

THE GEOMETRY OF THE MODULI SPACE OF CURVES AND ABELIAN VARIETIES

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ABSTRACT. Both algebraic curves and abelian varieties are basic objects in algebraic geometry and the study of their moduli spaces is a classical topic, with connections to number theory and other areas of geometry. In these expository notes, we discuss and compare the main geometric properties of moduli spaces of curves and abelian varieties, also in view of the relationships between them arising from Torelli and Prym-type maps.

1. THE MAIN PLAYERS

The aim of the present paper is to review and compare the geometry of the following two moduli spaces:

- (1) the moduli space $\mathcal{M}_{g,n}$ of smooth projective curves of genus g with n distinct marked points.
- (2) the moduli space \mathcal{A}_g of principally polarized abelian varieties.

The objects of $\mathcal{M}_{g,n}$ are $n + 1$ -tuples (C, p_1, \dots, p_n) where C is a smooth irreducible projective variety of dimension 1 and p_1, \dots, p_n are distinct points on C . If the Euler characteristic of $C \setminus \{p_1, \dots, p_n\}$ is negative, that is, if the integers $g, n \geq 0$ satisfy the condition $2g - 2 + n > 0$, then the automorphism group of any such n -pointed curve is finite and $\mathcal{M}_{g,n}$ exists as a smooth Deligne–Mumford stack of dimension $3g - 3 + n$. The coarse moduli space $M_{g,n}$ is a quasi-projective variety of the same dimension $3g - 3 + n$ with locally quotient singularities corresponding to the curves with non-trivial isomorphism group.

An abelian variety is simply a smooth projective variety A with a group structure on its set of points. In this review, we focus on the case of *principally polarized abelian varieties* (we will recall the definition in §5). This choice is motivated by the fact that most abelian varieties arising from geometrical constructions are principally polarized. This is the case, for instance, for Jacobian of curves and for intermediate Jacobians of threefolds. We will denote the moduli space of principally polarized abelian varieties of dimension g by \mathcal{A}_g . It is again a Deligne–Mumford stack, of dimension $g(g + 1)/2$, and its coarse moduli space A_g is a quasi-projective variety with locally quotient singularities.

Although the stacks $\mathcal{M}_{g,n}$ and \mathcal{A}_g are defined over \mathbb{Z} (see [DM69, FC90]) and have fascinating arithmetic properties, in this review we will restrict ourselves to the field \mathbb{C} of complex numbers unless specified otherwise.

The study of the geometry of $\mathcal{M}_{g,n}$ and \mathcal{A}_g is a very broad and rich subject. For this reason, this paper has no pretense of being a complete review. Its goal is rather to give something of the flavor of the techniques and kind of questions

that arise. For reviews on different sets of topics on moduli spaces of curves and abelian varieties, we refer the reader to [Oor81, Loo00, FP13, Har13] for curves and [vdGO99, HS02, Gru09, vdG13] for principally polarized abelian varieties.

The first two parts of this survey paper are devoted to the discussion of the geometry of $\mathcal{M}_{g,n}$ and \mathcal{A}_g , respectively. For both spaces, we will focus on two geometrical problems: the choice of an appropriate compactification and the birational geometry of the space. In the case of $\mathcal{M}_{g,n}$, we present the moduli space of Deligne–Mumford stable curves as the most natural compactification. For \mathcal{A}_g , we introduce the Baily–Borel–Satake compactification and the toroidal compactifications constructed in [AMRT75]. We will discuss the geometrical features of three toroidal compactifications: the perfect cone or first Voronoi compactification $\mathcal{A}_g^{\text{perf}}$, the second Voronoi compactification $\mathcal{A}_g^{\text{Vor}}$ and the matroidal partial compactification $\mathcal{A}_g^{\text{matr}}$. This illustrates the principle that the appropriate choice of a compactification for \mathcal{A}_g depends very much on the geometric properties one is interested in. As for birational geometry, we will mainly limit ourselves to a review of the results on the Kodaira dimension of both \mathcal{M}_g and \mathcal{A}_g and the related concept of *slope* of divisors. To get a more complete picture, we also give an overview of the progress in understanding the birational geometry of $\mathcal{M}_{0,n}$.

In the final part of the paper, we will tackle the relationship between moduli spaces of curves and abelian varieties that arise from the classical theory of Torelli and Prym maps, and describe the most important recent advance in the study of curve models for abelian varieties, i.e. the work [ADF⁺15] on the Prym–Tyurin–Kanev map to \mathcal{A}_6 .

The paper is structured as follows. The first part is devoted to the moduli space of curves. In §2 and §3 we describe its Deligne–Mumford compactification and introduce the tautological ring. The discussion of topics in the birational geometry of $\mathcal{M}_{g,n}$ is done in §4.

The second part of the paper starts in §5 with a review of the analytic construction of \mathcal{A}_g as a quotient of Siegel space by the symplectic group. The Baily–Borel–Satake compactification of \mathcal{A}_g and the Kodaira dimension of \mathcal{A}_g are discussed in §6 and §7, respectively. In §8, we introduce the toroidal compactifications of \mathcal{A}_g and in §§8.1–8.3 we outline their properties in the case of $\mathcal{A}_g^{\text{perf}}$, $\mathcal{A}_g^{\text{Vor}}$ and $\mathcal{A}_g^{\text{matr}}$.

The final part describes how to control the geometry of \mathcal{A}_g by means of the geometry of $\mathcal{M}_{g,n}$ and its covers. In §9 we review the Torelli and Prym constructions and in §10 we explain how they lead to curve models for abelian varieties of small dimension. Finally, in §11 we discuss the results from [ADF⁺15] for abelian 6-folds.

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2. DELIGNE–MUMFORD STABLE CURVES

With the obvious exception of $\mathcal{M}_{0,3}$, the moduli space $\mathcal{M}_{g,n}$ is never complete. However, there is a natural way to compactify it by considering moduli of *Deligne–Mumford stable n -pointed curves* [DM69].

