

A FAST ACTIVE SET BLOCK COORDINATE DESCENT ALGORITHM FOR ℓ_1 -REGULARIZED LEAST SQUARES*

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Abstract. The problem of finding sparse solutions to underdetermined systems of linear equations arises in several applications (e.g., signal and image processing, compressive sensing, statistical inference). A standard tool for dealing with sparse recovery is the ℓ_1 -regularized least squares approach that has been recently attracting the attention of many researchers. In this paper, we describe an active set estimate (i.e., an estimate of the indices of the zero variables in the optimal solution) for the considered problem that tries to quickly identify as many active variables as possible at a given point, while guaranteeing that some approximate optimality conditions are satisfied. A relevant feature of the estimate is that it gives a significant reduction of the objective function when setting to zero all those variables estimated to be active. This enables us to easily embed it into a given globally converging algorithmic framework. In particular, we include our estimate into a block coordinate descent algorithm for ℓ_1 -regularized least squares, analyze the convergence properties of this new active set method, and prove that its basic version converges with a linear rate. Finally, we report some numerical results showing the effectiveness of the approach.

Key words. ℓ_1 -regularized least squares, active set, sparse optimization

AMS subject classifications. 65K05, 90C25, 90C06

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1. Introduction. The problem of finding sparse solutions to large underdetermined linear systems of equations has received a lot of attention in the last decades. This is due to the fact that several real-world applications can be formulated as linear inverse problems. A standard approach is the so called ℓ_2 - ℓ_1 unconstrained optimization problem

$$(1.1) \quad \min_{x \in \mathbb{R}^n} \frac{1}{2} \|Ax - b\|^2 + \tau \|x\|_1,$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ ($m < n$), and $\tau \in \mathbb{R}^+$. We denote by $\|\cdot\|$ the standard ℓ_2 norm and by $\|\cdot\|_1$ the ℓ_1 norm defined as $\|x\|_1 = \sum_{i=1}^n |x_i|$.

Several classes of algorithms have been proposed for the solution of problem (1.1). Among the others, we would like to recall iterative shrinkage/thresholding (IST) methods (see, e.g., [3, 4, 7, 9, 27]), augmented Lagrangian approaches (see, e.g., [2]), second-order methods (see, e.g., [5, 15]), sequential deterministic (see, e.g., [25, 26, 32]) and stochastic (see, e.g., [14, 22] and references therein) block coordinate approaches, parallel deterministic (see, e.g., [13] and references therein) and stochastic (see, e.g., [8, 23] and references therein) block coordinate approaches, and active set strategies (see, e.g., [16, 28, 29]).

The main feature of this class of problems is the fact that the optimal solution is usually very sparse (i.e., it has many zero components). Then, quickly building

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and/or correctly identifying the active set (i.e., the subset of zero components in an optimal solution) for problem (1.1) is becoming a crucial task in the context of big data optimization, since it can guarantee relevant savings in terms of CPU time. As a very straightforward example, we can consider a huge scale problem having a solution with just a few nonzero components. In this case, both the fast construction and the correct identification of the active set can considerably reduce the complexity of the problem, thus also giving us the chance to use more sophisticated optimization methods than the ones usually adopted. Various attempts have been made in order to use active set techniques in the context of ℓ_1 -regularized problems.

In [28, 29], Wen et al. proposed a two-stage algorithm, FPC-AS, where an estimate of the active variables set is driven by using a first-order iterative shrinkage method.

In [30], a block coordinate relaxation approach with proximal linearized subproblems yields convergence to critical points, while identification of the optimal manifold (under a nondegeneracy condition) allows acceleration techniques to be applied on a reduced space.

In [18], the authors solve an ℓ_1 -regularized log determinant program related to the problem of sparse inverse covariance matrix estimation combining a second-order approach with a technique to correctly identify the active set.

An efficient version of the two-block nonlinear constrained Gauss–Seidel algorithm that at each iteration fixes some variables to zero according to a simple active set rule has been proposed in [21] for solving ℓ_1 -regularized least squares.

In a recent paper [5], Byrd et al. described an interesting family of second-order methods for ℓ_1 -regularized convex problems. Those methods combined a semi-smooth Newton approach with a mechanism to identify the active manifold in the given problem.

In the case one wants to solve very large problems, block coordinate descent algorithms (both sequential and parallel) represent a very good alternative and, sometimes, the best possible answer [26]. An interesting coordinate descent algorithm combining a Newton step with a line search technique was described by Yuan et al. in [31]. In this context, the authors also proposed a shrinking technique (i.e., a heuristic strategy that tries to fix to zero a subset of variables according to a certain rule), which can be seen as a way to identify the active variables. In [26], some ideas on how to speed up their block coordinate descent algorithm by including an active set identification strategy are described, but no theoretical analysis is given for the resulting approach.

What we want to highlight here is that all the approaches listed above, except the one described in [5], estimate the final active set by using the current active set and perform subspace minimization on the remaining variables. In [5], the authors define an estimate that performs multiple changes in the active manifold by also including variables that are nonzero at a given point and satisfy some specific condition. Since this active set mechanism, due to the aggressive changes in the index set, can cause cycling, including the estimate into a globally converging algorithmic framework is not always straightforward.

In this work, we adapt the active set estimate proposed in [12] for constrained optimization problems to the ℓ_1 -regularized least squares case. Our estimate, similarly to the one proposed in [5], does not focus only on the zero variables of a given point. Instead it tries to quickly identify as many active variables as possible (including the nonzero variables of the point), while guaranteeing that some approximate optimality conditions are satisfied.

The main feature of the proposed active set strategy is that a significant reduction of the objective function is obtained when setting to zero all those variables estimated active. This global property, which is strongly related to the fact that the components estimated active satisfy an approximate optimality condition, makes it easy to use the estimate in a given globally converging algorithmic framework.

Furthermore, inspired by the papers [26, 31, 32], we describe a new block coordinate descent algorithm that embeds the considered active set estimate. At each iteration, the method first sets to zero the active variables, then uses a decomposition strategy for updating a bunch of the nonactive ones. On the one hand, decomposing the nonactive variables enables us to handle huge scale problems that other active set approaches cannot solve in reasonable time. On the other hand, since the subproblems analyzed at every iteration explicitly take into account the ℓ_1 -norm, the proposed algorithmic framework does not require a sign identification strategy (for the nonactive variables), which is typically needed when using other active set methods from the literature.

The paper is organized as follows. In section 3, we introduce our active set strategy. In section 4, we describe the active set coordinate descent algorithm, and prove its convergence. We further analyze the convergence rate of the algorithm. In section 5, we report some numerical results showing the effectiveness of the approach. Finally, we draw some conclusions in section 6.

2. Notation and preliminary results. Throughout the paper we denote by $f(x)$, $q(x)$, $g(x)$, and H the original function in problem (1.1), the quadratic term of the objective function in problem (1.1), the n gradient vector, and the $n \times n$ Hessian matrix of $\frac{1}{2}\|Ax - b\|^2$, respectively. Explicitly

$$q(x) = \frac{1}{2}\|Ax - b\|^2, \quad g(x) = A^\top(Ax - b), \quad H = A^\top A.$$

Given a matrix $Q \in \mathbb{R}^{n \times n}$, we further denote by $\lambda_{max}(Q)$ and $\lambda_{min}(Q)$ the maximum and the minimum eigenvalue of the matrix Q , respectively. Furthermore, with I we indicate the set of indices $I = \{1, \dots, n\}$, and with $Q_{I_j I_j}$ we indicate the submatrix of Q whose rows and columns indices are in $I_j \subseteq I$. We also report the optimality conditions for problem (1.1):

PROPOSITION 2.1. $x^* \in \mathbb{R}^n$ is an optimal solution of problem (1.1) if and only if

$$(2.1) \quad \begin{cases} x_i^* > 0, & g_i(x^*) + \tau = 0, \\ x_i^* < 0, & g_i(x^*) - \tau = 0, \\ x_i^* = 0, & -\tau \leq g_i(x^*) \leq \tau. \end{cases}$$

Furthermore, we define a continuous function $\Phi_i(x)$ that measures the violation of the optimality conditions in x_i (and is connected to the Gauss–Southwell-r rule proposed in [26]), that is

$$(2.2) \quad \Phi_i(x) = -\text{mid} \left\{ \frac{g_i(x) - \tau}{H_{ii}}, x_i, \frac{g_i(x) + \tau}{H_{ii}} \right\},$$

where $\text{mid}\{a, b, c\}$ indicates the median of a , b , c .

Finally, we recall the concept of strict complementarity.

DEFINITION 2.2. *Strict complementarity holds if, for any $x_i^* = 0$, we have*

$$(2.3) \quad -\tau < g_i(x^*) < \tau.$$

3. Active set estimate. All the algorithms that adopt active set strategies need to estimate a particular subset of components of the optimal solution x^* . In nonlinear constrained minimization problems, for example, using an active set strategy usually means correctly identifying the set of active constraints at the solution. In our context, we deal with problem (1.1) and the active set is considered as the subset of zero components of x^* .

DEFINITION 3.1. *Let $x^* \in \mathbb{R}^n$ be an optimal solution for problem (1.1). We define the active set as follows:*

$$(3.1) \quad \bar{\mathcal{A}}(x^*) = \{i \in I : x_i^* = 0\}.$$

We further define as a nonactive set the complementary set of $\bar{\mathcal{A}}(x^*)$,

$$(3.2) \quad \bar{\mathcal{N}}(x^*) = I \setminus \bar{\mathcal{A}}(x^*) = \{i \in \{1, \dots, n\} : x_i^* \neq 0\}.$$

In order to get an estimate of the active set we rewrite problem (1.1) as a box constrained programming problem and we use similar ideas to those proposed in [10].

Problem (1.1) can be equivalently rewritten as follows:

$$(3.3) \quad \begin{aligned} \min \quad & \frac{1}{2} \|A(u - v) - b\|^2 + \tau \sum_{i=1}^n (u_i + v_i), \\ & u \geq 0 \\ & v \geq 0, \end{aligned}$$

where $u, v \in \mathbb{R}^n$. Indeed, we can transform a solution $x^* \in \mathbb{R}^n$ of problem (1.1) into a solution $(u^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^n$ of (3.3) by using the following transformation:

$$u^* = \max(0, x^*), \quad v^* = \max(0, -x^*).$$

Equivalently, we can transform a solution $(u^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^n$ of (3.3) into a solution $x^* \in \mathbb{R}^n$ of problem (1.1) by using the following transformation:

$$x^* = u^* - v^*.$$

The Lagrangian function associated with (3.3) is

$$\mathcal{L}(u, v, \lambda, \mu) = \frac{1}{2} \|A(u - v) - b\|^2 + \tau \sum_{i=1}^n (u_i + v_i) - \lambda^\top u - \mu^\top v,$$

with $\lambda, \mu \in \mathbb{R}^n$ vectors of Lagrangian multipliers. Let $(u^*, v^*, \lambda^*, \mu^*)$ be an optimal solution of problem (3.3). Then, from necessary optimality conditions, we have

$$(3.4) \quad \begin{aligned} \lambda_i^* &= g_i(u^* - v^*) + \tau = g_i(x^*) + \tau, \\ \mu_i^* &= \tau - g_i(u^* - v^*) = \tau - g_i(x^*). \end{aligned}$$

From (3.4), we can introduce the following two multiplier functions,

$$(3.5) \quad \begin{aligned} \lambda_i(u, v) &= g_i(u - v) + \tau, \\ \mu_i(u, v) &= \tau - g_i(u - v). \end{aligned}$$

By means of the multiplier functions, we can recall the nonactive set estimate $\mathcal{N}(u, v)$ and active set estimate $\mathcal{A}(u, v)$ proposed in the field of constrained smooth optimization (see [12] and references therein):

$$(3.6) \quad \mathcal{N}(u, v) = \{i : u_i > \epsilon \lambda_i(u, v)\} \cup \{i : v_i > \epsilon \mu_i(u, v)\},$$

$$(3.7) \quad \mathcal{A}(u, v) = I \setminus \mathcal{N}(u, v),$$

where ϵ is a positive scalar.

