

Lindenbaum's Lemma via Open Induction

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Abstract

With Raoult's Open Induction in place of Zorn's Lemma, we do a perhaps more perspicuous proof of Lindenbaum's Lemma for not necessarily countable languages of first-order predicate logic. We generally work for and with classical logic, but say what can be achieved for intuitionistic logic, which prompts the natural generalizations for distributive and complete lattices.

1 Introduction

It is not uncommon in mathematics that a concrete theorem admits an elegant but highly abstract proof by some transfinite method, typically Zorn's Lemma (ZL) combined with a proof by contradiction: under the hypothesis that there is any counterexample at all, by ZL there exists a maximal or minimal counterexample, which helps to the desired contradiction. Unfortunately, one thus virtually loses the computational information given by the input data, and the proof fails to produce an algorithm for computing the output data.

Some theorems of this kind [1, 5, 6, 10, 12, 18, 24] have already proved to follow in a direct and elementary way from (a variant of) the principle of Open Induction (OI) distinguished by Raoult [18]. Although fully-fledged OI is equivalent to ZL with classical logic [12], using OI rather than ZL has some advantage at least when the original statement is sufficiently concrete. In this case there is in fact some evidence that—in Hilbert's terms—the ideal objects characteristic of any invocation of ZL can be eliminated, and that one can get by with finite means only. Under sufficiently concrete circumstances, for example, OI can be reduced to ordinary mathematical induction or even, by fixing the size of the data under consideration, an entirely first-order proof can be obtained.

As a classical equivalent of ZL, OI cannot be considered a theorem from any constructive perspective; whence we prefer to call OI rather a principle than a lemma. Since, however, OI is a form of induction, it not only looks less harmful than ZL already at first glance, but also allows for a computational interpretation. Moreover, one can extract the computational content from proofs in which OI is used in combination with intuitionistic logic, which in turn can be made possible just by moving

away from the notorious proofs by contradiction with ZL.¹

In logic, semantic arguments based on Gödel’s Completeness Theorem (CT) give short and elegant proofs of purely syntactical results. To prove, say, a conservation theorem, with CT at hand it suffices to show that every model of the base theory can be extended to a model of the extended theory. Although this proof technique is non-constructive a priori, similar arguments are valid constructively [4]. Constructive completeness theorems for intuitionistic logic have been proved and applied e.g. in [2, 3, 7, 8, 22].

The aforementioned uses of OI suggest the possibility of getting some form of CT by means of OI. In this paper we re-prove Lindenbaum’s Lemma (LL) [26] in contrapositive form [23], the novelty being that we work with OI in place of ZL. We then discuss how to carry this over to intuitionistic logic, which leads us to the natural generalizations to distributive and complete lattices.

In spite of the method of proof by cases we need to employ, which is essentially classical, our approach with OI appears to be somewhat more direct than the usual one with ZL. Thanks to the move from ZL to OI, we indeed expect that, as in the case studies mentioned above, one will eventually be able to do with finite methods only, at least when dealing with some concrete application of CT or LL. Evidence for the latter claim is yet to be given, though there already is strong motivation: while the conservation of extensions of Peano Arithmetic by a Tarskian truth predicate was first dealt with from a semantic point of view [14, 9], a syntactical proof has become possible [15] by adapting just those semantic methods [14].

Conventions Let X be a set. The *complement* of a subset P of X is denoted by $-P$, that is,

$$-P = \{x \in X : x \notin P\}.$$

We say that a subset P of X is *inhabited* (rather than non-empty) if P has an element.²

Adopting a handy notation used by Sambin, we denote by $M \overset{\circ}{\cap} N$ (rather than $M \cap N \neq \emptyset$) that the intersection of the subsets M, N of X is inhabited, that is,

$$M \overset{\circ}{\cap} N \iff \exists x \in X (x \in M \wedge x \in N).$$

For example, $M \overset{\circ}{\cap} M$ says that M is inhabited; $\{x, y\} \overset{\circ}{\cap} N$ means that either $x \in N$ or $y \in N$; and $\neg(M \overset{\circ}{\cap} N)$ is tantamount to each of $M \subseteq -N$ and $N \subseteq -M$. Denoting the subset relation by \subseteq , we only write $M \subset N$ if M is a *proper* subset of N , that is, $-M \overset{\circ}{\cap} N$. We sometimes view a subset P of X as a unary predicate, and write $P(x)$ in place of $x \in P$ for any $x \in X$.