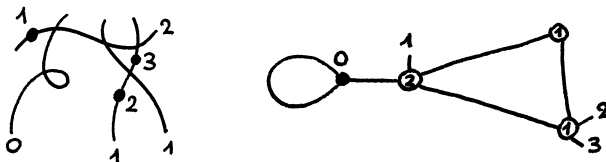


FIGURE 1. A stable curve of genus 6 with 3 marked points and the corresponding stable graph

A stable curve of genus g with n marked point is an $n + 1$ -tuple (C, p_1, \dots, p_n) where

- the curve C is a connected nodal curve of arithmetic genus g
- the points p_1, \dots, p_n are distinct points in the non-singular locus of C
- every irreducible component T of C satisfies the stability condition

$$\chi(T \setminus (C_{\text{sing}} \cup \{p_1, \dots, p_n\})) < 0.$$

The stability condition above is equivalent to requiring that each component of geometric genus 0 contains at least 3 special points, which can be either marked points or singular points of C .

The moduli space $\overline{\mathcal{M}}_{g,n}$ is then a proper Deligne–Mumford stack of dimension $3g - 3 + n$, defined over \mathbb{Z} . Its coarse moduli space $\overline{M}_{g,n}$ is a projective variety [Knu83].

An important feature of $\overline{\mathcal{M}}_{g,n}$ is its stratification by topological type. Namely, with each stable curve C with n marked points we can associate a labeled graph $\Gamma = \Gamma_C$ with n half-edges, called the *dual graph* of C . The dual graph Γ encodes all combinatorial information on the genera of the components of C and the position of the singular points, as follows:

- the *vertices* $v(C')$ of Γ are in bijection with the irreducible components C' of C ;
- each vertex $v(C')$ is *labeled* by the genus $g(C')$ of the corresponding components;
- the edges joining $v(C')$ and $v(C'')$ correspond to the singular points that C' and C'' have in common;
- there are n numbered half-edges (or leaves) l_1, \dots, l_n ; the half-edge l_j starts at the vertex $v(C')$ if and only if the j th marked point lies on the component C' .

In particular, two n -marked complex stable curves C and D have the same topological type if and only if their dual graphs Γ_C and Γ_D are isomorphic.

Let us observe that it is easy to reconstruct the (arithmetic) genus of C from its dual graph: it is equal to the genus of the graph plus the sum of the labels on its vertices. For instance, the stable curve in Figure 1 has genus 6, because the genus of the underlying graph is 2 and the labels on the vertices add up to 4.

If Γ is a stable graph of genus g with n half-edges, then the associated *gluing morphism* is

$$\text{gluing}_{\Gamma} : \prod_{v \in V} \overline{\mathcal{M}}_{g(v), e(v)+l(v)} \rightarrow \overline{\mathcal{M}}_{g,n},$$

where for every vertex v of Γ we denoted by $e(v)$ and $l(v)$ the number of edges and the number of leaves, respectively, starting at v . Then the image of gluing_Γ is the closure in $\overline{\mathcal{M}}_{g,n}$ of the locus of curves with dual graph Γ .

For instance, for a graph Γ with two vertices joined by one edge, we get the gluing map

$$\overline{\mathcal{M}}_{g_1, n_1+1} \times \overline{\mathcal{M}}_{g_2, n_2+1} \rightarrow \overline{\mathcal{M}}_{g_1+g_2, n_1+n_2}$$

whose image is a boundary divisor of $\overline{\mathcal{M}}_{g,n}$ with $g = g_1 + g_2$ and $n = n_1 + n_2$.

In this way we can describe explicitly all *boundary components* of $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$. In the case of $\overline{\mathcal{M}}_g$, they are

- the divisor Δ_0 whose general element is an irreducible nodal curve of geometric genus $g - 1$;
- for all i , $1 \leq i \leq \lfloor \frac{g}{2} \rfloor$, the divisor Δ_i whose elements are the union of a curve of arithmetic genus i and a curve of arithmetic genus $g - i$ intersecting at a point.

The description in the case of $\overline{\mathcal{M}}_{g,n}$ with $n \geq 0$ is analogous, but requires to keep track of how the marked points are distributed on the components. Furthermore, if there are at least 2 marked points, also curves with a smooth genus 0 component and a genus g component occur as general elements of divisors in $\mathcal{M}_{g,n}$.

3. TAUTOLOGICAL CLASSES ON THE MODULI SPACE OF CURVES

The study of the Chow (and the cohomology) ring of $\overline{\mathcal{M}}_{g,n}$ was initiated by Mumford in [Mum83b]. Mumford was interested in studying the rational Chow ring of $\overline{\mathcal{M}}_{g,n}$. As $\overline{\mathcal{M}}_{g,n}$ is a stack, its coarse moduli space $\overline{M}_{g,n}$ is singular in general, so Mumford had to prove first that the singularities of $\overline{\mathcal{M}}_{g,n}$ are sufficiently mild that it makes sense to consider the Chow ring with rational coefficients. By work of Looijenga [Loo94] and Boggi–Pikaart [BP00], it is now known that $\overline{\mathcal{M}}_{g,n}$ is the quotient stack of a smooth projective scheme by a finite group. This makes the construction of rational Chow groups and cohomology groups easier, because it reduces it to the construction of classes on the smooth cover that are invariant under the action of a finite group.

The definition of $\overline{\mathcal{M}}_{g,n}$ as a moduli space can be used to construct classes in its Chow ring. Natural examples include:

- The ψ -classes ψ_1, \dots, ψ_n obtained as the first Chern class of the cotangent line bundle at the i th marked point. For $i = 1, \dots, n$, let us denote by s_i the section of the universal family corresponding to the i th marked point:

$$\mathcal{C}_{g,n} \begin{array}{c} \xrightarrow{\pi} \\ \xleftarrow{s_i} \end{array} B$$

then the class ψ_i is defined by $\psi_i := c_1(s_i^* \omega_\pi) \in A^1(\overline{\mathcal{M}}_{g,n})$.

- the λ -classes, defined as the Chern classes $\lambda_i = c_i(\mathbf{E})$ of the Hodge bundle $\mathbf{E} = \pi_* \omega_\pi$, whose fiber over C is $H^0(C, \omega_C)$.
- the κ -classes, also known as Mumford–Morita–Miller-classes, defined as $\kappa_j := p_*(\psi_{n+1}^{j+1}) \in A^j(\overline{\mathcal{M}}_{g,n})$ for the forgetful map $p : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$.

