded right inverse $\mathcal{Z}^- : L^2_s(\Gamma) \to L^2_{s+\frac{1}{2}}(\mathfrak{D})$. In addition, the space $L^2_{s+\frac{1}{2};0}(\mathfrak{D})$ is the kernel of the trace operator $\operatorname{Tr}^- : L^2_{s+\frac{1}{2}}(\mathfrak{D}) \to L^2_s(\Gamma)$. The following operator is also well defined, linear and bounded:*

$$\operatorname{Tr}^- : L^2_r(\mathfrak{D}) \to L^2_1(\Gamma), \qquad r > \frac{3}{2}. \tag{2.9}$$

A similar result holds for the trace operators defined on Sobolev spaces of vector and tensor fields. For brevity, we use the same notation for them as in Lemma 2.1, but their meaning will be understood from the context.

2.1. The conormal derivative for the Stokes system on Sobolev spaces

Let $s \in [0,1]$. Let $d\sigma$ be the surface measure on Γ. Let ν denote the outward unit normal, which is defined a.e. with respect to $d\sigma$ on Γ. Note that $\nu \in L^\infty(\Gamma,\mathbb{R}^n)$.

The result below defines the *conormal derivative* for the Stokes system on Sobolev spaces as it has been introduced by Mitrea and Wright in [40, Theorem 10.4.1] (see also [36, Proposition 3.6], [23, Lemma 2.2] for the extension to the Brinkman operators in Lipschitz domains on compact Riemannian manifolds, and [34, Definition 3.1]):

Lemma 2.2. *Let $\mathfrak{D} \subseteq \mathbb{R}^n$ $(n \geq 2)$ be a bounded Lipschitz domain with boundary Γ. Then for any $s \in (0,1)$ the* conormal derivative *operator*[3] $\partial^-_\nu : L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathcal{L}_0) \to L^2_{s-1}(\Gamma,\mathbb{R}^n)$*, given by*

$$\langle\partial^-_\nu(\mathbf{u},\pi),\Psi\rangle_\Gamma := 2\langle\mathbb{E}(\mathbf{u}),\mathbb{E}(\mathcal{Z}^-\Psi)\rangle_{\mathfrak{D}} - \langle\pi,\operatorname{div}(\mathcal{Z}^-\Psi)\rangle_{\mathfrak{D}},\ \forall\ \Psi \in L^2_{1-s}(\Gamma,\mathbb{R}^n) \tag{2.10}$$

is well defined, linear and bounded, where $\mathbb{E}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right)$ and $(\nabla\mathbf{u})^\top$ is the transpose of $\nabla\mathbf{u} = \left(\frac{\partial u_j}{\partial x_k}\right)_{j,k=1,\dots,n}$. In addition, for all $(\mathbf{u},\pi) \in L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathcal{L}_0)$, one has the Green formula

$$2\langle E_{jk}(\mathbf{u}),E_{jk}(\mathbf{w})\rangle_{\mathfrak{D}} = \langle\pi,\operatorname{div}\mathbf{w}\rangle_{\mathfrak{D}} + \langle\partial^-_\nu(\mathbf{u},\pi),\operatorname{Tr}^-\mathbf{w}\rangle_\Gamma,\ \forall\ \mathbf{w} \in L^2_{\frac{3}{2}-s}(\mathfrak{D},\mathbb{R}^n). \tag{2.11}$$

[3] Hereafter one uses the Einstein repeated-index summation rule. Also $E_{jk}(\mathbf{u})$ are the components of $\mathbb{E}(\mathbf{u})$.

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2.2. Generalized Brinkman system and the corresponding conormal derivative

Let $\mathcal{P} \in L^\infty(\mathfrak{D}, \mathbb{R}^n \otimes \mathbb{R}^n)$ be a matrix-valued function with the entries $\mathcal{P}_{ij} \in L^\infty(\mathfrak{D})$, $i, j = 1, \ldots, n$, such that

$$\langle \mathcal{P}(\mathbf{x})\xi, \xi \rangle := \sum_{i,j=1}^{n} \mathcal{P}_{ij}(\mathbf{x})\xi_i \xi_j \geq 0, \quad \forall\ \xi \in \mathbb{R}^n \tag{2.12}$$

for almost all $\mathbf{x} \in \mathfrak{D}$. The condition (2.12) implies that

$$\langle \mathcal{P}\mathbf{v}, \mathbf{v} \rangle_{\mathfrak{D}} \geq 0, \quad \forall\ \mathbf{v} \in L^2(\mathfrak{D}, \mathbb{R}^n). \tag{2.13}$$

In the sequel, we use the same notation for the matrix value function $\mathcal{P}$ and the corresponding multiplication operator $\mathcal{M}_{\mathcal{P}} : L^2(\mathfrak{D}, \mathbb{R}^n) \to L^2(\mathfrak{D}, \mathbb{R}^n)$, $\mathcal{M}_{\mathcal{P}}(\mathbf{v}) = \mathcal{P}\mathbf{v}$. Then the *generalized Brinkman* operator, i.e., the following L^∞-perturbation of the Stokes operator[4]

$$\mathcal{B}_{\mathcal{P}} := \begin{pmatrix} -(\triangle - \mathcal{P}) & \nabla \\ \mathrm{div} & 0 \end{pmatrix} : L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D}) \to L^2_{s-\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D}) \tag{2.14}$$

is well defined, linear and bounded, for any $s \in (0, 1)$.

Let us now mention the significance of the conormal derivative

$$\mathrm{Tr}^-\left(-\pi\mathbb{I} + 2\mathbb{E}(\mathbf{u})\right)\nu \ \ \text{a.e. on}\ \ \Gamma \tag{2.15}$$

when the following Sobolev space is involved:

$$\begin{aligned} \mathfrak{B}^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathcal{L}_{\mathcal{P}}) := \big\{ (\mathbf{u}, \pi, \mathbf{f}, g) \in L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D}) \times L^2_{s-\frac{3}{2};0}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D}) : \\ \mathcal{L}_{\mathcal{P}}(\mathbf{u}, \pi) = \mathbf{f}|_{\mathfrak{D}} \ \text{and}\ \ \mathrm{div}\ \mathbf{u} = g \ \text{in}\ \mathfrak{D} \big\}, \end{aligned} \tag{2.16}$$

where

$$\mathcal{L}_{\mathcal{P}}(\mathbf{u}, \pi) := (\triangle - \mathcal{P})\mathbf{u} - \nabla\pi. \tag{2.17}$$

Then we have the following result (see also Lemma 2.2 for the Stokes system).

Lemma 2.3. *Let $\mathfrak{D}$ be a bounded Lipschitz domain in $\mathbb{R}^n$ $(n \geq 2)$ with boundary Γ. Let $s \in (0, 1)$. Then the operator*

$$\begin{aligned} &\partial^-_{\nu;\mathcal{P}} : \mathfrak{B}^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathcal{L}_{\mathcal{P}}) \to L^2_{s-1}(\Gamma, \mathbb{R}^n), \\ &\mathfrak{B}^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathcal{L}_{\mathcal{P}}) \ni (\mathbf{u}, \pi, \mathbf{f}, g) \longmapsto \partial^-_{\nu;\mathcal{P}}(\mathbf{u}, \pi)_{\mathbf{f},g} \in L^2_{s-1}(\Gamma, \mathbb{R}^n), \end{aligned} \tag{2.18}$$

given by

$$\begin{aligned} \left\langle \partial^-_{\nu;\mathcal{P}}(\mathbf{u}, \pi)_{\mathbf{f},g}, \Phi \right\rangle_{\Gamma} &:= 2\langle \mathbb{E}(\mathbf{u}), \mathbb{E}(\mathcal{Z}^-\Phi) \rangle_{\mathfrak{D}} - \langle \pi, \mathrm{div}(\mathcal{Z}^-\Phi) \rangle_{\mathfrak{D}} + \langle \nabla g, \mathcal{Z}^-\Phi \rangle_{\mathfrak{D}} \\ &\quad + \langle \mathbf{f}, \mathcal{Z}^-\Phi \rangle_{\mathfrak{D}} + \langle \mathcal{P}\mathbf{u}, \mathcal{Z}^-\Phi \rangle_{\mathfrak{D}}, \ \forall\ \Phi \in L^2_{1-s}(\Gamma, \mathbb{R}^n) \end{aligned} \tag{2.19}$$

is well defined and bounded. In addition, for any $(\mathbf{u}, \pi, \mathbf{f}, g) \in \mathfrak{B}^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathcal{L}_{\mathcal{P}})$, one has the Green formula

$$\begin{aligned} \left\langle \partial^-_{\nu;\mathcal{P}}(\mathbf{u}, \pi)_{\mathbf{f},g}, \mathrm{Tr}^-\ \mathbf{w} \right\rangle_{\Gamma} &= 2\langle \mathbb{E}(\mathbf{u}), \mathbb{E}(\mathbf{w}) \rangle_{\mathfrak{D}} - \langle \pi, \mathrm{div}(\mathbf{w}) \rangle_{\mathfrak{D}} + \langle \nabla g, \mathbf{w} \rangle_{\mathfrak{D}} \\ &\quad + \langle \mathbf{f}, \mathbf{w} \rangle_{\mathfrak{D}} + \langle \mathcal{P}\mathbf{u}, \mathbf{w} \rangle_{\mathfrak{D}}, \ \forall\ \mathbf{w} \in L^2_{\frac{3}{2}-s}(\mathfrak{D}, \mathbb{R}^n). \end{aligned} \tag{2.20}$$

Proof. Since $\mathcal{P} \in L^\infty(\mathfrak{D}, \mathbb{R}^n \otimes \mathbb{R}^n)$ the last duality pairing in the right-hand side of (2.19) is well defined. Also, by [36, (3.11), (3.13)], $L^2_{\frac{1}{2}-s}(\mathfrak{D}) = L^2_{\frac{1}{2}-s;0}(\mathfrak{D})$ and, by duality, $L^2_{s-\frac{1}{2}}(\mathfrak{D}) = L^2_{s-\frac{1}{2};0}(\mathfrak{D})$. In addition, the property [36, (3.14)] implies that $\nabla g \in L^2_{s-\frac{3}{2};0}(\mathfrak{D}, \mathbb{R}^n) = \left(L^2_{\frac{3}{2}-s}(\mathfrak{D}, \mathbb{R}^n)\right)'$, and hence, the third duality pairing is well defined. All other duality pairings are also well defined. Hence, the operator $\partial^-_{\nu;\mathcal{P}}$ given

[4] In the special case $\mathcal{P} = \lambda\mathbb{I}$, $\lambda > 0$, (2.14) reduces to the well-known Brinkman operator that describes the flows of viscous incompressible fluids in porous media (see, e.g., [22, 25] for further details).

by (2.18), (2.19) is well defined. The boundedness of $\partial^-_{\nu;\mathcal{P}}$ and the formula (2.20) can be obtained with similar arguments as for [40, Proposition 10.2.1, Theorem 10.4.1]. Also, let us mention the important property that the definition of $\partial^-_{\nu;\mathcal{P}}$ is independent of the choice of a bounded right inverse $\mathcal{Z}^-$ of the trace operator Tr^-. Such a property can be obtained with arguments similar to those in the proof of [34, Theorem 3.2]. We omit these arguments for the sake of brevity. □

Let us now consider the Sobolev space

$$\mathfrak{L}^2_{s+\frac{1}{2}}(\mathfrak{D},\mathcal{L}_{\mathcal{P}}) := \big\{(\mathbf{u},\pi,\mathbf{f}) : \mathbf{u}\in L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n),\ \pi\in L^2_{s-\frac{1}{2}}(\mathfrak{D}),\ \mathbf{f}\in L^2_{s-\frac{3}{2};0}(\mathfrak{D},\mathbb{R}^n) \\ \text{such that } \mathcal{L}_{\mathcal{P}}(\mathbf{u},\pi)=\mathbf{f}|_{\mathfrak{D}} \text{ and } \operatorname{div}\mathbf{u}=0 \text{ in } \mathfrak{D}\big\}. \tag{2.21}$$

The following useful result is a direct consequence of Lemma 2.3 in the special case $g=0$.

Corollary 2.4. *Let $\mathfrak{D}$ be a bounded Lipschitz domain in $\mathbb{R}^n$ $(n\geq 2)$ with boundary Γ. Let $s\in(0,1)$. Then the* conormal derivative *operator*

$$\begin{aligned}&\partial^-_{\nu;\mathcal{P}} : \mathfrak{L}^2_{s+\frac{1}{2}}(\mathfrak{D},\mathcal{L}_{\mathcal{P}})\to L^2_{s-1}(\Gamma,\mathbb{R}^n),\\ &\mathfrak{L}^2_{s+\frac{1}{2}}(\mathfrak{D},\mathcal{L}_{\mathcal{P}})\ni(\mathbf{u},\pi,\mathbf{f})\longmapsto \partial^-_{\nu;\mathcal{P}}(\mathbf{u},\pi)_{\mathbf{f}}\in L^2_{s-1}(\Gamma,\mathbb{R}^n),\end{aligned} \tag{2.22}$$

given by

$$\left\langle \partial^-_{\nu;\mathcal{P}}(\mathbf{u},\pi)_{\mathbf{f}},\Phi\right\rangle_{\Gamma} := 2\langle\mathbb{E}(\mathbf{u}),\mathbb{E}(\mathcal{Z}^-\Phi)_{\mathfrak{D}} - \langle\pi,\operatorname{div}(\mathcal{Z}^-\Phi)\rangle_{\mathfrak{D}} + \langle\mathcal{P}\mathbf{u},\mathcal{Z}^-\Phi\rangle_{\mathfrak{D}} + \langle\mathbf{f},\mathcal{Z}^-\Phi\rangle_{\mathfrak{D}}, \tag{2.23}$$

for any $\Phi\in L^2_{1-s}(\Gamma,\mathbb{R}^n)$, is well defined and bounded. Also, for all $(\mathbf{u},\pi,\mathbf{f})\in\mathfrak{L}^2_{s+\frac{1}{2}}(\mathfrak{D},\mathcal{L}_{\mathcal{P}})$ and $\mathbf{w}\in L^2_{\frac{3}{2}-s}(\mathfrak{D},\mathbb{R}^n)$, one has the Green formula:

$$\langle\partial^-_{\nu;\mathcal{P}}(\mathbf{u},\pi)_{\mathbf{f}},\mathrm{Tr}^-\ \mathbf{w}\rangle_{\Gamma} = 2\langle\mathbb{E}(\mathbf{u}),\mathbb{E}(\mathbf{w})\rangle_{\mathfrak{D}} - \langle\pi,\operatorname{div}\ \mathbf{w}\rangle_{\mathfrak{D}} + \langle\mathbf{f},\mathbf{w}\rangle_{\mathfrak{D}} + \langle\mathcal{P}\mathbf{u},\mathbf{w}\rangle_{\mathfrak{D}}. \tag{2.24}$$

Remark 2.5. (a) For $s\in(0,1)$, the conormal derivative $\partial^+_{\nu;\mathcal{P}}$, corresponding to $\mathfrak{D}_+ := \mathbb{R}^n\setminus\overline{\mathfrak{D}}$, can be defined by a variational formula similar to (2.19), by using a linear and continuous right inverse $\mathcal{Z}^+ : L^2_s(\Gamma,\mathbb{R}^n)\to L^2_{s+\frac{1}{2}}(\mathbb{R}^n,\mathbb{R}^n)$ of the trace operator $\mathrm{Tr} : L^2_{s+\frac{1}{2}}(\mathbb{R}^n,\mathbb{R}^n)\to L^2_s(\Gamma,\mathbb{R}^n)$ such that the supports of the images of $\mathcal{Z}^+$ are contained in a ball which contains $\overline{\mathfrak{D}}$ (for also [6,34]).

(b) Next, for $\mathcal{P}=0$, we use the short notation $\partial^-_{\nu}(\mathbf{u},\pi)_{\mathbf{f},g}$, and, for $\mathcal{P}=0$, $\mathbf{f}=\mathbf{0}$ and $g=0$, the notation $\partial^-_{\nu}(\mathbf{u},\pi)$.

3. Layer potential operators for the Stokes system

Let us denote by $\mathcal{G}(\cdot,\cdot)\in\mathcal{D}'(\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n\otimes\mathbb{R}^n)$ and $\Pi(\cdot,\cdot)\in\mathcal{D}'(\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$ the fundamental tensor and the fundamental vector, respectively, for the Stokes system in $\mathbb{R}^n$, $n\geq 2$. Therefore,[5]

$$\triangle_{\mathbf{x}}\mathcal{G}(\mathbf{x},\mathbf{y}) - \nabla_{\mathbf{x}}\Pi(\mathbf{x},\mathbf{y}) = -\delta_{\mathbf{y}}(\mathbf{x})\mathbb{I},\quad \operatorname{div}_{\mathbf{x}}\mathcal{G}(\mathbf{x},\mathbf{y})=0, \tag{3.1}$$

where $\mathbb{I}$ is the identity matrix and $\delta_{\mathbf{y}}$ is the Dirac distribution with mass at $\mathbf{y}$. Note that (see, e.g., [25, p. 38, 39]):

$$\begin{aligned}&\mathcal{G}_{jk}(\mathbf{x}) = \frac{1}{2\omega_n}\left\{\frac{\delta_{jk}}{(n-2)|\mathbf{x}|^{n-2}} + \frac{x_jx_k}{|\mathbf{x}|^n}\right\},\quad \Pi_j(\mathbf{x}) = \frac{1}{\omega_n}\frac{x_j}{|\mathbf{x}|^n},\quad n\geq 3\\ &\mathcal{G}_{jk}(\mathbf{x}) = \frac{1}{4\pi}\left(\frac{x_jx_k}{|\mathbf{x}|^2} - \delta_{jk}\left(\ln|\mathbf{x}| + \ln\alpha_0\right)\right),\quad \Pi_j(\mathbf{x}) = \frac{1}{2\pi}\frac{x_j}{|\mathbf{x}|^2},\ n=2,\end{aligned} \tag{3.2}$$

[5] The subscript $\mathbf{x}$ added to an operator shows that the operator acts with respect to $\mathbf{x}$.

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where ω_n is the area of the unit sphere in $\mathbb{R}^n$ and $\alpha_0 > 0$ is a constant (for details about the choice of such a constant, we refer the reader to [22, Appendix] and [48, (3.4)]). The components of the stress and pressure tensors $\mathbf{S}$ and Λ are given by (see [25, p. 38, 39, 132]):

$$S_{jk\ell}(\mathbf{x}) = -\Pi_j(\mathbf{x})\delta_{k\ell} + \frac{\partial \mathcal{G}_{jk}(\mathbf{x})}{\partial x_\ell} + \frac{\partial \mathcal{G}_{\ell k}(\mathbf{x})}{\partial x_j} = -\frac{n}{\omega_n}\frac{x_j x_k x_\ell}{|\mathbf{x}|^{n+2}},$$

$$\Lambda_{jk}(\mathbf{x},\mathbf{y}) = -\frac{2}{\omega_n}\left(-\frac{\delta_{jk}}{|\mathbf{x}|^n} + n\frac{x_j x_k}{|\mathbf{x}|^{n+2}}\right), \tag{3.3}$$

$$\triangle_{\mathbf{x}} S_{jk\ell}(\mathbf{y},\mathbf{x}) - \frac{\partial \Lambda_{j\ell}(\mathbf{x},\mathbf{y})}{\partial x_k} = 0, \quad \frac{\partial S_{jk\ell}(\mathbf{y},\mathbf{x})}{\partial x_k} = 0 \ \text{ for } \mathbf{x} \neq \mathbf{y}. \tag{3.4}$$

3.1. The single- and double-layer potential operators

We now assume that $\mathfrak{D} := \mathfrak{D}_- \subseteq \mathbb{R}^n$ $(n \geq 2)$ is a bounded Lipschitz domain with connected boundary Γ. Let $\mathfrak{D}_+ := \mathbb{R}^n \setminus \overline{\mathfrak{D}}$. Let $r \in [0,1]$. If $\mathbf{g} \in L^2_{r-1}(\Gamma, \mathbb{R}^n)$, the single-layer potential for the Stokes system $\mathbf{V}_\Gamma \mathbf{g}$ and the corresponding pressure potential $\mathcal{Q}^s_\Gamma \mathbf{g}$ are given by

$$\left(\mathbf{V}_\Gamma \mathbf{g}\right)(\mathbf{x}) := \left\langle \mathcal{G}(\mathbf{x},\cdot), \mathbf{g}\right\rangle_\Gamma, \quad \left(\mathcal{Q}^s_\Gamma \mathbf{g}\right)(\mathbf{x}) := \left\langle \Pi(\mathbf{x},\cdot), \mathbf{g}\right\rangle_\Gamma, \ \mathbf{x} \in \mathbb{R}^n \setminus \Gamma. \tag{3.5}$$

Let ν_ℓ, $\ell = 1, \ldots, n$, be the components of the outward unit normal ν to Γ. Let $\mathbf{h} \in L^2_r(\Gamma, \mathbb{R}^n)$. Then the double-layer potential $\mathbf{W}_\Gamma \mathbf{h}$ and the corresponding pressure potential $\mathcal{Q}^d_\Gamma \mathbf{h}$ are given by

$$\left(\mathbf{W}_\Gamma \mathbf{h}\right)_k(\mathbf{x}) := \int_\Gamma S_{jk\ell}(\mathbf{y},\mathbf{x})\nu_\ell(\mathbf{y})h_j(\mathbf{y})d\sigma(\mathbf{y}), \ \left(\mathcal{Q}^d_\Gamma \mathbf{h}\right)(\mathbf{x}) := \int_\Gamma \Lambda_{j\ell}(\mathbf{x},\mathbf{y})\nu_\ell(\mathbf{y})h_j(\mathbf{y})d\sigma(\mathbf{y}), \ \mathbf{x} \in \mathbb{R}^n \setminus \Gamma. \tag{3.6}$$

In addition, the (principal value) boundary version of $\mathbf{W}_\Gamma \mathbf{h}$ is given for a.e. $\mathbf{x} \in \Gamma$ by

$$(\mathbf{K}_\Gamma \mathbf{h})_k(\mathbf{x}) := \text{p.v.} \int_\Gamma S_{jk\ell}(\mathbf{y},\mathbf{x})\nu_\ell(\mathbf{y})h_j(\mathbf{y})d\sigma(\mathbf{y}), \tag{3.7}$$

where the notation p.v. means the principal value of a singular integral operator.