We draw inspiration from (3.6) and (3.7) to propose the new estimates of active and nonactive set for problem (1.1). Indeed, by using the relations

$$u = \max(0, x) \quad \text{and} \quad v = \max(0, -x),$$

we can give the following definitions.

DEFINITION 3.2. *Let $x \in \mathbb{R}^n$. We define the following sets as an estimate of the nonactive and active variables sets:*

$$(3.8) \quad \mathcal{N}(x) = \{i : \max(0, x_i) > \epsilon(\tau + g_i(x))\} \cup \{i : \max(0, -x_i) > \epsilon(\tau - g_i(x))\},$$

$$(3.9) \quad \mathcal{A}(x) = I \setminus \mathcal{N}(x).$$

In the next subsections, we first discuss local and global properties of our estimate, then we compare it with other active set estimates.

3.1. Local properties of the active set estimate. Now, we describe some local properties (in the sense that those properties only hold in a neighborhood of a given point) of our active set estimate. In particular, the following theoretical result states that when the point is sufficiently close to an optimal solution the related active set estimate is a subset of the active set calculated in the optimal point (and it includes the optimal active variables that satisfy strict complementarity). Furthermore, when strict complementarity holds the active set estimate is actually equal to the optimal active set.

THEOREM 3.3. *Let $x^* \in \mathbb{R}^n$ be an optimal solution of problem (1.1). Then, there exists a neighborhood of x^* such that, for each x in this neighborhood, we have*

$$(3.10) \quad \bar{\mathcal{A}}^+(x^*) \subseteq \mathcal{A}(x) \subseteq \bar{\mathcal{A}}(x^*)$$

with $\bar{\mathcal{A}}^+(x^*) = \bar{\mathcal{A}}(x^*) \cap \{i : -\tau < g_i(x^*) < \tau\}$. Furthermore, if strict complementarity (2.2) holds in x^* , then there exists a neighborhood of x^* such that, for each x in this neighborhood, we have

$$(3.11) \quad \mathcal{A}(x) = \bar{\mathcal{A}}(x^*).$$

Proof. The proof follows from Theorem 2.1 in [12]. □

3.2. A global property of the active set estimate. Here, we analyze a global property of the active set estimate. In particular, we show that, for a suitably chosen value of the parameter ϵ appearing in Definition 3.2, by starting from a point $z \in \mathbb{R}^n$ and fixing to zero all variables whose indices belong to the active set estimate $\mathcal{A}(z)$, it is possible to obtain a significant decrease of the objective function. This property, which strongly depends on the specific structure of the problem under analysis,

represents a new interesting theoretical result, since it enables us to easily embed the active set estimate into any globally converging algorithmic framework (in the next section, we will show how to include it into a specific block coordinate descent method). Furthermore, the global property cannot be deduced from the theoretical results already reported in [12].

Assumption 1. Parameter ϵ appearing in Definition 3.2 satisfies the following condition:

$$(3.12) \quad 0 < \epsilon < \frac{1}{\lambda_{\max}(A^T A)}.$$

PROPOSITION 3.4. *Let Assumption 1 hold. Given a point $z \in \mathbb{R}^n$ and the related sets $\mathcal{A}(z)$ and $\mathcal{N}(z)$, let y be the point defined as*

$$y_{\mathcal{A}(z)} = 0, \quad y_{\mathcal{N}(z)} = z_{\mathcal{N}(z)}.$$

Then,

$$f(y) - f(z) \leq -\frac{1}{2\epsilon} \|y - z\|^2.$$

Proof. For the proof, see Appendix A. □

3.3. Comparison with other active set strategies. Our active set estimate is somehow related to those proposed, respectively, by Byrd et al. in [5] and by Yuan et al. in [31]. It is also connected in some way to the IST Algorithm (ISTA), see, e.g., [3, 9]. Indeed, an ISTA step can be seen as a simple way to set to zero the variables in the context of ℓ_1 -regularized least squares problems.

Here, we would like to point out the similarities and the differences between those strategies and the one we propose in the present paper.

First of all, we notice that, at a generic iteration k of a given algorithm, if x^k is the related iterate and $i \in I$ is an index estimated active by our estimate, that is,

$$i \in \mathcal{A}(x^k) = \{i : \max(0, x_i^k) \leq \epsilon(\tau + g_i(x^k))\} \cap \{i : \max(0, -x_i^k) \leq \epsilon(\tau - g_i(x^k))\},$$

it is equivalent to write

$$(3.13) \quad x_i^k \in [\epsilon(g_i(x^k) - \tau), \epsilon(g_i(x^k) + \tau)] \quad \text{and} \quad -\tau \leq g_i(x^k) \leq \tau,$$

which means that x_i^k is sufficiently small and satisfies the optimality condition associated with a zero component (see (2.1)). As we will see, the estimate, due to the way it is defined, tends to be more conservative than other active set strategies (i.e., it might set to zero slightly smaller sets of variables). On the other hand, the global property analyzed in the previous section (i.e., decrease of the objective function when setting to zero the active variables) seems to indicate that the estimate truly contains indices related to variables that will be active in the optimal solution. As we will see later on, this important property does not hold when considering the other active set strategies analyzed here.

In the block active set algorithm for quadratic ℓ_1 -regularized problems proposed in [5], the active set estimate, at a generic iteration k , can be rewritten in the following way:

$$\begin{aligned} \mathcal{A}_{Byrd}^k &= \{i : x_i^k = 0; g_i(x^k) \in (-\tau, \tau)\} \cup \{i : x_i^k < 0; \\ &g_i(x^k) = -\tau\} \cup \{i : x_i^k > 0; g_i(x^k) = \tau\}. \end{aligned}$$

Let $x^k \in \mathbb{R}^n$ and $i \in \{1, \dots, n\}$ be an index estimated active by our estimate, from (3.13), we get $g_i(x^k) \in [-\tau, \tau]$.

Then, in the case $x_i^k = 0$, $i \in \mathcal{A}_{Byrd}^k$ implies $i \in \mathcal{A}(x^k)$. In fact, let $i \in \mathcal{A}_{Byrd}^k$. If $x_i^k = 0$ we have $g_i(x^k) \in (-\tau, \tau)$ so that $i \in \mathcal{A}(x^k)$. It is easy to see that the other way around is not true.

Other differences between the two estimates come out when considering indices i such that $x_i^k \neq 0$. Let $i \in \mathcal{A}_{Byrd}^k$ and, in particular, $i \in \{i : x_i^k < 0; g_i(x^k) = -\tau\}$. If $|x_i^k| > \epsilon 2\tau$, then we get

$$\max(0, -x_i^k) = -x_i^k > \epsilon 2\tau = \epsilon(\tau - g_i(x^k)),$$

so that $i \notin \mathcal{A}(x^k)$. Using the same reasoning we can see that, in the case $i \in \mathcal{A}_{Byrd}^k$ and, in particular, $i \in \{i : x_i^k > 0; g_i(x^k) = \tau\}$, it can happen

$$\max(0, x_i^k) = x_i^k > \epsilon 2\tau = \epsilon(\tau + g_i(x^k)),$$

so that $i \notin \mathcal{A}(x^k)$.

In [31], the active set estimate is defined as follows,

$$(3.14) \quad \mathcal{A}_{Yuan}^k = \{i : x_i^k = 0; g_i(x^k) \in (-\tau + M^{k-1}, \tau - M^{k-1})\},$$

where M^{k-1} is a positive scalar that measures the violation of the optimality conditions. It is easy to see that our active set contains the one proposed in [31]. Furthermore, we have that variables contained in our estimate are not necessarily contained in the estimate (3.14). In particular, a big difference between our estimate and the one proposed in [31] is that we can also include variables that are nonzero at the current iterate.

As a final comparison, we would like to point out the differences between the ISTA strategy and our estimate. Consider the generic iteration of ISTA with the same ϵ used in our active set strategy:

$$(3.15) \quad x^{k+1} = \arg \min_x \left\{ g(x^k) + g(x^k)^\top (x - x^k) + \epsilon \|x - x^k\|^2 + \tau \|x\|_1 \right\}.$$

From the optimality conditions of the inner problem in (3.15), we have that the zero variables at x^{k+1} belong to the following set

$$(3.16) \quad \mathcal{A}_{ISTA}^k = \{i : \epsilon(-\tau + g_i(x^k)) \leq x_i^k \leq \epsilon(\tau + g_i(x^k))\}.$$

We can easily see that $\mathcal{A}(x^k) \subseteq \mathcal{A}_{ISTA}^k$. The opposite is not always true, apart from the variables $x_i^k = 0$. As a matter of fact, let us consider $x_i^k > 0$ and $i \in \mathcal{A}_{ISTA}^k$. Then, we have that

$$\begin{aligned} x_i^k \leq \epsilon(\tau + g_i(x^k)) &\Rightarrow i \in \{i : \max(0, x_i^k) \leq \epsilon(\tau + g_i(x^k))\}, \\ x_i^k \geq \epsilon(-\tau + g_i(x^k)) &\Rightarrow -x_i^k \leq \epsilon(\tau - g_i(x^k)). \end{aligned}$$

In order to have $i \in \mathcal{A}(x^k)$ it should be

$$\epsilon(\tau - g_i) \geq \max\{0, -x_i^k\} = 0;$$

that is a tighter requirement with respect to the one within \mathcal{A}_{ISTA}^k . Similar reasoning applies also to variables $x_i^k < 0$ with $i \in \mathcal{A}_{ISTA}^k$. We would also like to note that the

ISTA step might generate unnecessary projections of variables to zero, thus being not always effective as a tool for identifying the active set.

In this final remark, we show that, when using the active set strategies analyzed above, a sufficient decrease of the objective function cannot be guaranteed by setting to zero the variables in the active set (i.e., Proposition 3.4 does not hold). This fact makes it hard, in some cases, to include those active set strategies into a globally convergent algorithmic framework.