These classes are not independent. For instance, Mumford showed that the λ -classes can always be expressed as linear combinations of polynomials in the κ -classes. This can be proved by applying Grothendieck–Riemann–Roch to the universal curve π and to ω_π .

The classes above make sense for every value of g and n and can be considered both in the Chow and in the cohomology ring of $\overline{\mathcal{M}}_{g,n}$. They are special case of the so-called *tautological classes*.

The Chow ring of $\overline{\mathcal{M}}_{g,n}$ becomes quickly very complicated g and n increases, so that a complete study of it cannot be realistically expected for positive genus. Instead, for many geometric applications it is enough to know the *tautological ring*, which is generated by the tautological classes introduced above, the fundamental classes of the closure of the strata of $\overline{\mathcal{M}}_{g,n}$ in the stratification by topological type, and other classes that can be obtained from the ψ - and κ -classes by considering the natural gluing morphisms associated to any stable graph Γ of genus g with n half-edges. Let us recall that in the genus 0 case the Chow ring coincides with the tautological ring and is completely known by work of Keel [Kee92].

4. BIRATIONAL GEOMETRY OF $\mathcal{M}_{g,n}$

The key ingredients in the birational geometry of $\overline{\mathcal{M}}_{g,n}$ are the Hodge class $\lambda = \lambda_1$, the cotangent classes ψ_1, \dots, ψ_n and the boundary divisors. The main underlying question is how to describe as explicitly as possible the *pseudo-effective cone* and the *nef cone*. The former is the convex cone inside $N^1(\overline{\mathcal{M}}_{g,n}) := \text{Pic}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{Q} / \equiv$ spanned by the effective divisors; the latter is defined analogously, but by taking the cone generated by nef divisors, i.e. by divisors with non-negative intersection with all curves in $\overline{\mathcal{M}}_{g,n}$.

These questions have a completely different flavor depending on whether g is positive or equal to zero. For genus 0, the nice geometric properties of $\overline{\mathcal{M}}_{0,n}$ allow to reformulate birational geometry questions in a very combinatorial fashion. We will discuss this later in §4.2. When the genus g of the curves considered is positive and $n = 0$, a primary topic in the birational geometry is the study of the *slope* of divisors, which is directly connected to the Kodaira dimension of M_g . We will focus on this in §4.1. For a more complete review of results concerning the slope, we refer the reader to [CFM13].

As a complete discussion of the progress on the birational geometry of $\mathcal{M}_{g,n}$ is not within the scope of this review, we will not include other important topics on the birational geometry of $\mathcal{M}_{g,n}$, such as Mori theory and the Hassett–Keel log minimal Mori program. We refer the interested reader to the survey [FS13] and the references therein. For the most recent progress in the Hassett–Keel program, see also [AFSvdW16].

We would like to mention that for $g > 0$, the *contractions* of $\overline{\mathcal{M}}_{g,n}$, i.e. the non-constant morphisms with connected fibers from $\overline{\mathcal{M}}_{g,n}$ to projective varieties, are known. Specifically, Gibney, Keel and Morrison [GKM02] proved that each contraction of $\overline{\mathcal{M}}_g$ is a birational morphism with target a compactification of M_g . This extends to the case of $\overline{\mathcal{M}}_{g,n}$ with $g, n > 0$. There, contractions always factor through a forgetful map $\overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n'}$ with $n' < n$.

4.1. Birational geometry of $\overline{\mathcal{M}}_g$ with $g > 0$. In this subsection, we will assume $g \geq 2$ and (for simplicity) $n = 0$. Then every irreducible effective divisor D not contained in the boundary of $\overline{\mathcal{M}}_g$ can be written (up to numerical equivalence) as a linear combination of $\lambda := \lambda$ and the classes $\delta_0, \dots, \delta_{[\frac{g}{2}]}$ of the boundary divisors

as

$$D \equiv a\lambda - \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} b_i \delta_i$$

with $a, b_i \in \mathbb{Q}_{>0}$.

Definition 1. The *slope* of D is $s(D) := \frac{a}{\min_i b_i}$.

The slope of the canonical divisor is known. Namely, the class of the canonical divisor of the stack $\overline{\mathcal{M}}_g$ is known to be

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2 \sum_0^{[g/2]} \delta_i.$$

As the natural map $\overline{\mathcal{M}}_g \rightarrow \overline{M}_g$ to the coarse moduli spaces is ramified along the divisor Δ_1 of stable curves containing a subcurve of arithmetic genus 1, one has

$$K_{\overline{M}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2 \sum_{i \geq 1} \delta_i.$$

In particular, if D is an effective divisor with slope $s(D) < \frac{13}{2}$, we have

$$K_{\overline{M}_g} = D + a\lambda + \sum_{i \geq 0} b_i \delta_i \quad \text{with } a, b_i \in \mathbb{Q}_{>0},$$

i.e. $K_{\overline{M}_g}$ is the sum of the *effective* divisors D and $\sum_{i \geq 0} b_i \delta_i$ and the *big* divisor $a\lambda$. Hence $K_{\overline{M}_g}$ is ample and \overline{M}_g is of general type. Let us remark that this kind of approach relies on the fact that \overline{M}_g has canonical singularities for $g \geq 2$ [HM82], so that one can always extend the sections of $K_{\overline{M}_g}$ and its tensor powers defined over the smooth locus of \overline{M}_g to desingularizations of \overline{M}_g .

Divisors of small slope play a fundamental role in the proof that \overline{M}_g is of general type for $g \geq 24$ [HM82, Har84, EH87]. For instance, for curves of odd genus [EH87] considers the Brill–Noether divisor $\overline{M}_{g,d}^r$ in \overline{M}_g of curves admitting a g_d^r with $g - r(r+1)(g+r-d) = -1$. By Brill–Noether theory, the general curve of genus g does not admit a linear series of this dimension and degree, and one can prove that $\overline{M}_{g,d}^r$ is indeed a divisor in \overline{M}_g . By using the degeneration technique of *limit linear series*, Harris and Mumford could prove that the slope of $\overline{M}_{g,d}^r$ is smaller than $13/2$.

