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Partial Differential Equations — *A Harnack's inequality and Hölder continuity for solutions of mixed type evolution equations*, by Fabio Paronetto, communicated on 8 May 2015.

Abstract. — We define a homogeneous parabolic De Giorgi classes of order 2 which suits a mixed type class of evolution equations whose simplest example is $\mu(x)\frac{\partial u}{\partial t} - \Delta u = 0$ where μ can be positive, null and negative, so that elliptic-parabolic and forward-backward parabolic equations are included. For functions belonging to this class we prove local boundedness and show a Harnack inequality which, as by-products, gives Hölder-continuity, in particular in the interface I where μ change sign, and a maximum principle.

Key words: Mixed type equations, Harnack's inequality, Hölder-continuity

Mathematics Subject Classification: 35M10, 35B65

1. Introduction

The purpose of this paper is to announce and present in a very short version a result which is to appear (see [8]). Moreover the result we present here regards a simplified situation with respect to that considered in [8]. In [8] we consider equations like (2) satisfying

$$\begin{aligned} &(A(x,t,u,Du), Du) \geq c\lambda(x)|Du|^2, \\ &|A(x,t,u,Du)| \leq C\lambda(x)|Du|, \\ &|B(x,t,u,Du)| \leq M\lambda(x)|Du| \end{aligned} \tag{1}$$

with $\mu, \lambda \in L^1_{\mathrm{loc}}(\Omega)$, $\lambda > 0$ almost everywhere, while $\mathrm{sgn}(\mu)$ can change and be zero also in some set of positive measure. One simple example (to stress that we are interested in the changing type equations) is

$$\mu(x)\frac{\partial u}{\partial t} - \mathrm{div}(\lambda(x)Du) = 0.$$

Mixed type equations have been considered in connection with many applications: for instance random processes, kinetic theory, electron scattering. Among the first papers where some mixed type equation of this type was considered we recall [1], [7], [2]. In all these papers the coefficient μ is of the type

$$\mathrm{sgn}\, x|x|^p, \quad p \geq 0.$$

A general and noteworthy result, but confined to $\mu \geq 0$, is [11]. In [10] and [9] we give some existence results for a quite wider class of mixed type equations.

Coming back to the present paper, we refer for every details, for a wider list of references, for the more general results and for the proofs to the appearing paper [8] which has already been submitted. In the present paper, on the contrary, we focus our attention on a simpler situation where $\lambda \equiv 1$ and $\mu \in L^\infty(\Omega)$ instead of $\mu, \lambda \in L^1_{\mathrm{loc}}(\Omega)$. We consider a class of functions, a suitable De Giorgi class of functions, which in particular contains solutions of a wide class of equations, precisely equations like

$$\mu(x)\frac{\partial u}{\partial t} - \operatorname{div} A(x,t,u,Du) = B(x,t,u,Du) \tag{2}$$

with A and B satisfying

$$\begin{aligned} &(A(x,t,u,Du), Du) \geq c|Du|^2, \\ &|A(x,t,u,Du)| \leq C|Du|, \\ &|B(x,t,u,Du)| \leq M|Du| \end{aligned} \tag{3}$$

for some positive constants c, C, M.

We give a local boundedness result, a Harnack type inequality and a result of regularity (the functions we consider are locally Hölder continuous as a byproduct of the Harnack's inequality) for a particular class of functions containing the solutions of a mixed type equation like

$$\mu(x)\frac{\partial u}{\partial t} - \Delta u = 0$$

(but also of (2)). We recall that Harnack inequalities and results of Hölder continuity for elliptic and parabolic equations have been widely studied and here we recall brifly some of the people who started to study these topics: we recall De Giorgi and Nash for the regularity of the solutions, Hadamard, Pini, Moser, Aronson, Serrin, Trudinger for the Harnack inequality.

More recently a technique due to De Giorgi was adapted to the parabolic case by DiBenedetto, Gianazza and Vespri and, to prove our result, we adapt this technique, following [4] and [6] (see also [5] for a result regarding non-linear equations).