Remark 3.5. Proposition 3.4 does not hold for the active set strategies described above. This can be easily seen in the following case.

Let us assume that, at some iteration k , there exists only one index $\hat{i} \in \mathcal{A}_{B_{yrd}}^k$ with $x_{\hat{i}}^k > 0$, $H_{\hat{i}\hat{i}} > 0$, and $g_{\hat{i}}(x^k) = \tau$. Let $z = x^k$ and y be the point defined as $y_i = x_i^k$ for all $i \neq \hat{i}$, and $y_{\hat{i}} = 0$. Then,

$$f(y) = f(x^k) + (g_{\hat{i}}(x^k) - \tau)(y_{\hat{i}} - x_{\hat{i}}^k) + \frac{1}{2}(y_{\hat{i}} - x_{\hat{i}}^k)^2 H_{\hat{i}\hat{i}}.$$

Since $H_{\hat{i}\hat{i}} > 0$ and $g_{\hat{i}}(x^k) = \tau$, we have $f(y) - f(x^k) > 0$, so that by setting to zero the active variable we get an increase of the objective function value.

The same reasoning applies also to the ISTA step, assuming that at some iteration k , there exists only one index \hat{i} such that

$$\epsilon(-\tau + g_{\hat{i}}(x^k)) < x_{\hat{i}}^k < \epsilon(\tau + g_{\hat{i}}(x^k))$$

and $g_{\hat{i}}(x^k) = \tau$.

Finally, it is easy to notice that, at each iteration k , the active set estimate $\mathcal{A}_{Y_{uan}}^k$ defined in [31] only keeps fixed to zero, at iteration k , some of the variables that are already zero in x^k , thus not changing the objective function value.

4. A fast active set block coordinate descent algorithm. In this section, we describe our fast active set block coordinate descent algorithm (FAST-BCDA) and analyze its theoretical properties. The main idea behind the algorithm is that of exploiting as much as possible the good properties of our active set estimate, more specifically:

- the ability to identify, for k sufficiently large, the “strong” active variables (namely, those variables satisfying the strict complementarity, see Theorem 3.3);
- the ability to obtain, at each iteration, a sufficient decrease of the objective function, by fixing to zero those variables belonging to the active set estimate (see Proposition 3.4 of the previous section).

As we have seen in the previous section, the estimate, due to the way it is defined, tends to be more conservative than other active set strategies (i.e., it might set to zero a slightly smaller set of variables at each iteration). Anyway, since for each block we exactly solve an ℓ_1 -regularized subproblem, we can eventually force to zero some other variables in the nonactive set. Another important consequence of including the ℓ_1 -norm in the subproblems is that we do not need any sign identification strategy for the nonactive variables.

At each iteration k , the algorithm defines two sets $\mathcal{N}^k = \mathcal{N}(x^k)$, $\mathcal{A}^k = \mathcal{A}(x^k)$ and executes two steps:

- (1) it sets to zero all of the active variables;
- (2) it minimizes only over a subset of the nonactive variables, i.e., those which violate the optimality conditions the most.

More specifically, we consider the measure related to the violation of the optimality conditions reported in (2.2). We then sort in decreasing order the indices of nonactive variables (i.e., the set of indices \mathcal{N}^k) with respect to this measure and define the subset $\bar{\mathcal{N}}_{ord}^k \subseteq \mathcal{N}^k$ containing the first s sorted indices. The set $\bar{\mathcal{N}}_{ord}^k$ is then partitioned into q subsets I_1, \dots, I_q of cardinality r , such that $s = qr$. Then the algorithm performs q subiterations. At the j th subiteration the algorithm considers the set $I_j \subseteq \bar{\mathcal{N}}_{ord}^k$ and solves to optimality the subproblem we get from (1.1), by fixing all the variables but the ones whose indices belong to I_j . Below we report the scheme of the proposed algorithm (see Algorithm 1).

Algorithm 1. FAST-BCDA.

- 1 **Choose** $x^0 \in \mathbb{R}^n$, **Set** $k = 0$.
- 2 **For** $k = 0, 1 \dots$
- 3 **Compute** $\mathcal{A}^k, \mathcal{N}^k, \bar{\mathcal{N}}_{ord}^k$;
- 4 **Set** $y_{\mathcal{A}^k}^{0,k} = 0$ and $y_{\mathcal{N}^k}^{0,k} = x_{\mathcal{N}^k}^k$;
- 5 **For** $j = 1, \dots, q$
- 6 **Compute** $y_{I_j}^{j,k}$, with $I_j \subseteq \bar{\mathcal{N}}_{ord}^k$, solution of problem

$$\min_{w \in \mathbb{R}^r} g_{I_j}(y^{j-1,k})^\top (w - y_{I_j}^{j-1,k}) + \frac{1}{2}(w - y_{I_j}^{j-1,k})^\top H_{I_j I_j} (w - y_{I_j}^{j-1,k}) + \tau \|w\|_1$$

- 7 **Set** $y_i^{j,k} = y_i^{j-1,k}$ if $i \notin I_j$;
 - 8 **End For**
 - 9 **Set** $x^{k+1} = y^{q,k}$;
 - 10 **End For**
-

The convergence of FAST-BCDA is based on two important results. The first one is Proposition 3.4, which guarantees a sufficient decrease of the objective function by setting to zero the variables in the active set. The second one is reported in the proposition below. It shows that, despite the presence of the nonsmooth term, by exactly minimizing problem (1.1) with respect to a subset J of the variables (keeping all the other variables fixed), it is possible to get a sufficient decrease of the objective function in the case $\lambda_{min}(H_{JJ}) > 0$.

PROPOSITION 4.1. *Given a point $z \in \mathbb{R}^n$ and a set $J \subseteq I$, let $w^* \in \mathbb{R}^{|J|}$ be the solution of problem (1.1), where all variables but the ones whose indices belong to J are fixed to $z_{I \setminus J}$. Let $y \in \mathbb{R}^n$ be defined as*

$$y_J = w^*, \quad y_{I \setminus J} = z_{I \setminus J}.$$

Then we have

$$(4.1) \quad f(y) - f(z) \leq -\frac{1}{2} \lambda_{min}(H_{JJ}) \|y - z\|^2.$$

Proof. For the proof, see Appendix B. □

Now, we introduce an assumption that will enable us to prove global convergence of our algorithm.

Assumption 2. The matrix $A \in \mathbb{R}^{m \times n}$ satisfies the following condition

$$(4.2) \quad \min_J \lambda_{\min}((A^\top A)_{JJ}) \geq \sigma > 0,$$

where J is any subset of $\{1, \dots, n\}$ such that $|J| = r$, with r the cardinality of the blocks used in FAST-BCDA.

Remark 4.2. We notice that even though there are some similarities between condition (4.2) and the well-known restricted isometry property (RIP) condition with fixed order r (see, e.g., [6] for further details), condition (4.2) is weaker than the RIP condition.

Finally, we are ready to state the main result concerning the global convergence of FAST-BCDA.

THEOREM 4.3. *Let Assumptions 1 and 2 hold. Let $\{x^k\}$ be the sequence produced by Algorithm FAST-BCDA.*

Then, either an integer $\bar{k} \geq 0$ exists such that $x^{\bar{k}}$ is an optimal solution for problem (1.1), or the sequence $\{x^k\}$ is infinite and every limit point x^ of the sequence is an optimal point for problem (1.1).*

Proof. For the proof, see Appendix B. □

Now, we discuss Assumptions 1 and 2 that are needed to guarantee convergence of FAST-BCDA.

4.1. Comments on the assumptions. Assumption 1 requires the evaluation of $\lambda_{\max}(A^\top A)$, which is not always easily computable for large scale problems. Hence, we describe an updating rule for the parameter ϵ , that enables us to avoid any “a priori” assumption on ϵ .

In practice, at each iteration k we need to find the smallest $h \in \mathbb{N}$ such that the value $\epsilon = \theta^h \bar{\epsilon}$ and the corresponding sets \mathcal{A}^k , \mathcal{N}^k give a point

$$y_{\mathcal{A}^k}^{0,k} = 0 \quad \text{and} \quad y_{\mathcal{N}^k}^{0,k} = x_{\mathcal{N}^k}^k$$

satisfying

$$(4.3) \quad f(y^{0,k}) \leq f(x^k) - \gamma \|y^{0,k} - x^k\|^2$$

with $\gamma > 0$. Then, we can introduce a variation of FAST-BCDA, namely, FAST-BCDA- ϵ , that includes the updating rule for the parameter ϵ in its scheme, and prove its convergence.

THEOREM 4.4. *Let Assumption 2 hold. Let $\{x^k\}$ be the sequence produced by Algorithm FAST-BCDA- ϵ .*

Then, either an integer $\bar{k} \geq 0$ exists such that $x^{\bar{k}}$ is an optimal solution for problem (1.1), or the sequence $\{x^k\}$ is infinite and every limit point x^ of the sequence is an optimal point for problem (1.1).*

Proof. The proof follows by repeating the same arguments of the proof of Theorem 4.3 by replacing the relation (B.10) with (4.3). □

Assumption 2, which we need to satisfy in order to guarantee convergence of both FAST-BCDA and FAST-BCDA- ϵ , is often met in practice if we consider blocks of 1 or 2

variables (i.e., r equal to 1 or 2). Indeed, when solving blocks of 1 variable, we need to guarantee that any column A_j of matrix A is such that

$$\|A_j\|^2 \geq \sigma > 0.$$

This is often the case when dealing with overcomplete dictionaries for signal/image reconstruction (as the columns of matrix A are usually normalized; see, e.g., [1]). When using 2-dimensional blocks, we want no parallel columns in matrix A . This is a quite common requirement in the context of overcomplete dictionaries (as it corresponds to asking that mutual coherence is lower than 1; see, e.g., [1]). Furthermore, the solution of 1-dimensional block subproblems can be determined in closed form by means of the well-known scalar soft-threshold function (see, e.g., [3, 27]). Similarly, we can express in closed form the solution of 2-dimensional block subproblems.

Summarizing, thanks to the possibility of using an updating rule for ϵ , and due to the fact that we only use blocks of dimensions 1 or 2 in our algorithm, we have that Assumptions 1 and 2 are quite reasonable in practice.

4.2. Convergence rate analysis. Here, we report a result related to the convergence rate of FAST-BCDA with 1-dimensional blocks (namely, FAST-1CDA). In particular, we show that it converges at a linear rate. In order to prove the result, we make an assumption that is common when analyzing the convergence rate of both algorithms for ℓ_1 -regularized problems (see, e.g., [17]) and algorithms for general problems (see, e.g., [20]).

Assumption 3. Let $\{x^k\}$ be the sequence generated by FAST-1CDA. We have that

$$(4.4) \quad \lim_{k \rightarrow \infty} x^k = x^*,$$

where x^* is an optimal point of problem (1.1).

Now, we state the theoretical result related to the linear convergence.

