Spectral filtering for the resolution of the Gibbs phenomenon in MPI applications

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Abstract

Polynomial interpolation on the node points of Lissajous curves using Chebyshev series is an e effective way for a fast image reconstruction in Magnetic Particle Imaging. Due to the nature of spectral methods, a Gibbs phenomenon occurs in the reconstructed image if the underlying function has discontinuities. A possible solution for this problem are spectral filtering methods acting on the coefficients of the interpolating polynomial.

In this work, after a description of the Gibbs phenomenon in two dimensions, we present an adaptive spectral filtering process for the resolution of this phenomenon and for an improved approximation of the underlying function or image. In this adaptive filtering technique, the spectral filter depends on the distance of a spatial point to the nearest discontinuity. We show the effectiveness of this filtering approach in theory, in numerical simulations as well as in the application in Magnetic Particle Imaging.

Interpolation and approximation on Lissajous curves

Let $\boldsymbol{n} = (n_1, n_2) \in \mathbb{N}^2$ be a vector of relatively prime integers and $\epsilon \in \{1, 2\}$. We consider the two-dimensional Lissajous curves

$$\gamma_\epsilon^{(\boldsymbol{n})} : [0, 2\pi] \to [-1, 1]^2 \ , \ \gamma_\epsilon^{(\boldsymbol{n})}(t) := (\cos(n_2 t), \cos(n_1 t - (\epsilon - 1)\pi/(2n_2))) \ . \tag{1}$$

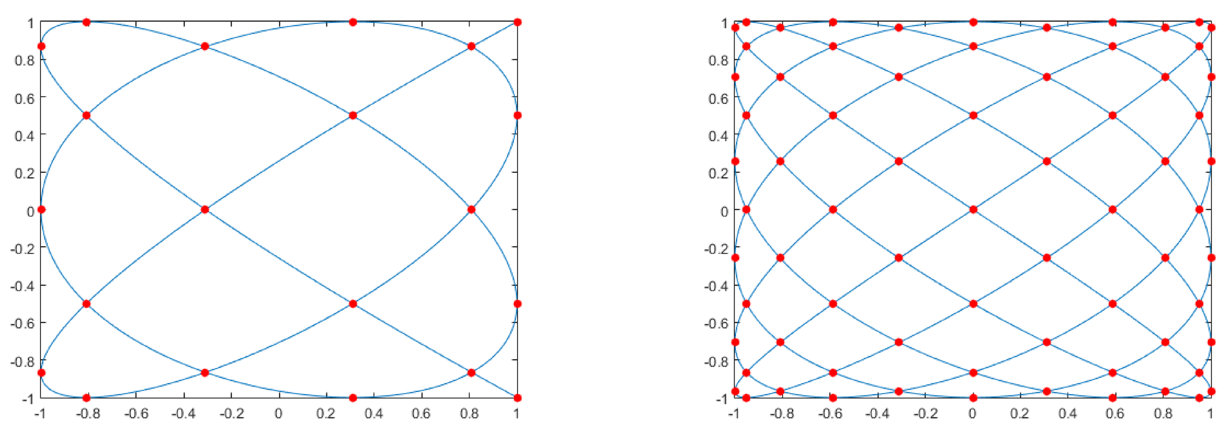
The curve $\gamma_\epsilon^{(\boldsymbol{n})}$ is called *degenerate* if $\epsilon = 1$, and *non-degenerate* otherwise.

Definition 1. *The set of Lissajous node points associated to the curve* $\gamma_\epsilon^{(\boldsymbol{n})}$ *is given by*

$$\mathrm{LS}_\epsilon^{(\boldsymbol{n})} := \left\{ \gamma_\epsilon^{(\boldsymbol{n})}\left(\tfrac{\pi k}{\epsilon n_1 n_2}\right) : \quad k = 0, \ldots, 2\epsilon n_1 n_2 - 1 \right\} \ . \tag{2}$$

We further define the following index set associated to the Lissajous nodes

$$\Gamma^{(\epsilon \boldsymbol{n})} := \left\{ (i,j) \in \mathbb{N}_0^2 \ : \ \frac{i}{\epsilon n_1} + \frac{j}{\epsilon n_2} < 1 \right\} \cup \{(0, \epsilon n_2)\} \ . \tag{3}$$



(a) The degenerate curve $\gamma_1^{(5,6)}$ **(b)** The non-degenerate curve $\gamma_2^{(5,6)}$

We focus on the non-degenerate case and define $\Pi^{(2\boldsymbol{n})} := \mathrm{span}\{T_i(x)T_j(y) : (i,j) \in \Gamma^{(2\boldsymbol{n})}\}$, where $T_i(x) = \cos(i \arccos(x))$ is the *i-th Chebyshev polynomial of the first kind*. We have