By (3.1) and (3.4), the pairs $(\mathbf{V}_\Gamma \mathbf{g}, \mathcal{Q}^s_\Gamma \mathbf{g})$ and $(\mathbf{W}_\Gamma \mathbf{h}, \mathcal{Q}^d_\Gamma \mathbf{h})$ satisfy the Stokes system in $\mathbb{R}^n \setminus \Gamma$.

As usual, denote by $\partial^\pm_\nu(\mathbf{V}_\Gamma \mathbf{g}, \mathcal{Q}^s_\Gamma \mathbf{g})$ the conormal derivatives of the layer potentials $\mathbf{V}_\Gamma \mathbf{g}$ and $\mathcal{Q}^s_\Gamma \mathbf{g}$, with a similar interpretation for $\partial^\pm_\nu(\mathbf{W}_\Gamma \mathbf{h}, \mathcal{Q}^d_\Gamma \mathbf{h})$.

The main properties of layer potentials for the Stokes system are given below (see [13], [40, Proposition 10.5.2, Theorem 10.5.3]):

Lemma 3.1. *Let $\mathfrak{D} := \mathfrak{D}_- \subseteq \mathbb{R}^n$ $(n \geq 2)$ be a bounded Lipschitz domain with connected boundary Γ, and let $\mathfrak{D}_+ := \mathbb{R}^n \setminus \overline{\mathfrak{D}}$. Let $s \in [0,1]$. Then for all $\mathbf{h} \in L^2_s(\Gamma, \mathbb{R}^n)$ and $\mathbf{g} \in L^2_{s-1}(\Gamma, \mathbb{R}^n)$, the following relations hold a.e. on Γ:*

$$\mathrm{Tr}^+\left(\mathbf{V}_\Gamma \mathbf{g}\right) = \mathrm{Tr}^-\left(\mathbf{V}_\Gamma \mathbf{g}\right) := \mathcal{V}_\Gamma \mathbf{g}, \ \mathrm{Tr}^\pm(\mathbf{W}_\Gamma \mathbf{h}) = \left(\pm\frac{1}{2}\mathbb{I} + \mathbf{K}_\Gamma\right)\mathbf{h}, \tag{3.8}$$

$$\partial^\pm_\nu\left(\mathbf{V}_\Gamma \mathbf{g}, \mathcal{Q}^s_\Gamma \mathbf{g}\right) = \left(\mp\frac{1}{2}\mathbb{I} + \mathbf{K}^*_\Gamma\right)\mathbf{g}, \ \partial^+_\nu\left(\mathbf{W}_\Gamma \mathbf{h}, \mathcal{Q}^d_\Gamma \mathbf{h}\right) = \partial^-_\nu\left(\mathbf{W}_\Gamma \mathbf{h}, \mathcal{Q}^d_\Gamma \mathbf{h}\right) := \mathbf{D}_\Gamma \mathbf{h}, \tag{3.9}$$

where $\mathbf{K}^*_\Gamma$ is the formal transpose of $\mathbf{K}_\Gamma$. In addition, the following operators

$$\mathcal{V}_\Gamma : L^2_{s-1}(\Gamma, \mathbb{R}^n) \to L^2_s(\Gamma, \mathbb{R}^n), \ \mathbf{K}_\Gamma : L^2_s(\Gamma, \mathbb{R}^n) \to L^2_s(\Gamma, \mathbb{R}^n),$$
$$\mathbf{K}^*_\Gamma : L^2_{s-1}(\Gamma, \mathbb{R}^n) \to L^2_{s-1}(\Gamma, \mathbb{R}^n), \ \mathbf{D}_\Gamma : L^2_s(\Gamma, \mathbb{R}^n) \to L^2_{s-1}(\Gamma, \mathbb{R}^n),$$

are well defined, linear and continuous. Also, $\mathcal{V}_{\Gamma} : L^2_{s-1}(\Gamma,\mathbb{R}^n) \to L^2_s(\Gamma,\mathbb{R}^n)$ *is a Fredholm operator with index zero having the kernel*

$$\mathrm{Ker}\left\{\mathcal{V}_{\Gamma} : L^2_{s-1}(\Gamma,\mathbb{R}^n) \to L^2_s(\Gamma,\mathbb{R}^n)\right\} := \left\{\varphi \in L^2_{s-1}(\Gamma,\mathbb{R}^n) : \mathcal{V}_{\Gamma}\varphi = 0 \ \textit{a.e. on}\ \Gamma\right\} = \mathbb{R}\nu. \tag{3.10}$$

For the property (3.10), we refer the reader to [40, Theorems 5.4.1, 5.4.3, 10.5.1] and [22, (A.27)].

A useful result for the next arguments is the following[6] (see, e.g., [40, Lemma 11.9.21], [12]):

Proposition 3.2. *Let* X_j, Y_j, $j = 1, 2$, *be Banach spaces such that the inclusions* $X_1 \hookrightarrow X_2$, $Y_1 \hookrightarrow Y_2$ *are continuous. Let the latter of the inclusions has dense range. Assume that* $T \in \mathcal{L}(X_1,Y_1) \cap \mathcal{L}(X_2,Y_2)$ *is Fredholm, as an operator defined on the space* X_1 *and on the space* X_2*, respectively. If the condition* $\mathrm{index}(T : X_1 \to Y_1) = \mathrm{index}(T : X_2 \to Y_2)$ *holds, then* $\mathrm{Ker}(T : X_1 \to Y_1) = \mathrm{Ker}(T : X_2 \to Y_2)$.

In the sequel, we remove the superscript $-$ from the operators Tr^-, $\mathcal{Z}^-$, $\partial^-_{\nu;\mathcal{P}}(\mathbf{u},\pi)_{\mathbf{f},g}$ and $\partial^-_{\nu}(\mathbf{u},\pi)_{\mathbf{f},g}$.

4. The Poisson problem for the generalized Brinkman system with Dirichlet boundary condition

The main purpose of this section is to show the existence of a solution of the Poisson problem for a semilinear Brinkman system with Dirichlet boundary condition and data in L^2-based Sobolev spaces.

4.1. The linear Poisson problem with Dirichlet boundary condition for the generalized Brinkman system

First, we show the well-posedness of the linear Poisson problem for the generalized Brinkman system in Lipschitz domains in $\mathbb{R}^n$ $(n \geq 2)$ with Dirichlet boundary condition and data in L^2-based Sobolev spaces.

Theorem 4.1. *Let* $\mathfrak{D} \subseteq \mathbb{R}^n$ $(n \geq 2)$ *be a bounded Lipschitz domain with connected boundary* Γ*. Assume that the matrix-valued function* $\mathcal{P} \in L^\infty(\mathfrak{D},\mathbb{R}^n \otimes \mathbb{R}^n)$ *satisfies the nonnegativity condition* (2.12)*. For* $s \in (0,1)$*, consider the linear Poisson problem with Dirichlet boundary condition for the generalized Brinkman system:*

$$\begin{cases} \triangle\mathbf{u} - \mathcal{P}\mathbf{u} - \nabla\pi = \mathbf{f} \in L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n), \\ \mathrm{div}\ \mathbf{u} = g \in L^2_{s-\frac{1}{2}}(\mathfrak{D}), \\ \mathrm{Tr}\ \mathbf{u} = \mathbf{h} \in L^2_s(\Gamma,\mathbb{R}^n), \\ \langle \pi, 1\rangle_{\mathfrak{D}} = 0, \end{cases} \tag{4.1}$$

subject to the necessary condition

$$\langle \nu, \mathbf{h}\rangle_{\Gamma} = \langle g, 1\rangle_{\mathfrak{D}}. \tag{4.2}$$

Then, there exists a constant $C \equiv C(\mathcal{P}, s, \mathfrak{D}) > 0$*, independent of* $\mathbf{f}$, g *and* $\mathbf{h}$*, such that the Poisson problem* (4.1) *has a unique solution* $(\mathbf{u},\pi) \in L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D})$*, which satisfies the inequality*

$$\|\mathbf{u}\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)} + \|\pi\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})} \leq C\Big(\|\mathbf{f}\|_{L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)} + \|g\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})} + \|\mathbf{h}\|_{L^2_s(\Gamma,\mathbb{R}^n)}\Big). \tag{4.3}$$

Proof. Let us consider the matrix operator

$$\mathfrak{B}_{\mathcal{P}} : L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D}) \to L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D}) \times L^2_s(\Gamma,\mathbb{R}^n),\ \mathfrak{B}_{\mathcal{P}} := \begin{pmatrix} \triangle - \mathcal{P} & -\nabla \\ \mathrm{div} & 0 \\ \mathrm{Tr} & 0 \end{pmatrix}. \tag{4.4}$$

We show that $\mathfrak{B}_{\mathcal{P}}$ is an isomorphism on a subspace of $L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D})$. First, note that

[6] If X and Y are Banach spaces, then $\mathcal{L}(X,Y)$ is the set of linear and bounded operators from X to Y.

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$$\mathfrak{B}_{\mathcal{P}} = \mathfrak{B}_0 + \mathfrak{P}, \tag{4.5}$$

where

$$\mathfrak{B}_0 : L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D}) \to L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})\times L^2_s(\Gamma,\mathbb{R}^n),\ \ \mathfrak{B}_0 := \begin{pmatrix} \triangle & -\nabla \\ \mathrm{div} & 0 \\ \mathrm{Tr} & 0 \end{pmatrix}, \tag{4.6}$$

$$\mathfrak{P} : L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D}) \to L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})\times L^2_s(\Gamma,\mathbb{R}^n),\ \ \mathfrak{P} := \begin{pmatrix} -\mathcal{P} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.7}$$

By [40, Theorem 10.6.2], [12, Theorem 5.6], the Poisson problem for the Stokes system is well-posed. Therefore, $\mathfrak{B}_0 : L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D}) \to L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})\times L^2_s(\Gamma,\mathbb{R}^n)$ is a Fredholm operator with index zero. In addition, the operator $\mathfrak{P} : L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D}) \to L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})\times L^2_s(\Gamma,\mathbb{R}^n)$ is compact, as the compactness of the product $L^\infty(\mathfrak{D},\mathbb{R}^n\otimes\mathbb{R}^n)\cdot L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\hookrightarrow L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)$ shows. Hence, $\mathfrak{B}_{\mathcal{P}} : L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D}) \to L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})\times L^2_s(\Gamma,\mathbb{R}^n)$ is a Fredholm operator with index zero, for any $s\in(0,1)$. Such a property and Proposition 3.2 imply that

$$\begin{aligned} &\mathrm{Ker}\Big(\mathfrak{B}_{\mathcal{P}} : L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D}) \to L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})\times L^2_s(\Gamma,\mathbb{R}^n)\Big) \\ &\quad = \mathrm{Ker}\Big(\mathfrak{B}_{\mathcal{P}} : L^2_1(\mathfrak{D},\mathbb{R}^n)\times L^2(\mathfrak{D}) \to L^2_{-1}(\mathfrak{D},\mathbb{R}^n)\times L^2(\mathfrak{D})\times L^2_{\frac{1}{2}}(\Gamma,\mathbb{R}^n)\Big),\ \forall\ s\in(0,1). \end{aligned} \tag{4.8}$$

In addition, by using the Green formula (2.20), we obtain that

$$\mathrm{Ker}\Big(\mathfrak{B}_{\mathcal{P}} : L^2_1(\mathfrak{D},\mathbb{R}^n)\times L^2(\mathfrak{D}) \to L^2_{-1}(\mathfrak{D},\mathbb{R}^n)\times L^2(\mathfrak{D})\times L^2_{\frac{1}{2}}(\Gamma,\mathbb{R}^n)\Big) = \{\mathbf{0}\}\times\mathbb{R}. \tag{4.9}$$

By (4.8) and (4.9), we find that the kernel of $\mathfrak{B}_{\mathcal{P}} : L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D}) \to L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})\times L^2_s(\Gamma,\mathbb{R}^n)$ is $\{\mathbf{0}\}\times\mathbb{R}$, for any $s\in(0,1)$. Hence, the range of $\mathfrak{B}_{\mathcal{P}}$ has the codimension one in $\mathcal{Y}_s := L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})\times L^2_s(\Gamma,\mathbb{R}^n)$. On the other hand, the Divergence Theorem yields that the range of $\mathfrak{B}_{\mathcal{P}}$ is contained in the subspace

$$\tilde{\mathcal{Z}}_s := \Big\{(\mathbf{F},G,\mathbf{H})\in L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})\times L^2_s(\Gamma,\mathbb{R}^n) : \langle G,1\rangle_{\mathfrak{D}} = \langle \nu,\mathbf{H}\rangle_{\Gamma}\Big\} \tag{4.10}$$

of codimension one in $\mathcal{Y}_s$. Thus, for any $s\in(0,1)$, the range of $\mathfrak{B}_{\mathcal{P}}$ is $\tilde{\mathcal{Z}}_s$, and its kernel is the set $\{\mathbf{0}\}\times\mathbb{R}$. Consequently, for any $s\in(0,1)$ and for all $(\mathbf{f},g,\mathbf{h})\in L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})\times L^2_s(\Gamma,\mathbb{R}^n)$, satisfying the condition (4.2), there exists a pair $(\mathbf{u},\pi)\in L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})$ such that

$$\begin{cases} (\triangle-\mathcal{P})\mathbf{u}-\nabla\pi=\mathbf{f},\ \mathrm{div}\ \mathbf{u}=g\ \ \text{in}\ \ \mathfrak{D}, \\ \mathrm{Tr}\ \mathbf{u}=\mathbf{h}\ \ \text{on}\ \ \Gamma. \end{cases} \tag{4.11}$$

If we require the condition $\langle\pi,1\rangle_{\mathfrak{D}}=0$, then the solution becomes unique. Hence, the problem (4.1) has a unique solution $(\mathbf{u},\pi)\in\tilde{\mathcal{X}}_s$, where

$$\tilde{\mathcal{X}}_s := \big\{(\mathbf{v},q)\in L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D}) : \langle q,1\rangle_{\mathfrak{D}}=0\big\}. \tag{4.12}$$

Consequently, the operator $\mathfrak{B}_{\mathcal{P}} : \tilde{\mathcal{X}}_s\to\tilde{\mathcal{Z}}_s$ is an isomorphism.

In addition, there exist two constants $c>0$ and $C\equiv C(\mathcal{P},s,\mathfrak{D})>0$ such that

$$\begin{aligned} \|(\mathbf{u},\pi)\|_{\tilde{\mathcal{X}}_s} &= \|\mathfrak{B}_{\mathcal{P}}^{-1}(\mathbf{f},g,\mathbf{h})^\top\|_{\tilde{\mathcal{X}}_s} \\ &\le c\|\mathfrak{B}_{\mathcal{P}}^{-1}\|_{\mathcal{L}(\tilde{\mathcal{Z}}_s,\tilde{\mathcal{X}}_s)}\|(\mathbf{f},g,\mathbf{h})\|_{\tilde{\mathcal{Z}}_s} \\ &\le C\Big(\|\mathbf{f}\|_{L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)}+\|g\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})}+\|\mathbf{h}\|_{L^2_s(\Gamma,\mathbb{R}^n)}\Big), \end{aligned} \tag{4.13}$$

where $\tilde{\mathcal{Z}}_s$ is the space defined in (4.10). Hence, we have obtained the inequality (4.3), as asserted. □

Next, we consider the operators

$$\begin{aligned} &\mathfrak{L}_1 : \mathcal{X}_s \to L^2_{s-\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n),\ \mathfrak{L}_1(\mathbf{u}, \pi) := \triangle \mathbf{u} - \mathcal{P}\mathbf{u} - \nabla \pi, \\ &\mathfrak{L}_2 : \mathcal{X}_s \to L^2_{s-\frac{1}{2}}(\mathfrak{D}),\ \mathfrak{L}_2(\mathbf{u}, \pi) := \mathrm{div}\ \mathbf{u}, \\ &\mathfrak{L}_3 : \mathcal{X}_s \to L^2_s(\Gamma, \mathbb{R}^n),\ \mathfrak{L}_3(\mathbf{u}, \pi) := \mathrm{Tr}\ \mathbf{u}, \end{aligned} \tag{4.14}$$

where

$$\mathcal{X}_s := L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D}), \quad \mathcal{Y}_s := L^2_{s-\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D}) \times L^2_s(\Gamma, \mathbb{R}^n). \tag{4.15}$$

Recalling that $\tilde{\mathcal{X}}_s$ is the space defined in (4.12), we show the following result.

Lemma 4.2. *Let $\mathfrak{D} \subseteq \mathbb{R}^n$ $(n \geq 2)$ be a bounded Lipschitz domain with connected boundary Γ. Let $s \in (0,1)$ and $a \in (0, \infty)$. Then, there exists a constant $C \equiv C(a, s, \mathfrak{D}) > 0$ such that*

$$\|(\mathbf{u}, \pi)\|_{\tilde{\mathcal{X}}_s} \leq C\Big(\|\mathfrak{L}_1(\mathbf{u}, \pi)\|_{L^2_{s-\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n)} + \|\mathfrak{L}_2(\mathbf{u}, \pi)\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})} + \|\mathfrak{L}_3(\mathbf{u}, \pi)\|_{L^2_s(\Gamma, \mathbb{R}^n)}\Big), \tag{4.16}$$

for all $(\mathbf{u}, \pi) \in \tilde{\mathcal{X}}_s$ and for each matrix-valued function $\mathcal{P} \in L^\infty(\mathfrak{D}, \mathbb{R}^n \otimes \mathbb{R}^n)$, which satisfies the nonnegativity condition (2.12) and the inequality

$$\|\mathcal{P}\|_{L^\infty(\mathfrak{D}, \mathbb{R}^n \otimes \mathbb{R}^n)} \leq a. \tag{4.17}$$

Proof. Let us assume by contradiction that such a constant C does not exist. Thus, we assume that the inequality (4.16) does not hold. Then, there exist two sequences $\{(\mathbf{u}_j, \pi_j)\}_{j \in \mathbb{N}}$ in $\tilde{\mathcal{X}}_s$ and $\{\mathcal{P}_j\}_{j \in \mathbb{N}}$ in $L^\infty(\mathfrak{D}, \mathbb{R}^n \otimes \mathbb{R}^n)$, such that $\mathcal{P}_j$ satisfies the nonnegativity condition (2.12) and the inequalities

$$\|\mathcal{P}_j\|_{L^\infty(\mathfrak{D}, \mathbb{R}^n \otimes \mathbb{R}^n)} \leq a, \quad \forall\ j \geq 1, \tag{4.18}$$

$$\|(\mathbf{u}_j, \pi_j)\|_{\tilde{\mathcal{X}}_s} > j\Big(\|(\triangle - \mathcal{P}_j)\mathbf{u}_j - \nabla \pi_j\|_{L^2_{s-\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n)} + \|\mathfrak{L}_2(\mathbf{u}_j, \pi_j)\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})} + \|\mathfrak{L}_3(\mathbf{u}_j, \pi_j)\|_{L^2_s(\Gamma, \mathbb{R}^n)}\Big), j \geq 1. \tag{4.19}$$

Let $(\mathbf{w}_j, r_j) \in \tilde{\mathcal{X}}_s$ be such that

$$(\mathbf{w}_j, r_j) := \frac{1}{\|(\mathbf{u}_j, \pi_j)\|_{\tilde{\mathcal{X}}_s}}(\mathbf{u}_j, \pi_j), \quad j \geq 1. \tag{4.20}$$

Thus, $\|(\mathbf{w}_j, r_j)\|_{\tilde{\mathcal{X}}_s} = 1$ and, for any $j \geq 1$,

$$j^{-1} > \|(\triangle - \mathcal{P}_j)\mathbf{w}_j - \nabla r_j\|_{L^2_{s-\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n)} + \|\mathfrak{L}_2(\mathbf{w}_j, r_j)\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})} + \|\mathfrak{L}_3(\mathbf{w}_j, r_j)\|_{L^2_s(\Gamma, \mathbb{R}^n)}. \tag{4.21}$$

On the other hand, by the Banach–Alaoglu Theorem (cf. [5, Chap. 5, Sect. 3]), the closed ball of radius a in the space $L^\infty(\mathfrak{D}, \mathbb{R}^n \otimes \mathbb{R}^n)$, which is the dual of the separable Banach space $L^1(\mathfrak{D}, \mathbb{R}^n \otimes \mathbb{R}^n)$, is sequentially compact in the weak-* topology. Since the sequence $\{\mathcal{P}_j\}_{j \in \mathbb{N}}$ is bounded in the space $L^\infty(\mathfrak{D}, \mathbb{R}^n \otimes \mathbb{R}^n)$, as each term $\mathcal{P}_j$ belongs to the closed ball of radius a of this space (see (4.18)), we then can select a weak-* convergent subsequence $\{\mathcal{P}_{j_k}\}_{k \in \mathbb{N}}$ of $\{\mathcal{P}_j\}_{j \in \mathbb{N}}$ with the limit in the same closed ball. Therefore, there exists $\mathcal{P}_0 \in L^\infty(\mathfrak{D}, \mathbb{R}^n \otimes \mathbb{R}^n)$ such that $\|\mathcal{P}_0\|_{L^\infty(\mathfrak{D}, \mathbb{R}^n \otimes \mathbb{R}^n)} \leq a$ and

$$\lim_{k \to \infty} \mathcal{P}_{j_k}(\varphi) = \mathcal{P}_0(\varphi), \quad \forall\ \varphi \in L^1(\mathfrak{D}, \mathbb{R}^n \otimes \mathbb{R}^n), \tag{4.22}$$

where

$$\mathcal{P}_{j_k}(\varphi) := \int_{\mathfrak{D}} \mathcal{P}_{j_k}(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}.$$

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In addition, $\mathcal{P}_0$ satisfies the nonnegativity condition (2.13). Indeed, for any $\mathbf{v} \in L^2(\mathfrak{D}, \mathbb{R}^n)$, we have $v_r v_s \in L^1(\mathfrak{D})$ for all $r, s = 1, \ldots, n$, and accordingly the condition (4.22) implies that

$$\lim_{k\to\infty} \langle \mathcal{P}_{j_k}\mathbf{v}, \mathbf{v}\rangle_{\mathfrak{D}} = \lim_{k\to\infty} \int\limits_{\mathfrak{D}} (\mathcal{P}_{j_k})_{rs}\, v_r v_s d\mathbf{x} = \int\limits_{\mathfrak{D}} (\mathcal{P}_0)_{rs}\, v_r v_s d\mathbf{x}, \tag{4.23}$$

where $(\mathcal{P}_{j_k})_{rs}$ are the components of $\mathcal{P}_{j_k}$, and $(\mathcal{P}_0)_{rs}$ are the components of $\mathcal{P}_0$, $r, s = 1, \ldots, n$. Since each $\mathcal{P}_{j_k} \in L^\infty(\mathfrak{D}, \mathbb{R}^n \otimes \mathbb{R}^n)$ satisfies the nonnegativity condition (2.12), the limit in (4.23) is nonnegative as well.