THEOREM 4.5. *Let Assumptions 1, 2, and 3 hold. Let $\{x^k\}$ be the sequence generated by FAST-1CDA.*

Then $\{f(x^k)\}$ converges at least Q -linearly to f^ , where $f^* = f(x^*)$. Furthermore, $\{x^k\}$ converges at least R -linearly to x^* .*

Proof. For the proof, see Appendix C.

5. Numerical results. In this section, we report the numerical experiments related to FAST-BCDA. We implemented our method in MATLAB, and considered four different versions of it in the experiments:

- FAST-1CDA and FAST-2CDA, basic versions of FAST-BCDA where blocks of dimension 1 and 2 are, respectively, considered;
- FAST-1CDA-E and FAST-2CDA-E, “enhanced” versions of FAST-BCDA where again blocks of dimension 1 and 2 are, respectively, considered (see subsection 5.1 for further details).

We first analyzed the performance of these four versions of our algorithm. Then, we compared the best one with other algorithms for ℓ_1 -regularized least squares problems. Namely, we compared FAST-2CDA-E with ISTA [3, 9], FISTA [3], PSSgb [24], SpARSA [27], and FPC_AS [28].

All the tests were performed on an Intel Xeon(R) CPU E5-1650 v2 3.50 GHz using MATLAB R2011b.

We considered two different testing problems of the form (1.1), commonly used for software benchmarking (see, e.g., [28, 15]). In particular, we generated artificial signals of dimension $n = 2^{14}, 2^{15}, 2^{16}, 2^{17}$, with a number of observations $m = n/4$ and we set the number of nonzeros $T = \text{round}(\rho m)$ with $\rho = \{0.01, 0.03, 0.05, 0.07, 0.1\}$. The two test problems (P1 and P2) differ in the way matrix A is generated:

P1: Considering \bar{A} as the Gaussian matrix whose elements are generated independently and identically distributed from the normal distribution $\mathcal{N}(0, 1)$, the matrix A was generated by scaling the columns of \bar{A} .

P2: Considering \bar{A} as the matrix generated by using the MATLAB command

$$A = \text{sprand}(m, n, \text{density}),$$

with density = 0.5, the matrix A was generated by scaling the columns of \bar{A} . We would like to note that the Hessian matrices $A^T A$ related to instances of problem P1 have most of the mass on the diagonal. Then, those instances are in general easier to solve than the ones of problem P2.

Once the matrix A was generated, the true signal x^* was built as a vector with T randomly placed ± 1 spikes, with zero in the other components. Finally, for all problems, the vector of observations b was chosen as $b = Ax^* + \eta$, where η is a Gaussian white noise vector, with variance 10^{-3} . We set $\tau = 0.1 \|A^T b\|_\infty$ as in [2, 27]. We produced ten different random instances for each problem, for a total of 400 instances. The comparison of the overall computational effort is carried out by using the performance profiles proposed by Dolan and Moré in [11], plotting graphs in a logarithmic scale.

For the value of s (number of nonactive variables to be used in \bar{N}_{ord}) we set $s = \text{round}(0.8T)$ for FAST-1CDA and $s = \text{round}(0.65T)$ for FAST-2CDA (these s values are the ones that guarantee the best performances among the ones we tried). Concerning the choice of the ϵ parameter used in the active set estimate, the easiest choice is that of setting ϵ to a fixed value. We tested several values and obtained the best results with $\epsilon = 10^{-4}$ and $\epsilon = 10^{-5}$ for FAST-1CDA and FAST-2CDA, respectively. We further tested an implementation of both FAST-1CDA- ϵ and FAST-2CDA- ϵ . Since there were no significant improvements in the performance, we decided to keep the ϵ value fixed.

We would also like to spend a few words about the criterion for choosing the variables in \bar{N}_{ord}^k . In some cases, we found it to be more efficient using the following measure:

$$(5.1) \quad \begin{array}{ll} |g_i(x^k) + \tau| & \text{if } x_i^k > 0, \\ |g_i(x^k) - \tau| & \text{if } x_i^k < 0, \\ \max\{0, -(g_i(x^k) + \tau), g_i(x^k) - \tau\} & \text{if } x_i^k = 0 \end{array}$$

in place of the one reported in (2.2), which we considered for proving the theoretical results. The main feature of this new measure is that it only takes into account first-order information (while (2.2) considers also proximity of the component value to zero). Anyway, replacing (2.2) with the new measure is not a big deal, since convergence can still be proved using (5.1). Furthermore, the linear rate can be easily obtained assuming that strict complementarity holds. Intuitively, considering only first-order information in the choice of the variables should make more sense in our context, since proximity to zero is already taken into account when using the estimate to select the active variables.

5.1. Enhanced version of FAST-BCDA. By running our codes, we noticed that the cardinality of the set related to the nonactive variables decreases quickly as the

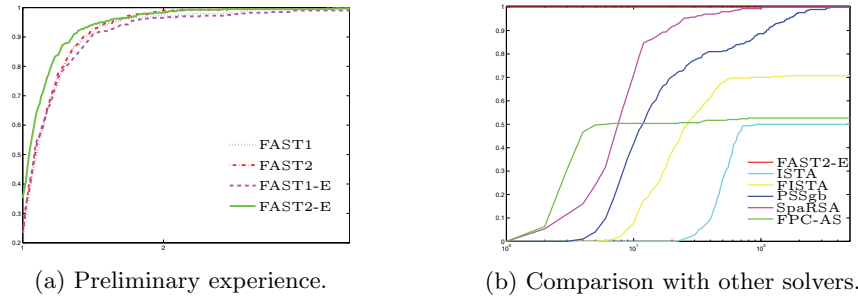


FIG. 1. Performance profiles on all instances (CPU time).

iterations go by. In general, very few iterations are needed to obtain the real nonactive set. By this evidence, and keeping in mind the theoretical result reported in section 3, we decided to develop an “enhanced” version of our algorithms, taking inspiration from the second stage of the FPC-AS algorithm [28]. Once a “good” estimate \mathcal{N}^k of $\mathcal{N}(x^*)$ was obtained, we solved the following smooth optimization subproblem

$$\begin{aligned} \min \quad & \frac{1}{2} \|Ax - b\|^2 + \tau \text{sign}(x_{\mathcal{N}^k})^\top x_{\mathcal{N}^k} \\ \text{s.t.} \quad & x_i = 0, \quad i \in \mathcal{A}^k. \end{aligned}$$

In practice, we considered an estimate \mathcal{N}^k “good” if both there are no changes in the cardinality of the set with respect to the last two iterations, and $|\mathcal{N}^k|$ is lower than or equal to a certain threshold ξ (we fixed $\xi = 0.05n$ in our experiments).

5.2. Preliminary experiments. In order to pick the best version among the four we developed, we preliminarily compared the performance of FAST-1CDA (FAST1), FAST-2CDA (FAST2), FAST-1CDA-E (FAST1-E), and FAST-2CDA-E (FAST2-E). In Figure 1(a), we report the performance profiles with respect to the CPU time.

As we can see, even if the four version of FAST-BCDA have similar behavior, FAST-2CDA-E is the one that gives the overall best performance. We then choose FAST-2CDA-E as the algorithm to be compared with the other state-of-the art algorithms for ℓ_1 -regularized problems.

5.3. Comparison with other algorithms. In this section, we report the numerical experience related to the comparison of FAST-2CDA-E with ISTA [3, 9], FISTA [3], PSSgb [24], SpaRSA [27], and FPC_AS [28].

In our tests, we first ran FAST-2CDA to obtain a target objective function value, then ran the other algorithms until each of them reached the given target (see, e.g., [27]). Any run exceeding the limit of 1000 iterations is considered failure. Default values were used for all parameters in SpaRSA [27] and FPC_AS [28]. For PSSgb [24] we considered the two-metric projection method and we set the parameter `options.quadraticInit` to 1, since this setting can achieve better performance for problems where backtracking steps are required on each iteration (see <http://www.cs.ubc.ca/~schmidtm/Software/thesis.html>). In all codes, we considered the null vector as the starting point and all matrices were stored explicitly. In Figure 1(b), we report the plot of the performance profiles related to the CPU time for all instances. From these profiles it is clear that FAST-2CDA-E outperforms all the other algorithms and that SpaRSA and PSSgb are the two best competitors. We then further compare, in Figure 2, FAST-2CDA-E, SpaRSA, and PSSgb reporting the

box plots related to the distribution of the CPU time. On each box, the central mark is the median, the edges of the box are the 25th and 75th percentiles, the whiskers extend to the most extreme data points not considered outliers, and outliers are plotted individually. In particular, Figure 2 shows the plots related to the distribution of

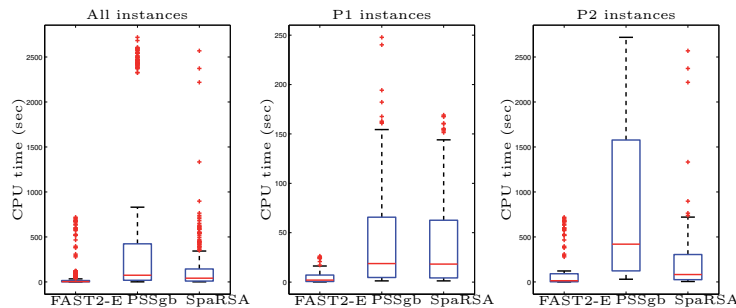


FIG. 2. Box plots (CPU time).

the CPU time for all instances, for P1 instances, and for P2 instances, respectively. For what concerns P1 instances, SpaRSA and PSSgb show similar behavior, while observing the plot related to P2 instances SpaRSA shows a better performance. For both classes, FAST-2CDA-E shows the lowest median. As a further comparison among FAST-2CDA-E, SpaRSA, and PSSgb, we report in Figures 3 and 4, the plots of the relative error versus the CPU time for the P1 and the P2 instances, respectively. In each plot, the curves are averaged over the ten runs for fixed ρ and n . Observing these plots, we notice that FAST-2CDA-E is able to reach better solutions with lower CPU time.

5.4. Real examples. In this subsection, we test the efficiency of our algorithm on realistic image reconstruction problems. We considered six images: a SheppLogan phantom available through the MATLAB Image Processing Toolbox and five widely used images downloaded from <http://dsp.rice.edu/cscamera> (the letter R, the mandrill, the dice, the ball, the mug). Each image has 128×128 pixels. We followed the procedure described in [28] to generate the instances (i.e., matrix A and vector b). What we want to highlight here is that the optimal solutions are unknown. Hence the reconstructed images can only be compared by visual inspection. Also in this case, we first ran FAST-2CDA to obtain a target objective function value, then ran the other algorithms until each of them reached the given target. The CPU time needed for reconstructing the images is reported in Table 1. In Figure 5, we report the images of the dice and of the mandrill reconstructed by FAST-2CDA-E, PSSgb, and SpaRSA. In Table 1, we can easily see that FAST-BCDA was faster in all problems.