As already said, in [8] we consider a more general situation, but here we confine to state the result for this simpler situation.

The main results are Theorem 3.1 and Theorem 4.1. Theorem 3.1 is needed also to start to prove the Harnack inequality stated in Theorem 4.1. Theorem 4.2 is just a corollary of Theorem 4.1, where is stressed the behaviour of u in the points of the interfaces where μ changes sign.

Finally, in the last sections, some important consequences of the Harnack inequality are stated, together with some examples.

2. Notations, prelimanary definitions and assumptions

Consider the coefficient μ appearing in (2). Even if we consider the equation with $(x,t) \in \Omega \times (0,T)$, and in particular $x \in \Omega \subset \mathbf{R^n}$, for simplicity we suppose that

$$\mu : \mathbf{R^n} \to \mathbf{R},$$

i.e. μ defined in the whole $\mathbf{R^n}$. Consider

$$\Omega_+ := \{x \in \Omega \,|\, \mu(x) > 0\}, \quad \Omega_- := \{x \in \Omega \,|\, \mu(x) < 0\} \quad \text{and}$$
$$\Omega_0 := \Omega \setminus (\Omega_+ \cup \Omega_-)$$

and define

$$I_+ = \partial\Omega_+ \cap \Omega, \quad I_- = \partial\Omega_- \cap \Omega, \quad I_0 = \partial\Omega_0 \cap \Omega, \quad I := I_+ \cup I_- \cup I_0.$$

We will suppose that Ω_+, Ω_- and $\Omega_0 \setminus I_0$ are the union of a finite number of open and connected subsets of Ω. In the same way for every ball $B_\rho(x_o)$ we will write

$$B_\rho^+(x_o) := B_\rho(x_o) \cap \Omega_+, \quad B_\rho^-(x_o) := B_\rho(x_o) \cap \Omega_-, \quad B_\rho^0(x_o) := B_\rho(x_o) \cap \Omega_0.$$

Then we define the positive (almost everywhere) weight

$$\tilde{\mu} := \begin{cases} \mu_+ & \text{in } \Omega_+, \\ \mu_- & \text{in } \Omega_-, \\ 1 & \text{in } \Omega_0 \end{cases}$$

where μ_+ denotes the positive part of μ and μ_- the negative part of μ, i.e.

$$\mu_+(x) := \max\{\mu(x), 0\} \geq 0, \quad \mu_-(x) := \max\{-\mu(x), 0\} \geq 0.$$

The extension with 1 in $\{x \in \Omega \,|\, \mu(x) = 0\}$, thanks to assumptions (H.2) and (H.3), permits to have the Sobolev-Poincaré inequality stated in Theorem 2.1.

First we introduce the Sobolev-type spaces we will need. Given a $\mu \in L^\infty(\Omega)$ we define $L^2(\Omega, \tilde{\mu})$ as the closure of $C^1(\overline{\Omega})$ with respect to the norm

$$\left(\int_\Omega u^2(x)\tilde{\mu}\,dx\right)^{1/2}$$

and $H^1(\Omega, \tilde{\mu}, 1)$ the closure of $C^1(\overline{\Omega})$ with respect to the norm

$$\left(\int_\Omega u^2(x)\tilde{\mu}\,dx + \int_\Omega |Du|^2(x)\,dx\right)^{1/2}.$$

In this case the notion of $u = 0$ in $\partial\Omega$ coincides with the classical trace in $H^1(\Omega)$.