However, the Brill–Noether divisors in general are not the effective divisors with the smallest possible slope. In particular, Farkas [Far09b] constructed an infinite sequence of divisors of slope $< 6 + \frac{12}{g+1}$ using the theory of syzygies. Moreover (see [Far09a]), in the case $g = 22$ the closure of the locus of curves with a linear series L of degree 25 and at least 7 linearly independent independent sections such that the natural map

$$\text{Sym}^2(H^0(L)) \rightarrow H^0(L^{\otimes 2})$$

is *not* injective is a divisor D_{22} with slope $s(D_{22}) = \frac{17121}{2636} < \frac{13}{2}$. This implies that \overline{M}_{22} is of general type. For \overline{M}_{23} , instead, it is known [Far00] that the Kodaira dimension is at least 2.

The approach for the case in which g is small is completely different and is based on the possibility of finding explicit geometric models for a general curve of a fixed low genus. For instance, the unirationality of \overline{M}_g for $2 \leq g \leq 10$ is a classical result,

whose proof is due to Severi and completed in modern mathematical language by Arbarello and Sernesi [AS79]. By work of several authors, most notably Chang and Ran, it is now known that $\overline{\mathcal{M}}_g$ is unirational for $g \leq 14$ and has negative Kodaira dimension for $g = 15, 16$. The problem of determining the Kodaira dimension in the intermediate cases $17 \leq g \leq 21$ is to my knowledge still open.

4.2. Birational geometry of $\overline{\mathcal{M}}_{0,n}$. The situation is very different for the case $g = 0$. First of all, in this case $\overline{\mathcal{M}}_{0,n}$ is always a fine moduli space, so that one does not need to distinguish between the stack and its coarse moduli space. Moreover, the spaces $\overline{\mathcal{M}}_{0,n}$ are smooth and can be described very explicitly as a blow-up of \mathbb{P}^{n-3} using Kapranov's construction [Kap93]. Also the stratification by topological type is simpler to describe. Since all irreducible components of a rational stable curve are rational, a stable graph of genus 0 is simply a tree with numbered leaves, such that each vertex has valence at least 3.

For $g = 0$ the space $N^1(\overline{\mathcal{M}}_{0,n})$ is generated by the classes of the boundary divisors [Kee92]. Hence, the question of describing the effective cone of $\overline{\mathcal{M}}_{0,n}$ can be translated into the combinatorics of stable trees — and can be restated as asking which linear combination of boundary divisors are effective. By work of Keel and Vermeire [Ver02], it is definitely known that for $n \geq 6$ there are effective divisors that are *not* an effective sum of boundary divisors. Hence, the boundary divisors are not the only divisors generating extremal rays in the effective cone of $\overline{\mathcal{M}}_{0,n}$.

However, it is not clear how to construct the additional generators. A breakthrough in this direction was given by Castravet and Tevelev's construction [CT13] of extremal divisors associated with certain combinatorial structures known as *hypertrees*. As the number of such hypertree divisors increases very rapidly with n , it was natural to expect that hypertree divisors and their pullbacks could provide a unified description of all missing extremal rays of the effective cone in $N^1(\overline{\mathcal{M}}_{0,n})$. This conjectural expectation has been disproved by Opie [Opi16]. In her work, she considered a family of rigid, effective divisors in $\overline{\mathcal{M}}_{1,n}$ constructed by Chen and Coskun in [CC14] and their pull-back to $\overline{\mathcal{M}}_{0,n+2}$ under the gluing map identifying the $n+1$ st and $n+2$ nd marked points. Such divisors can be constructed as follows. Let $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ be a sequence of integers with $a_1 + \dots + a_n = 0$ and $\gcd(a_1, \dots, a_n) = 1$. For any $(C, p_1, \dots, p_{n+2}) \in \overline{\mathcal{M}}_{0,n+2}$, let us denote by \tilde{C} the curve obtained by identifying the last two marked points. Then the associated divisor D_a is the closure of the locus in $\overline{\mathcal{M}}_{0,n+2}$ of rational n -pointed curves (C, p_1, \dots, p_{n+2}) such that the linear combination $a_1[p_1] + \dots + a_n[p_n] = 0$ vanishes in $\text{Pic}^0(\tilde{C})$. In the special case $a = (n, 1, -1, \dots, -1)$, one obtains in this way an extremal divisor which is not a hypertree divisor or the pull-back of a hypertree divisor.

The fact that the structure of $\overline{\mathcal{M}}_{0,n}$ can be described very explicitly raises the hope that $\overline{\mathcal{M}}_{0,n}$ shares most of the properties of a toric variety. For instance, in [HK00, Question 3.2] Hu and Keel pose the question of whether $\overline{\mathcal{M}}_{0,n}$ is a Mori dream space, a class of varieties X with mild singularities (specifically, \mathbb{Q} -factorial) with $\text{Pic}(X) \otimes \mathbb{Q} = N^1(X)$ and whose Cox ring is finitely generated. In particular, the nef cone of X should be generated by finitely many semiample divisors. (A semiample divisor is simply a divisor D such that $|mD|$ is base-point free for some $m > 0$).

Recently, Castravet and Tevelev [CT15] were able to answer this question in a negative sense: even in characteristic 0, the coarse moduli space $\overline{M}_{0,n}$ is *not* a Mori dream space for $n \geq 134$. Instead of working directly with $\overline{M}_{0,n}$, Castravet and Tevelev consider a different compactification of $M_{0,n}$, the Losev–Manin space \overline{LM}_n introduced in [LM00], which is a smooth toric variety of dimension $n - 3$. By a direct construction, they provide surjective morphisms from a certain projective modification of \overline{LM}_{n+1} to $\overline{M}_{0,n}$ and from $\overline{M}_{0,n}$ to the blow-up of \overline{LM}_n at the identity element e of the $n - 3$ -dimensional torus. In this way, on the one hand one obtains that if the blow-up of \overline{LM}_n is a Mori dream space for some value of n , then all $\overline{M}_{0,m}$ with $m < n$ are Mori dream spaces as well. On the other hand, if $\text{Bl}_e \overline{LM}_n$ is not a Mori dream space, also $\overline{M}_{0,n}$ is not a Mori dream space. Using a construction in toric geometry, the authors relate the birational geometry of the blow-up of \overline{LM}_n to that of the blow-up of the weighted projective plane $\mathbb{P}(a, b, c)$ whenever the integer n is of the form $n = a + b + c + 8$ with $a, b, c > 0$ coprime integers. This allows them to apply a result of Goto, Nishida and Watanabe that ensures that over a field of characteristic 0 the blow-up $\text{Bl}_2 \mathbb{P}(a, b, c)$ is not a Mori dream space when a, b and c are integers of a certain form.