Last but not least, if X is a poset, then every subset of X is tacitly endowed with the induced order; and a *chain* in X is a totally ordered, inhabited subset. Let us stress that we need to require that every chain be inhabited, to have that the supersets of a fixed set be closed under forming unions of chains.

2 Open Induction

Raoult’s principle of Open Induction is essentially the logical contrapositive of *Zorn’s Lemma* (ZL). To briefly explain this, let (X, \leq) be a poset; and recall that X is *chain-complete* if every chain $C \subseteq X$ has a least upper bound $\bigvee C \in X$. The form of ZL we have in mind reads as follows:

Let X be chain-complete. If X is inhabited, then X has a maximal element.

¹ The authors are most grateful to the anonymous referee for hinting at this issue.

² This and other choices of terminology typical for constructive settings are made to prepare for Section 3.3.

We cannot avoid requiring X to be inhabited, for in this paper the empty set cannot be admitted as a chain (see above). With classical logic we can rewrite ZL in an equivalent contrapositive form:

Let X be chain-complete. A subset Q of X is empty whenever

1. Q has no maximal elements and
2. Q is closed, that is, $C \subseteq Q \Rightarrow \bigvee C \in Q$ for every chain C .

In fact, the form of ZL we have given first can be relativised equivalently as follows:

If X is chain-complete, and $P \subseteq X$ inhabited and closed, then P has a maximal element.

Conditions 1 and 2 above have the following well-known dual forms (think of Q as $-P$):

Definition 1. *Let X be a chain-complete poset. A subset P of X is*

1. *progressive if $(\forall x \in X)(x > a \Rightarrow x \in P) \Rightarrow a \in P$ for every $a \in X$,*
2. *open if $\bigvee C \in P \Rightarrow C \not\Downarrow P$ for every chain $C \subseteq X$.*

With classical logic, in fact, P is progressive precisely when $-P$ has no maximal elements, and P is open precisely when $-P$ is closed. Note also that if P is progressive, then P is satisfied by every maximal element of X , and thus by the greatest element of X whenever this exists.

Open predicates form a topology. We pause to recall this standard fact, if only for the reader's convenience. We write $\uparrow x$ for the set of all y with $x \leq y$.

Lemma 2. *Let A and C be an open subset and a chain, respectively, of a chain-complete poset. If $\bigvee C \in A$, then $C \cap \uparrow c \subseteq A$ for some $c \in C$.*

Proof. Suppose, towards a contradiction, that for every $c \in C$ there exists $c' \in C$ such that $c \leq c'$ and $c' \notin A$. This gives $\bigvee(C - A) = \bigvee C$. So $\bigvee(C - A) \in A$ and hence $(C - A) \not\Downarrow A$, a contradiction. \square

While the proof involves classical logic, the lemma itself holds trivially if A is even *Scott open*, that is, open and upwards closed.

Proposition 3. *The open subsets of a chain-complete poset are closed under finite intersections and arbitrary unions.*

Proof. Closure under union is clear. Now let A and B be open and assume that $\bigvee C \in A \cap B$ for a given chain C . By the previous lemma, there exists $c \in C$ such that $C \cap \uparrow c \subseteq A$. Clearly, $C \cap \uparrow c$ is a chain and $\bigvee(C \cap \uparrow c) = \bigvee C$. So $\bigvee(C \cap \uparrow c) \in B$ and hence $(C \cap \uparrow c) \not\Downarrow B$. Hence $C \not\Downarrow (A \cap B)$. \square

After this digression on topology, we recall from Raoult [18] the principle we use in place of ZL:

Open Induction (OI) *Let X be a chain-complete poset. If P is a progressive and open predicate on X , then $P(x)$ for all $x \in X$.*

With classical logic (taking P and Q as complements of each other), OI is equivalent to one of the equivalents (“if Q is a closed subset of a chain-complete poset X , and Q has no maximal elements, then Q is empty”) of ZL we have displayed before; whence OI is equivalent to ZL [12].