We assume that $\tilde{\mu}$ is *doubling*, i.e that there is a constant $\mathfrak{q} > 1$ such that

$$\text{(H.0)} \qquad \int_{B_{2\rho}(\bar{x})} \tilde{\mu}(x)\,dx \leq \mathfrak{q} \int_{B_\rho(\bar{x})} \tilde{\mu}(x)\,dx$$

for every ball $B_{2\rho}(\bar{x}) \subset \Omega$. For a fixed positive ε and A a subset of $\mathbf{R^n}$ we define

$$A^{\varepsilon} := \{x \in \Omega \mid \mathrm{dist}(x, A) < \varepsilon\}.$$

The assumptions about the function μ are the following: we will suppose the existence of two positive constants K_2, K_3 (the choice of the *names* of the constants is the same considered in [8]), a constant $q > 2$, $\varsigma \in (0, 1)$, $\mathfrak{q} > 1$ (for simplicity the same as in (H.0)) such that:

(H.1) - $\mu \in L^{\infty}(\Omega)$,

(H.2) - $\displaystyle \frac{r}{|B_r(\bar{x})|^{1/2}} \Big(\int_{B_r(\bar{x})} \tilde{\mu}(x)\, dx \Big)^{1/q} \leq K_2 \frac{R}{|B_R(\bar{x})|^{1/2}} \Big(\int_{B_R(\bar{x})} \tilde{\mu}(x)\, dx \Big)^{1/q}$

(H.3) - $\displaystyle \frac{1}{|S|^{\varsigma}} \int_S \tilde{\mu}(x)\, dx \leq K_3 \frac{1}{|B|^{\varsigma}} \int_B \tilde{\mu}(x)\, dx$

(H.4) - $\left| \begin{array}{ll} \mu_+(B^+_{2\rho}(x)) \leq \mathfrak{q}\mu_+(B_\rho(x)) & \text{for every } x \in \Omega_+ \cup I_+, \\ \mu_-(B^-_{2\rho}(y)) \leq \mathfrak{q}\mu_-(B_\rho(y)) & \text{for every } y \in \Omega_- \cup I_-, \\ |B^0_{2\rho}(z)| \leq \mathfrak{q}|B^0_\rho(z)| & \text{for every } z \in \Omega_0 \cup I_0, \end{array} \right.$

(H.5) - I is a such that $\displaystyle \lim_{\varepsilon \to 0^+} |I^{\varepsilon}| = 0$,

where (H.2) has to hold for every pairs of concentric balls $B_r(\bar{x}) \subset B_R(\bar{x}) \subset \mathbf{R^n}$ with $r < R$, (H.3) has to hold for every ball $B \subset \mathbf{R^n}$ and every measurable set $S \subset B$, (H.4) has to hold for every $\rho > 0$ for which $B_{2\rho}(x) \subset \Omega$.

By (H.2) one can prove (see [8]) that there is $\alpha \in (0, 1)$ such that

$$\text{(H.2)}' \text{ - } \frac{r^{\alpha}}{|B_r(\bar{x})|^{1/2}} \Big(\int_{B_r(\bar{x})} \tilde{\mu}(x)\, dx \Big)^{1/q} \leq K_2 \frac{R^{\alpha}}{|B_R(\bar{x})|^{1/2}} \Big(\int_{B_R(\bar{x})} \tilde{\mu}(x)\, dx \Big)^{1/q}.$$

A $\alpha < 1$ is needed twice in the proof of Theorem 7.1 in [8] (Theorem 4.1 below).

By (H.3) one can prove (see [8]) that there are two positive constants τ and κ such that

$$\frac{|S|}{|B|} \leq \kappa \Big(\frac{\tilde{\mu}(S)}{\tilde{\mu}(B)} \Big)^{\tau}, \quad \frac{\tilde{\mu}(S)}{\tilde{\mu}(B)} \leq \kappa \Big(\frac{|S|}{|B|} \Big)^{\tau} \tag{4}$$

for every measurable $S \subset B$, for every B ball of $\mathbf{R^n}$ (this is needed in the proof of Lemma 6.6 in [8]).

The following result is needed (we refer to [3] for the proof).