6. Conclusions. In this paper, we devised an active set block coordinate descent method (FAST-BCDA) for solving ℓ_1 -regularized least squares problems. The way the active set estimate is calculated guarantees a sufficient decrease in the objective function at every iteration when setting to zero the variables estimated active. Furthermore, since the subproblems related to the blocks explicitly take into account the ℓ_1 -norm, the proposed algorithmic framework does not require a sign identification strategy for the nonactive variables.

Global convergence of the method is established. A linear convergence result is also proved. Numerical results are presented to verify the practical efficiency of the

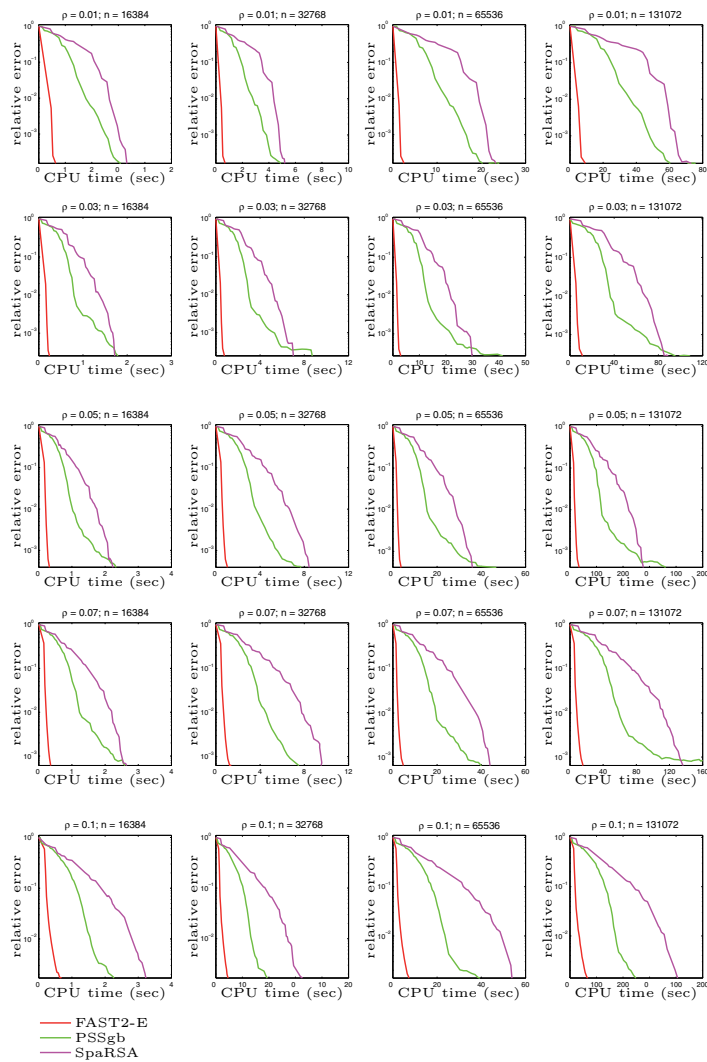


FIG. 3. Relative error versus CPU time—P1 instances.

method, and they indicate that FAST-BCDA compares favorably with other state-of-the-art techniques.

We further would like to note that the proposed active set strategy is independent from the specific algorithm we have designed and can be easily included into other algorithms for ℓ_1 -regularized least squares, both sequential and parallel, to improve their performance. We finally highlight that the algorithmic scheme we described can be easily modified in order to work in a parallel fashion. Future work will be devoted to adapt the presented approach to handle convex ℓ_1 -regularized problems.

Appendix A. Main theoretical result related to the active set estimate.

Here, we prove the main theoretical result related to the active set estimate.

Proof of Proposition 3.4. We first define the sets $\mathcal{N} = \mathcal{N}(z)$ and $\mathcal{A} = \mathcal{A}(z)$. By taking into account the definitions of the sets \mathcal{A} and \mathcal{N} and the points y and z , we

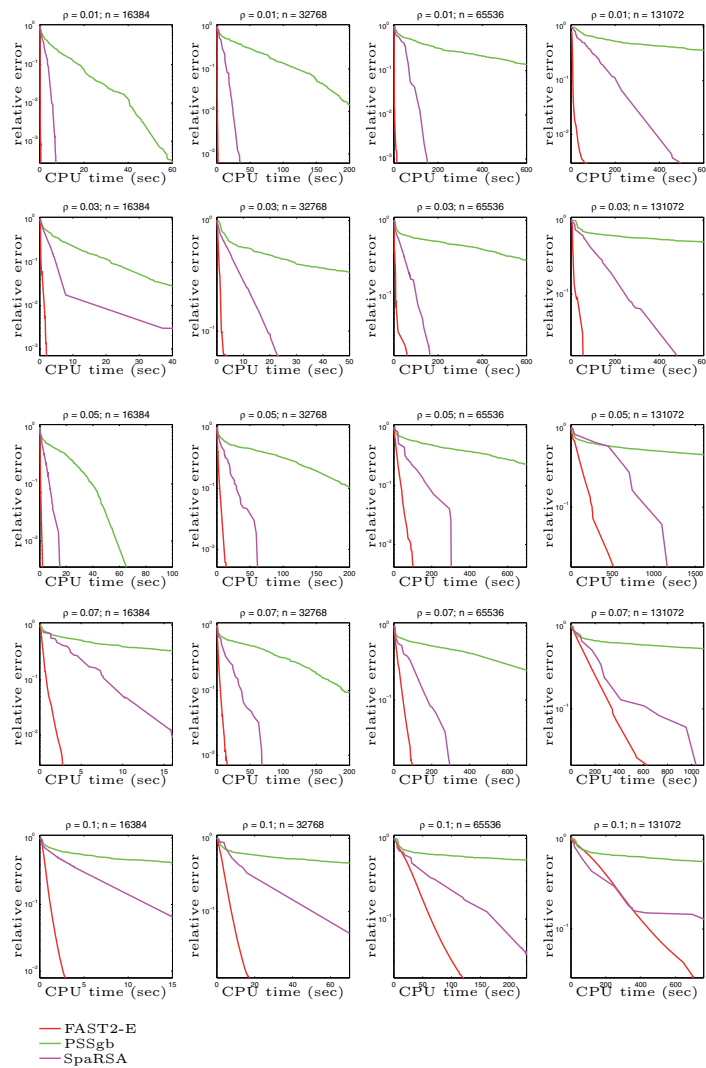


FIG. 4. Relative error versus CPU time—P2 instances.

have

$$(A.1) \quad f(y) = q(y) + \tau \sum_{i=1}^n \text{sign}(y_i) y_i = q(y) + \tau \sum_{i \in \mathcal{N}} \text{sign}(y_i) y_i + \tau \sum_{i \in \mathcal{A}} \text{sign}(z_i) y_i$$

from which

$$f(y) = f(z) + (g_{\mathcal{A}}(z) + \tau S_{\mathcal{A}} e)^{\top} (y - z)_{\mathcal{A}} + \frac{1}{2} (y - z)_{\mathcal{A}}^{\top} H_{\mathcal{A}\mathcal{A}} (y - z)_{\mathcal{A}},$$

where $e \in \mathbb{R}^{|\mathcal{A}|}$ is the unit vector, and $S_{\mathcal{A}}$ is the diagonal matrix defined as

$$S_{\mathcal{A}} = \text{Diag}(\text{sign}(z_{\mathcal{A}}))$$

with the function $\text{sign}(\cdot)$ intended componentwise.

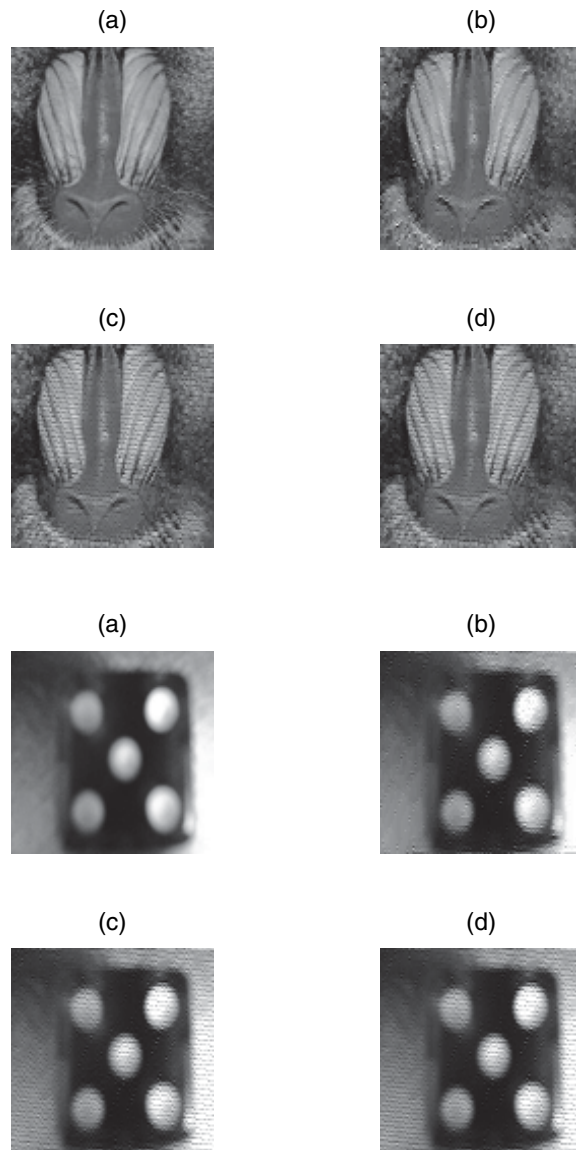


FIG. 5. *Real examples experiment: (a) original image, (b) FAST-2CDA-E reconstruction, (c) PSSgb reconstruction, (d) SpaRSA reconstruction.*

Since $H = A^\top A$ we have that the following inequality holds:

$$f(y) \leq f(z) + (g_{\mathcal{A}}(z) + \tau S_{\mathcal{A}} e)^\top (y - z)_{\mathcal{A}} + \frac{\lambda_{\max}(A^\top A)}{2} \|(y - z)_{\mathcal{A}}\|^2.$$

Recalling (3.12) we obtain

$$(A.2) \quad f(y) \leq f(z) + (g_{\mathcal{A}}(z) + \tau S_{\mathcal{A}} e)^\top (y - z)_{\mathcal{A}} + \frac{1}{2\epsilon} \|(y - z)_{\mathcal{A}}\|^2.$$

TABLE 1
Real examples experiment—CPU time.