In the sequel, as in [24], by *induction for P and X* we mean the following principle:

If P is progressive, then $(\forall x \in X)P(x)$.

Classically, induction holds for *every* P precisely when X is *well-founded* in the sense that every inhabited predicate on X has a maximal element. This is known as *Transfinite Induction* (TI), and is implied by OI. In fact, if the poset X is well-founded, then every chain in X has a greatest element; whence X is chain-complete, and every predicate P on X is open. Unlike OI, TI is provable in **ZF**.

In some important cases, moreover, induction is provable by mathematical induction only. For instance, induction holds whenever X is

1. a tree (with the root as top element) or, more generally, a forest;
2. a finite poset.

In either case there is no need to have that X be well-founded or chain-complete, let alone that P be open. To see this, let P be a progressive predicate on a poset X . Recall first that $P(y)$ whenever y is a maximal element of X . If X is a forest, to prove $P(x)$ for any $x \in X$ one can do induction on the distance from x to the root y of the tree of X to which x belongs, for which $P(y)$ as this y is maximal. If X is finite and inhabited, then X has a maximal element y , for which $P(y)$ and which y we thus may remove from X ; whence induction on the size of X applies.

The second instance above also was the outcome of a reduction of OI in a concrete situation [24]. The following is yet another special case of induction provable by mathematical induction.

Example 4. *Let P be a progressive predicate on \mathbb{N} . If $P \not\leq I$ for every infinite $I \subseteq \mathbb{N}$, then $P = \mathbb{N}$.*

Proof. First, $\neg P$ must be finite, since otherwise, by hypothesis, we would have $P \not\leq \neg P$, which is impossible. We now can show that $\neg P$ is empty. If $\neg P$ were inhabited, then $\neg P$ would have a greatest element m , for which $n \in P$ for all $n > m$. But P is progressive and thus $m \in P$, a contradiction. \square

This example can also be seen as a case of OI, as follows. Consider the chain-complete poset $X = \mathbb{N} \cup \{\infty\}$, that is, the ordinal $\omega + 1$. Given P as above, the property $Q = P \cup \{\infty\}$ is progressive, because so is P , and also open. To check this, let C be a chain in X with $\bigvee C \in Q$. If $\bigvee C \in \mathbb{N}$, then C is finite and hence $\bigvee C \in C$; therefore $C \not\leq Q$. On the other hand, if $\bigvee C = \infty$, then either $\infty \in C$, and hence $C \not\leq Q$, or C is an infinite subset of \mathbb{N} , in which case $C \not\leq P$ by hypothesis.

3 Lindenbaum's Lemma

Let \mathcal{L} be a first-order language, which need not be countable. We write $\mathcal{S}_{\mathcal{L}}$ for the set of sentences over \mathcal{L} . Let \vdash denote the deducibility relation of classical predicate logic.³

The *deductive closure* of $\Gamma \subseteq \mathcal{S}_{\mathcal{L}}$ is

$$\bar{\Gamma} = \{\varphi \in \mathcal{S}_{\mathcal{L}} \mid \Gamma \vdash \varphi\}. \quad (1)$$

We note in passing that $\Gamma \mapsto \bar{\Gamma}$ defines a closure operator on the subsets of $\mathcal{S}_{\mathcal{L}}$. This means that (i) $\Gamma \subseteq \bar{\Gamma}$, (ii) $\Gamma \subseteq \Gamma'$ implies $\bar{\Gamma} \subseteq \bar{\Gamma}'$ and (iii) $\bar{\bar{\Gamma}} = \bar{\Gamma}$, for all $\Gamma, \Gamma' \subseteq \mathcal{S}_{\mathcal{L}}$. Note that the latter says that $\bar{\Gamma} \vdash \varphi$ is tantamount to $\Gamma \vdash \varphi$. This closure operator, however, is not topological, that is, it does not preserve finite unions. In fact, $\bar{\emptyset}$ is not empty and $\overline{\Gamma_1 \cup \Gamma_2}$ is, in general, greater than $\bar{\Gamma}_1 \cup \bar{\Gamma}_2$.