THEOREM 2.1. *Suppose* (H.0) *and* (H.2) *are satisfied and consider* $\rho > 0$, $x_0 \in \mathbf{R^n}$. *Then there is a constant* γ_1 *depending* (*only*) *on* n, q, K_2, $\mathfrak{q}$ *such that*

$$\left[\frac{1}{\tilde{\mu}(B_\rho)} \int_{B_\rho} |u(x)|^q \tilde{\mu}(x)\, dx \right]^{1/q} \leq \gamma_1 \rho \left[\frac{1}{|B_\rho|} \int_{B_\rho} |Du(x)|^2\, dx \right]^{1/2} \tag{5}$$

for every u Lipschitz continuous function defined in $B_\rho = B_\rho(x_0)$, with either support contained in $B_\rho(x_0)$ or with null mean value.

The constant $q > 2$ appearing in (H.2) is the same appearing in (5). Thanks to this inequality one can prove the following useful result, which is indeed a corollary.

THEOREM 2.2. *Assume the same assumptions of Theorem 2.1. There is $\kappa > 1$ such that for every $s_1, s_2 \in (0, T)$, for every family of open sets $A(t)$, $t \in (s_1, s_2)$ in such a way $E = \bigcup_{t\in(s_1,s_2)} A(t)$ is an open subset of $B_\rho \times (s_1, s_2)$ and for every $v \in C^0([s_1, s_2]; L^2(B_\rho, \tilde{\mu})) \cap L^2(s_1, s_2; H_0^1(B_\rho, \tilde{\mu}, 1))$ it holds*

$$\begin{aligned}\frac{1}{v(B_\rho)} &\iint_E |u|^{2\kappa}(x,t)v(x)\,dx\,dt \\ &\le \gamma_1^2 \rho^2 \left(\frac{1}{\tilde{\mu}(B_\rho)}\right)^{\kappa-1} \left(\sup_{s_1<t<s_2} \int_{A(t)} |u|^2(x,t)\tilde{\mu}(x)\,dx\right)^{\kappa-1} \\ &\quad \cdot \frac{1}{|B_\rho|} \int_{s_1}^{s_2} \int_{B_\rho} |Du|^2(x,t)\,dx\,dt\end{aligned}$$

where the inequality holds both with $v = \tilde{\mu}$ and $v \equiv 1$.

Now we define the De Giorgi class $DG(\Omega, T, \mu, \gamma)$ as the class of functions

$$u \in L^2_{\mathrm{loc}}(0, T; H^1_{\mathrm{loc}}(\Omega, \tilde{\mu}, 1)) \cap L^\infty_{\mathrm{loc}}((0, T); L^2_{\mathrm{loc}}(\Omega, \tilde{\mu}))$$

satisfying

$$\begin{aligned}&\int_{B_\rho(x_0)} (u-k)_\pm^2(x,t_2)\zeta^2(x,t_2)\mu_+(x)\,dx + \int_{B_\rho(x_0)} (u-k)_\pm^2(x,t_1)\zeta^2(x,t_1)\mu_-(x)\,dx \\ &\qquad + \int_{t_1}^{t_2} \int_{B_\rho(x_0)} |D(u-k)_\pm|^2 \zeta^2\,dx\,dt \\ &\quad \le \gamma \int_{t_1}^{t_2} \int_{B_\rho(x_0)} (u-k)_\pm^2 (|D\zeta|^2 + \zeta\zeta_t \mu)\,dx\,dt \\ &\qquad + \int_{B_\rho(x_0)} (u-k)_\pm^2(x,t_2)\zeta^2(x,t_2)\mu_-(x)\,dx \\ &\qquad + \int_{B_\rho(x_0)} (u-k)_\pm^2(x,t_1)\zeta^2(x,t_1)\mu_+(x)\,dx\end{aligned}$$

for every $k \in \mathbf{R}$, every $\zeta \in \mathrm{Lip}(\Omega \times (0, T))$ such that $\zeta(\cdot, t) \in \mathrm{Lip}_c(\Omega)$ for every $t \in (0, T)$ and for every $B_\rho(x_0) \times (t_1, t_2) \subset \Omega \times (0, T)$.