On the other hand, since the embedding $\tilde{\mathcal{X}}_s \hookrightarrow \tilde{\mathcal{X}}_t$ is compact whenever $t, s \in (0,1)$, $t < s$ (see, e.g., [19, Theorem 7.10]), there exists a subsequence $\{(\mathbf{w}_{j_k}, r_{j_k})\}_{k\in\mathbb{N}}$ of the bounded sequence $\{(\mathbf{w}_j, r_j)\}_{j\in\mathbb{N}}$ of $\tilde{\mathcal{X}}_s$ and an element $(\mathbf{w}, r) \in \tilde{\mathcal{X}}_t$ such that

$$\|(\mathbf{w}_{j_k}, r_{j_k}) - (\mathbf{w}, r)\|_{\tilde{\mathcal{X}}_t} \to 0 \quad \text{as} \ \ k \to \infty. \tag{4.24}$$

Recall that $\tilde{\mathcal{X}}_t = \big\{(\mathbf{v}, q) \in L^2_{t+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{t-\frac{1}{2}}(\mathfrak{D}) : \ \langle q, 1\rangle_{\mathfrak{D}} = 0\big\}$.

Taking into account of the relations (4.18), (4.22) and (4.24) (and, possibly, extracting further subsequences of $\{\mathcal{P}_{j_k}\}_{k\in\mathbb{N}}$ and $\{\mathbf{w}_{j_k}\}_{k\in\mathbb{N}}$ denoted, for the sake of brevity, as the sequences), one obtains that

$$\lim_{k\to\infty} \mathcal{P}_{j_k}\mathbf{w}_{j_k} = \mathcal{P}_0\mathbf{w}, \tag{4.25}$$

weakly in $L^2(\mathfrak{D}, \mathbb{R}^n)$ and accordingly, in the sense of distributions in $\mathfrak{D}$. Indeed, for any $\varphi \in L^2(\mathfrak{D}, \mathbb{R}^n)$, one has the equality

$$\int\limits_{\mathfrak{D}} \langle \mathcal{P}_{j_k}\mathbf{w}_{j_k} - \mathcal{P}_0\mathbf{w}, \varphi\rangle\, d\mathbf{x} = \int\limits_{\mathfrak{D}} (\mathcal{P}_{j_k} - \mathcal{P}_0)_{rs} w_r \varphi_s d\mathbf{x} + \int\limits_{\mathfrak{D}} \Big\langle \mathcal{P}_{j_k}(\mathbf{w}_{j_k} - \mathbf{w}), \varphi \Big\rangle\, d\mathbf{x}.$$

The first integral in the right-hand side of the above equality tends to zero, as (4.22) and the property $w_r\varphi_s \in L^1(\mathfrak{D})$ show. In addition, the properties (4.18) and (4.24) imply that the second integral also tends to zero as $k \to \infty$.

By (4.24), the continuous embedding of $\tilde{\mathcal{X}}_t$ into the space of distributions, and by (4.25), we have

$$\lim_{k\to\infty} ((\triangle - \mathcal{P}_{j_k})\mathbf{w}_{j_k} - \nabla r_{j_k}) = (\triangle - \mathcal{P}_0)\mathbf{w} - \nabla r \tag{4.26}$$

in the sense of distributions in $\mathfrak{D}$. In addition, we obtain the limiting relation

$$\lim_{k\to\infty} \operatorname{div}\, \mathbf{w}_{j_k} = \operatorname{div}\, \mathbf{w} \tag{4.27}$$

in $L^2_{t-\frac{1}{2}}(\mathfrak{D})$ and accordingly in the sense of distributions in $\mathfrak{D}$. Also, we have the limiting relation

$$\lim_{k\to\infty} \operatorname{Tr}\, \mathbf{w}_{j_k} = \operatorname{Tr}\, \mathbf{w} \tag{4.28}$$

in $L^2_t(\Gamma, \mathbb{R}^n)$ and accordingly in the sense of distributions in Γ.

By (4.21), $\{(\triangle - \mathcal{P}_{j_k})\mathbf{w}_{j_k} - \nabla r_{j_k}\}_{k\in\mathbb{N}}$ converges to zero in $L^2_{s-\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n)$ and accordingly, in the sense of distributions in $\mathfrak{D}$. Comparing this result with (4.26), we find that

$$(\triangle - \mathcal{P}_0)\mathbf{w} - \nabla r = \mathbf{0} \quad \text{in} \ \ \mathfrak{D}. \tag{4.29}$$

Similarly, we get $\operatorname{div}\, \mathbf{w} = 0$ in $\mathfrak{D}$, $\operatorname{Tr}\, \mathbf{w} = \mathbf{0}$ on Γ, and $\langle r, 1\rangle_{\mathfrak{D}} = 0$. Consequently, the pair $(\mathbf{w}, r) \in \tilde{\mathcal{X}}_t$ is a solution of the homogeneous problem for the generalized Brinkman system

$$\begin{cases} \triangle\mathbf{w} - \mathcal{P}_0\mathbf{w} - \nabla r = \mathbf{0} & \text{in} \ \ \mathfrak{D}, \\ \operatorname{div}\, \mathbf{w} = 0 & \text{in} \ \ \mathfrak{D}, \\ \operatorname{Tr}\, \mathbf{w} = \mathbf{0} & \text{on} \ \ \Gamma, \\ \langle r, 1\rangle_{\mathfrak{D}} = 0. \end{cases} \tag{4.30}$$

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The uniqueness of the solution to this problem in the space $\mathcal{X}_t := L^2_{t+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{t-\frac{1}{2}}(\mathfrak{D})$ (see Theorem 4.1) implies that $(\mathbf{w},r)=(\mathbf{0},0)$. Then, by (4.24), we obtain the limiting relations

$$\|\mathbf{w}_{j_k}\|_{L^2_{t+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)}\to 0,\quad \|r_{j_k}\|_{L^2_{t-\frac{1}{2}}(\mathfrak{D})}\to 0\quad \text{as}\quad k\to\infty. \tag{4.31}$$

Combining (4.31) with the uniform boundedness of the sequence $\{\mathcal{P}_{j_k}\}_{k\in\mathbb{N}}$ in $L^\infty(\mathfrak{D},\mathbb{R}^n\otimes\mathbb{R}^n)$, we obtain the limiting relation

$$\lim_{k\to\infty}\mathcal{P}_{j_k}\mathbf{w}_{j_k}=\mathbf{0}\qquad \text{in}\quad L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n). \tag{4.32}$$

Indeed, there exists a constant $c\equiv c(\mathfrak{D},s)>0$, such that

$$\begin{aligned}\|\mathcal{P}_{j_k}\mathbf{w}_{j_k}\|_{L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)} &\le c\|\mathcal{P}_{j_k}\mathbf{w}_{j_k}\|_{L^2(\mathfrak{D},\mathbb{R}^n)}\\ &\le c\|\mathcal{P}_{j_k}\|_{L^\infty(\mathfrak{D},\mathbb{R}^n\otimes\mathbb{R}^n)}\|\mathbf{w}_{j_k}\|_{L^2(\mathfrak{D},\mathbb{R}^n)}\\ &\le ca\|\mathbf{w}_{j_k}\|_{L^2_{t+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)}\to 0 \text{ as } k\to\infty.\end{aligned} \tag{4.33}$$

Now, by (4.21) and (4.32), we get $\triangle\mathbf{w}_{j_k}-\nabla r_{j_k}\to\mathbf{0}$ in $L^2_{s-\frac{3}{2}}(\mathfrak{D})$, $\mathrm{Tr}\ \mathbf{w}_{j_k}\to\mathbf{0}$ in $L^2_s(\Gamma,\mathbb{R}^n)$, as $k\to\infty$. Therefore,

$$\begin{cases}\triangle\mathbf{w}_{j_k}-\nabla r_{j_k}\to\mathbf{0}\ \text{ in }\ L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)\\ \mathrm{div}\ \mathbf{w}_{j_k}\to 0\ \text{ in }\ L^2_{s-\frac{1}{2}}(\mathfrak{D})\\ \mathrm{Tr}\ \mathbf{w}_{j_k}\to\mathbf{0}\ \text{ in }\ L^2_s(\Gamma,\mathbb{R}^n)\end{cases}\qquad \text{as}\quad k\to\infty. \tag{4.34}$$

Finally, by exploiting the well-posedness of the Dirichlet problem for the Stokes system in the space $\tilde{\mathcal{X}}_s := \{(\mathbf{v},q)\in L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D}) : \langle q,1\rangle_{\mathfrak{D}}=0\}$ (see [40, Theorem 10.6.2]), we obtain the limiting relation

$$\|(\mathbf{w}_{j_k},r_{j_k})\|_{\tilde{\mathcal{X}}_s}\to 0\quad \text{as}\quad k\to\infty, \tag{4.35}$$

which contradicts the choice of the sequence $\{(\mathbf{w}_{j_k},r_{j_k})\}_{k\ge 1}$ in $\tilde{\mathcal{X}}_s$, i.e., the relation $\|(\mathbf{w}_{j_k},r_{j_k})\|_{\tilde{\mathcal{X}}_s}=1$ for any $k\ge 1$. Thus, the proof is complete. □

4.2. Poisson problem for the semilinear Brinkman system with Dirichlet boundary condition

Next, we introduce the semilinear Poisson problem with Dirichlet boundary condition in L^2-based Sobolev spaces on the Lipschitz domain $\mathfrak{D}\subseteq\mathbb{R}^n$. We take $s\in(\frac{1}{2},1)$, and we consider a function $\mathcal{P}\in L^\infty(\mathfrak{D}\times\mathbb{R}^n\times\mathbb{R},\mathbb{R}^n\otimes\mathbb{R}^n)$, which satisfies the Carathéodory condition, i.e., $\mathcal{P}(\cdot,\mathbf{v},\xi)$ is measurable for almost all $(\mathbf{v},\xi)\in\mathbb{R}^n\times\mathbb{R}$ and $\mathcal{P}(\mathbf{x},\cdot,\cdot)$ is continuous for all $\mathbf{x}\in\mathfrak{D}$. In addition, we assume that $\mathcal{P}$ satisfies the following nonnegativity condition: There exists a subset $N_{\mathcal{P}}$ of measure zero of $\mathfrak{D}$ such that

$$\langle\mathcal{P}(\mathbf{x},\mathbf{v},\xi)\mathbf{b},\mathbf{b}\rangle\ge 0,\quad \forall\ \mathbf{b}\in\mathbb{R}^n,\ (\mathbf{x},\mathbf{v},\xi)\in(\mathfrak{D}\setminus N_{\mathcal{P}})\times\mathbb{R}^n\times\mathbb{R}. \tag{4.36}$$

Finally, we assume that $(\mathbf{f},g,\mathbf{h})\in L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})\times L^2_s(\Gamma,\mathbb{R}^n)$ satisfies the compatibility condition

$$\langle\nu,\mathbf{h}\rangle_{\Gamma}=\langle g,1\rangle_{\mathfrak{D}}, \tag{4.37}$$

and we consider the semilinear Poisson problem

$$\begin{cases}\big(\triangle-\mathcal{P}\big(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x})\big)\big)\,\mathbf{u}-\nabla\pi=\mathbf{f} & \text{in }\ \mathfrak{D}\\ \mathrm{div}\ \mathbf{u}=g & \text{in }\ \mathfrak{D}\\ \mathrm{Tr}\ \mathbf{u}=\mathbf{h} & \text{on }\ \Gamma,\\ \langle\pi,1\rangle_{\mathfrak{D}}=0\end{cases} \tag{4.38}$$

with the unknown $(\mathbf{u},\pi)\in\mathcal{X}_s := L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})$. In order to have an existence result for the problem (4.38), we resort to the well-known Schauder Fixed Point Theorem (see, e.g., [16, Theorem 11.1]):

Theorem 4.3. *Let K be a closed convex subset of a Banach space X. If $T : K \to K$ is a continuous mapping such that $T(K)$ is a relatively compact subset of K, then T has a fixed point.*

Then, we prove the following existence result.

Theorem 4.4. *Let $\mathfrak{D} \subseteq \mathbb{R}^n\,(n \geq 2)$ be a bounded Lipschitz domain with connected boundary Γ. Let $a > 0$ and $s \in (\frac{1}{2}, 1)$. Then, there exists a constant $C \equiv C(a, s, \mathfrak{D}) > 0$ such that for each $(\mathbf{f}, g, \mathbf{h}) \in L^2_{s-\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D}) \times L^2_s(\Gamma, \mathbb{R}^n)$ satisfying the compatibility condition (4.37) and for each essentially bounded Carathéodory function $\mathcal{P}$ from $\mathfrak{D} \times \mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}^n \otimes \mathbb{R}^n$ satisfying the nonnegativity condition (4.36) and the inequality*

$$\|\mathcal{P}\|_{L^\infty(\mathfrak{D}\times\mathbb{R}^n\times\mathbb{R},\mathbb{R}^n\otimes\mathbb{R}^n)} \leq a, \tag{4.39}$$

the semilinear Poisson problem (4.38) has at least a solution $(\mathbf{u}, \pi) \in \mathcal{X}_s$ such that

$$\|(\mathbf{u}, \pi)\|_{\mathcal{X}_s} \leq C\Big(\|\mathbf{f}\|_{L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)} + \|g\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})} + \|\mathbf{h}\|_{L^2_s(\Gamma,\mathbb{R}^n)}\Big). \tag{4.40}$$

Proof. For a fixed $(\mathbf{u}, \pi) \in \tilde{\mathcal{X}}_s$, where $\tilde{\mathcal{X}}_s$ is the space defined in (4.12), we first consider the auxiliary linear Poisson problem with Dirichlet boundary condition

$$\begin{cases} \big(\triangle - \mathcal{P}\big(\mathbf{x}, \mathbf{u}(\mathbf{x}), \pi(\mathbf{x})\big)\big)\,\mathbf{v} - \nabla\zeta = \mathbf{f} \in L^2_{s-\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n), \\ \mathrm{div}\ \mathbf{v} = g \in L^2_{s-\frac{1}{2}}(\mathfrak{D}), \\ \mathrm{Tr}\ \mathbf{v} = \mathbf{h} \in L^2_s(\Gamma, \mathbb{R}^n). \end{cases} \tag{4.41}$$

Note that $\mathbf{f}$, g and $\mathbf{h}$ are the given data of the semilinear Poisson problem (4.38). By Theorem 4.1, there exists a constant $C \equiv C(a, s, \mathfrak{D}) > 0$ such that the problem (4.41) has a unique solution $(\mathbf{v}, \zeta) \in \tilde{\mathcal{X}}_s$, which satisfies the inequality [see (4.16)]

$$\|(\mathbf{v}, \zeta)\|_{\tilde{\mathcal{X}}_s} \leq C\Big(\|\big(\triangle - \mathcal{P}\big(\mathbf{x}, \mathbf{u}(\mathbf{x}), \pi(\mathbf{x})\big)\big)\mathbf{v} - \nabla\zeta\|_{L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)} + \|\mathfrak{L}_2(\mathbf{v}, \zeta)\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})} + \|\mathfrak{L}_3(\mathbf{v}, \zeta)\|_{L^2_s(\Gamma,\mathbb{R}^n)}\Big) \tag{4.42}$$

where $\mathfrak{L}_2$ and $\mathfrak{L}_3$ are the operators given in (4.14). By (4.41) and (4.42), we obtain that

$$\|(\mathbf{v}, \zeta)\|_{\tilde{\mathcal{X}}_s} \leq A, \tag{4.43}$$

where

$$A := C\Big(\|\mathbf{f}\|_{L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)} + \|g\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})} + \|\mathbf{h}\|_{L^2_s(\Gamma,\mathbb{R}^n)}\Big) > 0. \tag{4.44}$$

Therefore, $(\mathbf{v}, \zeta) \in B_A$, where $B_A := \{z \in \tilde{\mathcal{X}}_s : \|z\|_{\tilde{\mathcal{X}}_s} \leq A\}$. We now consider the nonlinear operator

$$\mathcal{T}_{\mathbf{f},g,\mathbf{h}} : B_A \to B_A, \quad B_A \ni (\mathbf{u}, \pi) \overset{\mathcal{T}_{\mathbf{f},g,\mathbf{h}}}{\longmapsto} (\mathbf{v}, \zeta), \tag{4.45}$$

which associates to $(\mathbf{u}, \pi) \in B_A$ the unique solution $(\mathbf{v}, \zeta) \in B_A$ of the linear Poisson problem of Dirichlet type (4.41). Such an operator is well defined, as the inequality (4.43) shows. We now turn to show that $\mathcal{T}_{\mathbf{f},g,\mathbf{h}} : B_A \to B_A$ is continuous and compact.

Let $\{(\mathbf{u}_j, \pi_j)\}_{j\in\mathbb{N}}$ be a sequence in $\big(B_A, \|\cdot\|_{\tilde{\mathcal{X}}_s}\big)$, and let $t \in (\frac{1}{2}, 1)$, $t < s$. Since the embedding $\tilde{\mathcal{X}}_s \hookrightarrow \tilde{\mathcal{X}}_t$ is compact, there exists a subsequence $\{(\mathbf{u}_{j_k}, \pi_{j_k})\}_{k\in\mathbb{N}}$ of $\{(\mathbf{u}_j, \pi_j)\}_{j\in\mathbb{N}}$ that converges to an element $(\tilde{\mathbf{u}}, \tilde{\pi}) \in \tilde{\mathcal{X}}_t$, i.e.,

$$\|(\mathbf{u}_{j_k}, \pi_{j_k}) - (\tilde{\mathbf{u}}, \tilde{\pi})\|_{\tilde{\mathcal{X}}_t} \to 0 \qquad \text{as } k \to \infty. \tag{4.46}$$

In addition, since $\tilde{\mathcal{X}}_s$ is a reflexive Banach space (as a closed subspace of the reflexive Banach space $\mathcal{X}_s$), we can select a further subsequence of the bounded sequence $\{(\mathbf{u}_{j_k}, \pi_{j_k})\}_{k\in\mathbb{N}}$ in B_A, still denoted by $\{(\mathbf{u}_{j_k}, \pi_{j_k})\}_{k\in\mathbb{N}}$, which converges weakly to an element $(\mathbf{u}_0, \pi_0) \in B_A$, i.e.,

$$\langle \varphi, (\mathbf{u}_{j_k}, \pi_{j_k}) \rangle_{\mathfrak{D}} - \langle \varphi, (\mathbf{u}_0, \pi_0) \rangle_{\mathfrak{D}} \to 0, \ \ \forall\, \varphi \in \left(\tilde{\mathcal{X}}_s\right)'. \tag{4.47}$$

By (4.47) and the property that the convergence in norm of $\tilde{\mathcal{X}}_t$ implies the weak convergence, we obtain for any $\varphi \in \left(\tilde{\mathcal{X}}_t\right)' \hookrightarrow \left(\tilde{\mathcal{X}}_s\right)'$ that

$$\langle \varphi, (\mathbf{u}_0, \pi_0) - (\tilde{\mathbf{u}}, \tilde{\pi}) \rangle_{\mathfrak{D}} = \langle \varphi, (\mathbf{u}_0, \pi_0) - (\mathbf{u}_{j_k}, \pi_{j_k}) \rangle_{\mathfrak{D}} + \langle \varphi, (\mathbf{u}_{j_k}, \pi_{j_k}) - (\tilde{\mathbf{u}}, \tilde{\pi}) \rangle_{\mathfrak{D}} \to 0 \ \text{ as } \ k \to \infty. \tag{4.48}$$

Therefore, $(\mathbf{u}_0, \pi_0) = (\tilde{\mathbf{u}}, \tilde{\pi})$. Consequently, the proof of the continuity and compactness of the operator $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}$ in $(B_A, \|\cdot\|_{\tilde{\mathcal{X}}_s})$ reduces to the continuity of $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}$ from $(B_A, \|\cdot\|_{\tilde{\mathcal{X}}_t})$ to $(B_A, \|\cdot\|_{\tilde{\mathcal{X}}_s})$ whenever $\frac{1}{2} < t < s < 1$.