Lemma 1. *For all bivariate polynomials P satisfying* $\langle P, T_{2kn_1} T_{2kn_2} \rangle = 0$, $k \in \mathbb{N}$, *the following identity holds:*

$$\frac{1}{\pi^2} \int_{-1}^{1} \int_{-1}^{1} P(x,y)\omega(x,y)\mathrm{d}x\mathrm{d}y = \frac{1}{2\pi} \int_0^{2\pi} P(\gamma_2^{(\boldsymbol{n})}(t))\mathrm{d}t. \tag{4}$$

From this identity, we obtain a quadrature formula on the set $\mathrm{LS}_2^{(\boldsymbol{n})}$ and a unique polynomial interpolant of a given function f in the space $\Pi^{(2\boldsymbol{n})}$ as

$$\mathcal{L}^{(\boldsymbol{n})} f(x,y) = \sum_{(i,j)\in\Gamma^{(2\boldsymbol{n})}} c_{ij} T_i(x) T_j(y) \ . \tag{5}$$

The coefficients c_{ij} are uniquely given by the values of the function f on the point set $\mathrm{LS}_2^{(\boldsymbol{n})}$. For $\epsilon = 1$ and $\boldsymbol{n} = (n, n+1)$ this scheme corresponds to polynomial interpolation on the Padua points, see [3]. Using the change of variables $x = \cos(t)$, $y = \cos(s)$, and expanding the index set we can reformulate the Chebyshev series in (5) to the Fourier series

$$\mathcal{L}^{(\boldsymbol{n})} f(t,s) = \sum_{(i,j)\in\Gamma_S^{(2\boldsymbol{n})}} \tilde{c}_{ij}\, e_i(t) e_j(s) \ , \quad \text{where } e_j(s) = e^{\mathrm{i}jt}. \tag{6}$$

Fourier series and the Gibbs phenomenon

Definition 2. *Let* $f : \mathbb{R}^\nu \to \mathbb{R}$, $f \in L^1_{2\pi}(\mathbb{R}^\nu)$. *The multi-dimensional Fourier series of f in complex form is defined as*

$$Sf(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^\nu} c_{\boldsymbol{n}}(f) e_{\boldsymbol{n}}(\boldsymbol{x}) \quad , \quad \boldsymbol{x} \in \mathbb{R}^\nu \ , \tag{7}$$

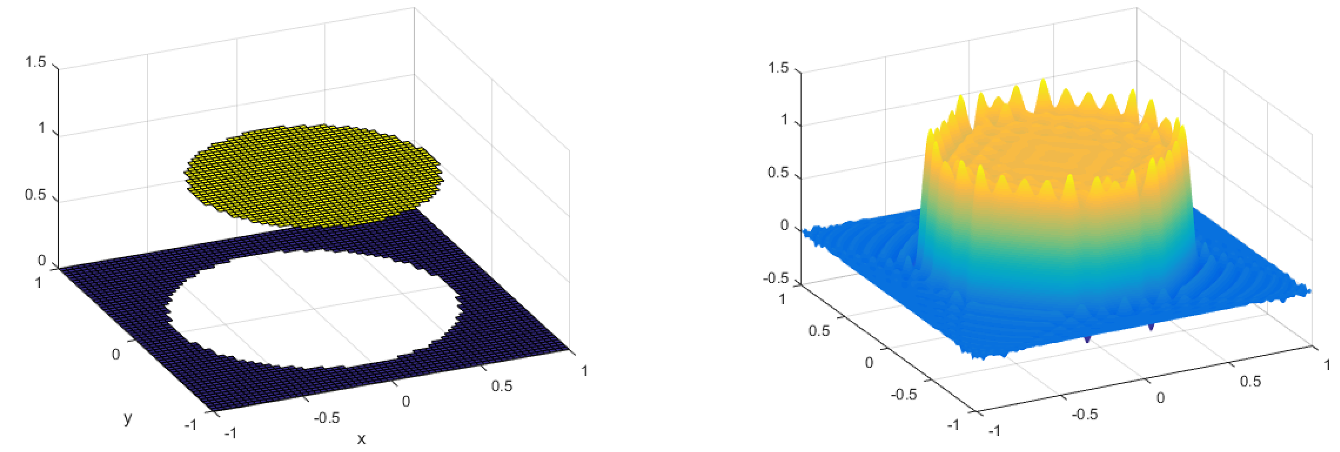
where for every $\boldsymbol{n} \in \mathbb{Z}^\nu$

$$c_{\boldsymbol{n}}(f) = (2\pi)^{-\nu} \int_{(-\pi,\pi)^\nu} f(\boldsymbol{x}) \overline{e_{\boldsymbol{n}}(\boldsymbol{x})} \mathrm{d}\boldsymbol{x} \ . \tag{8}$$