³ We could equally have worked for and with propositional logic, with arbitrary formulas in place of sentences.

3.1 Types of theories

A *theory* is a set of sentences that equals its deductive closure. Conversely, with the concept of theory at hand one can characterise the one of deductive closure. In fact, $\bar{\Gamma}$ is the smallest theory in the language \mathcal{L} which contains the given set $\Gamma \subseteq \mathcal{S}_{\mathcal{L}}$, which is to say that $\bar{\Gamma}$ equals the intersection of all theories in the language \mathcal{L} which contain Γ . In view of the impredicative character of all this, we have preferred to define $\bar{\Gamma}$ by (1), from which one can of course prove these characterisations.

We next recollect a few well-known features of theories, which are scattered across the literature.

Remark 5. *A set of sentences Γ is a theory if and only if the following hold for all $\varphi, \psi \in \mathcal{S}_{\mathcal{L}}$:*

1. $\top \in \Gamma$;
2. if $\{\varphi, \psi\} \subseteq \Gamma$, then $\varphi \wedge \psi \in \Gamma$;
3. if $\varphi \in \Gamma$ and $\varphi \vdash \psi$, then $\psi \in \Gamma$.

So theories correspond to filters of the Lindenbaum algebra, the quotient of $\mathcal{S}_{\mathcal{L}}$ modulo equivalence.

As usual, a set of sentences Γ is *consistent* if $\Gamma \not\vdash \perp$. A theory Γ is consistent if and only if it is *proper*: that is, $\Gamma \subset \mathcal{S}_{\mathcal{L}}$. So consistent theories correspond to proper filters of the Lindenbaum algebra.

We further recall that a theory Γ is

- *complete* if $\{\varphi, \neg\varphi\} \not\subseteq \Gamma$ for every $\varphi \in \mathcal{S}_{\mathcal{L}}$;
- *prime* if $\varphi \vee \psi \in \Gamma$ implies $\{\varphi, \psi\} \not\subseteq \Gamma$ for every $\{\varphi, \psi\} \subseteq \mathcal{S}_{\mathcal{L}}$.

Although the next lemma is well-known, we give a proof for the sake of later inspection (Section 3.3).

Lemma 6. *The complete consistent theories are exactly the proper prime theories. More precisely,*

1. *if a consistent theory Γ is complete, then Γ is a prime theory;*
2. *every prime theory Γ is complete.*

Proof. As for part 1, let Γ be a consistent theory. Assume that Γ is complete. To show that Γ is prime, let $\varphi \vee \psi \in \Gamma$. Since Γ is complete, we have as required either $\varphi \in \Gamma$ or $\psi \in \Gamma$. In fact, if otherwise both $\neg\varphi \in \Gamma$ and $\neg\psi \in \Gamma$, that is, $\neg(\varphi \vee \psi) \in \Gamma$, then $\perp \in \Gamma$, which is impossible in view of Γ being consistent. As for part 2, every prime theory Γ is complete, simply because $\varphi \vee \neg\varphi \in \Gamma$. \square

In other words, the ultrafilters of the Lindenbaum algebra are just its proper prime filters, which is no surprise as this is a Boolean algebra. As a digression, we next recall that these notions of theories are equivalent to yet another one—of which, however, we will not make any use in the sequel.

Remark 7. *Let Γ be a consistent set of sentences. The following conditions are equivalent:*

1. Γ is a complete theory.
- 2a. For every sentence ψ , if $\Gamma \cup \{\psi\}$ is consistent, then $\psi \in \Gamma$.
- 2b. For every sentence ψ , if $\psi \notin \Gamma$, then $\Gamma \cup \{\psi\}$ is inconsistent.
- 3a. If Γ' is a consistent theory with $\Gamma' \supseteq \Gamma$, then $\Gamma' = \Gamma$.
- 3b. If Γ' is a theory with $\Gamma' \supset \Gamma$, then Γ' is inconsistent.