Before we prove such a continuity, we show an intermediate statement. Indeed, we next turn to prove that the operator $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}$ is continuous from $(B_A, \|\cdot\|_{\tilde{\mathcal{X}}_t})$ to $(B_A, \|\cdot\|_{\tilde{\mathcal{X}}_t})$.

The continuity of the operator $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}$ from $(B_A, \|\cdot\|_{\tilde{\mathcal{X}}_t})$ to $(B_A, \|\cdot\|_{\tilde{\mathcal{X}}_t})$

Let $\{(\mathbf{u}_j, \pi_j)\}_{j \in \mathbb{N}}$ be a sequence in $(B_A, \|\cdot\|_{\tilde{\mathcal{X}}_t})$, which converges to $(\mathbf{u}, \pi) \in B_A$ in the $\tilde{\mathcal{X}}_t$-norm, i.e.,

$$\|(\mathbf{u}_j, \pi_j) - (\mathbf{u}, \pi)\|_{\tilde{\mathcal{X}}_t} \to 0 \ \text{ as } \ j \to \infty. \tag{4.49}$$

In particular, we note that for $\frac{1}{2} < t < s < 1$, the convergence in norm of $\mathcal{X}_t$ implies the L^2-convergence. Therefore, there exists a subsequence $\{(\mathbf{u}_{j_k}, \pi_{j_k})\}_{k \in \mathbb{N}}$ of the sequence $\{(\mathbf{u}_j, \pi_j)\}_{j \in \mathbb{N}}$, which converges to $(\mathbf{u}, \pi)$ a.e. in $\mathfrak{D}$, i.e.,

$$\lim_{k \to \infty} (\mathbf{u}_{j_k}, \pi_{j_k}) = (\mathbf{u}, \pi) \ \text{ a.e. in } \mathfrak{D}. \tag{4.50}$$

In addition, in view of the inequality (4.16), the sequence $\{(\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j), \mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j))\}_{j \in \mathbb{N}}$ is bounded in $\tilde{\mathcal{X}}_s$, where $\mathcal{T}_{\mathbf{f},g,\mathbf{h}} = (\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}, \mathcal{T}_{2;\mathbf{f},g,\mathbf{h}})$. Then, by the compactness of the embedding $\tilde{\mathcal{X}}_s \hookrightarrow \tilde{\mathcal{X}}_t$, possibly considering a subsequence, we can assume that $\{(\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k}, \pi_{j_k}), \mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k}, \pi_{j_k}))\}_{k \in \mathbb{N}}$ converges to an element $(\tilde{\mathbf{v}}, \tilde{\xi}) \in \tilde{\mathcal{X}}_t$. Thus,

$$\lim_{k \to \infty} \left\| (\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k}, \pi_{j_k}), \mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k}, \pi_{j_k})) - (\tilde{\mathbf{v}}, \tilde{\xi}) \right\|_{\tilde{\mathcal{X}}_t} = 0. \tag{4.51}$$

We now consider the semilinear Poisson problem

$$\begin{cases} \left(\triangle - \mathcal{P}\big(\mathbf{x}, \mathbf{u}_{j_k}(\mathbf{x}), \pi_{j_k}(\mathbf{x})\big)\right) \mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k}, \pi_{j_k}) - \nabla \mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k}, \pi_{j_k}) = \mathbf{f} & \text{in } \mathfrak{D}, \\ \operatorname{div} \mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k}, \pi_{j_k}) = g & \text{in } \mathfrak{D}, \\ \operatorname{Tr} \mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k}, \pi_{j_k}) = \mathbf{h} & \text{on } \Gamma, \end{cases} \tag{4.52}$$

and note that $\mathcal{P}\big(\mathbf{x}, \mathbf{u}_{j_k}, \pi_{j_k}\big) \mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k}, \pi_{j_k}) \in L^2(\mathfrak{D}, \mathbb{R}^n)$. In addition, by the uniform boundedness of $\mathcal{P}$ in $L^\infty(\mathfrak{D} \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n \otimes \mathbb{R}^n)$ and (4.45), the sequence $\{(\mathcal{P}\big(\mathbf{x}, \mathbf{u}_{j_k}, \pi_{j_k}\big) \mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k}, \pi_{j_k})\}_{k \in \mathbb{N}}$ is bounded in $L^2(\mathfrak{D}, \mathbb{R}^n)$. Then, possibly extracting a subsequence, still denoted as the sequence, we obtain the limiting relation

$$\lim_{k \to \infty} \mathcal{P}\big(\mathbf{x}, \mathbf{u}_{j_k}, \pi_{j_k}\big) \mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k}, \pi_{j_k}) = \mathcal{P}\big(\mathbf{x}, \mathbf{u}, \pi\big) \tilde{\mathbf{v}} \tag{4.53}$$

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in the weak-* topology of $L^2(\mathfrak{D},\mathbb{R}^n)$. Indeed, for any $\boldsymbol{\varphi}\in L^2(\mathfrak{D},\mathbb{R}^n)$, we have the inequality

$$\Big|\int_{\mathfrak{D}}\big\langle \mathcal{P}\big(\mathbf{x},\mathbf{u}_{j_k},\pi_{j_k}\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})-\mathcal{P}\big(\mathbf{x},\mathbf{u},\pi\big)\tilde{\mathbf{v}},\boldsymbol{\varphi}\big\rangle\,d\mathbf{x}\Big|$$
$$\leq \|\mathcal{P}\big(\cdot,\mathbf{u}_{j_k},\pi_{j_k}\big)\|_{L^\infty(\mathfrak{D}\times\mathbb{R}^n\times\mathbb{R},\mathbb{R}^n\otimes\mathbb{R}^n)}\int_{\mathfrak{D}}|\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})-\tilde{\mathbf{v}}||\boldsymbol{\varphi}|d\mathbf{x}$$
$$+\int_{\mathfrak{D}}|\tilde{\mathbf{v}}|\;|\boldsymbol{\varphi}|\;|\mathcal{P}\big(\mathbf{x},\mathbf{u}_{j_k},\pi_{j_k}\big)-\mathcal{P}\big(\mathbf{x},\mathbf{u},\pi\big)|d\mathbf{x}. \tag{4.54}$$

In addition, $|\mathcal{P}\big(\mathbf{x},\mathbf{u}_{j_k},\pi_{j_k}\big)-\mathcal{P}\big(\mathbf{x},\mathbf{u},\pi\big)|\leq 2\|\mathcal{P}\|_{L^\infty(\mathfrak{D}\times\mathbb{R}^n\times\mathbb{R},\mathbb{R}^n\otimes\mathbb{R}^n)}$ and, by the continuity of $\mathcal{P}(\mathbf{x},\mathbf{v},q)$ with respect to $(\mathbf{v},q)\in\mathbb{R}^n\times\mathbb{R}$, we have

$$\lim_{k\to\infty}|\tilde{\mathbf{v}}|\;|\varphi|\;|\mathcal{P}\big(\mathbf{x},\mathbf{u}_{j_k},\pi_{j_k}\big)-\mathcal{P}\big(\mathbf{x},\mathbf{u},\pi\big)|=0\ \text{ a.e. }\ \mathbf{x}\in\mathfrak{D}.$$

Then, by the Lebesgue Dominated Convergence Theorem (see, e.g., [42]), we deduce the limiting relation

$$\lim_{k\to\infty}\int_{\mathfrak{D}}|\tilde{\mathbf{v}}|\;|\varphi|\;|\mathcal{P}\big(\mathbf{x},\mathbf{u}_{j_k},\pi_{j_k}\big)-\mathcal{P}\big(\mathbf{x},\mathbf{u},\pi\big)|d\mathbf{x}=0. \tag{4.55}$$

It remains to prove that the first integral in the right-hand side of (4.54) tends to 0 as $k\to\infty$. To this aim, we use the Hölder inequality and the relation (4.51) and obtain a constant $c>0$ such that

$$\int_{\mathfrak{D}}|\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})-\tilde{\mathbf{v}}||\varphi|d\mathbf{x}\leq c\|\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})-\tilde{\mathbf{v}}\|_{L^2(\mathfrak{D},\mathbb{R}^n)}\|\varphi\|_{L^2(\mathfrak{D},\mathbb{R}^n)}$$
$$\leq c\|\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})-\tilde{\mathbf{v}}\|_{L^2_{t+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)}\|\varphi\|_{L^2(\mathfrak{D},\mathbb{R}^n)}\to 0 \text{ as } k\to\infty. \tag{4.56}$$

In view of (4.54), (4.55) and (4.56), we obtain the limiting relation

$$\lim_{k\to\infty}\int_{\mathfrak{D}}\big\langle\mathcal{P}\big(\mathbf{x},\mathbf{u}_{j_k},\pi_{j_k}\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})-\mathcal{P}\left(\mathbf{x},\mathbf{u},\pi\right)\tilde{\mathbf{v}},\varphi\big\rangle\,d\mathbf{x}=0,\quad\forall\,\varphi\in L^2(\mathfrak{D},\mathbb{R}^n),$$

which leads to the property (4.53). In addition, (4.51) implies that

$$\lim_{k\to\infty}(\triangle\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}\mathbf{u}_{j_k}-\nabla\mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}\mathbf{u}_{j_k})=\triangle\tilde{\mathbf{v}}-\nabla\tilde{\xi},\ \lim_{k\to\infty}\mathrm{div}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}\mathbf{u}_{j_k}=\mathrm{div}\ \tilde{\mathbf{v}},\ \lim_{k\to\infty}\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}\mathbf{u}_{j_k}=\mathrm{Tr}\ \tilde{\mathbf{v}}, \tag{4.57}$$

in the sense of distributions.

Now, by (4.52), (4.53) and (4.57), we obtain that $(\tilde{\mathbf{v}},\tilde{\xi})$ satisfies the linear Poisson problem

$$\begin{cases}\big(\triangle-\mathcal{P}\big(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x})\big)\big)\,\tilde{\mathbf{v}}-\nabla\tilde{\xi}=\mathbf{f} & \text{in }\mathfrak{D},\\ \mathrm{div}\ \tilde{\mathbf{v}}=g & \text{in }\mathfrak{D},\\ \mathrm{Tr}\ \tilde{\mathbf{v}}=\mathbf{h} & \text{on }\Gamma,\end{cases} \tag{4.58}$$

in the sense of distributions. On the other hand, in view of (4.41) and (4.45), we have

$$\begin{cases}\big(\triangle-\mathcal{P}\big(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x})\big)\big)\,\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)-\nabla\mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)=\mathbf{f} & \text{in }\mathfrak{D},\\ \mathrm{div}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)=g & \text{in }\mathfrak{D},\\ \mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)=\mathbf{h} & \text{on }\Gamma.\end{cases} \tag{4.59}$$

Then, comparing (4.58) and (4.59), and using the uniqueness of the solution to the linear Poisson problem for the generalized Brinkman system in the space $\mathcal{X}_t$ (see Theorem 4.1), we obtain

$$\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)=\tilde{\mathbf{v}},\ \ \mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)=\tilde{\xi}. \tag{4.60}$$

Consequently, we have shown that if $s > \frac{1}{2}$ and if $(\mathbf{u}_j, \pi_j) \to (\mathbf{u}, \pi)$ in $\tilde{\mathcal{X}}_t$, then there exists a subsequence $\{(\mathbf{u}_{j_k}, \pi_{j_k})\}_{k\in\mathbb{N}}$ of $\{(\mathbf{u}_j, \pi_j)\}_{j\in\mathbb{N}}$ such that

$$\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k}, \pi_{j_k}) \to \mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u}, \pi) \quad \text{in } \tilde{\mathcal{X}}_t. \tag{4.61}$$

By using the same method as above, we can show that each subsequence of $\{(\mathbf{u}_j, \pi_j)\}_{j\in\mathbb{N}}$ contains a further subsequence such that its image by the operator $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}$ converges to $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u}, \pi)$ in $\tilde{\mathcal{X}}_t$. Therefore,

$$\lim_{j\to\infty} \mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j) = \mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u}, \pi) \quad \text{in } \tilde{\mathcal{X}}_t. \tag{4.62}$$

The continuity of the operator $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}$ from $\left(B_A, \|\cdot\|_{\tilde{\mathcal{X}}_t}\right)$ to $\left(B_A, \|\cdot\|_{\tilde{\mathcal{X}}_s}\right)$

Next, we show that if $\{(\mathbf{u}_j, \pi_j)\}_{j\in\mathbb{N}}$ is a sequence in $(B_A, \|\cdot\|_{\tilde{\mathcal{X}}_s})$, which converges to $(\mathbf{u}, \pi) \in B_A$ in $\tilde{\mathcal{X}}_t$, then each subsequence of $\{\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j)\}_{j\in\mathbb{N}}$ has a further subsequence which converges to $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u}, \pi)$ in $\tilde{\mathcal{X}}_s$. To shorten our notation, we still denote by $\{(\mathbf{u}_j, \pi_j)\}_{j\in\mathbb{N}}$ a subsequence of $\{(\mathbf{u}_j, \pi_j)\}_{j\in\mathbb{N}}$.

To show the desired property, we now consider the Poisson problem

$$\begin{cases} \triangle \mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j) - \nabla \mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j) = \mathbf{f} + \mathcal{P}\big(\mathbf{x}, \mathbf{u}_j(\mathbf{x}), \pi_j(\mathbf{x})\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j) & \text{in } \mathfrak{D}, \\ \operatorname{div} \mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j) = g & \text{in } \mathfrak{D}, \\ \operatorname{Tr} \mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j) = \mathbf{h} & \text{on } \Gamma, \end{cases} \tag{4.63}$$

and we turn to prove the limiting relation

$$\lim_{j\to\infty} \mathcal{P}\big(\mathbf{x}, \mathbf{u}_j, \pi_j\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j) = \mathcal{P}\big(\mathbf{x}, \mathbf{u}, \pi\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}, \pi) \quad \text{in } L^2_{s-\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n). \tag{4.64}$$

Possibly selecting a further subsequence, we can assume that (4.50) holds (with $\mathbf{u}_j$ instead of $\mathbf{u}_{j_k}$). Next, we prove the limiting relation (4.64) by duality and by exploiting the equality $L^2_{s-\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n) = \Big(L^2_{\frac{3}{2}-s;0}(\mathfrak{D}, \mathbb{R}^n)\Big)'$. Indeed, for any $\Psi \in L^2_{\frac{3}{2}-s;0}(\mathfrak{D}, \mathbb{R}^n)$, we have

$$\begin{aligned} &\Big| \int_{\mathfrak{D}} \big\langle \mathcal{P}\big(\mathbf{x}, \mathbf{u}_j, \pi_j\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j) - \mathcal{P}\big(\mathbf{x}, \mathbf{u}, \pi\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}, \pi), \Psi \big\rangle \, d\mathbf{x} \Big| \\ &\quad \le \int_{\mathfrak{D}} \Big| \Big(\mathcal{P}\big(\mathbf{x}, \mathbf{u}_j, \pi_j\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j) - \mathcal{P}\big(\mathbf{x}, \mathbf{u}, \pi\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}, \pi) \Big) \Big| \, |\Psi| d\mathbf{x} \\ &\quad \le \int_{\mathfrak{D}} \big|\mathcal{P}\big(\mathbf{x}, \mathbf{u}_j, \pi_j\big)\big| \, |\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j) - \mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}, \pi)| \, |\Psi| d\mathbf{x} \\ &\qquad + \int_{\mathfrak{D}} \Big| \mathcal{P}\big(\mathbf{x}, \mathbf{u}_j, \pi_j\big) - \mathcal{P}\big(\mathbf{x}, \mathbf{u}, \pi\big) \Big| \, |\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}, \pi)| \, |\Psi| d\mathbf{x}. \end{aligned} \tag{4.65}$$

In addition, by using the Hölder inequality and the inequality (4.39), we obtain that

$$\begin{aligned} &\int_{\mathfrak{D}} \big|\mathcal{P}\big(\mathbf{x}, \mathbf{u}_j, \pi_j\big)\big| \, |\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j) - \mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}, \pi)| \, |\Psi| d\mathbf{x} \\ &\quad \le a \|\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j) - \mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}, \pi)\|_{L^2(\mathfrak{D}, \mathbb{R}^n)} \|\Psi\|_{L^2(\mathfrak{D}, \mathbb{R}^n)} \\ &\quad \le a' \|\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j, \pi_j) - \mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}, \pi)\|_{L^2_{t+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n)} \|\Psi\|_{L^2_{\frac{3}{2}-s;0}(\mathfrak{D}, \mathbb{R}^n)} \to 0 \quad \text{as } j \to \infty, \end{aligned}$$

with a constant $a' \equiv a'(\mathfrak{D},t) > 0$. Hence, for any $\Psi \in L^2_{\frac{3}{2}-s;0}(\mathfrak{D},\mathbb{R}^n)$, one has the limiting relation

$$\lim_{j\to\infty}\int_{\mathfrak{D}} \left|\mathcal{P}\big(\mathbf{x},\mathbf{u}_j,\pi_j\big)\right| \left|\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)-\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)\right| |\Psi| d\mathbf{x} = 0, \tag{4.66}$$

which holds uniformly when Ψ ranges in the unit ball of $L^2_{\frac{3}{2}-s;0}(\mathfrak{D},\mathbb{R}^n)$. On the other hand, in view of (4.50) and the property that $\mathcal{P}$ is a Carathéodory function, we obtain the limiting relation

$$\lim_{j\to\infty}\left|\mathcal{P}\big(\mathbf{x},\mathbf{u}_j,\pi_j\big)-\mathcal{P}\big(\mathbf{x},\mathbf{u},\pi\big)\right| = 0 \ \text{ a.e. } \mathbf{x}\in\mathfrak{D}.$$

Combining such a property with the Hölder inequality, the membership of $|\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)|$ in $L^2(\mathfrak{D})$, the inequality (4.39), and with the Lebesgue Dominated Convergence Theorem, one obtains the limiting relation

$$\lim_{j\to\infty}\int_{\mathfrak{D}} \left|\mathcal{P}\big(\mathbf{x},\mathbf{u}_j,\pi_j\big)-\mathcal{P}\big(\mathbf{x},\mathbf{u},\pi\big)\right| \left|\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)\right| |\Psi| d\mathbf{x} = 0, \tag{4.67}$$

which holds uniformly when Ψ ranges in the unit ball of $L^2_{\frac{3}{2}-s;0}(\mathfrak{D},\mathbb{R}^n)$. The limiting relations (4.65), (4.66) and (4.67) lead to the desired limiting relation (4.64). Hence, the right-hand side of the problem (4.63) converges to $\big(\mathbf{f}+\mathcal{P}\big(\mathbf{x},\mathbf{u},\pi\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi),g,\mathbf{h}\big)$ in the space $L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})\times L^2_s(\Gamma,\mathbb{R}^n)$. Then, the well-posedness of the linear Poisson problem for the Stokes system with Dirichlet condition in $\tilde{\mathcal{X}}_s$ (see [40, Theorem 10.6.2]) yields the desired property

$$\lim_{j\to\infty}\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j) = \mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi) \quad \text{in } \ \tilde{\mathcal{X}}_s. \tag{4.68}$$

Consequently, the nonlinear operator $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}: B_A \to B_A$ is continuous and compact, as asserted.

Existence of a solution to the semilinear Poisson problem (4.38)

Finally, the Schauder Fixed Point Theorem (see Theorem 4.3) applied to the continuous and compact nonlinear operator $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}: B_A \to B_A$, and to the closed, bounded and convex subset B_A of the Banach space $\tilde{\mathcal{X}}_s$, implies that $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}$ has a fixed point $(\mathbf{u},\pi)\in B_A$. This is a solution of the semilinear Poisson problem (4.38) in the space $\tilde{\mathcal{X}}_s$, which satisfies the inequality $\|(\mathbf{u},\pi)\|_{\tilde{\mathcal{X}}_s} \le A$, where A is the constant given by (4.44). Thus, the proof is complete. □

Remark 4.5. The results of Theorem 4.4 can be extended to other Sobolev and Besov spaces by using [40, Theorem 10.6.2], i.e., the well-posedness result in such spaces for the Poisson problem for the Stokes system with Dirichlet boundary condition, embedding results, as well as an argument similar to those in the proof of Theorem 4.4, which we omit for the sake of brevity.

5. The semilinear Brinkman system with nonlinear Robin condition

In this section, we show the existence of a solution of the Poisson problem for the generalized Brinkman system with nonlinear Robin boundary condition and data in L^2-based Sobolev spaces.

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5.1. The linear Poisson problem for the Stokes system with Robin boundary condition

Let us first prove the well-posedness of the Poisson problem for the Stokes system with Robin boundary condition, by using a single-layer potential approach. Note that the existence of a solution to a Robin problem for the Stokes system in a bounded or an exterior Lipschitz domain in $\mathbb{R}^n (n \geq 2)$, with a non-connected compact boundary, has been proved in [44, Theorem 4.1], by exploiting a double-layer potential approach. In particular, the Robin problem for the homogeneous Stokes system in a bounded domain $G \subseteq \mathbb{R}^3$ with Lyapunov boundary $\partial G \in C^{1,\alpha}$, $\alpha \in (0,1)$, and boundary data in $C^{\alpha}(\partial G, \mathbb{R}^3)$, or in $L^s(\partial G, \mathbb{R}^3)$, $s \in (1, \infty)$, has been studied in [32, Theorem 4.3].