Fast1	Fast2	Fast1-E	Fast2-E	ISTA	FISTA	PSSgb	SpaRSA	FPC_AS
2.18	2.46	2.54	2.02	34.65	9.26	5.01	5.08	10.46
1.65	1.67	1.51	1.95	73.01	16.31	7.77	14.48	12.87
1.86	1.97	1.91	1.65	78.05	18.41	7.90	15.56	14.69
3.52	2.05	2.13	2.32	63.13	12.69	6.51	7.53	9.77
2.29	2.12	1.79	2.16	51.12	13.79	6.36	11.13	9.33
4.12	4.21	4.16	2.41	56.66	12.09	6.69	7.12	9.72

Then, we can write

$$f(y) \leq f(z) + \left(g_{\mathcal{A}}(z) + \tau S_{\mathcal{A}} e + \frac{1}{\epsilon} (y - z)_{\mathcal{A}} \right)^{\top} (y - z)_{\mathcal{A}} - \frac{1}{2\epsilon} \|(y - z)_{\mathcal{A}}\|^2.$$

In order to prove the proposition, we need to show that

$$(A.3) \quad \left(g_{\mathcal{A}}(z) + \tau S_{\mathcal{A}} e + \frac{1}{\epsilon} (y - z)_{\mathcal{A}} \right)^{\top} (y - z)_{\mathcal{A}} \leq 0.$$

Inequality (A.3) follows from the fact that $\forall i \in \mathcal{A}$,

$$(A.4) \quad \left(g_i(z) + \tau \text{sign}(z_i) + \frac{1}{\epsilon} (y_i - z_i) \right)^{\top} (y_i - z_i) \leq 0.$$

We distinguish two cases:

- (a) If $z_i > 0$, we have that $\text{sign}(z_i) = 1$ and, since $y_i = 0$, $(y_i - z_i) \leq 0$.
Then, from the fact that $i \in \mathcal{A}$, we have

$$\begin{aligned} y_i &= 0, \\ z_i &\leq \epsilon (g_i(z) + \tau), \\ (z_i - y_i) &\leq \epsilon (g_i(z) + \tau), \\ \frac{1}{\epsilon} (z_i - y_i) &\leq g_i(z) + \tau, \end{aligned}$$

so that

$$g_i(z) + \tau + \frac{1}{\epsilon} (y_i - z_i) \geq 0$$

and (A.4) is satisfied.

- (b) If $z_i < 0$, we have that $\text{sign}(z_i) = -1$ and, since $y_i = 0$, $(y_i - z_i) \geq 0$.
Then, by reasoning as in case (a), from the fact that $i \in \mathcal{A}$, we can write

$$\begin{aligned} y_i &= 0, \\ -z_i &\leq \epsilon (\tau - g_i(z)), \\ (y_i - z_i) &\leq \epsilon (\tau - g_i(z)), \\ \frac{1}{\epsilon} (y_i - z_i) &\leq \tau - g_i(z) \end{aligned}$$

from which we have

$$g_i(z) - \tau + \frac{1}{\epsilon} (y_i - z_i) \leq 0.$$

Again, we have that (A.4) is satisfied. \square

Appendix B. Theoretical results related to the convergence analysis.

First, we prove the result that guarantees a sufficient decrease when minimizing with respect to a given block.

Proof of Proposition 4.1. Let us consider the subproblem obtained by fixing all variables in I but the ones whose indices belong to J to $z_{I \setminus J}$. Let $w^* \in \mathbb{R}^{|J|}$ be a solution of this subproblem.

We consider the set $J = \{j_1, \dots, j_{|J|}\}$ as the union of two sets

$$J = J_E \cup J_D,$$

where

$$J_E = J_{E^+} \cup J_{E^-}, \quad J_D = J_{D^+} \cup J_{D^-}$$

and

$$J_{D^+} = \{j_i \in J : \text{sign}(w_i^*) > 0\}, \quad J_{D^-} = \{j_i \in J : \text{sign}(w_i^*) < 0\},$$

$$J_{E^+} = \{j_i \in J : w_i^* = 0; \text{sign}(z_{j_i}) > 0\}, \quad J_{E^-} = \{j_i \in J : w_i^* = 0; \text{sign}(z_{j_i}) < 0\}.$$

Let $\tilde{f} : \mathbb{R}^{|J|} \rightarrow \mathbb{R}$ with $w \in \mathbb{R}^{|J|}$ be the following function,

$$\begin{aligned} \tilde{f}(w) = & q(z) + \tau \sum_{j \in I \setminus J} \text{sign}(z_j) z_j + g_J(z)^\top (w - z_J) + \frac{1}{2} (w - z_J)^\top H_{JJ} (w - z_J) \\ & + \tau \sum_{j_i \in J_E} \text{sign}(z_{j_i}) w_i + \tau \sum_{j_i \in J_D} \text{sign}(w_i^*) w_i. \end{aligned}$$

Then, w^* can be equivalently seen as the solution of the following problem

$$\begin{aligned} \min & \quad \tilde{f}(w) \\ \text{(B.1)} \quad \text{s.t.} & \quad w_i \geq 0 \quad \text{for } j_i \in J_{D^+} \cup J_{E^+}, \\ & \quad w_i \leq 0 \quad \text{for } j_i \in J_{D^-} \cup J_{E^-}. \end{aligned}$$

By introducing the diagonal matrix $S = \text{Diag}(s) \in \mathbb{R}^{|J| \times |J|}$, where $s \in \{-1, 0, 1\}^{|J|}$ is the vector defined as

$$s_i = \begin{cases} \text{sign}(w_i^*) & \text{if } j_i \in J_D, \\ \text{sign}(z_{j_i}) & \text{if } j_i \in J_E, \end{cases}$$

problem (B.1) can be written in a more compact form as

$$\begin{aligned} \min & \quad \tilde{f}(w) \\ \text{(B.2)} \quad \text{s.t.} & \quad Sw \geq 0. \end{aligned}$$

From the KKT condition for problem (B.2) at w^* we have

$$\text{(B.3)} \quad g_J(z) + H_{JJ}(w^* - z_J) + \tau s - S\lambda = 0,$$

where $\lambda \in \mathbb{R}^{|J|}$ is the vector of multipliers with respect to the constraints $Sw \geq 0$.

We now analyze (B.3) for each index $i \in J$. We distinguish two cases:

- $j_i \in J_D$. In this case we have that $s_i = \text{sign}(w_i^*)$ and $\lambda_i = 0$. Then, from (B.3) we have

$$(B.4) \quad g_{j_i}(z) + H_{j_i j_i}(w_i^* - z_{j_i}) + \tau s_i = 0.$$

- $j_i \in J_E$. In this case we have that $s_i = \text{sign}(z_{j_i})$ and $\lambda_i \geq 0$.

Therefore,

$$g_{j_i}(z) + H_{j_i j_i}(w_i^* - z_{j_i}) + \tau s_i \geq 0 \quad \text{if } s_i = \text{sign}(z_{j_i}) \geq 0,$$

$$g_{j_i}(z) + H_{j_i j_i}(w_i^* - z_{j_i}) + \tau s_i \leq 0 \quad \text{if } s_i = \text{sign}(z_{j_i}) \leq 0.$$

The previous inequalities and the fact that $w_i^* = 0$ for all $j_i \in J_E$ imply that, whatever is the sign of z_i , we have

$$(B.5) \quad \left(g_{j_i}(z) + H_{j_i j_i}(w_i^* - z_{j_i}) + \tau s_i \right) (w_i^* - z_{j_i}) \leq 0.$$

Taking into account (B.4) and (B.5), we have that

$$(B.6) \quad \left(g_J(z) + H_{JJ}(w^* - z_J) + \tau s \right)^\top (w^* - z_J) \leq 0.$$

Now, consider the difference between $\tilde{f}(w^*)$ and $\tilde{f}(z_J)$. We have that

$$\begin{aligned} \tilde{f}(w^*) - \tilde{f}(z_J) &= g_J(z)^\top (w^* - z_J) + \frac{1}{2} (w^* - z_J)^\top H_{JJ} (w^* - z_J) \\ &\quad + \tau \sum_{j_i \in J_E} \text{sign}(z_{j_i}) (w_i^* - z_{j_i}) + \tau \sum_{j_i \in J_D} \text{sign}(w_i^*) (w_i^* - z_{j_i}), \end{aligned}$$

which can be rewritten as

$$\tilde{f}(w^*) - \tilde{f}(z_J) = \left(g_J(z) + H_{JJ}(w^* - z_J) + \tau s \right)^\top (w^* - z_J) - \frac{1}{2} (w^* - z_J)^\top H_{JJ} (w^* - z_J).$$

Recalling (B.6) and the fact that $y_J = w^*$ we have

$$(B.7) \quad \tilde{f}(w^*) - \tilde{f}(z_J) \leq -\frac{1}{2} (w^* - z_J)^\top H_{JJ} (w^* - z_J) \leq -\frac{1}{2} \lambda_{\min}(H_{JJ}) \|y - z\|^2.$$

Since

$$q(y) = q(z) + g_J(z)^\top (y - z)_J + \frac{1}{2} (y - z)_J^\top H_{JJ} (y - z)_J,$$

by the definition of \tilde{f} we have that

$$(B.8) \quad \begin{aligned} f(y) &= q(y) + \tau \sum_{j=1}^n \text{sign}(y_j) y_j = q(y) + \tau \sum_{j \in I \setminus J} \text{sign}(z_j) z_j + \\ &\quad + \tau \sum_{j_i \in J_E} \text{sign}(z_{j_i}) w_i^* + \tau \sum_{j_i \in J_D} \text{sign}(w_i^*) w_i^* = \tilde{f}(w^*) \end{aligned}$$

and

$$(B.9) \quad \tilde{f}(z_J) = q(z) + \tau \sum_{j_i \in I \setminus J_D} \text{sign}(z_{j_i}) z_{j_i} + \tau \sum_{j_i \in J_D} \text{sign}(w_i^*) z_{j_i} \leq q(z) + \tau \|z\|_1 = f(z).$$

Now (B.7), (B.8), and (B.9) prove the proposition. \square

Then, we prove the main convergence result related to FAST-BCDA.

Proof of Theorem 4.3. We first prove that FAST-BCDA is well defined (in the sense that $x^{k+1} \neq x^k$ iff the point x^k is not an optimum). Let x^k not be optimum, then by contradiction we assume that $x^{k+1} = x^k$. Thus we have that either $\mathcal{N}(x^k) = \emptyset$, or for all $i \in \mathcal{A}(x^k)$, $x_i^k = 0$. This, in turns, implies that $x^k = 0$ and, by taking into account the definition of $\mathcal{A}(x^k)$, we have that x^k is optimal, thus getting a contradiction. The proof of the other implication easily follows from Propositions 3.4 and 4.1.