A typical choice of a function ζ will be done in such a way that $\zeta_t\mu \geq 0$ and in such a way that

$$|D\zeta|^2|B_\rho(x_0)| \sim \zeta\zeta_t\mu(B_\rho(x_0)) \sim \frac{1}{(R-r)^2}|B_\rho(x_0)|$$

and

$$t_2 - t_1 \sim \frac{\tilde{\mu}(B_\rho(x_0))}{|B_\rho(x_0)|}\rho^2$$

but for a detailed and clearer definition of the De Giorgi class we refer to [8]. From now on we will denote by h the following function

$$h(x_0,\rho) := \frac{\tilde{\mu}(B_\rho(x_0))}{|B_\rho(x_0)|}.$$

3. Local boundedness for functions in DG

By $|\cdot|$ we denote the n-dimensional Lebesgue measure.

Theorem 3.1. *Suppose $u \in DG(\Omega, T, \mu, \gamma)$ and consider $(x_0, t_0) \in \Omega \times (0, T)$. Then there is a constant c_∞ depending only on γ, γ_1, κ such that:*

i) *for every $B_R(x_0) \times (t_0, t_0 + h(x_0,R)R^2) \subset \Omega \times (0,T)$ if $\mu_+(B_R(x_0)) > 0$ we have*

$$\operatorname*{ess\,sup}_{B^+_{R/2}(x_0)\times\left(t_0+\frac{1}{2}h(x_0,R)R^2,\, t_0+h(x_0,R)R^2\right)} |u| \leq c_\infty\left[\frac{1}{h(x_0,R)R^2\tilde{\mu}(B_R(x_0))}\iint_{B^+_{\frac{3R}{2}}(x_0)\times(t_0,\, t_0+h(x_0,R)R^2)} u^2\mu_+ dx\,dt + \frac{1}{h(x_0,R)R^2|B_R(x_0)|}\iint_{B^+_{\frac{3R}{2}}(x_0)\times(t_0,\, t_0+h(x_0,R)R^2)} u^2\,dx\,dt\right]^{1/2};$$

ii) *for every $B_R(x_0) \times (t_0 - h(x_0,R)R^2, t_0) \subset \Omega \times (0,T)$ if $\mu_-(B_R(x_0)) > 0$ we have*

$$\operatorname*{ess\,sup}_{B^-_{R/2}(x_0)\times\left(t_0-h(x_0,R)R^2,\, t_0-\frac{1}{2}h(x_0,R)R^2\right)} |u| \leq c_\infty\left[\frac{1}{h(x_0,R)R^2\tilde{\mu}(B_R(x_0))}\iint_{B^-_{\frac{3R}{2}}(x_0)\times(t_0-h(x_0,R)R^2,\, t_0)} u^2\mu_-\,dx\,dt + \frac{1}{h(x_0,R)R^2|B_R(x_0)|}\iint_{B^-_{\frac{3R}{2}}(x_0)\times(t_0-h(x_0,R)R^2,\, t_0)} u^2\,dx\,dt\right]^{1/2};$$

iii) *for every* $B_R(x_0) \times (\sigma_1, \sigma_2) \subset \Omega \times (0,T)$, $\sigma_2 - \sigma_1 = R^2$, *if* $|B_R^0(x_0)| > 0$

$$\operatorname*{ess\,sup}_{B^0_{R/2}(x_0)\times(\sigma_1,\sigma_2)} |u| \le c_\infty \left(\frac{1}{R^2 |B_R(x_0)|} \iint_{B^0_{\frac{3R}{2}}(x_0)\times(\sigma_1,\sigma_2)} u^2 \, dx \, dt \right)^{1/2}.$$

4. The Harnack type inequality

Theorem 4.1. *Assume* $u \in DG(\Omega, T, \mu, \gamma)$, $u \ge 0$, $(x_o, t_o) \in \Omega \times (0,T)$ *and fix* $\rho > 0$.