Theorem 5.1. *Let $\mathfrak{D} \subseteq \mathbb{R}^n\, (n \geq 2)$ be a bounded Lipschitz domain with connected boundary Γ. Let $s \in (0,1)$. Let $\lambda \in L^{\infty}(\Gamma, \mathbb{R}^n \otimes \mathbb{R}^n)$ be a symmetric matrix-valued function, such that*

$$\langle \lambda \mathbf{v}, \mathbf{v} \rangle_{\Gamma} \geq 0,\ \forall\ \mathbf{v} \in L^2(\Gamma, \mathbb{R}^n) \ \text{ and } \ \langle \lambda \mathbf{v}, \mathbf{v} \rangle_{\Gamma} = 0 \iff \mathbf{v} = \mathbf{0}. \tag{5.1}$$

Then, there exists a constant $C \equiv C(\lambda, s, \mathfrak{D}) > 0$ such that the Poisson problem for the Stokes system with Robin boundary condition:

$$\begin{cases} \triangle \mathbf{v} - \nabla p = \mathbf{f}|_{\mathfrak{D}},\ \mathbf{f} \in L^2_{s-\frac{3}{2};0}(\mathfrak{D}, \mathbb{R}^n), \\ \operatorname{div}\, \mathbf{v} = g \in L^2_{s-\frac{1}{2}}(\mathfrak{D}), \\ \partial_{\nu}(\mathbf{v}, p)_{\mathbf{f},g} + \lambda \mathrm{Tr}\ \mathbf{v} = \mathbf{h} \in L^2_{s-1}(\Gamma, \mathbb{R}^n) \end{cases} \tag{5.2}$$

has a unique solution $(\mathbf{v}, p) \in L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D})$, which satisfies the inequality

$$\|\mathbf{v}\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n)} + \|p\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})} \leq C\Big(\|\mathbf{f}\|_{L^2_{s-\frac{3}{2};0}(\mathfrak{D}, \mathbb{R}^n)} + \|g\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})} + \|\mathbf{h}\|_{L^2_{s-1}(\Gamma, \mathbb{R}^n)} \Big). \tag{5.3}$$

Proof. First, we show that the problem (5.2) has at most one solution $(\mathbf{v}, p) \in \mathcal{X}_s$, where $\mathcal{X}_s := L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D})$. Indeed, assuming that the pair $(\mathbf{v}_0, p_0) \in \mathcal{X}_s$ is a solution of the homogeneous problem associated with (5.2), one has the layer potential representation (see, e.g., [40, (10.95)])

$$\mathbf{v}_0 = \mathbf{V}_{\Gamma} \left(\partial_{\nu}(\mathbf{v}_0, p_0) \right) - \mathbf{W}_{\Gamma} \left(\mathrm{Tr}\ \mathbf{v}_0 \right) = -\mathbf{V}_{\Gamma} \left(\lambda \mathrm{Tr}\ \mathbf{v}_0 \right) - \mathbf{W}_{\Gamma} \left(\mathrm{Tr}\ \mathbf{v}_0 \right) \ \text{ in } \ \mathfrak{D}, \tag{5.4}$$

which leads to the following equation with the unknown $\mathrm{Tr}\ \mathbf{v}_0 \in L^2_s(\Gamma, \mathbb{R}^n)$:

$$\left(\frac{1}{2}\mathbb{I} + \mathbf{K}_{\Gamma} + \mathcal{V}_{\Gamma} \lambda \right) \mathrm{Tr}\ \mathbf{v}_0 = \mathbf{0}. \tag{5.5}$$

Since $\frac{1}{2}\mathbb{I} + \mathbf{K}_{\Gamma} : L^2_s(\Gamma, \mathbb{R}^n) \to L^2_s(\Gamma, \mathbb{R}^n)$ is Fredholm with index zero (see, e.g.,[40, Theorem 10.5.3]) and $\mathcal{V}_{\Gamma}\lambda : L^2_s(\Gamma, \mathbb{R}^n) \to L^2_s(\Gamma, \mathbb{R}^n)$ is compact, the operator $\frac{1}{2}\mathbb{I} + \mathbf{K}_{\Gamma} + \mathcal{V}_{\Gamma}\lambda : L^2_s(\Gamma, \mathbb{R}^n) \to L^2_s(\Gamma, \mathbb{R}^n)$ is Fredholm with index zero as well, for any $s \in (0,1)$. Therefore, this operator is invertible if and only if

$$\mathrm{Ker}\left(\frac{1}{2}\mathbb{I} + \mathbf{K}^*_{\Gamma} + \lambda \mathcal{V}_{\Gamma} : L^2_{-s}(\Gamma, \mathbb{R}^n) \to L^2_{-s}(\Gamma, \mathbb{R}^n) \right) = \{\mathbf{0}\}. \tag{5.6}$$

On the other hand, by using again Proposition 3.2, we obtain the equality

$$\mathrm{Ker}\left(\frac{1}{2}\mathbb{I} + \mathbf{K}^*_{\Gamma} + \lambda \mathcal{V}_{\Gamma} : L^2_{-s}(\Gamma, \mathbb{R}^n) \to L^2_{-s}(\Gamma, \mathbb{R}^n) \right) = \mathrm{Ker}\left(\frac{1}{2}\mathbb{I} + \mathbf{K}^*_{\Gamma} + \lambda \mathcal{V}_{\Gamma} : L^2_{-\frac{1}{2}}(\Gamma, \mathbb{R}^n) \to L^2_{-\frac{1}{2}}(\Gamma, \mathbb{R}^n) \right), \tag{5.7}$$

for any $s \in (0,1)$. Hence, the proof of the property (5.6) reduces to show that

$$\mathrm{Ker}\left(\frac{1}{2}\mathbb{I} + \mathbf{K}^*_{\Gamma} + \lambda \mathcal{V}_{\Gamma} : L^2_{-\frac{1}{2}}(\Gamma, \mathbb{R}^n) \to L^2_{-\frac{1}{2}}(\Gamma, \mathbb{R}^n) \right) = \{\mathbf{0}\}. \tag{5.8}$$

This property follows by means of the Green formula (2.11) and standard arguments of the potential theory, which we omit for the sake of brevity. Consequently, $\frac{1}{2}\mathbb{I} + \mathbf{K}_{\Gamma} + \mathcal{V}_{\Gamma}\lambda : L^2_s(\Gamma, \mathbb{R}^n) \to L^2_s(\Gamma, \mathbb{R}^n)$ is an isomorphism for any $s \in (0,1)$. Hence, the equation (5.5) has only the solution $\mathrm{Tr}\ \mathbf{v}_0 = \mathbf{0}$. By (5.4) and by $\partial_\nu(\mathbf{v}_0,p_0)+\lambda\mathrm{Tr}\ \mathbf{v}_0=\mathbf{0}$, we obtain that $(\mathbf{v}_0,p_0)=(\mathbf{0},0)$. Therefore, the problem (5.2) has at most one solution. It remains to observe that the pair $(\mathbf{v},p)\in L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})$,

$$\begin{aligned}\mathbf{v}&:=\mathcal{N}_{\mathfrak{D}}(\mathbf{f}-\nabla g)+\nabla\mathcal{N}_{\triangle}g+\mathbf{V}_\Gamma\left(\tfrac{1}{2}\mathbb{I}+\mathbf{K}^*_\Gamma+\lambda\mathcal{V}_\Gamma\right)^{-1}\mathbf{h}_1,\\ p&:=\mathcal{Q}_{\mathfrak{D}}(\mathbf{f}-\nabla g)+\mathcal{Q}_\Gamma\left(\tfrac{1}{2}\mathbb{I}+\mathbf{K}^*_\Gamma+\lambda\mathcal{V}_\Gamma\right)^{-1}\mathbf{h}_1,\end{aligned}\tag{5.9}$$

is the unique solution of the Poisson problem with Robin boundary condition (5.2), where $\mathcal{N}_{\mathfrak{D}}$ and $\mathcal{Q}_{\mathfrak{D}}$ are the Newtonian potential and its corresponding pressure potential for the Stokes system in $\mathfrak{D}$, and $\mathcal{N}_\triangle$ is the Newtonian potential for the Laplace operator in $\mathfrak{D}$. In addition, we have that

$$\mathbf{h}_1:=\mathbf{h}-\partial_\nu\left(\mathcal{N}_{\mathfrak{D}}(\mathbf{f}-\nabla g),\mathcal{Q}_{\mathfrak{D}}(\mathbf{f}-\nabla g)\right)-\partial_\nu\left(\nabla\mathcal{N}_\triangle g,0\right)\in L^2_{s-1}(\Gamma,\mathbb{R}^n).$$

On the other hand, the boundedness of the involved layer potentials in (5.9) shows that this solution satisfies the estimate (5.3) in terms of data $(\mathbf{f},g,\mathbf{h})\in L^2_{s-\frac{3}{2};0}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})\times L^2_{s-1}(\Gamma,\mathbb{R}^n)$, with a constant $C\equiv C(\lambda,s,\mathfrak{D})>0$ independent of these data. □

5.2. The linear Poisson problem for the generalized Brinkman system with Robin boundary condition

Theorem 5.2. *Let $\mathfrak{D}\subseteq\mathbb{R}^n\,(n\geq 2)$ be a bounded Lipschitz domain with connected boundary Γ. Let $s\in(0,1)$. Let $\mathcal{P}\in L^\infty(\mathfrak{D},\mathbb{R}^n\otimes\mathbb{R}^n)$ be a matrix-valued function, which satisfies the nonnegativity condition (2.12), and let $\lambda\in L^\infty(\Gamma,\mathbb{R}^n\otimes\mathbb{R}^n)$ be a symmetric matrix-valued function, which satisfies the strong positivity condition (5.1). Then, there exists a constant $C\equiv C(\mathcal{P},\lambda,s,\mathfrak{D})>0$ such that the linear Poisson problem for the generalized Brinkman system with Robin boundary condition:*

$$\begin{cases}\triangle\mathbf{u}-\mathcal{P}\mathbf{u}-\nabla\pi=\mathbf{f}|_{\mathfrak{D}},\ \ \mathbf{f}\in L^2_{s-\frac{3}{2};0}(\mathfrak{D},\mathbb{R}^n),\\ \mathrm{div}\ \mathbf{u}=g\in L^2_{s-\frac{1}{2}}(\mathfrak{D}),\\ \partial_\nu(\mathbf{u},\pi)_{\mathbf{f}+\mathcal{P}\mathbf{u},g}+\lambda\mathrm{Tr}\ \mathbf{u}=\mathbf{h}\in L^2_{s-1}(\Gamma,\mathbb{R}^n)\end{cases}\tag{5.10}$$

has a unique solution $(\mathbf{u},\pi)\in L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})$, which satisfies the inequality

$$\|\mathbf{u}\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)}+\|\pi\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})}\leq C\Big(\|\mathbf{f}\|_{L^2_{s-\frac{3}{2};0}(\mathfrak{D},\mathbb{R}^n)}+\|g\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})}+\|\mathbf{h}\|_{L^2_{s-1}(\Gamma,\mathbb{R}^n)}\Big).\tag{5.11}$$

Proof. Let us consider the following operator associated with the Poisson problem (5.10):

$$A_{\lambda;\mathcal{P}}:\mathcal{X}_s\to\mathcal{W}_s,\quad A_{\lambda;\mathcal{P}}(\mathbf{u},\pi)=\left(\triangle\mathbf{u}-\mathcal{P}\mathbf{u}-\nabla\pi,\mathrm{div}\ \mathbf{u},\partial_\nu(\mathbf{u},\pi)_{\triangle\mathbf{u}-\nabla\pi,\ \mathrm{div}\ \mathbf{u}}+\lambda\mathrm{Tr}\ \mathbf{u}\right),\tag{5.12}$$

where

$$\mathcal{X}_s:=L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D}),\tag{5.13}$$

$$\mathcal{W}_s:=\left\{(\mathbf{F}|_{\mathfrak{D}},G,\mathbf{H}):\mathbf{F}\in L^2_{s-\frac{3}{2};0}(\mathfrak{D},\mathbb{R}^n),\ G\in L^2_{s-\frac{1}{2}}(\mathfrak{D}),\ \mathbf{H}\in L^2_{s-1}(\Gamma,\mathbb{R}^n)\right\}.\tag{5.14}$$

Note that for any $s\in(0,1)$, we have the equality (see, e.g., [36, (3.13)])

$$L^2_{s-\frac{3}{2};z}(\mathfrak{D})=L^2_{s-\frac{3}{2}}(\mathfrak{D}),\tag{5.15}$$

where

$$L^2_{s-\frac{3}{2};z}(\mathfrak{D}):=\left\{f\in\mathcal{D}'(\mathfrak{D}):\exists\ g\in L^2_{s-\frac{3}{2};0}(\mathfrak{D})\ \text{such that}\ f=g|_{\mathfrak{D}}\right\}.\tag{5.16}$$

Also, note that $\triangle\mathbf{v}-\mathcal{P}\mathbf{v}-\nabla q\in L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)$ for any $(\mathbf{v},q)\in\mathcal{X}_s$. In addition, by using Lemma 2.3 (see also Remark 2.5), we obtain the useful relation

$$\partial_{\nu;\mathcal{P}}(\mathbf{v},q)_{\mathbf{F},G}=\partial_{\nu}(\mathbf{v},q)_{\mathbf{F}+\mathcal{P}\mathbf{v},G}, \tag{5.17}$$

for any $(\mathbf{v},q,\mathbf{F},G)\in L^2_{s+\frac12}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac12}(\mathfrak{D})\times L^2_{s-\frac32;0}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac12}(\mathfrak{D})$ such that

$$\triangle\mathbf{v}-\mathcal{P}\mathbf{v}-\nabla q=\mathbf{F}|_{\mathfrak{D}},\quad \mathrm{div}\ \mathbf{v}=G\ \text{ in }\ \mathfrak{D}. \tag{5.18}$$

This relation has suggested the expression of the Robin condition in (5.10). Therefore, the operator $A_{\lambda;\mathcal{P}}$ given by (5.12) can be written as

$$A_{\lambda;\mathcal{P}}=\mathcal{A}_{\lambda}+\mathcal{C}_{\mathcal{P}}, \tag{5.19}$$

where

$$\mathcal{A}_{\lambda}:\mathcal{X}_s\to\mathcal{W}_s,\quad \mathcal{A}_{\lambda}(\mathbf{u},\pi):=\left(\triangle\mathbf{u}-\nabla\pi,\mathrm{div}\ \mathbf{u},\partial_{\nu}(\mathbf{u},\pi)_{\triangle\mathbf{u}-\nabla\pi,\ \mathrm{div}\ \mathbf{u}}+\lambda\mathrm{Tr}\ \mathbf{u}\right), \tag{5.20}$$

$$\mathcal{C}_{\mathcal{P}}:\mathcal{X}_s\to\mathcal{W}_s,\quad \mathcal{C}_{\mathcal{P}}(\mathbf{u},\pi):=(-\mathcal{P}\mathbf{u},0,\mathbf{0}). \tag{5.21}$$

The well-posedness of the Poisson problem for the Stokes system with Robin condition (5.2) (see Theorem 5.1) shows that for any $(\mathbf{F}|_{\mathfrak{D}},\mathbf{G},\mathbf{H})\in\mathcal{W}_s$, there is a unique pair $(\mathbf{v},p)\in\mathcal{X}_s$ such that

$$\triangle\mathbf{v}-\nabla p=\mathbf{F}|_{\mathfrak{D}},\ \mathrm{div}\ \mathbf{u}=G\ \text{ in }\ \mathfrak{D},\ \partial_{\nu}(\mathbf{v},p)_{\mathbf{F},G}+\lambda\mathrm{Tr}\ \mathbf{v}=\mathbf{H}\ \text{ on }\ \Gamma, \tag{5.22}$$

i.e., the associated operator $\mathcal{A}_{\lambda}:\mathcal{X}_s\to\mathcal{W}_s$ is an isomorphism, and hence Fredholm with index zero. In addition, since $\mathcal{P}\in L^{\infty}(\mathfrak{D},\mathbb{R}^n\otimes\mathbb{R}^n)$, the corresponding multiplication operator from $L^2_{s+\frac12}(\mathfrak{D},\mathbb{R}^n)$ to $L^2_{s-\frac32}(\mathfrak{D},\mathbb{R}^n)$, denoted in the same manner as the matrix-valued function $\mathcal{P}$, is compact. Indeed, the diagram

$$\begin{array}{ccc} L^2_{s+\frac12}(\mathfrak{D},\mathbb{R}^n) & \xrightarrow{\ \mathcal{P}\ } & L^2(\mathfrak{D},\mathbb{R}^n)\\ \mathcal{I}_{0;s-\frac32}\circ\mathcal{P}\Big\downarrow & & \Big\downarrow\mathcal{I}_{0;s-\frac32}\\ L^2_{s-\frac32}(\mathfrak{D},\mathbb{R}^n) & \xleftarrow{\ \mathbb{I}\ } & L^2_{s-\frac32}(\mathfrak{D},\mathbb{R}^n) \end{array} \tag{5.23}$$

is commutative and the imbedding of $L^2(\mathfrak{D},\mathbb{R}^n)$ into $L^2_{s-\frac32}(\mathfrak{D},\mathbb{R}^n)$ is compact, i.e., the inclusion operator $\mathcal{I}_{0;s-\frac32}:L^2(\mathfrak{D},\mathbb{R}^n)\to L^2_{s-\frac32}(\mathfrak{D},\mathbb{R}^n)$ is compact. Therefore, the operator $\mathcal{C}_{\mathcal{P}}:\mathcal{X}_s\to\mathcal{W}_s$ given by (5.21) is compact as well. Consequently, the operator $A_{\lambda;\mathcal{P}}=\mathcal{A}_{\lambda}+\mathcal{C}_{\mathcal{P}}:\mathcal{X}_s\to\mathcal{W}_s$ is Fredholm with index zero, for any $s\in(0,1)$. By Proposition 3.2, one then obtains the following equality

$$\mathrm{Ker}\left(A_{\lambda;\mathcal{P}}:\mathcal{X}_s\to\mathcal{W}_s\right)=\mathrm{Ker}\left(A_{\lambda;\mathcal{P}}:\mathcal{X}_{\frac12}\to\mathcal{W}_{\frac12}\right),\ \forall\ s\in(0,1). \tag{5.24}$$

Next, we turn to show that

$$\mathrm{Ker}\left(A_{\lambda;\mathcal{P}}:\mathcal{X}_{\frac12}\to\mathcal{W}_{\frac12}\right)=\{(\mathbf{0},0)\}. \tag{5.25}$$

To show this property, assume that $(\mathbf{u}_0,\pi_0)\in\mathrm{Ker}\left(A_{\lambda;\mathcal{P}}:\mathcal{X}_{\frac12}\to\mathcal{W}_{\frac12}\right)$. By Lemma 2.3, one has the identity

$$2\int_{\mathfrak{D}}E_{jk}(\mathbf{u}_0)E_{jk}(\mathbf{u}_0)d\mathbf{x}+\langle\mathcal{P}\mathbf{u}_0,\mathbf{u}_0\rangle_{\mathfrak{D}}=\langle\partial_{\nu}(\mathbf{u}_0,\pi_0)_{\mathcal{P}\mathbf{u}_0},\mathrm{Tr}\ \mathbf{u}_0\rangle_{\Gamma}=\langle-\lambda\mathrm{Tr}\ \mathbf{u}_0,\mathrm{Tr}\ \mathbf{u}_0\rangle_{\Gamma}, \tag{5.26}$$

where the left-hand side of (5.26) is nonnegative, as $\mathcal{P}\in L^{\infty}(\mathfrak{D},\mathbb{R}^n\otimes\mathbb{R}^n)$ satisfies the nonnegativity condition (2.12), and the right-hand side is less or equal to zero, as $\lambda\in L^{\infty}(\Gamma,\mathbb{R}^n\otimes\mathbb{R}^n)$ satisfies the strong positivity condition (5.1). Therefore,

$$E_{jk}(\mathbf{u}_0)=0\quad\text{in }\ \mathfrak{D},\ j,k=1,\ldots,n,\quad\text{and }\ \mathrm{Tr}\ \mathbf{u}_0=\mathbf{0}\quad\text{on }\ \Gamma. \tag{5.27}$$

The first condition in (5.27) implies that $\mathbf{u}_0$ is a rigid body motion field, i.e., $\mathbf{u}_0=\mathcal{A}\mathbf{x}+\mathbf{b}$, where $\mathbf{b}\in\mathbb{R}^n$ and $\mathcal{A}$ is a skew symmetric matrix ($\mathcal{A}^{\top}=-\mathcal{A}$) of type $n\times n$. But $\mathrm{Tr}\ \mathbf{u}_0=\mathbf{0}$ a.e. on Γ, and thus $\mathcal{A}=0$ and $\mathbf{b}=\mathbf{0}$, i.e., $\mathbf{u}_0=\mathbf{0}$ in $\mathfrak{D}$. This result combined with the generalized Brinkman equation $\triangle\mathbf{u}_0-\mathcal{P}\mathbf{u}_0-\nabla\pi_0=0$ implies that $\pi_0=c_0\in\mathbb{R}$ in $\mathfrak{D}$. However, the second condition in (5.27) implies that $\partial_{\nu}(\mathbf{u}_0,\pi_0)_{\mathcal{P}\mathbf{u}_0}=-\lambda\mathrm{Tr}\ \mathbf{u}_0=\mathbf{0}$ a.e. on Γ, and hence $c_0=0$. Therefore, $\mathbf{u}_0=\mathbf{0}$ and

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$\pi_0 = 0$ in $\mathfrak{D}$. This result shows the property (5.25). Then, by (5.24), the Fredholm operator with index zero $A_{\lambda;\mathcal{P}} : \mathcal{X}_s \to \mathcal{W}_s$ is one-to-one, i.e., an isomorphism, for any $s \in (0,1)$. This property implies that the linear Poisson problem for the generalized Brinkman system with Robin boundary condition (5.10) has a unique solution $(\mathbf{u}, \pi) \in L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D})$. In addition, the boundedness of the operator $A_{\lambda;\mathcal{P}} : \mathcal{X}_s \to \mathcal{W}_s$ and of the restriction operator from $L^2_{s-\frac{3}{2};0}(\mathfrak{D}, \mathbb{R}^n)$ to $L^2_{s-\frac{3}{2};z}(\mathfrak{D}, \mathbb{R}^n)$ (see, e.g., [36, 3.6]) implies that there exists a constant $C \equiv C(\mathcal{P}, \lambda, s, \mathfrak{D}) > 0$ such that

$$\begin{aligned} \|\mathbf{u}\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)} + \|\pi\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})} &= \|A^{-1}_{\lambda;\mathcal{P}}(\mathbf{f}|_{\mathfrak{D}}, g, \mathbf{h})\|_{\mathcal{X}_s} \\ &\le C\Big(\|\mathbf{f}\|_{L^2_{s-\frac{3}{2};0}(\mathfrak{D},\mathbb{R}^n)} + \|g\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})} + \|\mathbf{h}\|_{L^2_{s-1}(\Gamma,\mathbb{R}^n)}\Big). \end{aligned} \tag{5.28}$$

Hence, the solution $(\mathbf{u}, \pi)$ satisfies the desired estimate (5.11), and the proof is complete. □

Recalling that $\mathcal{X}_s$ is the space defined in (5.14), we now consider the operators

$$\begin{aligned} &\mathfrak{L}_{1;\mathfrak{R}} : \mathcal{X}_s \to L^2_{s-\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n),\ \mathfrak{L}_{1;\mathfrak{R}}(\mathbf{u}, \pi) := (\triangle - \mathcal{P})\mathbf{u} - \nabla\pi, \\ &\mathfrak{L}_{2;\mathfrak{R}} : \mathcal{X}_s \to L^2_{s-\frac{1}{2}}(\mathfrak{D}),\ \mathfrak{L}_{2;\mathfrak{R}}(\mathbf{u}, \pi) := \operatorname{div} \mathbf{u}, \\ &\mathfrak{L}_{3;\mathfrak{R}} : \mathcal{X}_s \to L^2_{s-1}(\Gamma, \mathbb{R}^n),\ \mathfrak{L}_{3;\mathfrak{R}}(\mathbf{u}, \pi) := \partial_\nu(\mathbf{u}, \pi)_{\mathfrak{L}_{1;\mathfrak{R}}(\mathbf{u},\pi)+\mathcal{P}\mathbf{u},\mathfrak{L}_{2;\mathfrak{R}}(\mathbf{u},\pi)} + \lambda \mathrm{Tr}\ \mathbf{u}. \end{aligned} \tag{5.29}$$

Then, we have the following result.