In all, a set of sequences Γ is a complete consistent theory precisely when Γ is a *maximal consistent* set of sequences, i.e. maximal among the consistent sets of sentences. Here maximality may be understood in the sense of any of the conditions 2a–b, 3a–b of Remark 7. In particular, the maximal consistent sets of sentences correspond to the ultrafilters of the Lindenbaum algebra [19].

A typical example of a maximal consistent theory is the theory $\text{Th}(\mathcal{M})$ of a model \mathcal{M} . By the Completeness Theorem, this is the only type of maximal consistent theory. In fact, if Γ is a consistent theory, then it has a model \mathcal{M} . So $\Gamma \subseteq \text{Th}(\mathcal{M})$, and if Γ is maximal consistent, then $\Gamma = \text{Th}(\mathcal{M})$.

3.2 Lindenbaum's Lemma with Open Induction

We now show how to prove Lindenbaum's Lemma with OI in place of ZL.

Theorem 8 (OI). *For each $\Gamma \cup \{\varphi\} \subseteq \mathcal{S}_{\mathcal{L}}$ the following are equivalent:*

- (i) $\Gamma \vdash \varphi$;
- (ii) $\Gamma \subseteq \Delta \Rightarrow \varphi \in \Delta$ for every proper prime theory $\Delta \subseteq \mathcal{S}_{\mathcal{L}}$.

Proof. The non-trivial implication using OI is the one from (ii) to (i). To this end, define

$$X = \{\Delta \subseteq \mathcal{S}_{\mathcal{L}} \mid \Delta \text{ is a theory and } \Gamma \subseteq \Delta\}.$$

Clearly $\bar{\Gamma}$ is the least element of X , partially ordered by inclusion.

Claim 1: (X, \subseteq) is chain-complete. To see this, let $\{\Delta_i \mid i \in I\}$ be a chain in X . We claim that $\Delta = \bigcup_{i \in I} \Delta_i$ belongs to X . Since $\Gamma \subseteq \Delta$, as I is inhabited, we only need to verify that Δ is a theory. If $\Delta \vdash \psi$, then $K \vdash \psi$ for some finite $K \subseteq \Delta$. So there is $k \in I$ such that $K \subseteq \Delta_k$. Hence $\Delta_k \vdash \psi$ and so $\psi \in \Delta_k \subseteq \Delta$. Note that the join of a chain of theories is its union.

Now let P be the predicate on X defined by

$$P(\Delta) \iff \varphi \in \Delta.$$

Claim 2: P is open. In fact, if $\varphi \in \bigcup_{i \in I} \Delta_i$, then $\varphi \in \Delta_i$ for some i .

Claim 3: P is progressive. Given $\Delta \in X$, we have to deduce $\varphi \in \Delta$ from the induction hypothesis that $\varphi \in \Delta'$ for every $\Delta' \in X$ such that $\Delta' \supset \Delta$. To this end we distinguish the following cases.

Case I: Δ is both proper and prime. In this case, (ii) applies to Δ ; whence $\varphi \in \Delta$.

Case II, the negation of Case I, is split into two subcases, as follows.

Subcase IIIa: Δ is not proper, i.e. $\Delta = \mathcal{S}_{\mathcal{L}}$, in which case trivially $\varphi \in \Delta$.

Subcase IIIb: Δ is not prime, i.e. there are $\psi_1, \psi_2 \in \mathcal{S}_{\mathcal{L}}$, both outside Δ , such that $\psi_1 \vee \psi_2 \in \Delta$. For $i = 1, 2$ we then have $\Delta \cup \{\psi_i\} \supset \Delta$ and thus $P(\overline{\Delta \cup \{\psi_i\}})$, that is, $\Delta, \psi_i \vdash \varphi$, by induction hypothesis. Hence $\Delta, \psi_1 \vee \psi_2 \vdash \varphi$ by disjunction elimination, and so eventually $\varphi \in \Delta$ because $\psi_1 \vee \psi_2 \in \Delta$.

With OI at hand we can conclude that $P(\Delta)$ for every $\Delta \in X$. In particular, P holds for the least element of X , that is, $P(\bar{\Gamma})$. By definition of P , this means $\Gamma \vdash \varphi$. \square

In view of Lemma 6, Theorem 8 can equivalently be put as follows.