Let $\{y^{h,k}\}$, with $h = 0, \dots, q$ be the sequence of points produced by Algorithm FAST-BCDA. By setting $y = y^{0,k}$ and $z = x^k$ in Proposition 3.4, we have

$$(B.10) \quad f(y^{0,k}) \leq f(x^k) - \frac{1}{2\epsilon} \|y^{0,k} - x^k\|^2.$$

By setting $y = y^{h+1,k}$ and $z = y^{h,k}$, for $h = 0, \dots, q - 1$ in Proposition 4.1, we have

$$(B.11) \quad f(y^{h+1,k}) \leq f(y^{h,k}) - \frac{\sigma}{2} \|y^{h+1,k} - y^{h,k}\|^2.$$

By using (B.10) and (B.11), we can write

$$(B.12) \quad f(x^{k+1}) \leq f(y^{q-1,k}) \leq \dots \leq f(y^{0,k}) \leq f(x^k),$$

from which we have

$$x^k \in \mathcal{L}^0 = \{x \in \mathbb{R}^n : f(x) \leq f(x^0)\}.$$

From the coercivity of the objective function of problem (1.1) we have that the level set \mathcal{L}^0 is compact. Hence, the sequence $\{x^k\}$ has at least a limit point and

$$(B.13) \quad \lim_{k \rightarrow \infty} (f(x^{k+1}) - f(x^k)) = 0.$$

Now, let x^* be any limit point of the sequence $\{x^k\}$ and $\{x^k\}_K$ be the subsequence such that

$$(B.14) \quad \lim_{k \rightarrow \infty, k \in K} x^k = x^*.$$

Let us assume, by contradiction, that x^* is not an optimal point of problem (1.1). By taking into account that inequality $\|\sum_{i=1}^l a_i\| \leq l \sum_{i=1}^l \|a_i\|^2$ holds for the squared norm of sums of l vectors a_i , and by recalling (B.10), (B.11), and (B.12), we have

$$(B.15) \quad f(x^{k+1}) \leq f(y^{0,k}) \leq f(x^k) - \frac{1}{2\epsilon} \|y^{0,k} - x^k\|^2,$$

$$(B.16) \quad f(x^{k+1}) \leq f(y^{h,k}) \leq f(x^k) - \frac{\sigma}{2} \|y^{h,k} - x^k\|^2$$

with $h = 1, \dots, q$.

Now, (B.13), (B.14), (B.15), and (B.16) imply

$$(B.17) \quad \lim_{k \rightarrow \infty, k \in K} y^{h,k} = x^*$$

for $h = 0, \dots, q$.

For every index $j \in \mathcal{A}^k$, we can define the point $\tilde{y}^{j,k}$ as follows:

$$(B.18) \quad \tilde{y}_i^{j,k} = \begin{cases} 0 & \text{if } i = j, \\ x_i^k & \text{otherwise.} \end{cases}$$

Recalling the definition of points $\tilde{y}^{j,k}$ and $y^{0,k}$, we have

$$\|\tilde{y}^{j,k} - x^k\|^2 = (\tilde{y}^{j,k} - x^k)_j^2 = (y^{0,k} - x^k)_j^2 \leq (y^{0,k} - x^k)_j^2 + \sum_{i \in \mathcal{A}^k, i \neq j} (x_i^k)^2 = \|y^{0,k} - x^k\|^2.$$

From the last inequality and (B.17) we obtain

$$(B.19) \quad \lim_{k \rightarrow \infty, k \in K} \tilde{y}^{j,k} = x^*$$

for all $j \in \mathcal{A}^k$.

To conclude the proof, we consider the function $\Phi_i(x)$, defined in (2.2), that measures the violation of the optimality conditions for a variable x_i .

Since, by contradiction, we assume that x^* is not an optimal point there must exist an index \hat{i} such that

$$(B.20) \quad |\Phi_{\hat{i}}(x^*)| > 0.$$

Taking into account that the number of possible different choices of \mathcal{A}^k and \mathcal{N}^k is finite, we can find a subset $\hat{K} \subseteq K \subseteq \{1, 2, 3, \dots\}$ such that $\mathcal{A}^k = \hat{\mathcal{A}}$ and $\mathcal{N}^k = \hat{\mathcal{N}}$ for all $k \in \hat{K}$. We can have two different cases: either $\hat{i} \in \hat{\mathcal{A}}$ or $\hat{i} \in \hat{\mathcal{N}}$ for k sufficiently large.

Suppose first that $\hat{i} \in \hat{\mathcal{A}}$ for k sufficiently large. Then, by Definition 3.1, we have for all $k \in \hat{K}$

$$\max\{0, x_{\hat{i}}^k\} \leq \epsilon (g_{\hat{i}}(x^k) + \tau) \quad \text{and} \quad \max\{0, -x_{\hat{i}}^k\} \leq \epsilon (\tau - g_{\hat{i}}(x^k)).$$

For all $k \in \hat{K}$, let $\tilde{y}^{\hat{i},k}$ be the point defined as in (B.18). By construction we have that

$$(B.21) \quad \tilde{y}_{\hat{i}}^{\hat{i},k} = 0.$$

Now we consider three different subcases:

(i) $x_{\hat{i}}^k > 0$. In this case, (B.18) and (B.21) imply

$$(B.22) \quad (\tilde{y}_{\hat{i}}^{\hat{i},k} - x_{\hat{i}}^k) \leq 0.$$

Recalling (3.12) in Assumption 1, there exists $\rho \geq 0$ such that

$$\epsilon \leq \frac{1}{H_{\hat{i}\hat{i}} + \rho}.$$

Furthermore, since $\hat{i} \in \hat{\mathcal{A}}$, we can write

$$\begin{aligned} x_{\hat{i}}^k &\leq \epsilon (g_{\hat{i}}(x^k) + \tau), \\ x_{\hat{i}}^k - \tilde{y}_{\hat{i}}^{\hat{i},k} &\leq \epsilon (g_{\hat{i}}(x^k) + \tau), \\ x_{\hat{i}}^k - \tilde{y}_{\hat{i}}^{\hat{i},k} &\leq \frac{1}{H_{\hat{i}\hat{i}} + \rho} (g_{\hat{i}}(x^k) + \tau). \end{aligned}$$

Then we have

$$(H_{\hat{i}\hat{i}} + \rho)(x_{\hat{i}}^k - \tilde{y}_{\hat{i}}^{\hat{i},k}) \leq g_{\hat{i}}(x^k) + \tau,$$

which can be rewritten as

$$g_{\hat{i}}(x^k) + H_{\hat{i}\hat{i}}(\tilde{y}_{\hat{i}}^{\hat{i},k} - x_{\hat{i}}^k) + \tau \geq \rho(x_{\hat{i}}^k - \tilde{y}_{\hat{i}}^{\hat{i},k}) \geq 0,$$

that is

$$(B.23) \quad g_i(\tilde{y}^{i,k}) + \tau \geq 0.$$

On the other hand, since

$$0 \leq \max\{0, -x_i^k\} \leq \epsilon(\tau - g_i(x^k))$$

we have that $g_i(x^k) - \tau \leq 0$ and, as $H_{ii} \geq 0$ and (B.22) holds, we get

$$(B.24) \quad g_i(\tilde{y}^{i,k}) - \tau = g_i(x^k) + H_{ii}(\tilde{y}_i^{i,k} - x_i^k) - \tau \leq 0.$$

By (B.21), (B.23), and (B.24), we have that

$$|\Phi_i(\tilde{y}^{i,k})| = 0.$$

Furthermore, by (B.19) and the continuity of Φ , we can write

$$|\Phi_i(x^*)| = 0.$$

Thus we get a contradiction with (B.20).

- (ii) $x_i^k < 0$. It is a verbatim repetition of the previous case.
- (iii) $x_i^k = 0$. Since $\hat{i} \in \hat{\mathcal{A}}$ we have

$$g_i(x^k) + \tau \geq 0 \quad \text{and} \quad -(g_i(x^k) - \tau) \geq 0,$$

which imply that

$$|\Phi_i(x^k)| = 0.$$

By the continuity of $\Phi(\cdot)$ and the fact that

$$\lim_{k \rightarrow \infty, k \in \tilde{K}} x^k = x^*,$$

we get a contradiction with (B.20).

Suppose now that $\hat{i} \in \hat{\mathcal{N}}$ for k sufficiently large. We can choose a further subsequence $\{x^k\}_{\tilde{K}}$ with $\tilde{K} \subseteq \hat{K}$ such that

$$|\Phi_{\hat{i}}(x^k)| = \max_{i \in \hat{\mathcal{N}}} |\Phi_i(x^k)| \quad \forall k \in \tilde{K}.$$

Hence,

$$(B.25) \quad |\Phi_{\hat{i}}(x^k)| \geq |\Phi_i(x^k)| \quad \forall k \in \tilde{K},$$

which, by continuity of $\Phi(\cdot)$, implies

$$(B.26) \quad |\Phi_{\hat{i}}(x^*)| \geq |\Phi_i(x^*)|.$$

Furthermore, the instructions of Algorithm FAST-BCDA guarantee that, for all $k \in \tilde{K}$, a set of indices I_{h_k} exists such that

$$\bar{i} \in I_{h_k} \subseteq \tilde{\mathcal{N}}_{ord}^k.$$

For all $k \in \tilde{K}$, Algorithm FAST-BCDA produces a vector $y^{h_k, k}$ by minimizing problem (1.1) with respect to all the variables whose indices belong to I_{h_k} . Therefore, the point $y^{h_k, k}$ satisfies

$$|\Phi_{\hat{i}}(y^{h_k, k})| = 0.$$

Furthermore, by (B.17), the continuity of $\Phi(\cdot)$, and taking into account (B.26), we can write

$$0 = |\Phi_{\hat{i}}(x^*)| \geq |\Phi_{\hat{i}}(x^*)|,$$

which contradicts (B.20). \square

Appendix C. Theoretical results related to the convergence rate analysis. Here, following the ideas in [19], we prove that the convergence rate of FAST-BCDA with 1-dimensional blocks (namely, FAST-1CDA) is linear. First, we try to better analyze the indices in the set $\mathcal{N}(x)$ by introducing the following two sets:

$$(C.1) \quad \mathcal{N}^+(x) = \{i \in \mathcal{N}(x) : g_i(x) \leq 0\} \text{ and } \mathcal{N}^-(x) = \{i \in \mathcal{N}(x) : g_i(x) > 0\}.$$

We further introduce the sets

$$(C.2) \quad \mathcal{E}^+(x^*) = \{i : x_i^* \geq 0, g_i(x^*) = -\tau\} \text{ and } \mathcal{E}^-(x^*) = \{i : x_i^* \leq 0, g_i(x^*) = \tau\},$$

which satisfy the following equality:

$$\mathcal{E}(x^*) = \mathcal{E}^+(x^*) \cup \mathcal{E}^-(x^*) = \bar{\mathcal{N}}(x^*) \cup \{i : x_i^* = 0, |g_i(x^*)| = \tau\}.$$

We further notice that

$$(C.3) \quad I = \bar{\mathcal{A}}^+(x^*) \cup \mathcal{E}^+(x^*) \cup \mathcal{E}^-(x^*).$$

We can finally prove a result that will be used in the convergence analysis.