Lemma 5.3. *Let $\mathfrak{D} \subseteq \mathbb{R}^n$ $(n \ge 2)$ be a bounded Lipschitz domain with connected boundary Γ. Let $s \in (0,1)$, α, $a \in (0, +\infty)$, $\alpha \le a$. Then, there exists a constant $C \equiv C(a, \alpha, s, \mathfrak{D}) > 0$ such that*

$$\|(\mathbf{u}, \pi)\|_{\mathcal{X}_s} \le C\Big(\|\mathfrak{L}_{1;\mathfrak{R}}(\mathbf{u}, \pi)\|_{L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)} + \|\mathfrak{L}_{2;\mathfrak{R}}(\mathbf{u}, \pi)\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})} + \|\mathfrak{L}_{3;\mathfrak{R}}(\mathbf{u}, \pi)\|_{L^2_{s-1}(\Gamma,\mathbb{R}^n)}\Big), \tag{5.30}$$

for all $(\mathbf{u}, \pi) \in \mathcal{X}_s$, for any $\mathcal{P} \in L^\infty(\mathfrak{D}, \mathbb{R}^n \otimes \mathbb{R}^n)$, which satisfies the nonnegativity condition (2.12) and the inequality

$$\|\mathcal{P}\|_{L^\infty(\mathfrak{D}\times\mathbb{R}^n\times\mathbb{R},\mathbb{R}^n\otimes\mathbb{R}^n)} \le a, \tag{5.31}$$

and for any symmetric matrix-valued function $\lambda \in L^\infty(\Gamma, \mathbb{R}^n \otimes \mathbb{R}^n)$, which satisfies the conditions

$$\langle\lambda\mathbf{v}, \mathbf{v}\rangle_\Gamma \ge \alpha\|\mathbf{v}\|^2_{L^2(\Gamma,\mathbb{R}^n)},\ \forall\ \mathbf{v} \in L^2(\Gamma, \mathbb{R}^n), \tag{5.32}$$

$$\|\lambda\|_{L^\infty(\Gamma,\mathbb{R}^n\otimes\mathbb{R}^n)} \le a. \tag{5.33}$$

The proof of Lemma 5.3 is based on the well-posedness result in Theorem 5.2 and on arguments similar to those in the proof of Lemma 4.2, which we omit for the sake of brevity.

5.3. Existence result for the Poisson problem for the semilinear Brinkman system with nonlinear Robin boundary condition

Next, we consider a semilinear Poisson problem with nonlinear Robin boundary condition in L^2-based Sobolev spaces on a bounded Lipschitz domain $\mathfrak{D} \subseteq \mathbb{R}^n$ $(n \ge 2)$. This problem requires to show the existence of a pair $(\mathbf{u}, \pi) \in L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D})$, such that:

$$\begin{cases} \big(\triangle - \mathcal{P}\big(\mathbf{x}, \mathbf{u}(\mathbf{x}), \pi(\mathbf{x})\big)\big)\,\mathbf{u} - \nabla\pi = \mathbf{f}|_{\mathfrak{D}},\ \mathbf{f} \in L^2_{s-\frac{3}{2};0}(\mathfrak{D}, \mathbb{R}^n), \\ \operatorname{div}\ \mathbf{u} = g \in L^2_{s-\frac{1}{2}}(\mathfrak{D}), \\ \partial_\nu(\mathbf{u}, \pi)_{\mathbf{f}+\mathcal{P}(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x}))\mathbf{u},\,g} + \lambda\,(\mathbf{x}, \mathrm{Tr}\ \mathbf{u}(\mathbf{x}))\ \mathrm{Tr}\ \mathbf{u} = \mathbf{h} \in L^2_{s-1}(\Gamma, \mathbb{R}^n). \end{cases} \tag{5.34}$$

Assume that $\mathcal{P} : \mathfrak{D} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \otimes \mathbb{R}^n$ and $\lambda : \Gamma \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ are two essentially bounded matrix-valued Carathéodory functions, such that $\mathcal{P}$ satisfies the nonnegativity condition (4.36) and λ

satisfies the following condition: There exists a constant $\alpha > 0$ and a subset N_Γ of measure zero of Γ such that

$$\langle \lambda(\mathbf{x},\mathbf{v})\mathbf{b},\mathbf{b}\rangle \geq \alpha|\mathbf{b}|^2,\ \forall\ \mathbf{b}\in\mathbb{R}^n,\ (\mathbf{x},\mathbf{v})\in(\Gamma\setminus N_\Gamma)\times\mathbb{R}^n. \tag{5.35}$$

Based on Lemma 5.3 and the Schauder Fixed Point Theorem (see Theorem 4.3), we obtain the following existence result for the semilinear Poisson problem (5.34).

Theorem 5.4. *Let $\mathfrak{D}\subseteq\mathbb{R}^n\,(n\geq 2)$ be a bounded Lipschitz domain with connected boundary Γ. Let $s\in(\frac{1}{2},1)$, $\alpha,\ a\in(0,+\infty)$, $\alpha\leq a$. Then, there exists a constant $C\equiv C(a,\alpha,s,\mathfrak{D})>0$ with the following property: For any $(\mathbf{f},g,\mathbf{h})\in L^2_{s-\frac{3}{2};0}(\mathfrak{D},\mathbb{R}^n)\times L^p_{s-\frac{1}{2}}(\mathfrak{D})\times L^2_{s-1}(\Gamma,\mathbb{R}^n)$, for any essentially bounded Carathéodory function $\mathcal{P}$ from $\mathfrak{D}\times\mathbb{R}^n\times\mathbb{R}$ to $\mathbb{R}^n\otimes\mathbb{R}^n$, satisfying the nonnegativity condition (4.36) and the inequality $\|\mathcal{P}\|_{L^\infty(\mathfrak{D}\times\mathbb{R}^n\times\mathbb{R},\mathbb{R}^n\otimes\mathbb{R}^n)}\leq a$, and for any essentially bounded Carathéodory function λ from $\Gamma\times\mathbb{R}^n$ to the set of symmetric elements of $\mathbb{R}^n\otimes\mathbb{R}^n$, satisfying the condition (5.35) and the inequality $\|\lambda\|_{L^\infty(\Gamma\times\mathbb{R}^n,\mathbb{R}^n\otimes\mathbb{R}^n)}\leq a$, there exists at least a solution $(\mathbf{u},\pi)\in L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})$ of the semilinear Poisson problem (5.34) such that*

$$\|(\mathbf{u},\pi)\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})}\leq C\Big(\|\mathbf{f}\|_{L^2_{s-\frac{3}{2};0}(\mathfrak{D},\mathbb{R}^n)}+\|g\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})}+\|\mathbf{h}\|_{L^2_{s-1}(\Gamma,\mathbb{R}^n)}\Big). \tag{5.36}$$

Proof. First, for a fixed $(\mathbf{u},\pi)\in\mathcal{X}_s$, where $\mathcal{X}_s=L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})$, we consider the auxiliary linear Poisson problem with the Robin boundary condition

$$\begin{cases}\big(\triangle-\mathcal{P}\big(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x})\big)\big)\,\mathbf{v}-\nabla\zeta=\mathbf{f}|_{\mathfrak{D}},\ \mathbf{f}\in L^2_{s-\frac{3}{2};0}(\mathfrak{D},\mathbb{R}^n),\\ \operatorname{div}\,\mathbf{v}=g\in L^2_{s-\frac{1}{2}}(\mathfrak{D}),\\ \partial_\nu(\mathbf{v},\zeta)_{\mathbf{f}+\mathcal{P}(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x}))\mathbf{v},\,g}+\lambda\left(\mathbf{x},\operatorname{Tr}\,\mathbf{u}(\mathbf{x})\right)\operatorname{Tr}\,\mathbf{v}=\mathbf{h}\in L^2_{s-1}(\Gamma,\mathbb{R}^n)\end{cases} \tag{5.37}$$

with the same given data $\mathbf{f}$, g and $\mathbf{h}$ as in the semilinear Poisson problem (5.34). This problem has a unique solution $(\mathbf{v},\zeta)\in\mathcal{X}_s$, which satisfies the inequality (see (5.30))

$$\begin{aligned}\|(\mathbf{v},\zeta)\|_{\mathcal{X}_s}\leq C\Big(&\|\big(\triangle-\mathcal{P}\big(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x})\big)\big)\mathbf{v}-\nabla\zeta\|_{L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)}+\|\operatorname{div}\,\mathbf{v}\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})}\\ &+\|\partial_\nu(\mathbf{v},\zeta)_{\mathbf{f}+\mathcal{P}(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x}))\mathbf{v},\,g}+\lambda\left(\mathbf{x},\operatorname{Tr}\,\mathbf{u}(\mathbf{x})\right)\operatorname{Tr}\,\mathbf{v}\|_{L^2_{s-1}(\Gamma,\mathbb{R}^n)}\Big)\end{aligned} \tag{5.38}$$

with some constant $C\equiv C(a,\alpha,s,\mathfrak{D})>0$. Let $\mathcal{R}_{\mathfrak{D}}\mathbf{v}:=\mathbf{v}|_{\mathfrak{D}}$ denote the operator of restriction to $\mathfrak{D}$. In view of (5.37) and by the boundedness of the operator $\mathcal{R}_{\mathfrak{D}}:L^2_{s-\frac{3}{2};0}(\mathfrak{D},\mathbb{R}^n)\to L^2_{s-\frac{3}{2};z}(\mathfrak{D},\mathbb{R}^n)$, where $L^2_{s-\frac{3}{2};z}(\mathfrak{D},\mathbb{R}^n):=\{\mathbf{F}=(F_1,\ldots,F_n):F_i\in L^2_{s-\frac{3}{2};z}(\mathfrak{D}),\ i=1,\ldots,n\}$ (see [36, (3.6),(3.12)]), the inequality (5.38) becomes

$$\|(\mathbf{v},\zeta)\|_{\mathcal{X}_s}\leq A, \tag{5.39}$$

where

$$A:=C\Big(\|\mathbf{f}\|_{L^2_{s-\frac{3}{2};0}(\mathfrak{D},\mathbb{R}^n)}+\|g\|_{L^2_{s-\frac{1}{2}}(\mathfrak{D})}+\|\mathbf{h}\|_{L^2_{s-1}(\Gamma,\mathbb{R}^n)}\Big)>0. \tag{5.40}$$

Therefore, $(\mathbf{v},\zeta)\in B_A$, where $B_A:=\{z\in\mathcal{X}_s:\|z\|_{\mathcal{X}_s}\leq A\}$. We now consider the nonlinear operator

$$\mathcal{T}_{\mathbf{f},g,\mathbf{h}}:B_A\to B_A,\quad B_A\ni(\mathbf{u},\pi)\stackrel{\mathcal{T}_{\mathbf{f},g,\mathbf{h}}}{\longmapsto}(\mathbf{v},\zeta), \tag{5.41}$$

which maps $(\mathbf{u},\pi)\in B_A$ to the unique solution $(\mathbf{v},\zeta)\in B_A$ of the linear Poisson problem with the Robin boundary condition (5.37). This operator is well defined, as follows from the a priori estimate (5.30) in the linear case. We now show that $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}:B_A\to B_A$ is a continuous and compact operator.

Let $\{(\mathbf{u}_j,\pi_j)\}_{j\in\mathbb{N}}$ be a bounded sequence in $(B_A,\|\cdot\|_{\mathcal{X}_s})$. Let $t\in(\frac{1}{2},1)$, $t<s$. Since the embedding $\mathcal{X}_s\hookrightarrow\mathcal{X}_t$ is compact, there exists a subsequence $\{(\mathbf{u}_{j_k},\pi_{j_k})\}_{k\in\mathbb{N}}$ of $\{(\mathbf{u}_j,\pi_j)\}_{j\in\mathbb{N}}$ that converges to an element $(\tilde{\mathbf{u}},\tilde{\pi})\in\mathcal{X}_t$, i.e.,

$$\|(\mathbf{u}_{j_k},\pi_{j_k})-(\tilde{\mathbf{u}},\tilde{\pi})\|_{\mathcal{X}_t}\to 0\qquad\text{as }k\to\infty. \tag{5.42}$$

In addition, since $\mathcal{X}_s$ is a reflexive Banach space, one can select a further subsequence of the bounded sequence $\{(\mathbf{u}_{j_k},\pi_{j_k})\}_{k\in\mathbb{N}}$ in $\mathcal{X}_s$, still denoted by $\{(\mathbf{u}_{j_k},\pi_{j_k})\}_{k\in\mathbb{N}}$, which converges weakly to an element $(\mathbf{u}_0,\pi_0)\in B_A$, i.e.,

$$\lim_{k\to\infty}\langle\varphi,(\mathbf{u}_{j_k},\pi_{j_k})-(\mathbf{u}_0,\pi_0)\rangle_{\mathfrak{D}}=0,\quad \forall\,\varphi\in(\mathcal{X}_s)'. \tag{5.43}$$

In view of (5.43) and the property that the convergence in norm of $\mathcal{X}_t$ implies the weak convergence, one obtains the equality $(\mathbf{u}_0,\pi_0)=(\tilde{\mathbf{u}},\tilde{\pi})$, which shows that the proof of compactness of the operator $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}$ on $(B_A,\|\cdot\|_{\mathcal{X}_s})$ reduces to the continuity of $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}$ from $(B_A,\|\cdot\|_{\mathcal{X}_t})$ to $(B_A,\|\cdot\|_{\mathcal{X}_s})$, whenever $\frac{1}{2}<t<s<1$.

Before we show such a continuity, we prove an intermediate statement. Indeed, we prove that $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}$ is continuous from $(B_A,\|\cdot\|_{\mathcal{X}_t})$ to $(B_A,\|\cdot\|_{\mathcal{X}_t})$.

The continuity of the operator $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}$ from $\left(B_A,\|\cdot\|_{\mathcal{X}_t}\right)$ to $\left(B_A,\|\cdot\|_{\mathcal{X}_t}\right)$

Let $\{(\mathbf{u}_j,\pi_j)\}_{j\in\mathbb{N}}$ be a sequence in B_A which converges to $(\mathbf{u},\pi)\in B_A$ with respect to the norm of $\mathcal{X}_t$, i.e.,

$$\|(\mathbf{u}_j,\pi_j)-(\mathbf{u},\pi)\|_{\mathcal{X}_t}\to 0\quad\text{as}\quad j\to\infty. \tag{5.44}$$

In particular, we note that for $\frac{1}{2}<t<s<1$, the convergence in norm of $\mathcal{X}_t$ implies the L^2-convergence. Then, one can extract a subsequence $\{(\mathbf{u}_{j_k},\pi_{j_k})\}_{k\in\mathbb{N}}$ of the sequence $\{(\mathbf{u}_j,\pi_j)\}_{j\in\mathbb{N}}$, which converges a.e. to $(\mathbf{u},\pi)$. Therefore,

$$\lim_{k\to\infty}(\mathbf{u}_{j_k},\pi_{j_k})=(\mathbf{u},\pi)\ \text{ a.e. in } \mathfrak{D}. \tag{5.45}$$

In addition, in view of (5.41), $\{(\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j),\mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j))\}_{j\in\mathbb{N}}\subseteq\mathcal{X}_s$ is a bounded sequence in $\mathcal{X}_s$, where $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}=(\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}},\mathcal{T}_{2;\mathbf{f},g,\mathbf{h}})$. Then, by the compactness of the embedding $\mathcal{X}_s\hookrightarrow\mathcal{X}_t$, possibly considering a subsequence, we can assume that $\{(\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}),\mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}))\}_{k\in\mathbb{N}}$ converges to an element $(\tilde{\mathbf{v}},\tilde{\xi})\in\mathcal{X}_t$. Thus,

$$\lim_{k\to\infty}\left\|\left(\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}),\mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})\right)-(\tilde{\mathbf{v}},\tilde{\xi})\right\|_{\mathcal{X}_t}=0. \tag{5.46}$$

We now consider the semilinear Poisson problem

$$\begin{cases}\left(\triangle-\mathcal{P}\big(\mathbf{x},\mathbf{u}_{j_k}(\mathbf{x}),\pi_{j_k}(\mathbf{x})\big)\right)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})-\nabla\mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})=\mathbf{f}|_{\mathfrak{D}},\\ \operatorname{div}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})=g \text{ in } \mathfrak{D},\\ \partial_\nu\big(\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}),\mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})\big)_{\mathbf{f}+\mathcal{P}(\mathbf{x},\mathbf{u}_{j_k}(\mathbf{x}),\pi_{j_k}(\mathbf{x}))\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}),\,g}\\ \qquad+\lambda\left(\mathbf{x},\operatorname{Tr}\,\mathbf{u}_{j_k}(\mathbf{x})\right)\operatorname{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})=\mathbf{h} \text{ on } \Gamma.\end{cases} \tag{5.47}$$

Note that $\mathcal{P}\big(\mathbf{x},\mathbf{u}_{j_k},\pi_{j_k}\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})\in L^2(\mathfrak{D},\mathbb{R}^n)$. Since $\mathcal{P}$ is a Carathéodory function, the inequality $\|\mathcal{P}\|_{L^\infty(\mathfrak{D}\times\mathbb{R}^n\times\mathbb{R},\mathbb{R}^n\otimes\mathbb{R}^n)}\le a$ and (5.41) imply that the sequence $\{\mathcal{P}\big(\mathbf{x},\mathbf{u}_{j_k},\pi_{j_k}\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})\}_{k\in\mathbb{N}}$ is bounded in $L^2(\mathfrak{D},\mathbb{R}^n)$. Then, possibly selecting a subsequence, we obtain the limiting relation

$$\lim_{k\to\infty}\mathcal{P}\big(\mathbf{x},\mathbf{u}_{j_k},\pi_{j_k}\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})=\mathcal{P}\big(\mathbf{x},\mathbf{u},\pi\big)\tilde{\mathbf{v}} \tag{5.48}$$

in the weak-* topology of $L^2(\mathfrak{D},\mathbb{R}^n)$ (see the proof of the property (4.53)). By (5.44) we also have

$$\|\operatorname{Tr}\,\mathbf{u}_{j_k}-\operatorname{Tr}\,\mathbf{u}\|_{L^2_t(\Gamma,\mathbb{R}^n)}\to 0\ \text{ as }\ k\to\infty.$$

Then, possibly selecting a subsequence, we can assume that $\lim_{k\to\infty}\operatorname{Tr}\,\mathbf{u}_{j_k}=\operatorname{Tr}\,\mathbf{u}$ a.e. on Γ. Since $\lambda(\cdot,\cdot)$ is a Carathéodory function, we deduce that $\lim_{k\to\infty}\lambda\left(\mathbf{x},\operatorname{Tr}\,\mathbf{u}_{j_k}(\mathbf{x})\right)=\lambda\left(\mathbf{x},\operatorname{Tr}\,\mathbf{u}(\mathbf{x})\right)$ a.a. $\mathbf{x}\in\Gamma$. In addition, λ is essentially bounded, and then, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{k\to\infty}\lambda\left(\mathbf{x},\operatorname{Tr}\,\mathbf{u}_{j_k}(\mathbf{x})\right)=\lambda\left(\mathbf{x},\operatorname{Tr}\,\mathbf{u}(\mathbf{x})\right)\quad\text{in}\quad L^2(\Gamma).$$

By (5.46), we have $\lim_{k\to\infty} \mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}) = \mathrm{Tr}\ \tilde{\mathbf{v}}$ in $L_t^2(\Gamma,\mathbb{R}^n) \hookrightarrow L^2(\Gamma,\mathbb{R}^n)$. Thus,

$$\lim_{k\to\infty} \lambda(\mathbf{x}, \mathrm{Tr}\ \mathbf{u}_{j_k}(\mathbf{x}))\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}) = \lambda(\mathbf{x}, \mathrm{Tr}\ \mathbf{u}(\mathbf{x}))\mathrm{Tr}\ \tilde{\mathbf{v}} \ \text{ in } \ L^1(\Gamma,\mathbb{R}^n) \tag{5.49}$$

and hence in the sense of distributions in Γ.