Corollary 9 (OI). *For each $\Gamma \cup \{\varphi\} \subseteq \mathcal{S}_{\mathcal{L}}$ the following are equivalent:*

- (i) $\Gamma \vdash \varphi$;
- (ii) $\Gamma \subseteq \Delta \Rightarrow \varphi \in \Delta$ for every complete consistent theory $\Delta \subseteq \mathcal{S}_{\mathcal{L}}$.

To prepare for Section 3.3, we sketch how a proof of Corollary 9 can be obtained from the above proof of Theorem 8. The only modifications are required in Case II of the proof of Claim 3, which again is to be split into two subcases.

Subcase IIIa': Δ is inconsistent, that is, $\perp \in \Delta$. Hence again $\Delta = \mathcal{S}_{\mathcal{L}}$ and thus trivially $\varphi \in \Delta$.

Subcase IIIb': Δ is not complete, i.e. there is $\psi \in \mathcal{S}_{\mathcal{L}}$ such that both ψ and $\neg\psi$ lie outside Δ . Hence $\Delta \cup \{\psi\} \supset \Delta$ and $\Delta \cup \{\neg\psi\} \supset \Delta$, and thus $P(\overline{\Delta \cup \{\psi\}})$ and $P(\overline{\Delta \cup \{\neg\psi\}})$ by induction hypothesis, which is to say that $\Delta, \psi \vdash \varphi$ and $\Delta, \neg\psi \vdash \varphi$. In all, $\Delta, \psi \vee \neg\psi \vdash \varphi$ by disjunction elimination, and so $\varphi \in \Delta$ simply because $\psi \vee \neg\psi \in \Delta$, as Δ is a classical theory.

3.3 Intuitionistic logic

As usual, \perp stands for absurdity, $\neg\varphi$ for $\varphi \rightarrow \perp$ and \top for $\neg\perp$. We write EFQ and TND for the axioms *ex falso quodlibet* $\perp \rightarrow \psi$ and *tertium non datur* $\psi \vee \neg\psi$ restricted to sentences ψ .

Convention *If we say that a statement holds in a certain logic, then we mean that this statement can be proved for deducibility within this logic and with the very same logic used in the meta-language. Likewise, if we say that we use EFQ and/or TND, then we mean this in the meta-language, too.*

3.3.1 Excluded middle

Remark 5 holds in intuitionistic logic, too. This also is the case for some but not all implications within Remark 7—in which, however, we are not interested, for we have focussed on proper prime and complete consistent theories rather than maximal consistent sets of sentences.

Let us turn our attention to Lemma 6 instead. Trivially, every consistent theory is proper. By EFQ every proper theory is consistent; and again by EFQ every complete consistent theory is prime. In intuitionistic logic, in particular, Corollary 9 implies Theorem 8. In intuitionistic logic, however, one cannot prove that every prime theory is complete, for which—as in the proof of Lemma 6—one needs TND. Here is a characteristic example.

The smallest theory $\bar{\emptyset}$ is consistent and prime in intuitionistic logic, thanks to Gentzen's *Hauptsatz* and the disjunction property, respectively. In particular, Theorem 8 holds for $\Gamma = \emptyset$. On the other hand, $\bar{\emptyset}$ cannot be proved to be complete in intuitionistic logic, as this would just mean to postulate TND.

This example also shows that intuitionistic logic has not enough complete theories to make Corollary 9 hold for $\Gamma = \emptyset$, which would indeed entail TND. In fact, if ψ is any sentence, then $\varphi \equiv \psi \vee \neg\psi$ belongs to every complete theory Δ , since either $\psi \in \Delta$ or $\neg\psi \in \Delta$. So Corollary 9 is definitely too strong to hold in intuitionistic logic.