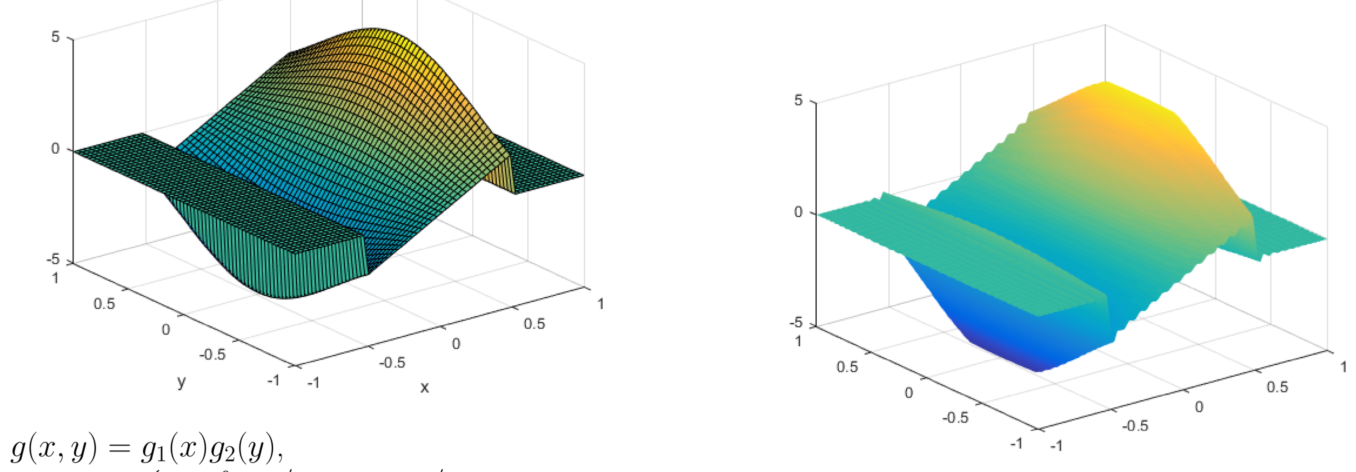
It is well-known that

$$\lim_{\boldsymbol{n}\to\infty} c_{\boldsymbol{n}}(f) = 0 \ , \tag{9}$$

where $\boldsymbol{n} \to \infty$ is understood in the sense that $\max\{n_1, n_2, \ldots, n_\nu\} \to \infty$.
If f is a discontinuous and piecewise differentiable function then the decay rate of the coefficients is only of first order and a Gibbs phenomenon appears.



(a) $f(x,y) = \mathbb{1}_{[0,1]}(x^2 + y^2)$ **(b)** Fourier approximation of f with Gibbs phenomenon.



(a) $g(x,y) = g_1(x) g_2(y)$,
where $g_1(x) = \begin{cases} x & \text{if } -1/2 \le x \le 1/2 \ , \\ 0 & \text{otherwise} \ , \end{cases}$
$g_2(y) = 10e^{-y^2}$

(b) Fourier approximation of g with Gibbs phenomenon.

Adaptive spectral filtering

In order to diminish the Gibbs effect and to obtain a faster convergence outside the discontinuities, we proceed as follows:

- A real and even function σ is called a spectral filter of order p if:
  (a) $\sigma \in C^{p-1}(\mathbb{R})$.
  (b) $\sigma(0) = 1$ and $\sigma^{(l)}(0) = 0$ for $1 \le l \le p-1$.
  (c) $\sigma(\eta) = 0$ for $|\eta| \ge 1$.

  Two well-known examples are the Lanczos filter $\sigma(\eta) = \frac{\sin(\pi\eta)}{\pi\eta}$ (first order) and the raised cosine filter $\sigma(\eta) = \frac{1}{2}(1 + \cos(\pi\eta))$ (second order).
  Setting $\sigma_k = \sigma(|k|/N)$, we consider the filtered series $S_N^\sigma f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \boldsymbol{\sigma_k} c_{\boldsymbol{k}}(f) e_{\boldsymbol{k}}(\boldsymbol{x})$, where $\boldsymbol{\sigma_k} = \sigma_{k_1}\sigma_{k_2}$ is a two-dimensional tensor product filter.
- We use an edge-detector to find the discontinuities of the function f (e.g. Canny).
- Defining

$$\sigma^p(x) = \begin{cases} \exp\left(\dfrac{x^p}{x^2 - 1}\right) & |x| < 1 \ , \\ 0 & |x| \ge 1 \ , \end{cases} \tag{10}$$

  we look for an adaptive tensor-product filter $\boldsymbol{\sigma^p}(\mathbf{x}) = \sigma^{p_1}(x_1)\ \sigma^{p_2}(x_2)$ which depends on the physical position of the points with respect to the related nearest discontinuity. It is known that the operation of such filters is equivalent to mollification in the physical space. Indeed, defining the mollifier

$$\Phi^\sigma(y) := \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \sigma_k e_k(y) \tag{11}$$