i) *Suppose* $x_o \in \Omega_+ \cup I_+$. *For every* $\vartheta_+ \in (0,1]$ *for which* $B_{5\rho}(x_o) \times [t_o - h(x_o,\rho)\rho^2, t_o + 16h(x_o,4\rho)\rho^2 + \vartheta_+ h(x_o,\rho)\rho^2] \subset \Omega \times (0,T)$ *there exists* $c_+ > 0$ *depending (only) on* γ_1, γ, $\mathfrak{q}$, κ, α, $\varkappa$, τ, K_2, K_3, q, ς, ϑ_+ *such that*

$$u(x_o, t_o) \le c_+ \inf_{B^+_\rho(x_o)} u(x, t_o + \vartheta_+ \rho^2 h(x_o, \rho)).$$

ii) *Suppose* $x_o \in \Omega_- \cup I_-$. *For every* $\vartheta_- \in (0,1]$ *for which* $B_{5\rho}(x_o) \times [t_o - 16h(x_o,4\rho)\rho^2 + \vartheta_- h(x_o,\rho)\rho^2, t_o + h(x_o,\rho)\rho^2] \subset \Omega \times (0,T)$ *there exists* $c_- > 0$ *depending (only) on* γ_1, γ, $\mathfrak{q}$, κ, α, $\varkappa$, τ, K_2, K_3, q, ς, ϑ_- *such that*

$$u(x_o, t_o) \le c_- \inf_{B^-_\rho(x_o)} u(x, t_o - \vartheta_- \rho^2 h(x_o, \rho)).$$

iii) *Suppose* $x_o \in \Omega_0 \cup I_0$. *Suppose* $B_{5\rho}(x_o) \times [t_o - 16h(x_o,4\rho)\rho^2, t_o + 16h(x_o,4\rho)\rho^2] \subset \Omega \times (0,T)$. *For every* s_1, s_2 *for which* $s_2 - t_o = t_o - s_1 \le 16h(x_o,4\rho)\rho^2$, *suppose* $s_2 - t_o = t_o - s_1 = \omega h(x_o,4\rho)\rho^2$ *for* $\omega \in (0,16]$, *there is* c_0 *depending (only) on* K_2, K_3, q, ς, κ, γ_1, γ, ω, $h(x_o,4\rho)$, $\mathfrak{q}$ *such that*

$$\sup_{B^+_\rho(x_o)\times[s_1,s_2]} u \le c_0 \inf_{B^+_\rho(x_o)\times[s_1,s_2]} u.$$

iv) *Suppose* $B_{5\rho}(x_o) \subset \Omega_0$. *Then there is* c *depending (only) on* K_2, K_3, q, ς, κ, γ_1, γ, $\mathfrak{q}$ *such that for almost every* $t \in (0,T)$

$$\sup_{B_\rho(x_o)} u(\cdot, t) \le c \inf_{B_\rho(x_o)} u(\cdot, t).$$

The following result, as already said, is in fact a corollary of the previous result. We state it just to stress the result for points in the interface I where μ changes sign.

Theorem 4.2. *Assume* $u \in DG(\Omega, T, \mu, \gamma)$, $u \ge 0$. *Fix* $\rho > 0$ *and* $\vartheta \in (0,1]$ *for which* $B_{5\rho}(x_o) \times [t_o - 16h(x_o,4\rho)\rho^2 - \vartheta h(x_o,\rho)\rho^2, t_o + 16h(x_o,4\rho)\rho^2 + \vartheta h(x_o,\rho)\rho^2] \subset \Omega \times (0,T)$. *Suppose* $x_o \in I$. *Then there exists* $c > 0$ *depending on* γ_1, γ, $\mathfrak{q}$, κ, α, $\varkappa$, τ, K_1, K_2, K_3, q, ς, ϑ_+ *such that*

$$u(x_o, t_o) \le c \inf_{B_\rho(x_o)} \tilde{u}(x)$$

where

$$\tilde{u}(x) = \begin{cases} u(x, t_o + \vartheta h(x_o, \rho)\rho^2) & \text{if } x \in B_\rho^+(x_o) \\ u(x, t_o - \vartheta h(x_o, \rho)\rho^2) & \text{if } x \in B_\rho^-(x_o) \end{cases} \quad \text{if } x_o \in I_+ \cap I_-,$$