Before we study the status of Theorem 8 in intuitionistic logic, let us make a digression in the spirit of constructive reverse mathematics [13]. We say that a theory Γ is *Boolean* if $\psi \vee \neg\psi \in \Gamma$ for every sentence ψ . In classical logic every theory is Boolean, for which TND suffices. Independent of that, every complete theory is Boolean, as we have just stated—even in intuitionistic logic where conversely every Boolean theory is complete provided that it is prime. In intuitionistic logic, in particular, the complete consistent theories are exactly the Boolean proper prime theories. Now we can sum up.

Proposition 10. *In intuitionistic logic the following are equivalent.*

1. *TND holds.*
2. *Every theory is Boolean.*

3. Every prime theory is complete.
4. The smallest theory $\bar{0}$ is complete.
5. Theorem 8 implies Corollary 9.
6. Corollary 9 holds for $\Gamma = \emptyset$.

We recommend to proceed alongside the following paths: $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 6$ and $3 \Rightarrow 5 \Rightarrow 6 \Rightarrow 1$.

3.3.2 Proof by cases

We next inspect the proofs of Theorem 8 and Corollary 9. In the proof of Corollary 9 sketched before, we have used EFQ and TND to settle Subcase IIa' and Subcase IIb', respectively. This parallels our use of these principles, noticed above, during the proof of the parts of Lemma 6 that are relevant for deducing Corollary 9 from Theorem 8.

In the proof of Theorem 8, on the other hand, we did not need EFQ and TND to settle Subcase IIa and Subcase IIb. Yet in the proof of Theorem 8, and likewise in the one of Corollary 9, we had to distinguish Case I from Case II, actually from Subcase IIa and Subcase IIb. To control this instance of a proof by cases, which is essentially classical, we make it explicit following [16].

To this end we say that a finitely axiomatisable theory Γ has a *strong primality test* if for every theory Δ with $\Delta \supseteq \Gamma$ one can do a proof by cases of the type incriminated above. By this we mean that one can tell whether Δ is proper and prime, and if this is not the case, then one either knows that $\Delta = \mathcal{S}_{\mathcal{L}}$ or else one has at hand $\psi_1, \psi_2 \in \mathcal{S}_{\mathcal{L}}$, both lying outside Δ , such that $\psi_1 \vee \psi_2 \in \Delta$.

Hence the following variant of Theorem 8 equally holds for intuitionistic logic.

Theorem 11 (OI). *If a finitely axiomatisable theory $\Gamma \subseteq \mathcal{S}_{\mathcal{L}}$ admits a strong primality test, then for each $\varphi \in \mathcal{S}_{\mathcal{L}}$ the following are equivalent:*

- (i) $\varphi \in \Gamma$;
- (ii) $\Gamma \subseteq \Delta \Rightarrow \varphi \in \Delta$ for every proper prime theory $\Delta \subseteq \mathcal{S}_{\mathcal{L}}$.

Two choices have made possible to move to intuitionistic logic in the foregoing: first, to work with proper prime theories rather than complete consistent theories; secondly, to assume a strong primality test. Note that neither choice would make any difference for classical logic.

Although the assumption of a strong primality test may somehow look like cheating, it is of interest to review the situation in commutative algebra by which it has been inspired:

1. The appropriate analogue of a strong primality test is constructively provable for a sufficiently rich class of rings, the so-called fully Lasker-Noether rings [16].
2. If Krull's Lemma (KL), the counterpart of Theorem 8, is used to give a short and elegant proof of a certain theorem of concrete character, by reduction to the case of integral domains, then a constructive proof of the theorem is possible [17, 24] under decidability assumptions that are
 - (a) more elementary than a strong primality test and
 - (b) can be eliminated by basic proof-theoretic means.

For the time being, however, we do not know whether any part of this method can be carried over from algebra to logic. But KL and LL together, in the form of Theorem 8, have given rise to a universal Krull-Lindenbaum Theorem that equally follows from OI [20], which in turn has prompted a fairly general, constructive and syntactical conservation theorem for abstract entailment relations [21].

4 Related results for lattices

4.1 Distributive lattices

The proof given above of Theorem 8—which, as we have seen, in fact proves Theorem 11—can be generalized from the Lindenbaum algebra to arbitrary distributive lattices.