Now let $\mathcal{Z} : L^2_{1-t}(\Gamma,\mathbb{R}^n) \to L^2_{\frac{3}{2}-t}(\mathfrak{D},\mathbb{R}^n)$ be a right inverse of the non-tangential trace operator $\mathrm{Tr} : L^2_{\frac{3}{2}-t}(\mathfrak{D},\mathbb{R}^n) \to L^2_{1-t}(\Gamma,\mathbb{R}^n)$. Then for any $k\in\mathbb{N}$ we have (see (2.19))

$$\begin{aligned} &\left\langle \partial_\nu\left(\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}), \mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})\right)_{\mathbf{f}+\mathcal{P}\left(\mathbf{x},\mathbf{u}_{j_k}(\mathbf{x}),\pi_{j_k}(\mathbf{x})\right)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}),g}, \Phi\right\rangle_\Gamma \\ &= 2\langle \mathbb{E}(\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})), \mathbb{E}(\mathcal{Z}\Phi)\rangle_{\mathfrak{D}} - \langle \mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}), \mathrm{div}\mathcal{Z}\Phi)\rangle_{\mathfrak{D}} + \langle \nabla g, \mathcal{Z}\Phi\rangle_{\mathfrak{D}} \\ &+ \langle \mathbf{f}, \mathcal{Z}\Phi\rangle_{\mathfrak{D}} + \int_{\mathfrak{D}} \left\langle \mathcal{P}\left(\mathbf{x},\mathbf{u}_{j_k}(\mathbf{x}),\pi_{j_k}(\mathbf{x})\right)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k}(\mathbf{x}),\pi_{j_k}(\mathbf{x})), (\mathcal{Z}\Phi)(\mathbf{x})\right\rangle d\mathbf{x}, \end{aligned} \tag{5.50}$$

for all $\Phi \in C^\infty_{\mathrm{comp}}(\Gamma,\mathbb{R}^n)$. Also, if $\Phi \in C^\infty_{\mathrm{comp}}(\Gamma,\mathbb{R}^n)$ then $\mathcal{Z}\Phi \in L^2_{\frac{3}{2}-t}(\mathfrak{D},\mathbb{R}^n) \hookrightarrow L^2(\mathfrak{D},\mathbb{R}^n)$, $\mathbb{E}(\mathcal{Z}\Phi) \in L^2_{\frac{1}{2}-t}(\mathfrak{D},\mathbb{R}^n\otimes\mathbb{R}^n)$ and $\mathrm{div}(\mathcal{Z}\Phi) \in L^2_{\frac{1}{2}-t}(\mathfrak{D})$.

Now, by (5.46), we have

$$\lim_{k\to\infty} \mathbb{E}(\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})) = \mathbb{E}\tilde{\mathbf{v}} \ \text{ in } \ L^2_{t-\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n\otimes\mathbb{R}^n), \ \ \lim_{k\to\infty} \mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}) = \tilde{\xi} \ \text{ in } \ L^2_{t-\frac{1}{2}}(\mathfrak{D}),$$

and, thus, the limiting relations (5.48), (5.49) and the equality (5.50) imply that

$$\begin{aligned} &\lim_{k\to\infty} \Big(\partial_\nu\left(\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}), \mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})\right)_{\mathbf{f}+\mathcal{P}\left(\mathbf{x},\mathbf{u}_{j_k}(\mathbf{x}),\pi_{j_k}(\mathbf{x})\right)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}),\, g} \\ &\quad + \lambda\left(\mathbf{x}, \mathrm{Tr}\ \mathbf{u}_{j_k}(\mathbf{x})\right)\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})\Big) = \partial_\nu(\tilde{\mathbf{v}},\tilde{\xi})_{\mathbf{f}+\mathcal{P}(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x}))\tilde{\mathbf{v}},g} + \lambda\left(\mathbf{x}, \mathrm{Tr}\ \mathbf{u}(\mathbf{x})\right)\mathrm{Tr}\ \tilde{\mathbf{v}} \end{aligned} \tag{5.51}$$

in the sense of distributions in Γ. Also, by the limiting relation (5.46), we have

$$\lim_{k\to\infty} \left(\triangle\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}) - \nabla\mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})\right) = \triangle\tilde{\mathbf{v}} - \nabla\tilde{\xi}, \ \lim_{k\to\infty} \mathrm{div}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}) = \mathrm{div}\ \tilde{\mathbf{v}} \tag{5.52}$$

in the sense of distributions in $\mathfrak{D}$.

By (5.47)–(5.52), we obtain that $(\tilde{\mathbf{v}},\tilde{\xi})$ satisfies the linear Poisson problem with Robin boundary condition

$$\begin{cases} \left(\triangle - \mathcal{P}\left(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x})\right)\right)\tilde{\mathbf{v}} - \nabla\tilde{\xi} = \mathbf{f}|_{\mathfrak{D}} & \text{in } \mathfrak{D}, \\ \mathrm{div}\ \tilde{\mathbf{v}} = g & \text{in } \mathfrak{D}, \\ \partial_\nu\left(\tilde{\mathbf{v}},\tilde{\xi}\right)_{\mathbf{f}+\mathcal{P}(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x}))\tilde{\mathbf{v}},g} + \lambda\left(\mathbf{x}, \mathrm{Tr}\ \mathbf{u}(\mathbf{x})\right)\mathrm{Tr}\ \tilde{\mathbf{v}} = \mathbf{h} & \text{on } \Gamma \end{cases} \tag{5.53}$$

in the sense of distributions.

On the other hand, in view of (5.37) and (5.41), we have

$$\begin{cases} \left(\triangle - \mathcal{P}\left(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x})\right)\right)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi) - \nabla\mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi) = \mathbf{f}|_{\mathfrak{D}} \text{ in } \mathfrak{D}, \\ \mathrm{div}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi) = g \text{ in } \mathfrak{D}, \\ \partial_\nu\left(\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi), \mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)\right)_{\mathbf{f}+\mathcal{P}(\mathbf{x},\mathbf{u},\pi)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi),g} + \lambda\left(\mathbf{x}, \mathrm{Tr}\ \mathbf{u}(\mathbf{x})\right)\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi) = \mathbf{h} \text{ on } \Gamma. \end{cases} \tag{5.54}$$

Then, by (5.53) and (5.54), Theorem 5.2 implies that

$$\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi) = \tilde{\mathbf{v}}, \ \mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi) = \tilde{\xi}. \tag{5.55}$$

Consequently, for $s\in(\frac{1}{2},1)$ given, we have shown that if $(\mathbf{u}_j,\pi_j) \to (\mathbf{u},\pi)$ in B_A, with respect to the norm of $\mathcal{X}_t$, then there exists a subsequence $\{(\mathbf{u}_{j_k},\pi_{j_k})\}_{k\in\mathbb{N}}$ of $\{(\mathbf{u}_j,\pi_j)\}_{j\in\mathbb{N}}$ such that

$$\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}) \to \mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi) \quad \text{in } \mathcal{X}_t. \tag{5.56}$$

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In fact, we can show that each subsequence of $\{(\mathbf{u}_j,\pi_j)\}_{j\in\mathbb{N}}$ contains a further subsequence such that its image by the operator $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}$ converges to $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)$ with respect to the norm of $\mathcal{X}_t$. Therefore, we obtain the limiting relation

$$\lim_{j\to\infty}\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)=\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)\quad \text{in } \mathcal{X}_t. \tag{5.57}$$

The continuity of the operator $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}$ from $(B_A,\|\cdot\|_{\mathcal{X}_t})$ to $(B_A,\|\cdot\|_{\mathcal{X}_s})$

Next, we show that if $\{(\mathbf{u}_j,\pi_j)\}_{j\in\mathbb{N}}$ is a sequence in B_A which converges to $(\mathbf{u},\pi)\in B_A$, with respect to the norm of $\mathcal{X}_t$, then $\{\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)\}_{j\in\mathbb{N}}$ converges to $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)$ with respect to the norm of $\mathcal{X}_s$.

To do so, we first observe that the definition of the operator $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}$ and the formula (5.17) imply

$$\begin{cases}\triangle\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)-\nabla\mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)=\mathbf{f}|_{\mathfrak{D}}+\mathcal{P}\big(\mathbf{x},\mathbf{u}_j(\mathbf{x}),\pi_j(\mathbf{x})\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j),\\ \operatorname{div}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)=g \quad \text{in } \mathfrak{D},\\ \partial_\nu\big(\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j),\mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)\big)_{\mathbf{f}+\mathcal{P}(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x}))\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi),g}+\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)\\ =-\partial_\nu(0,0)\mathcal{P}(\mathbf{x},\mathbf{u}_j(\mathbf{x}),\pi_j(\mathbf{x}))\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)-\mathcal{P}(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x}))\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi),0\\ \quad+\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)-\lambda(\mathbf{x},\mathrm{Tr}\ \mathbf{u}_j(\mathbf{x}))\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)+\mathbf{h} \quad \text{on } \Gamma.\end{cases} \tag{5.58}$$

By using arguments similar to those in the proof of the limiting relation (4.64), we can prove that

$$\lim_{j\to\infty}\mathcal{P}\big(\mathbf{x},\mathbf{u}_j,\pi_j\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)=\mathcal{P}\big(\mathbf{x},\mathbf{u},\pi\big)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)\quad \text{in } L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n). \tag{5.59}$$

In addition, by the convergence of $\{(\mathbf{u}_j,\pi_j)\}_{j\in\mathbb{N}}$ to $(\mathbf{u},\pi)$ in $\mathcal{X}_t$, and by the definition (2.19) of the conormal derivative and by (5.59), we obtain the limiting relations

$$\begin{aligned}&\lim_{j\to\infty}\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)=\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)\ \text{ in } L^2_t(\Gamma,\mathbb{R}^n)\hookrightarrow L^2_{s-1}(\Gamma,\mathbb{R}^n),\\ &\lim_{j\to\infty}\{\partial_\nu(0,0)\mathcal{P}(\mathbf{x},\mathbf{u}_j(\mathbf{x}),\pi_j(\mathbf{x}))\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)-\mathcal{P}(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x}))\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi),0\}=0\ in\ L^2_{s-1}(\Gamma,\mathbb{R}^n).\end{aligned} \tag{5.60}$$

Then the Sobolev Embedding Theorem implies the limiting relations

$$\begin{aligned}&\lim_{j\to\infty}\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)=\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)\ \text{ in } L^{\frac{2(n-1)}{n-1-2t}}(\Gamma,\mathbb{R}^n),\quad \text{if } n\geq 3\\ &\lim_{j\to\infty}\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)=\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)\ \text{ in } L^{\infty}(\Gamma,\mathbb{R}^n),\qquad \text{if } n=2.\end{aligned} \tag{5.61}$$

On the other hand, by the convergence of $\{\mathrm{Tr}\ \mathbf{u}_j\}_{j\in\mathbb{N}}$ to $\mathrm{Tr}\ \mathbf{u}$ in $L^2_t(\Gamma,\mathbb{R}^n)\hookrightarrow L^2(\Gamma,\mathbb{R}^n)$, there exists a subsequence $\{\mathbf{u}_{j_k}\}_{k\in\mathbb{N}}$ of $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ such that $\lim_{k\to\infty}\mathrm{Tr}\ \mathbf{u}_{j_k}=\mathrm{Tr}\ \mathbf{u}$ a.e. on Γ. Now, if $n\geq 3$, we choose $t^*\in(2,+\infty)$ such that $\frac{(n-1)-2t}{2(n-1)}+\frac{1}{t^*}<\frac{1}{2}$. Instead, if $n=2$, we choose $t^*\in(2,+\infty)$ arbitrarily. Since λ is essentially bounded, the Dominated Convergence Theorem yields the limiting relation

$$\lim_{k\to\infty}\lambda(\mathbf{x},\mathrm{Tr}\ \mathbf{u}_{j_k}(\mathbf{x}))=\lambda(\mathbf{x},\mathrm{Tr}\ \mathbf{u}(\mathbf{x}))\quad \text{in } L^{t^*}(\Gamma,\mathbb{R}^n\otimes\mathbb{R}^n). \tag{5.62}$$

Then, by (5.61), (5.62) and the Hölder inequality, we deduce that

$$\lim_{k\to\infty}\lambda(\mathbf{x},\mathrm{Tr}\ \mathbf{u}_{j_k}(\mathbf{x}))\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})=\lambda(\mathbf{x},\mathrm{Tr}\ \mathbf{u}(\mathbf{x}))\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)\text{ in } L^2(\Gamma,\mathbb{R}^n). \tag{5.63}$$

Moreover, we know that $L^2(\Gamma,\mathbb{R}^n\otimes\mathbb{R}^n)\hookrightarrow L^2_{s-1}(\Gamma,\mathbb{R}^n\otimes\mathbb{R}^n)$.

By (5.59), (5.60) and (5.63), the right-hand side of (5.58) (with $\mathbf{u}_{j_k}$ instead of $\mathbf{u}_j$) converges to

$$\Big(\mathbf{f}|_{\mathfrak{D}}+\mathcal{P}(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x}))\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi),g,\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)-\lambda(\mathbf{x},\mathrm{Tr}\ \mathbf{u}(\mathbf{x}))\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)+\mathbf{h}\Big)$$

in $L^2_{s-\frac{3}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})\times L^2_{s-1}(\Gamma,\mathbb{R}^n)$. Also, by Theorem 5.1, the linear Poisson problem for the Stokes system with Robin boundary condition

$$\begin{cases}\triangle \mathbf{v}-\nabla q=\mathbf{f}|_{\mathfrak{D}}+\mathcal{P}(\mathbf{x},\mathbf{u}(\mathbf{x}),\pi(\mathbf{x}))\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi),\\ \operatorname{div}\,\mathbf{v}=g \quad \text{ in } \ \mathfrak{D},\\ \partial_\nu(\mathbf{v},q)_{\mathbf{f}+\mathcal{P}(\mathbf{x},\mathbf{u},\pi)\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi),g}+\mathrm{Tr}\ \mathbf{v}=\mathfrak{R}_0,\end{cases} \tag{5.64}$$

where

$$\mathfrak{R}_0:=\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)-\lambda(\mathbf{x},\mathrm{Tr}\ \mathbf{u}(\mathbf{x}))\mathrm{Tr}\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)+\mathbf{h}\in L^2_{s-1}(\Gamma,\mathbb{R}^n),$$

is well-posed in the space $\mathcal{X}_s$. Therefore, the following limiting relation holds

$$\lim_{k\to\infty}(\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}),\mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k}))=(\mathcal{T}_{1;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi),\mathcal{T}_{2;\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi))\ \text{ in } \mathcal{X}_s, \tag{5.65}$$

i.e., $\lim_{k\to\infty}\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u}_{j_k},\pi_{j_k})=\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)$ in $\mathcal{X}_s$. By the same method, we can show that each subsequence of $\{(\mathbf{u}_j,\pi_j)\}_{j\in\mathbb{N}}$ has a further subsequence such that its image by $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}$ converges to $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)$ in $\mathcal{X}_s$. Hence, $\lim_{j\to\infty}\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u}_j,\pi_j)=\mathcal{T}_{\mathbf{f},g,\mathbf{h}}(\mathbf{u},\pi)$ in $\mathcal{X}_s$. Consequently, the operator $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}:B_A\to B_A$ is continuous and compact, as desired.

Finally, the Schauder Fixed Point Theorem (see Theorem 4.3) shows that the nonlinear operator $\mathcal{T}_{\mathbf{f},g,\mathbf{h}}:B_A\to B_A$ has a fixed point $(\mathbf{u},\pi)$ in the closed, bounded and convex subset B_A of the Banach space $\mathcal{X}_s$. Such a fixed point is a solution of the semilinear Poisson problem (5.34) in the space $\mathcal{X}_s$, which satisfies the inequality $\|(\mathbf{u},\pi)\|_{\mathcal{X}_s}\le A$, where A is the constant given by (5.40). □

6. The semilinear Darcy–Forchheimer–Brinkman model

The semilinear Poisson problems studied in this paper have been suggested by the semilinear system

$$\triangle\mathbf{u}-(\alpha\mathbf{u}+k|\mathbf{u}|\mathbf{u})-\nabla\pi=\mathbf{0},\ \operatorname{div}\ \mathbf{u}=0, \tag{6.1}$$

where $\alpha,k>0$ are given constants. For $n=2,3$, the first equation in (6.1) is a generalization of the Darcy and Brinkman equations for viscous incompressible flows in saturated porous media, called the *semilinear Darcy–Forchheimer–Brinkman equation* (for more details see, e.g., [3,41]).

6.1. The Dirichlet problem for the semilinear Darcy–Forchheimer–Brinkman system

Let $s\in(\frac{1}{2},1)$. We consider the space

$$L^2_{s;\nu}(\Gamma,\mathbb{R}^n):=\left\{\mathbf{F}\in L^2_s(\Gamma,\mathbb{R}^n):\int\limits_\Gamma\langle\nu,\mathbf{F}\rangle d\sigma=0\right\}.$$

Note that for $n\le 4$, the map which takes $(\mathbf{x},\mathbf{v},\xi)$ to $\alpha\mathbf{v}+k|\mathbf{v}|\mathbf{v}$ is not essentially bounded on $\mathfrak{D}\times\mathbb{R}^n\times\mathbb{R}$. Hence, the result of Theorem 4.4 cannot be applied to the Dirichlet problem for the semilinear Darcy–Forchheimer–Brinkman system (6.1). However, by exploiting an idea similar to that of Russo and Tartaglione [44, Theorem 5.1], which gives the existence of a solution of the Robin problem for the Navier–Stokes system on a Lipschitz (or C^1) domain in $\mathbb{R}^3$ (for related results, see [12, Theorems 7.1 and 7.3] and [4, Theorems 25 and 26, Lemma 29]), we obtain the following result.

Theorem 6.1. *Let $n\le 4$. Let $\mathfrak{D}\subseteq\mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary Γ. Let $s\in(\frac{1}{2},1)$. Let $\alpha,k>0$ be given constants. Then, there exist two constants $\tilde{\alpha}_0,\gamma>0$, which depend* *only on $\mathfrak{D}$, α, k and s, such that the Dirichlet problem for the semilinear Darcy–Forchheimer–Brinkman system*

$$\begin{cases} \triangle \mathbf{u} - \alpha \mathbf{u} - k|\mathbf{u}|\mathbf{u} - \nabla \pi = \mathbf{0} & in\ \ \mathfrak{D}, \\ \mathrm{div}\ \mathbf{u} = 0 & in\ \ \mathfrak{D}, \\ \mathrm{Tr}\ \mathbf{u} = \mathbf{h} \in L^2_{s;\nu}(\Gamma, \mathbb{R}^n), \end{cases} \tag{6.2}$$

with $\|\mathbf{h}\|_{L^2_{s;\nu}(\Gamma,\mathbb{R}^n)} \leq \tilde{\alpha}_0$, has a unique solution $(\mathbf{u}, \pi) \in L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D})$, which satisfies the inequality $\|\mathbf{u}\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)} \leq \gamma$.

Proof. First, note that for $n \leq 4$ and $s \in (\frac{1}{2}, 1)$, the Sobolev Embedding Theorem yields the continuous embeddings

$$L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \hookrightarrow L^2_1(\mathfrak{D}, \mathbb{R}^n) \hookrightarrow L^{p^*}(\mathfrak{D}, \mathbb{R}^n) \hookrightarrow L^4(\mathfrak{D}, \mathbb{R}^n), \tag{6.3}$$

where the first of them is compact. In addition, $p^* = \frac{2n}{n-2} \geq 4$ for $2 < n \leq 4$, while, for $n = 2$, we choose $p^* \geq 4$ arbitrarily. Indeed, if $n = 2$, the embedding $L^2_1(\mathfrak{D}, \mathbb{R}^n) \hookrightarrow L^q(\mathfrak{D}, \mathbb{R}^n)$ is continuous for any $q \geq 1$. Therefore, there exists a constant $c_* = c_*(s, \mathfrak{D}) > 0$ such that

$$\|\ |\mathbf{v}|\mathbf{v}\ \|_{L^2(\mathfrak{D},\mathbb{R}^n)} = \|\mathbf{v}\|^2_{L^4(\mathfrak{D},\mathbb{R}^n)} \leq c_* \|\mathbf{v}\|^2_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)},\ \forall\ \mathbf{v} \in L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n). \tag{6.4}$$

Hence, $|\mathbf{v}|\mathbf{v} \in L^2(\mathfrak{D}, \mathbb{R}^n) \hookrightarrow L^2_{s-\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n)$ for any $\mathbf{v} \in L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n)$.