Hu and Keel’s original motivation in studying Mori dream spaces (see [HK00, §3]) was to get a better understanding of the cone of effective divisors in $\overline{M}_{0,n}$. There is a long standing conjecture about this, which generalizes to $g \geq 0$ as well.

Conjecture 2 (F-conjecture). *A divisor D on $\overline{M}_{g,n}$ is nef if and only if $D \cdot C \geq 0$ holds for all irreducible curves C in the stratification by topological type.*

The 1-dimensional strata in the stratification by topological type are also known as *F-curves*. Basically, there are two types of F-curves. The first type of curve is parametrized by $\overline{\mathcal{M}}_{1,1}$. Let us call a stable curve of type *maximally degenerate* if its dual graph is a trivalent graph with only genus 0 vertices. Such maximally degenerate curves correspond to the dimension 0 strata in the stratification by topological type. Let us fix a stable maximally degenerate stable curve of genus $g - 1$ and $n + 1$ marked points and attach to it an elliptic curve E at the $n + 1$ st marked point. Letting E move in $\overline{\mathcal{M}}_{1,1}$ gives a one-dimensional stratum of $\overline{M}_{g,n}$.

The other type of F-curves is parametrized by $\overline{\mathcal{M}}_{0,4}$. One starts by fixing two maximally degenerate one-pointed stable curves C_1 and C_2 of genus i and $g - 1 - i$, respectively. Then for every $(C, p_1, p_2, p_3, p_4) \in \overline{\mathcal{M}}_{0,4}$ one considers the curve obtained by attaching C_1 to C at p_1 and C_2 to C at p_2 , and identifying p_3 and p_4 to obtain a nodal curve. One can also start by choosing a maximally degenerate two-pointed stable curve (C', q_1, q_2) . Then one attaches C to C' by identifying p_i with q_i for $i = 1, 2$ and p_3 with p_4 to create a node.

The F-conjecture is still open. However, to settle it, it would be sufficient to prove it in the case of genus 0, where so many combinatorial tools are available:

Theorem 3 ([GKM02]). *If the F-conjecture holds for $\overline{M}_{0,n}$ for all $n \geq 3$, then it holds for $\overline{M}_{g,n}$ for all values of g and n .*

5. CONSTRUCTION OF \mathcal{A}_g

Abelian varieties are smooth projective varieties A with a group structure (defined by regular morphisms) on their set of points. In moduli theory, one always fixes an ample line bundle L on A , i.e. a *polarization*. This rigidifies the moduli problem, because the automorphism of the pair (A, L) is always finite, even when A

has a positive-dimensional automorphism group. In this review we will concentrate on the case of principal polarizations.

Definition 4. A *principally polarized abelian variety* (for short, ppav) is a pair (A, L) where A is an abelian variety and L is a *principal polarization*, i.e. an ample line bundle on A with one-dimensional space of sections, considered up to translation on A .

Using complex analysis, it is easy to construct a parameter space for principally polarized abelian varieties, at least, if one works over the complex numbers. An attractive feature of this classical construction is that it works in the same way for all values of g .

The starting point is that an abelian variety over \mathbb{C} is the same as a complex torus together with a polarization. In particular, every g -dimensional abelian variety is the quotient of \mathbb{C}^g by a lattice and without loss of generality one can restrict to lattices of the form

$$\Lambda_\tau := \mathbb{Z}^g + \mathbb{Z}\tau_1 + \cdots + \mathbb{Z}\tau_g$$

with $\tau_1, \dots, \tau_g \in \mathbb{C}^g$. By Riemann's bilinear relations, to obtain an algebraic torus the $g \times g$ matrix $\tau = (\tau_1, \dots, \tau_g)$ must be an element of *Siegel space*

$$\mathbb{H}_g := \{\text{complex symmetric } g \times g\text{-matrices with positive definite imaginary part}\}.$$

For matrices $\tau \in \mathbb{H}_g$ the theta function

$$\theta(\tau, z) = \sum_{m \in \mathbb{Z}^g} \exp(\pi i m^t \tau m + 2\pi i n^t z)$$

is well-defined. Its zero locus $\Theta_\tau \subset A_\tau := \mathbb{C}^g / \Lambda_\tau$ is the divisor defining a principal polarization on A_τ . Two principally polarized abelian varieties A_τ and $A_{\tau'}$ with $\tau, \tau' \in \mathbb{H}_g$ are isomorphic if and only if

$$\tau' = (A\tau + B)(C\tau + D)^{-1}$$

for a symplectic matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, i.e. an element of the symplectic group

$$\mathrm{Sp}(2g, \mathbb{Z}) := \{M \in \mathrm{Mat}(2g \times 2g, \mathbb{Z}) \mid MJM^t = J\},$$

where $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$ is the matrix defining the standard symplectic form on \mathbb{Z}^{2g} .

One can use this construction to identify \mathcal{A}_g with the quotient $\mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathbb{H}_g$. This description shows that \mathcal{A}_g is intimately related to the space of Siegel modular forms, i.e. of holomorphic maps on \mathbb{H}_g that behave well with respect to the $\mathrm{Sp}(2g, \mathbb{Z})$ -action.

There is also a natural construction that yields a finite cover of \mathcal{A}_g which is a smooth scheme. Namely, one can rigidify the moduli problem by looking at moduli of ppav with a level n structure.

Definition 5. A (principal) level n structure on an abelian variety A is an isomorphism $\alpha : (\mathbb{Z}/n\mathbb{Z})^{2g} \rightarrow A[n]$, where $A[n]$ denotes the subgroup of A consisting of the points whose order divides n .

Then one can realize the moduli space of ppav with a level n structure as the quotient

$$\mathcal{A}_{g,n} = \Gamma_g(n) \backslash \mathbb{H}_g,$$

of Siegel space by the *level n subgroup* $\Gamma_g(n) := \{M \in \mathrm{Sp}(2g, \mathbb{Z}) \mid M \equiv I_{2g} \pmod{n}\}$.