$$\tilde{u}(x) = \begin{cases} u(x, t_o + \vartheta h(x_o, \rho)\rho^2) & \text{if } x \in B_\rho^+(x_o) \\ u(x, t_o) & \text{if } x \in B_\rho^0(x_o) \end{cases} \quad \text{if } x_o \in I_+ \cap I_0,$$

$$\tilde{u}(x) = \begin{cases} u(x, t_o - \vartheta h(x_o, \rho)\rho^2) & \text{if } x \in B_\rho^-(x_o) \\ u(x, t_o) & \text{if } x \in B_\rho^0(x_o) \end{cases} \quad \text{if } x_o \in I_- \cap I_0,$$

$$\tilde{u}(x) = \begin{cases} u(x, t_o + \vartheta h(x_o, \rho)\rho^2) & \text{if } x \in B_\rho^+(x_o) \\ u(x, t_o - \vartheta h(x_o, \rho)\rho^2) & \text{if } x \in B_\rho^-(x_o) \\ u(x, t_o) & \text{if } x \in B_\rho^0(x_o). \end{cases} \quad \text{if } x_o \in I_+ \cap I_- \cap I_0.$$

5. Some consequences of the Harnack inequality

An important and standard consequence for a function satisfying a Harnack's inequality is Hölder-continuity. By classical computations and assuming (if necessary taking γ bigger)

$$\frac{\gamma}{\gamma - 1} < 2$$

one can get that if $u \in DG(\Omega, T, \mu, \gamma)$ then u is locally α-Hölder continuous with respect to x and $\alpha/2$-Hölder continuous with respect to t, where $\alpha = \left(\log_2 \frac{\gamma}{\gamma-1}\right)$, in $(\Omega_+ \cup \Omega_- \cup I) \times (0, T)$. As regards Ω_0 we can only get that for every $t \in (0, T)$ $u(\cdot, t)$ is locally α-Hölder continuous in Ω_0. Notice that in the interface I separating Ω_0 and $\Omega_+ \cup \Omega_-$ the function u is regular also with respect to t.

Another consequence is a strong maximum pronciple, which one can get, again by standard arguement. If, for instance, we suppose $x_o \in I_+ \cap I_0 \cap I_-$ (and again with obvious generalization in the other cases) we could briefly state the maximum principles as follows: suppose $(x_o, t_o) \in \Omega \times (0, T)$ is a maximum point for u in a set

$$\begin{gathered} (B_\rho^+(x_o) \times (t_o - \vartheta h(x_o, \rho)\rho^2, t_o + \vartheta h(x_o, \rho)\rho^2)) \cup (B_\rho^0(x_o) \times \{t_o\}) \\ \cup (\bigcup B_\rho^-(x_o) \times (t_o - \vartheta h(x_o, \rho)\rho^2, t_o + \vartheta h(x_o, \rho)\rho^2)) \end{gathered}$$

for some $\vartheta \in (0, 1]$, then u is constant in the set

$$\begin{gathered} (B_\rho^+(x_o) \times (t_o - \vartheta h(x_o, \rho)\rho^2, t_o]) \cup (B_\rho^0(x_o) \times \{t_o\}) \\ \cup (\bigcup B_\rho^-(x_o) \times [t_o, t_o + \vartheta h(x_o, \rho)\rho^2)). \end{gathered}$$