THEOREM C.1. *Let $x^* \in \mathbb{R}^n$ be a solution of problem (1.1). Then, there exists a neighborhood of x^* such that, for each x in this neighborhood, we have*

$$(C.4) \quad \mathcal{N}^+(x) \subseteq \mathcal{E}^+(x^*),$$

$$(C.5) \quad \mathcal{N}^-(x) \subseteq \mathcal{E}^-(x^*).$$

Proof. Let us assume there exists a sequence $\{\epsilon^k\}$, $\epsilon^k \rightarrow 0$, a related sequence of neighborhoods $\{\mathcal{B}(x^*, \epsilon^k)\}$, and a sequence of points $\{x^k\}$ such that $x^k \in \mathcal{B}(x^*, \epsilon^k)$ for all k , satisfying

$$\mathcal{N}^+(x^k) \not\subseteq \mathcal{E}^+(x^*).$$

Then, since the number of indices is finite, there exist subsequences $\{\epsilon^k\}_K$ and $\{\mathcal{B}(x^*, \epsilon^k)\}_K$ such that an index \hat{i} can be found, satisfying the following:

$$\hat{i} \in \mathcal{N}^+(x^k), \quad \hat{i} \notin \mathcal{E}^+(x^*).$$

From Theorem 3.3, for k sufficiently large,

$$\mathcal{N}(x^k) \subseteq \mathcal{E}^+(x^*) \cup \mathcal{E}^-(x^*).$$

Therefore, we have that

$$\hat{i} \in \mathcal{E}^-(x^*) \text{ and } g_{\hat{i}}(x^*) = \tau. \quad \square$$

By continuity of the gradient, $g_{\hat{i}}(x^k) > 0$ for k sufficiently large. On the other hand, since $\hat{i} \in \mathcal{N}^+(x^k)$, we have $g_{\hat{i}}(x^k) \leq 0$. This gives a contradiction, and proves (C.4). A similar reasoning can be used for proving (C.5).

Finally, we report another theoretical result that is used in the convergence rate analysis.

PROPOSITION C.2. *Let Assumption 3 hold. Then, there exists a \bar{k} such that*

- (a) $x_i^k = 0, \quad i \in \bar{\mathcal{A}}^+(x^*),$
- (b) $-sign(g_i(x^*)) x_i^k \geq 0, \quad i \in \mathcal{E}(x^*),$

for all $k \geq \bar{k}$.

Proof. (a) Recalling (3.10), for k sufficiently large, we have

$$\bar{\mathcal{A}}^+(x^*) \subseteq \mathcal{A}^k.$$

Therefore, by taking into account the steps of the FAST-1CDA algorithm, we have

$$(C.6) \quad x_i^{k+1} = 0, \quad i \in \bar{\mathcal{A}}^+(x^*).$$

Furthermore, by continuity of g and (4.4), we obtain

$$(C.7) \quad \tau + g_i(x^{k+1}) > 0 \quad \text{and} \quad \tau - g_i(x^{k+1}) > 0,$$

and we can write

$$i \in \mathcal{A}^{k+1}.$$

Hence, (C.6) and (C.7) still hold for x^{k+2} , and so on.

(b) Let us consider an index $i \in \mathcal{E}(x^*)$. By contradiction, we assume that there exists a subsequence $K = \{k_1, k_2, \dots\}$ such that

$$(C.8) \quad -sign(g_i(x^*)) x_i^k < 0$$

for all $k \in K$. Without any loss of generality, we can consider another subsequence $\hat{K} = \{\bar{k}_1, \bar{k}_2, \dots\}$ related to K , such that $i \in \mathcal{N}^{\bar{k}_j}$ and

$$(C.9) \quad x_i^{k_j} = -sign\left(g_i(x^{\bar{k}_j}) - H_{ii}x_i^{\bar{k}_j}\right) \frac{\max\left\{\left|g_i(x^{\bar{k}_j}) - H_{ii}x_i^{\bar{k}_j}\right| - \tau, 0\right\}}{H_{ii}}$$

for all $k_j \in K$ and $\bar{k}_j \in \hat{K}$.

If $i \in \mathcal{E}(x^*) \setminus \bar{\mathcal{N}}(x^*)$, when j is sufficiently large, we have by continuity of g , (4.4), and (C.9)

$$-sign(g_i(x^*)) x_i^{k_j} \geq 0,$$

which contradicts (C.8).

If $i \in \bar{\mathcal{N}}(x^*)$, when j is sufficiently large, we have by continuity of g , (3.8), and (4.4)

$$-sign(g_i(x^*)) x_i^{k_j} \geq 0,$$

and again we get a contradiction with (C.8). □

Now, we prove that the algorithm converges at a linear rate.

Proof of Theorem 4.5. First of all, for ease of notation we set $\bar{\mathcal{A}}^+ = \bar{\mathcal{A}}^+(x^*)$, and $\mathcal{E} = \mathcal{E}(x^*)$. Without any loss of generality, we can assume $|\mathcal{N}_{ord}^k| = 1$ for all k . We then notice that the objective function $f(x)$ can be rewritten as follows:

$$f(x) = q(x) + \tau \sum_{i \in \bar{\mathcal{A}}^+} \text{sign}(x_i)x_i + \tau \sum_{i \in \mathcal{E}} \text{sign}(x_i)x_i.$$

We further introduce the function

$$F(x) = q(x) - \sum_{i \in \mathcal{E}} \text{sign}(g_i(x^*))x_i.$$

By taking into account Proposition C.2, we have, for k sufficiently large,

$$(C.10) \quad f(x^k) = F(x^k) \quad \text{and} \quad f(x^*) = F(x^*).$$

Furthermore, when k is sufficiently large, by the definition of $\bar{\mathcal{A}}^+$ and \mathcal{E} , and recalling again Proposition C.2, we can write

$$(C.11) \quad x_{\bar{\mathcal{A}}^+}^k = x_{\bar{\mathcal{A}}^+}^*,$$

$$(C.12) \quad \nabla_i F(x^*) = 0 \quad \forall i \in \mathcal{E}.$$

Then, by considering (C.3), (C.11), and (C.12), it follows

$$\begin{aligned} F(x^k) - F(x^*) &= \nabla F(x^*)^\top (x^k - x^*) + \frac{1}{2} (x^k - x^*)^\top \nabla^2 F(x^*) (x^k - x^*) \\ &= \frac{1}{2} (x^k - x^*)^\top \nabla^2 F(x^*) (x^k - x^*) \leq \frac{\lambda_{\max}(\nabla^2 F(x^*))}{2} \|x^k - x^*\|^2, \end{aligned}$$

and, taking into account (C.10), we can write

$$(C.13) \quad f(x^k) - f(x^*) \leq \rho \|x^k - x^*\|^2$$

with $\rho > 0$. Then, recalling Theorem 3.3 and (C.1), for k sufficiently large the problem we actually solve is

$$(C.14) \quad \begin{aligned} \min \quad & \tilde{F}(x) = \frac{1}{2} \|Ax - b\|^2 - \tau \text{sign}(g_{\mathcal{N}^k}(x^k))^\top x_{\mathcal{N}^k}, \\ & x_{\mathcal{A}^k} = 0, \\ & -\text{sign}(g_i(x^k)) x_i \geq 0 \quad i \in \mathcal{N}^k. \end{aligned}$$

Now, let $y^{0,k}$ be the point obtained at step 4 of Algorithm 1 (i.e., after fixing to the active variables zero) and $y_s^{0,k}$ the component that most violates condition (2.2) in the nonactive set. We notice that finding the most violating variable according to condition (2.2) is equivalent, when considering problem (C.14), to getting the component that most violates the following condition

$$|x_i - [x_i - \nabla_i \tilde{F}(x)]_+|;$$

see [19] for further details. Thus, we can write

$$\begin{aligned}
 \frac{1}{\sqrt{|\mathcal{N}^k|}} \|y^{0,k} - [y^{0,k} - \nabla \tilde{F}(y^{0,k})]_+\| &\leq |y_s^{0,k} - [y_s^{0,k} - \nabla_s \tilde{F}(y^{0,k})]_+| \\
 &= |y_s^{0,k} - [y_s^{0,k} - \nabla_s \tilde{F}(y^{0,k})]_+ - x_s^{k+1}| \\
 &\quad + |x_s^{k+1} - \nabla_s \tilde{F}(x^{k+1})| \\
 &\leq 2|y_s^{0,k} - x_s^{k+1}| + |\nabla_s \tilde{F}(y^{0,k}) - \nabla_s \tilde{F}(x^{k+1})| \\
 &\leq 2\|y^{0,k} - x^{k+1}\| + \|\nabla \tilde{F}(y^{0,k}) - \nabla \tilde{F}(x^{k+1})\| \\
 &\leq M\|y^{0,k} - x^{k+1}\| = M\|x^k - x^{k+1}\|_{\mathcal{N}^k},
 \end{aligned}
 \tag{C.15}$$

where $[\cdot]_+$ is the projection on the set of inequalities in problem (C.14), and $M = \max\{2, L\}$ with L the Lipschitz constant of $\nabla \tilde{F}$. By using Propositions 3.4 and 4.1, we can also write

$$f(x^k) - f(x^{k+1}) \geq \delta \|x^{k+1} - x^k\|^2
 \tag{C.16}$$

with $\delta > 0$. By taking into account inequality (C.15) and the definition of $y^{0,k}$, we can write, for k sufficiently large,

$$\begin{aligned}
 \|x^{k+1} - x^k\|^2 &= \|x^k - x^*\|_{\mathcal{A}^k}^2 + \|y^{0,k} - x^{k+1}\|_{\mathcal{N}^k}^2 \\
 &\geq \|x^k - x^*\|_{\mathcal{A}^k}^2 + \frac{1}{M\sqrt{|\mathcal{N}^k|}} \|y^{0,k} - [y^{0,k} - \nabla \tilde{F}(y^{0,k})]_+\|^2.
 \end{aligned}
 \tag{C.17}$$

Now, considering Theorem 2.1 in [19] we have, for k sufficiently large,

$$\sigma \|y^{0,k} - [y^{0,k} - \nabla \tilde{F}(y^{0,k})]_+\| \geq \|y^{0,k} - x^*\| = \|x^k - x^*\|_{\mathcal{N}^k}$$

with $\sigma > 0$. Therefore, by taking into account inequality (C.17), we can write

$$\|x^{k+1} - x^k\|^2 \geq \|x^k - x^*\|_{\mathcal{A}^k}^2 + \gamma \|x^k - x^*\|_{\mathcal{N}^k}^2 \geq \tilde{\gamma} \|x^k - x^*\|^2
 \tag{C.18}$$

with $\tilde{\gamma} > 0$. By combining inequalities (C.13), (C.16), and (C.18), we can write

$$f(x^k) - f(x^*) \leq c_1 (f(x^k) - f(x^{k+1}))
 \tag{C.19}$$

with $c_1 > 1$. After rearranging the terms in (C.19), we obtain

$$f(x^{k+1}) - f(x^*) \leq c_2 (f(x^k) - f(x^*))$$

with $c_2 = \left(1 - \frac{1}{c_1}\right) < 1$. Then, $\{f(x^k)\}$ converges at least linearly to f^* .

Finally, by using (C.16) and Lemma 3.1 in [19] we get that the sequence $\{x^k\}$ converges at least linearly to x^* . \square

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