Serre [Ser62] proved that a ppav \mathcal{A} with a level n structure α has no non-trivial automorphisms if $n \geq 3$. In particular, the natural map $\mathcal{A}_{g,n} \rightarrow \mathcal{A}_g$ realizes \mathcal{A}_g as the quotient of the smooth scheme $\mathcal{A}_{g,n}$ by the action of a finite group.

6. THE HODGE CLASSES AND THE BAILY–BOREL–SATAKE COMPACTIFICATION

The existence of a universal family

$$\mathcal{X}_g \begin{array}{c} \xrightarrow{\pi} \\ \xleftarrow{s} \end{array} \mathcal{A}_g$$

over the stack \mathcal{A}_g allows to define a natural vector bundle, called the *Hodge bundle*

$$\mathbb{E} := \pi_*(\Omega_{\mathcal{X}_g/\mathcal{A}_g}^1) = s^*(\Omega_{\mathcal{X}_g/\mathcal{A}_g}^1)$$

The Chern classes $\lambda_i := c_i(\mathbb{E}) \in A^i(\mathcal{A}_g)$ are called the Hodge classes. They generate a subring of $A^\bullet(\mathcal{A}_g)$ known as the tautological subring. This is very well understood (see [vdG99]).

First, the relations between λ -classes are given by

$$(1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots + (-1)^g \lambda_g) = 1,$$

and

$$\lambda_g = 0.$$

Both relations can be proved using Grothendieck–Riemann–Roch, for the normalized universal theta divisor on \mathcal{X}_g in the former case and for the structure sheaf of \mathcal{X}_g in the latter case, respectively.

Furthermore, the class λ_1 — or, equivalently, the determinant of the Hodge bundle \mathbb{E} — is ample on \mathcal{A}_g . Over \mathbb{C} one can use the description of \mathcal{A}_g as the quotient $\mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathbb{H}_g$ to see that the pull-back of the sections of $(\det \mathbb{E})^k$ to \mathbb{H}_g give weight k Siegel modular forms, i.e. functions $f : \mathbb{H}_g \rightarrow \mathbb{C}$ satisfying

$$f((A\tau + B)(C\tau + D)^{-1}) = \det(C\tau + D)^k f(\tau)$$

for all symplectic matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z})$. Baily and Borel proved in [BB66] that modular forms of sufficiently high degree define an embedding of \mathcal{A}_g into projective space. Its image is the minimal compactification of \mathcal{A}_g , known as the *Baily–Borel–Satake compactification* \mathcal{A}_g^S . The same compactification can be constructed by considering the structure of \mathcal{A}_g as a locally symmetric space, and indeed it is the first case in which a Baily–Borel compactification was constructed.

7. KODAIRA DIMENSION OF \mathcal{A}_g AND MUMFORD’S PARTIAL COMPACTIFICATION

Set-theoretically, we can stratify \mathcal{A}_g^S as the disjoint union of moduli spaces of ppav of dimension $\leq g$:

$$\mathcal{A}_g^S = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_0.$$

This stratification is directly related to the structure of the Borel subgroups of the symplectic group, but also has an algebro-geometric counterpart in the semistable reduction theorem [FC90, I.2.6], asserting that 1-dimensional families of abelian varieties can always be completed (up to finite base change) to a family of *semiabelian varieties*, i.e. smooth group schemes obtained as extensions of the form

$$0 \rightarrow (\mathbb{G}_m)^r \rightarrow S \rightarrow A \rightarrow 0$$

where \mathbb{G}_m denotes the multiplicative group and A is an abelian variety of dimension $g - r$. The integer r is called the *torus rank* of S .

In particular, if we consider a curve $B \subset \mathcal{A}_g$, the intersection of $\overline{B} \subset \mathcal{A}_g^S$ with the boundary of \mathcal{A}_g^S is determined by its abelian part A . However, all information about the extension disappears in \mathcal{A}_g^S , so that \mathcal{A}_g^S is not a moduli space of degenerations of abelian varieties. For instance, if one wants to encode the geometry of rank 1 degenerations of abelian varieties, the additional data needed is a description of the line bundle over the $g - 1$ -dimensional abelian variety A obtained by partially compactifying S . As line bundles over A are the points of the dual abelian variety \hat{A} , which is canonically isomorphic to A by the principal polarization, rank 1 semiabelian varieties of dimension g are parametrized by a finite quotient of the universal abelian variety \mathcal{X}_{g-1} over \mathcal{A}_{g-1} , which has indeed dimension $g(g+1)/2 - 1$. More specifically, a line bundle L and its inverse L^{-1} define the same semiabelian variety, so that rank 1 degenerations are parametrized by $\mathcal{X}_{g-1}/\pm 1$. This suggests that the boundary of a modular compactification of \mathcal{A}_g should have codimension 1. However, the boundary of \mathcal{A}_g^S has codimension g . Furthermore, although \mathcal{A}_g^S is known to be normal, its boundary is highly singular.

The partial compactification \mathcal{A}'_g obtained by adding the divisor $\mathcal{X}_{g-1}/\pm 1$ to \mathcal{A}_g is studied in [Mum83a]. There, it is used to gain information on the Kodaira dimension of \mathcal{A}_g . The strategy is similar to the one outlined in the case of the moduli space of curves. The rational Picard group of the coarse moduli space A'_g of \mathcal{A}'_g is generated by $\lambda := \lambda_1$ and the class δ of the boundary. This allows to define the *slope* of a divisor $a\lambda - b\delta$ in A'_g as the ratio a/b . One can compute directly that the class of the canonical divisor is $(g+1)\lambda - \delta$.

Hence, if one can find an effective divisor of the form $a\lambda - b\delta$ with $a/b > g+1$, one automatically obtains that A'_g , and hence \mathcal{A}_g , is of general type. In [Mum83a], Mumford considers the Andreotti–Mayer locus N_0 , i.e. the closure in A'_g of the locus of abelian varieties whose Θ -divisor is singular, and proves that its slope is $6 + 12/(g+1)$. This implies that \mathcal{A}_g is of general type for $g \geq 7$. This refines previous results by Tai in [Tai82], who had already proved that \mathcal{A}_g was of general type for $g \geq 9$ by studying explicitly the spaces of sections of $K_{\mathcal{A}_g}^{\otimes N}$.