Let $(\mathcal{G}_\alpha, \Pi_\alpha)$ be the fundamental solution of the Brinkman system in $\mathbb{R}^n$, i.e.,

$$(\triangle_{\mathbf{x}} - \alpha \mathbb{I})\mathcal{G}_\alpha(\mathbf{x}, \mathbf{y}) - \nabla_{\mathbf{x}} \Pi_\alpha(\mathbf{x}, \mathbf{y}) = -\delta_{\mathbf{y}}(\mathbf{x})\mathbb{I},\ \ \mathrm{div}_{\mathbf{x}} \mathcal{G}_\alpha(\mathbf{x}, \mathbf{y}) = 0, \tag{6.5}$$

where $\mathbb{I}$ is the identity matrix and $\delta_{\mathbf{y}}$ is the Dirac distribution with mass at $\mathbf{y}$. The components of $\mathcal{G}_\alpha$ and those of Π_α are given in [50, Chapter 2] and [25, Chapter 2]. Now, for a fixed $\mathbf{u} \in L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n)$, such that div $\mathbf{u} = 0$ in $\mathfrak{D}$, consider the potentials on $\mathfrak{D}$ with the density $k|\mathbf{u}|\mathbf{u}$, given by

$$\mathfrak{N}_\alpha(\mathbf{u})(\mathbf{x}) = -\left\langle \mathcal{G}_\alpha(\mathbf{x}, \cdot), k|\mathbf{u}|\mathbf{u} \right\rangle_{\mathfrak{D}},\ \ \mathfrak{Q}_\alpha(\mathbf{u})(\mathbf{x}) = -\left\langle \Pi_\alpha(\mathbf{x}, \cdot), k|\mathbf{u}|\mathbf{u} \right\rangle_{\mathfrak{D}}. \tag{6.6}$$

Let us mention the following relation

$$\mathfrak{N}_\alpha = \mathcal{N}_{\alpha;\mathfrak{D}} \mathcal{I}_{\mathfrak{D}} : L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \to L^2_2(\mathfrak{D}, \mathbb{R}^n), \tag{6.7}$$

where

$$\mathcal{N}_{\alpha;\mathfrak{D}} : L^2(\mathfrak{D}, \mathbb{R}^n) \to L^2_2(\mathfrak{D}, \mathbb{R}^n),\ \ (\mathcal{N}_{\alpha;\mathfrak{D}} \mathbf{f})(\mathbf{x}) = -\left\langle \mathcal{G}_\alpha(\mathbf{x}, \cdot), \mathbf{f} \right\rangle_{\mathfrak{D}},\ \ \mathbf{x} \in \mathfrak{D} \tag{6.8}$$

is the Newtonian potential operator in $\mathfrak{D}$, and

$$\mathcal{I}_{\mathfrak{D}} : L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \to L^2(\mathfrak{D}, \mathbb{R}^n),\ \ \mathcal{I}_{\mathfrak{D}}(\mathbf{v}) := k|\mathbf{v}|\mathbf{v}.$$

Note that for $s \in (\frac{1}{2}, 1)$ and $n \leq 4$, the embedding $L^2_{s+\frac{1}{2}}(\mathfrak{D}) \hookrightarrow L^4(\mathfrak{D})$ is compact. Then, one can prove that the nonlinear operator $\mathfrak{N}_\alpha : L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \to L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n)$ is continuous and compact (see also [43, p. 483] and the argument below (6.17)). Also, for a fixed $\mathbf{u} \in L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n)$, such that div $\mathbf{u} = 0$ in $\mathfrak{D}$, we have

$$\begin{cases} (\triangle - \alpha \mathbb{I})\mathfrak{N}_\alpha(\mathbf{u}) - \nabla \mathfrak{Q}_\alpha(\mathbf{u}) = k|\mathbf{u}|\mathbf{u} \in L^2_{s-\frac{3}{2}}(\mathfrak{D}, \mathbb{R}^n), \\ \mathrm{div}\ \mathfrak{N}_\alpha(\mathbf{u}) = 0\ \ \text{in}\ \ \mathfrak{D}, \\ \mathrm{Tr}\,(\mathfrak{N}_\alpha(\mathbf{u})) \in L^2_{s;\nu}(\mathfrak{D}, \mathbb{R}^n). \end{cases} \tag{6.9}$$

Let $(\mathfrak{M}_\alpha(\mathbf{u}), \mathfrak{P}_\alpha(\mathbf{u})) \in L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D})$ be the unique solution (up to a constant pressure) of the Dirichlet problem[7]

$$\begin{cases} (\triangle - \alpha\mathbb{I})\mathfrak{M}_\alpha(\mathbf{u}) - \mathfrak{P}_\alpha(\mathbf{u}) = \mathbf{0} \ \text{ in } \ \mathfrak{D}, \\ \operatorname{div} \mathfrak{M}_\alpha(\mathbf{u}) = 0 \ \text{ in } \ \mathfrak{D}, \\ \operatorname{Tr}(\mathfrak{M}_\alpha(\mathbf{u})) = -\operatorname{Tr}(\mathfrak{N}_\alpha(\mathbf{u})) \in L^2_{s;\nu}(\mathfrak{D}, \mathbb{R}^n). \end{cases} \tag{6.10}$$

In addition, there exist two constants $C_i' \equiv C_i'(s, \alpha, \mathfrak{D}) > 0$, $i = 0, 1$, such that

$$\|\mathfrak{M}_\alpha(\mathbf{u})\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)} \leq C_0' \|\operatorname{Tr}(\mathfrak{N}_\alpha(\mathbf{u}))\|_{L^2_{s;\nu}(\Gamma,\mathbb{R}^n)} \leq C_1' \|\mathfrak{N}_\alpha(\mathbf{u})\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)}. \tag{6.11}$$

Moreover, there exists a constant $C_2 \equiv C_2(s, \alpha, \mathfrak{D}) > 0$ such that the Dirichlet problem

$$\begin{cases} (\triangle - \alpha\mathbb{I})\mathbf{u}_0 - \nabla\pi_0 = \mathbf{0} \ \text{ in } \ \mathfrak{D}, \\ \operatorname{div} \mathbf{u}_0 = 0 \ \text{ in } \ \mathfrak{D}, \\ \operatorname{Tr} \mathbf{u}_0 = \mathbf{h} \in L^2_{s;\nu}(\mathfrak{D}, \mathbb{R}^n). \end{cases} \tag{6.12}$$

has a unique solution $(\mathbf{u}_0, \pi_0) \in L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D})$ (up to a constant pressure), which satisfies the inequality

$$\|\mathbf{u}_0\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)} \leq C_2 \|\mathbf{h}\|_{L^2_{s;\nu}(\Gamma,\mathbb{R}^n)}. \tag{6.13}$$

We now consider the nonlinear operator

$$\mathfrak{F} : L^2_{s+\frac{1}{2};*}(\mathfrak{D}, \mathbb{R}^n) \to L^2_{s+\frac{1}{2};*}(\mathfrak{D}, \mathbb{R}^n), \quad \mathfrak{F}(\mathbf{v}) := \mathbf{u}_0 + \mathfrak{M}_\alpha(\mathbf{v}) + \mathfrak{N}_\alpha(\mathbf{v}), \tag{6.14}$$

and, for $\mathbf{u} \in L^2_{s+\frac{1}{2};*}(\mathfrak{D}, \mathbb{R}^n)$ fixed, we define the pressure term $\pi = \pi(\mathbf{u})$,

$$\pi := \pi_0 + \mathfrak{P}_\alpha(\mathbf{u}) + \mathfrak{Q}_\alpha(\mathbf{u}) \in L^2_{s-\frac{1}{2}}(\mathfrak{D}), \tag{6.15}$$

where

$$L^2_{s+\frac{1}{2};*}(\mathfrak{D}, \mathbb{R}^n) := \left\{ \mathbf{v} \in L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) : \operatorname{div} \mathbf{v} = 0 \text{ in } \mathfrak{D} \right\}.$$

For a fixed $\mathbf{u} \in L^2_{s+\frac{1}{2};*}(\mathfrak{D}, \mathbb{R}^n)$, the pair $(\mathfrak{F}(\mathbf{u}), \pi) \in L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \times L^2_{s-\frac{1}{2}}(\mathfrak{D})$ is, in view of (6.9), (6.10) and (6.12), a solution of the Dirichlet problem

$$\begin{cases} (\triangle - \alpha\mathbb{I})\mathfrak{F}(\mathbf{u}) - k|\mathbf{u}|\mathbf{u} - \nabla\pi = \mathbf{0} & \text{in } \ \mathfrak{D}, \\ \operatorname{div} \mathfrak{F}(\mathbf{u}) = 0 & \text{in } \ \mathfrak{D}, \\ \operatorname{Tr}(\mathfrak{F}(\mathbf{u})) = \mathbf{h} \in L^2_{s;\nu}(\mathfrak{D}, \mathbb{R}^n). \end{cases} \tag{6.16}$$

Consequently, a fixed point $\mathbf{u} \in L^2_{s+\frac{1}{2};*}(\mathfrak{D}, \mathbb{R}^n)$ of the operator $\mathfrak{F}$ together with the associated pressure π given by (6.15) determine a solution of the Dirichlet problem for the semilinear Darcy–Forchheimer–Brinkman system (6.2). We now turn to show that $\mathfrak{F}$ maps a suitable closed ball B_γ of the space $L^2_{s+\frac{1}{2};*}(\mathfrak{D}, \mathbb{R}^n)$ to B_γ.

The decomposition (6.7) of the nonlinear operator $\mathfrak{N}_\alpha : L^2_{s+\frac{1}{2}}(\mathfrak{D}, \mathbb{R}^n) \to L^2_2(\mathfrak{D}, \mathbb{R}^n)$, the boundedness of the linear operator $\mathcal{N}_{\alpha;\mathfrak{D}} : L^2(\mathfrak{D}, \mathbb{R}^n) \to L^2_2(\mathfrak{D}, \mathbb{R}^n)$ given by (6.8) (see, e.g., [14, Proposition 2.1] in the case of the Laplace equation, while for the Brinkman system, the boundedness of the Newtonian operator $\mathcal{N}_{\alpha;\mathfrak{D}}$ can be obtained by using properties of Calderón-Zygmund operators, namely [47, Theorem

[7] The well-posedness result of the Dirichlet problem for the Brinkman system in a Lipschitz domain with boundary data in Sobolev spaces follows from Theorem 4.1, by considering $\mathcal{P} = \alpha\mathbb{I}$, $\mathbf{f} = \mathbf{0}$ and $g = 0$ in (4.1) (see also [40, Theorem 10.6.2] in the case of the Stokes system).

2, Chapter II]), the continuity of the embedding $L_2^2(\mathfrak{D},\mathbb{R}^n)\hookrightarrow L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)$ and the inequality (6.4) yield the inequalities

$$\begin{aligned}\|\mathfrak{N}_\alpha(\mathbf{v})\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)} &= \|\mathcal{N}_{\alpha;\mathfrak{D}}(k|\mathbf{v}|\mathbf{v})\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)} \le c_{0;*}\|\mathcal{N}_{\alpha;\mathfrak{D}}(k|\mathbf{v}|\mathbf{v})\|_{L^2_2(\mathfrak{D},\mathbb{R}^n)}\\ &\le c_{1;*}\|\ |\mathbf{v}|\mathbf{v}\ \|_{L^2(\mathfrak{D},\mathbb{R}^n)} \le c_{2;*}\|\mathbf{v}\|^2_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)},\end{aligned}\tag{6.17}$$

with some constants $c_{0;*}\equiv c_{0;*}(s,\mathfrak{D})>0$ and $c_{j;*}\equiv c_{j;*}(s,k,\alpha,\mathfrak{D})>0$, $j=1,2$. In addition, the nonlinear operators $\mathfrak{N}_\alpha: L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)\to L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)$ and $\mathfrak{M}_\alpha: L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)\to L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)$ are compact and continuous. To prove the continuity of $\mathfrak{N}_\alpha: L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)\to L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)$, we first show the continuity of $\mathfrak{N}_\alpha$ from $L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)$ to $L^2_{2;*}(\mathfrak{D},\mathbb{R}^n):=\big\{\mathbf{v}\in L^2_2(\mathfrak{D},\mathbb{R}^n): \operatorname{div}\mathbf{v}=0 \text{ in } \mathfrak{D}\big\}$. Let $\big\{\mathbf{v}_j\big\}_{j\in\mathbb{N}}$ be a convergent sequence in $L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)$ to an element $\mathbf{v}\in L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)$. Then, the continuity of the embedding $L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)\hookrightarrow L^4(\mathfrak{D},\mathbb{R}^3)$, the integral form (6.8) of the operator $\mathfrak{N}_\alpha$ and the Hölder inequality show that there exists some constant $c_{3;*}>0$, such that

$$\|\mathfrak{N}_\alpha(\mathbf{v}_j)-\mathfrak{N}_\alpha(\mathbf{v})\|_{L^2_2(\mathfrak{D},\mathbb{R}^n)}\le c_{3;*}\|\mathbf{v}_j-\mathbf{v}\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)}\Big(\|\mathbf{v}_j\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)}+\|\mathbf{v}\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)}\Big)\to 0 \text{ as } j\to\infty.$$

Thus, $\mathfrak{N}_\alpha: L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)\to L^2_{2;*}(\mathfrak{D},\mathbb{R}^n)$ is continuous. Then, the compactness of the embedding $L^2_{2;*}(\mathfrak{D},\mathbb{R}^n)\hookrightarrow L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)$ yields that the nonlinear operator $\mathfrak{N}_\alpha: L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)\to L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)$ is continuous and compact. In addition, the nonlinear operator $\mathfrak{M}_\alpha: L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)\to L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)$ is also continuous and compact, as (6.10) and the relation $(\mathfrak{M}_\alpha(\mathbf{v}),(\mathfrak{P}_\alpha(\mathbf{v}))=\mathfrak{B}_\alpha^{-1}\left(\mathbf{0},0,-\mathrm{Tr}(\mathfrak{N}_\alpha(\mathbf{v}))\right)^\top$ show, where $\mathfrak{B}_\alpha$ is the isomorphism given by (4.4) with $\mathcal{P}=\alpha\mathbb{I}$. Consequently, the nonlinear operator $\mathfrak{F}: L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)\to L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)$ given by (6.14) is continuous and compact as well.

Now, by (6.11), (6.13), (6.14) and (6.17), there exist some constants $C\equiv C(s,\alpha,\mathfrak{D})>0$ and $C_*\equiv C_*(k,s,\alpha,\mathfrak{D})>0$ such that

$$\|\mathfrak{F}(\mathbf{v})\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)}\le C\|\mathbf{h}\|_{L^2_{s;\nu}(\Gamma,\mathbb{R}^n)}+C_*\|\mathbf{v}\|^2_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)},\ \forall\ \mathbf{v}\in L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n).\tag{6.18}$$

By using an argument similar to that in the proof of [44, Theorem 5.1] (see also [43, p. 506], [45]), we assume that the norm of the given datum $\mathbf{h}\in L^2_{s;\nu}(\Gamma,\mathbb{R}^n)$ is small, such that

$$\|\mathbf{h}\|_{L^2_{s;\nu}(\Gamma,\mathbb{R}^n)}\le\tilde{\alpha}_0,\quad \tilde{\alpha}_0:=\frac{1}{CC_*(2+\beta)^2},\tag{6.19}$$

with some constant $\beta>0$. Also, consider the closed ball

$$B_\gamma:=\big\{\mathbf{v}\in L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n):\operatorname{div}\mathbf{v}=0 \text{ in } \mathfrak{D},\ \|\mathbf{v}\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)}\le\gamma\big\},\ \gamma:=\frac{1}{C_*(2+\beta)}>0.\tag{6.20}$$

By (6.18) and (6.19), one has $\|\mathfrak{F}(\mathbf{u})\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)}\le\gamma$ for any $\mathbf{u}\in B_\gamma$, and hence $\mathfrak{F}$ maps the closed ball B_γ to B_γ. In addition, we have shown that $\mathfrak{F}: L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)\to L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)$ is continuous and compact. Hence, $\mathfrak{F}: B_\gamma\to B_\gamma$ is also continuous and compact. Then, by the Schauder Fixed Point Theorem, $\mathfrak{F}$ has a fixed point $\mathbf{u}\in B_\gamma$, and the pair $(\mathbf{u},\pi)\in B_\gamma\times L^2_{s-\frac{1}{2}}(\mathfrak{D})$, with π given by (6.15), is a solution of the Dirichlet problem (6.2). We now turn to show that for a given boundary datum $\mathbf{h}$ such that $\|\mathbf{h}\|_{L^2_{s;\nu}(\Gamma,\mathbb{R}^n)}$ is sufficiently small (i.e., for a special choice of the constant β), the solution of the Dirichlet problem (6.2) is unique. To do so, we note that the inequality (6.11) and the argument before (6.17) imply that there exist two constants $C_0\equiv C_0(k,s,\alpha,\mathfrak{D})>0$ and $C_{*;s+\frac{1}{2}}\equiv C_{*;s+\frac{1}{2}}(s,\mathfrak{D})>0$ such that the map

$\mathfrak{F}: L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n) \to L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)$ given by (6.14) satisfies the inequalities

$$\begin{aligned}\|\mathfrak{F}(\mathbf{v})-\mathfrak{F}(\mathbf{w})\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)} &\le \|\mathfrak{N}_\alpha(\mathbf{v})-\mathfrak{N}_\alpha(\mathbf{w})\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)} + \|\mathfrak{M}_\alpha(\mathbf{v})-\mathfrak{M}_\alpha(\mathbf{w})\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)}\\ &\le C_0\|\mathbf{v}|\mathbf{v}|-|\mathbf{w}|\mathbf{w}\|_{L^2(\mathfrak{D},\mathbb{R}^n)}\\ &\le C_0 C^2_{*;s+\frac{1}{2}}\|\mathbf{v}-\mathbf{w}\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)}\Big(\|\mathbf{v}\|_{L_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)}+\|\mathbf{w}\|_{L_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)}\Big), \end{aligned} \tag{6.21}$$

for all $\mathbf{v},\mathbf{w} \in L^2_{s+\frac{1}{2};*}(\mathfrak{D},\mathbb{R}^n)$. Consequently,

$$\|\mathfrak{F}(\mathbf{v})-\mathfrak{F}(\mathbf{w})\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)} \le 2\gamma C_0 C^2_{*;s+\frac{1}{2}}\|\mathbf{v}-\mathbf{w}\|_{L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)},\quad \forall\, \mathbf{v},\mathbf{w}\in B_\gamma, \tag{6.22}$$

where γ is defined in (6.20). If we choose the constant $\beta > 0$ in the expression of γ such that

$$(2+\beta)^{-1} < C_*\big(2C_0C^2_{*;s+\frac{1}{2}}\big)^{-1}, \tag{6.23}$$

then $2\gamma C_0 C^2_{*;s+\frac{1}{2}} < 1$. Therefore, for $n \le 4$, $s \in (\frac{1}{2},1)$ and for a constant $\beta > 0$ as in (6.23), the map $\mathfrak{F}: B_\gamma \to B_\gamma$ is a contraction in B_γ. Then, the Banach-Caccioppoli Contraction Theorem implies that $\mathfrak{F}$ has a unique fixed point $\mathbf{u}\in B_\gamma$. In addition, the pair $(\mathbf{u},\pi)\in B_\gamma \times L^2_{s-\frac{1}{2}}(\mathfrak{D})$, with π given by (6.15), is a solution of the semilinear Dirichlet problem (6.2). We now turn to show that such a solution is unique (up to a constant pressure) in $B_\gamma \times L^2_{s-\frac{1}{2}}(\mathfrak{D})$. To do so, we assume that $(\mathbf{v},q)\in B_\gamma\times L^2_{s-\frac{1}{2}}(\mathfrak{D})$ is another solution of the problem (6.2), and let $(\mathfrak{F}(\mathbf{v}),p)$, where $\mathfrak{F}(\mathbf{v})$ and $p=\pi(\mathbf{v})$ are defined as in (6.14) and (6.15), respectively. Then, $\mathfrak{F}(\mathbf{v})\in B_\gamma$, and we obtain the problem

$$\begin{cases}(\triangle-\alpha\mathbb{I})(\mathfrak{F}(\mathbf{v})-\mathbf{v})-\nabla(p-q)=\mathbf{0} & \text{in } \mathfrak{D},\\ \operatorname{div}(\mathfrak{F}(\mathbf{v})-\mathbf{v})=0 & \text{in } \mathfrak{D},\\ \operatorname{Tr}(\mathfrak{F}(\mathbf{v})-\mathbf{v})=\mathbf{0} & \text{on } \Gamma.\end{cases} \tag{6.24}$$

By Theorem 4.1, (6.24) has the unique solution $(\mathfrak{F}(\mathbf{v})-\mathbf{v},p-q)=(\mathbf{0},0)$ (up to a constant pressure) in $L^2_{s+\frac{1}{2}}(\mathfrak{D},\mathbb{R}^n)\times L^2_{s-\frac{1}{2}}(\mathfrak{D})$, i.e., $\mathfrak{F}(\mathbf{v})=\mathbf{v}$. Consequently, $\mathbf{v}=\mathbf{u}$, as $\mathfrak{F}$ has a unique fixed point in B_γ. Thus, the proof is complete. □

Remark 6.2. If $n\in\{2,3\}$, the existence statement of Theorem 6.1 holds also for any $s\in\big[\frac{1}{2},1\big)$. The proof of such a result is based on the Sobolev Embedding Theorem and on arguments similar to those for Theorem 6.1, which we omit for sake of brevity.

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