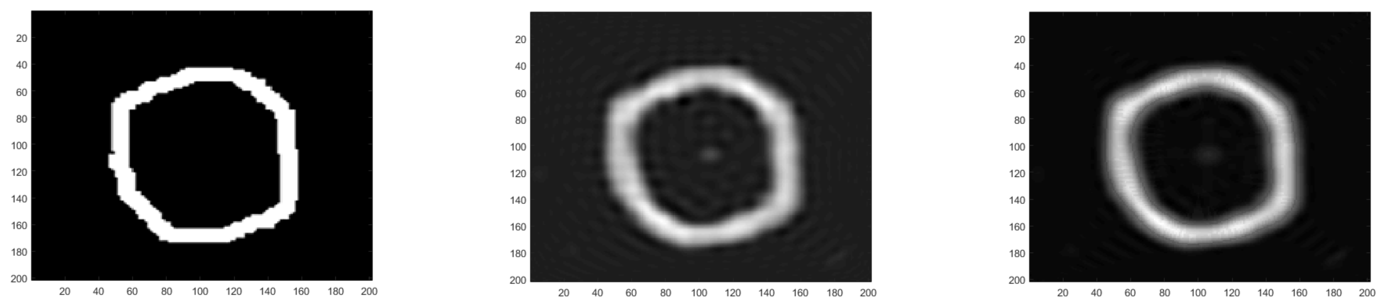
  we can write

$$S_N^\sigma f(x) \equiv f * \Phi^\sigma(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi^\sigma(y) f(x-y)\mathrm{d}y \ . \tag{12}$$

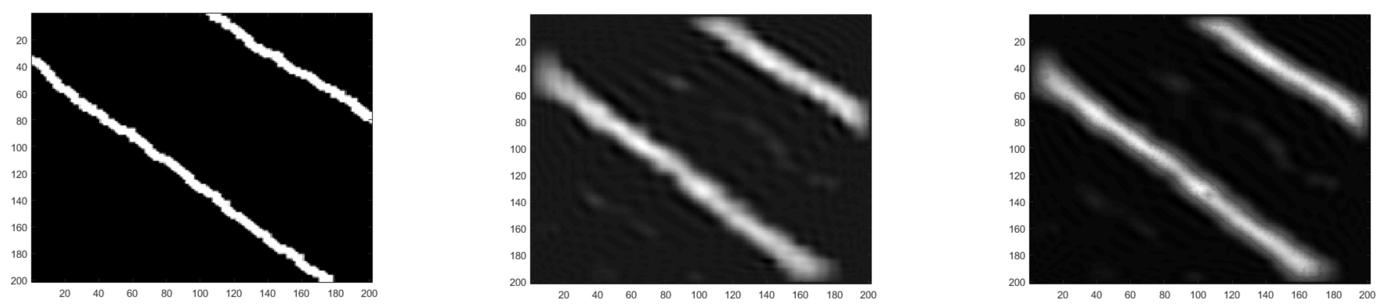
  For the $\mathbf{x}$ dependent parameter $\mathbf{p}$ we finally use

$$\boldsymbol{p}(\boldsymbol{x}) = ((N\eta_1^* d_1(x_1))^{1/2}, (N\eta_2^* d_2(x_2))^{1/2}) \ . \tag{13}$$

Applications in Magnetic Particle Imaging



(a) Original phantom **(b)** Simulated reconstruction from Lissajous sampling, see [6]. $SSIM = 0.6653$. **(c)** Reconstructed image after adaptive spectral filtering. $SSIM = 0.7017$



(a) Original phantom **(b)** Simulated reconstruction from Lissajous sampling, see [6]. $SSIM = 0.6155$. **(c)** econstructed image after adaptive spectral filtering. $SSIM = 0.6506$

Work in progress

We are working on a three-dimensional extension, considering 3D Lissajous curve and applying the adaptive filtering process, which is suitable due to its tensor product structure.

Conclusion

- On this poster, we gave a short introduction to polynomial interpolation and approximation on Lissajous curves and illustrated that a Gibbs phenomenon occurs, if the underlying function is not continuous.
- We showed how this Gibbs phenomenon can be described in the framework of Fourier series
- We showed how adaptive filtering techniques developed in the one-dimensional setting can be extended to higher dimensional cases
- We applied adaptive filtering techniques to improve the interpolation procedure in the reconstruction process of magnetic particle images.

Acknowledgment

This work has been supported by the University of Padova Project CPDA124755 "Multivariate approximation with applications to image reconstruction".
The authors gratefully acknowledge the financial support of the German Research Foundation (DFG, grant number ER 777/1-1).

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