6. Examples

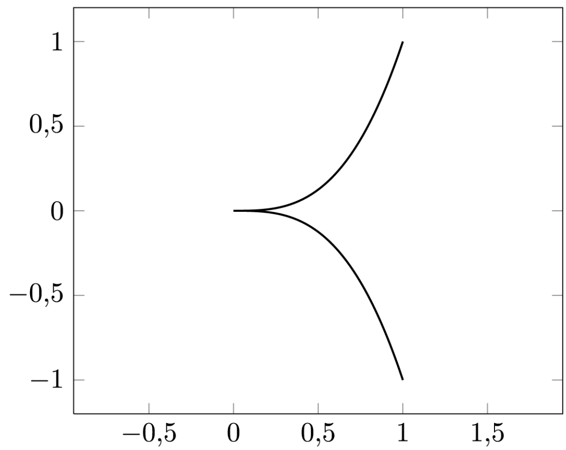
Here we give a couple of examples of possible interfaces we manage to treat, without giving some counterexamples for the ones we are not able to treat. So it is not clear if the conditions we require are sharp or not.

For a more detailed series of examples we refer to [8]. Anyway here we propose two possible bad behaviours of the interface where we get, as a particular case, Hölder continuity for the solutions.

The first one is the following: suppose (part of) the interface is that in the picture below and the vertex is the point $(0,0)$ and suppose $\mu = 1$ on the left of the interface and $\mu = -1$ on the right. To require that μ_+ and μ_- satisfy (H.4) means

$$\mu_+(B_{2\rho}(0,0)) \le \mathfrak{q}\mu_+(B_\rho((0,0))),$$
$$\mu_-(B_{2\rho}(0,0)) \le \mathfrak{q}\mu_-(B_\rho((0,0))).$$

One can verify that not every cusp is admitted and satisfies the above conditions. For instance, if the curve in the picture is the union of the graphs of $f(x) = x^n$ and $g(x) = -x^n$ for $x \in [0, L]$, $L > 0$, and $n \in \mathbf{N}$, $n \ge 1$, the above conditions are satisfied, while if, for instance, the curve in the picture is the union of the graphs of $f(x) = e^{-1/x}$ and $g(x) = -e^{-1/x}$ the above inequalities does not hold any more.



Another example we show is the following: again for simplicity suppose $|\mu| \equiv \lambda \equiv 1$ in $\mathbf{R}^2$ and suppose $\mu \equiv 1$ in the region above the graphic of f, which we will call Ω_+, and $\mu \equiv -1$ in the region below the graphic of f, which we will call Ω_-, where

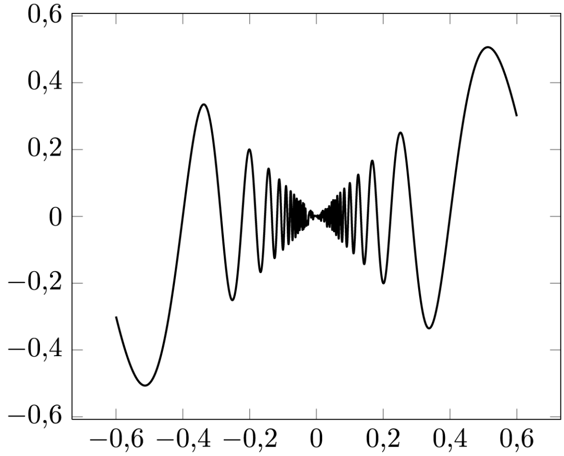
$$f(y) = y\cos\frac{1}{y} \quad (f(0) = 0).$$

In spite of the fact that the length of the graphic inside the ball $B := B_1(0,0)$ is infinite, the measure (the 2-dimensional Lebesgue measure $\mathscr{L}^2$) of the ε-neighbourhood of I is of order ε and then going to zero when $\varepsilon \to 0^+$. Moreover, due to the simmetry of the graphic of f we have that

$$\mu_+(B_{2\rho}(0,0)) = \frac{1}{2}\mathscr{L}^2(B_{2\rho}(0,0)) \leq \frac{1}{2}c\mathscr{L}^2(B_\rho(0,0)) = \frac{1}{2}c\mu_+(B_\rho(0,0))$$

where c denotes the doubling constant for $\mathscr{L}^2$. Therefore also in this case assumptions (H.4) and (H.5) are satisfied and even if I is not rectifiable can be an admissible interface.

In the point $(0,0)$ a solution is then Hölder continuous.



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