8. TOROIDAL COMPACTIFICATIONS OF \mathcal{A}_g

Ideally one would like to be able to construct a compactification of \mathcal{A}_g which has properties similar to those of $\overline{\mathcal{M}}_{g,n}$: a compactification with a modular interpretation and which is smooth as an algebraic stack. In pursuit of a compactification with better properties than \mathcal{A}_g^S , Ash, Mumford, Rapoport and Tai [AMRT75] introduced the toroidal compactifications of \mathcal{A}_g . The intuitive idea behind this is that degenerations of elements τ of Siegel space have imaginary part which is *positive semidefinite*. Indeed, the combinatorial gadget needed to define a toroidal compactification of \mathcal{A}_g is a decomposition into convex polyhedral cones of the space $\text{Sym}_{\geq 0}^2(\mathbb{R}^g)$ of positive semidefinite quadratic forms with rational radical. This cone decomposition Σ should satisfy certain admissibility properties. In particular, the cone decomposition Σ should define a stratification of $\text{Sym}_{\geq 0}^2(\mathbb{R}^g)$ into open convex polyhedral cones, it should be $\text{GL}(g, \mathbb{Z})$ -invariant and contain only finitely many $\text{GL}(g, \mathbb{Z})$ -isomorphism classes of cones. Then, toroidal geometry provides a recipe to construct a toroidal compactification \mathcal{A}_g^Σ of \mathcal{A}_g . This compactification is not

necessarily a moduli space; however, it possesses a universal family of semiabelian varieties.

The compactification \mathcal{A}_g^Σ has a natural stratification. Namely, it can be written as the disjoint union of locally closed strata $\beta(\sigma)$ corresponding to the $\mathrm{GL}(g, \mathbb{Z})$ -orbits of cones $\sigma \in \Sigma$. If we denote the maximal rank of a quadratic form in Σ and the real dimension of the cone σ by r and d , respectively, then $\beta(\sigma)$ has complex codimension d in \mathcal{A}_g^Σ is the finite quotient of a torus bundle over the fiber product over \mathcal{A}_g of r copies of \mathcal{X}_{g-r} . For instance, rank 0 degenerations are just abelian varieties of dimension g and we have $\beta(0) = \mathcal{A}_g$. As already described, rank 1 degenerations give a divisor in \mathcal{A}_g^Σ , namely $\beta(\mathbb{R}_{>0}x_1^2) \cong \mathcal{X}_{g-1}/\pm 1$. The situation becomes more complicated for rank 2 degenerations. In this case, up to $\mathrm{GL}(2, \mathbb{Z})$ -equivalence there are two cones that one needs to consider, namely the two-dimensional cone $\sigma_{1+1} := \mathbb{R}_{>0}x_1^2 + \mathbb{R}_{>0}x_2^2$ and the three-dimensional cone $\sigma_{C_3} := \mathbb{R}_{>0}x_1^2 + \mathbb{R}_{>0}x_2^2 + \mathbb{R}_{>0}(x_1 - x_2)^2$. They correspond to different types of degenerations of abelian varieties.

Among the explicitly known families of admissible cone decompositions (see e.g. [Nam80]), the most useful for geometric applications are the *perfect cone decomposition* Σ^{perf} (also known as *first Voronoi decomposition*), the *second Voronoi decomposition* Σ^{Vor} and the *central cone decomposition* Σ^{centr} . They give rise to the perfect cone compactification $\mathcal{A}_g^{\mathrm{perf}}$, the second Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}$ and the central cone compactification $\mathcal{A}_g^{\mathrm{centr}}$, respectively. Each of these compactifications can be seen as the “most natural one” from a different geometrical point of view. For instance, Igusa [Igu67] proved that the central cone compactification can be characterized as the normalization of the blow-up of \mathcal{A}_g^Σ along the boundary. The other two compactifications will be considered in the next sections.

It is important to remark that all of these compactifications are singular if g is sufficiently large. Their existence can be detected using toric geometry. Let us recall that a convex cone $\sigma \subset \mathrm{Sym}^2(\mathbb{R}^g)$ is called *basic* if the extremal rays of σ are a \mathbb{Z} -basis of $\mathrm{Sym}^2(\mathbb{Z}^g)$. If the extremal rays of σ form a \mathbb{Q} -basis of $\mathrm{Sym}^2(\mathbb{Q}^g)$, the cone is called *simplicial*. In toric geometry, non-basic cones give rise to singular toric varieties and simplicial cones correspond to the existence of locally quotient singularities. This description of the singularities extends to the toroidal compactifications \mathcal{A}_g^Σ . For g sufficiently large, all cone decompositions mentioned above contain non-simplicial cones. Therefore, they give rise to algebraic stacks \mathcal{A}_g^Σ with singularities which are not locally quotient ones. However, all cone decompositions above can be refined to obtain smooth toroidal compactifications. The existence of these smooth toroidal compactification is very important from a theoretical point of view and is one of the fundamental ingredients of [FC90].

8.1. Perfect cone compactification. The perfect cone decomposition Σ^{perf} is easy to define. Let $Q \in \mathrm{Sym}_{>0}^2(\mathbb{R}^g)$ be a positive definite quadratic form. Let us denote by $M(Q)$ the finite and non-empty set of non-trivial elements of the lattice \mathbb{Z}^g on which Q attains its minimal value:

$$M(Q) = \left\{ \xi_0 \in \mathbb{Z}^g \mid Q(\xi_0) = \min_{\xi \in \mathbb{Z}^g \setminus \{0\}} Q(\xi) \right\}.$$

Then Σ^{perf} is the set of all cones of the form

$$\sigma(Q) := \sum_{\xi_0 \in M(Q)} \mathbb{R}_{>0}\xi_0^2$$

with $Q \in \Sigma_{>0}(\mathbb{R}^g)$. In particular, cones in Σ^{perf} are generated by quadratic forms of rank 1, so that up to the $\text{GL}(g, \mathbb{Z})$ -action there is only one 1-dimensional cone in Σ^{perf} , namely, the one spanned by x_1^2 . As a consequence, the boundary of $\mathcal{A}_g^{\text{perf}}$ is irreducible. This makes the birational geometry of its coarse moduli space A_g^{perf} a natural object of study, because the rational Picard group of A_g^{perf} then coincides with that of the partial compactification \mathcal{A}'_g , generated by λ and δ .