Proposition 12 (OI). *In a distributive lattice, every filter is the intersection of the (proper) prime filters above.*

Proof. Let (S, \leq) be a distributive lattice. We show that every filter $A \subseteq S$ is the intersection of all proper prime filters U with $A \subseteq U$. In other words, given $b \in S$, we show that $b \in A$ follows from the assumption that $A \subseteq U \Rightarrow b \in U$ for every proper prime filter U .

To this end, we consider

$$X = \{B \subseteq S \mid B \text{ is a filter and } A \subseteq B\}.$$

Partially ordered by inclusion, this X is chain-complete: e.g., the union of a chain of filters is a filter.

Let P be the predicate on X defined by

$$P(B) \iff b \in B.$$

Note that P is open, as joins of chains in X are given by unions.

We now prove that P is progressive, that is, $P(B)$ follows from the assumption that $P(B')$ holds for all $B' \in X$ with $B' \supset B$. To this end we distinguish two cases. If B is proper and prime, then we use the hypothesis $A \subseteq U \Rightarrow b \in U$ for $U = B$. Otherwise, either B is improper, i.e. $B = S$, in which case $b \in B$ anyway; or B is not prime. In the latter case there must be $c, d \notin B$ such that $c \vee d \in B$. Consider the filters B_c and B_d generated by $B \cup \{c\}$ and $B \cup \{d\}$, respectively. Since these filters are strictly larger than B , by assumption both $P(B_c)$ and $P(B_d)$, that is, $b \in B_c \cap B_d$. This means that either $b \in B$, and we are done, or there are b_c and b_d in B with $b \geq b_c \wedge c$ and $b \geq b_d \wedge d$. Hence

$$b = b \vee b \geq (b_c \wedge c) \vee (b_d \wedge d) = (b_c \vee b_d) \wedge (b_c \vee d) \wedge (c \vee b_d) \wedge (c \vee d) \in B,$$

because $b_c, b_d, c \vee d \in B$ and B is a filter. So $b \in B$ as well, that is, $P(B)$.

By OI, the predicate P holds for all $U \in X$. In particular, $P(A)$, that is, $b \in A$. □

Corollary 13. *For all $a, b \in S$,*

$$a \leq b \text{ if and only if } a \in U \Rightarrow b \in U \text{ for all (proper) prime filters } U \subseteq S.$$

Proof. Apply the previous proposition to the principal filter $\uparrow a$. □

4.2 Complete lattices

In the case of complete lattices, it is natural to try to replace prime filters with completely-prime filters. Recall that a filter U in a complete lattice S is *completely-prime* if $\bigvee T \in U$ implies $T \not\subseteq U$ for every $T \subseteq S$. One cannot, however, expect a similar proposition to hold true in general. For instance, in the case of locales (frames), the statement “every filter is an intersection of completely-prime filters” would imply the so-called property of spatiality, which is simply not true in general. Recall that a locale is spatial when $a \leq b$ holds if and only if every “point” in a also belongs to b , where the notion of a point for a locale is equivalent to that of a completely-prime filter. The regular open sets of, say,

the real line form a locale in which joins are interiors of closures of unions. Such a locale has no point (though having many filters, of course: e.g. the principal ones).

For complete lattices we must content ourselves with the following. Note that Proposition 12 remains true, by a similar proof, when distributivity is dropped, but filters are replaced by upward closed sets (*upsets*). The same result extends to the complete case as well.

Proposition 14 (OI). *Every upset in a complete lattice is the intersection of the (proper) completely-prime upsets above.*

As for the proof of Proposition 12, the crucial case is the one in which the upset B is not completely-prime, i.e. in which there is $\{c_i : i \in I\} \subseteq S$, disjoint from B , with $\bigvee_{i \in I} c_i \in B$. Now if I is empty, then $B = S$ and so $b \in B$ for the $b \in S$ under consideration. If I is inhabited, for each $i \in I$ the upset $B_i = \uparrow(B \cup \{c_i\})$ is strictly larger than B and hence contains b by hypothesis. So either $b \in B$ or $b \geq c_i$ for all $i \in I$, that is, $b \geq \bigvee_{i \in I} c_i \in B$, and thus $b \in B$.

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