In [SB06], Shepherd–Barron uses this property to prove that A_g^{perf} is the canonical model of \mathcal{A}_g for $g \geq 12$. Let us recall that a canonical model of a quasi-projective variety Y is a normal complex projective variety birational to Y , with canonical singularities and ample canonical class. Shepherd–Barron’s result relies on first characterizing the cone of ample divisors on A_g^{perf} and then proving that A_g^{perf} has canonical singularities for $g \geq 5$. Unfortunately, the proof that A_g^{perf} has canonical singularities has a gap; the interested reader can now refer to the preprint [ASB16] for how to complete the proof.

8.2. Second Voronoi compactification. The second Voronoi decomposition is combinatorially more involved than the perfect cone decomposition. Its definition relies on the fact that each positive semidefinite form Q has an associated Deloné decomposition, which subdivides the points in \mathbb{R}^g according to which lattice point is nearest according to the metric associated to Q .

More precisely, let $\alpha \in \mathbb{R}^g \setminus \{0\}$ be arbitrary and let us denote by $m_Q(\alpha)$ the minimal value of $Q(\xi - \alpha)$ for $\xi \in \mathbb{Z}^g \setminus \{0\}$. Then a subset of \mathbb{R}^g is called a *Deloné cell* if it is the convex element of the set of elements $a \in \mathbb{Z}^g \setminus \{0\}$ such that $Q(a - \alpha) = m_Q(\alpha)$ holds. The Deloné cells form a polyhedral subdivision of \mathbb{R}^g , invariant under translation by elements of the lattice \mathbb{Z}^g : this is called the *Deloné decomposition* corresponding to Q . If D is a fixed Deloné decomposition, then the set of all positive semidefinite quadratic forms with Deloné decomposition equal to D forms a convex polyhedral cone σ_D . Then one can define the second Voronoi decomposition as

$$\Sigma^{\text{Vor}} = \{\sigma_D \mid D \text{ is a Deloné decomposition}\}.$$

The reason why the second Voronoi compactification $\mathcal{A}_g^{\text{Vor}}$ is important geometrically is that it is the only compactification of \mathcal{A}_g for which an interpretation as a moduli space is known. Namely, by work of Alexeev [Ale02], it can be described as (the normalization of) a component of the moduli space of semiabelic pairs. A more precise description of this component using the language of logarithmic geometry was given by Olsson in [Ols08].

8.3. Matroidal cones. As mentioned before, the combinatorial structure of $\mathcal{A}_g^{\text{perf}}$ and $\mathcal{A}_g^{\text{Vor}}$ is very complicated. A complete classification for $\text{GL}(g, \mathbb{Z})$ -orbits of cones in Σ^{Vor} is known only for $g \leq 5$; for $g = 6$ the number of orbits is known to be larger than 250,000 (see [Val03, Ch. 4]). For perfect cones the situation is better and a classification is known for $g \leq 7$ (see [EVGS13] and references therein). Hence, for none of these compactifications the combinatorics can be expressed as explicitly as the combinatorics of stable graphs in the case of moduli of curves. Nevertheless, there is a partial compactification which is particularly nice from this point of view.

A $g \times n$ matrix with integer coefficients is called *totally unimodular* if each every square submatrix has determinant -1 , 0 or 1 . A cone in $\text{Sym}_{\geq 0}^2(\mathbb{R}^g)$ is called *matroidal* if up to the $\text{GL}(g, \mathbb{Z})$ -action it is spanned by the rank 1 forms given by

the columns of a totally unimodular matrix. The set of all matroidal cones gives rise to a polyhedral cone decomposition Σ^{matr} whose union is strictly contained in the set of positive semidefinite quadratic forms with rational radical. As such, it gives rise to a partial compactification $\mathcal{A}_g^{\text{matr}}$ of \mathcal{A}_g . Furthermore, matroidal cones and totally unimodular matrices can be reinterpreted in combinatorial terms as regular matroids [Oxl11]. One can even prove that matroidal cones are always simplicial, so that the stack $\mathcal{A}_g^{\text{matr}}$ has only locally quotient singularities.

Matroidal cones can also be characterized in a completely different way. Indeed, building on Alexeev and Brunyate's paper [AB11], Melo and Viviani [MV12] proved that Σ^{matr} equals the intersection of Σ^{perf} and Σ^{Vor} . Geometrically, this means that $\mathcal{A}_g^{\text{matr}}$ is the largest open subset that can be naturally embedded in both the perfect cone and the second Voronoi compactification.

9. TORELLI AND PRYM MAP

So far, we have seen an overview of the geometry of \mathcal{A}_g and $\mathcal{M}_{g,n}$, with special attention to their birational geometry and the choice of a compactification. Of the two spaces, the construction of \mathcal{A}_g over the complex numbers was particularly explicit and had the advantage of providing a direct relationship between forms and functions on \mathcal{A}_g and modular forms for the symplectic group. Other constructions, like the choice of a compactification, are more straightforward in the case of the moduli space of curves. On the other hand, the study of the birational geometry of \mathcal{A}_g and $\mathcal{M}_{g,n}$ uses techniques that are similar. In the case of $\mathcal{M}_{g,n}$, one profits from the wealth of explicit constructions coming from the geometry of curves. In the case of \mathcal{A}_g , the relation with Siegel space and the symplectic group can be useful in allowing an analytic approach to the problem.

However, it is important to keep in mind how intimately related the two spaces are. First and foremost, for every non-singular curve C the degree zero Picard group $JC := \text{Pic}^0(C)$ is an abelian variety, with a natural choice of principal polarization given by taking the theta divisor Θ_C , defined as the image of $\text{Sym}^{g-1} C \subset \text{Pic}^{g-1}(C)$ under any translation map $\text{Pic}^{g-1}(C) \xrightarrow{\cong} \text{Pic}^0(C)$. This gives rise to the classical *Torelli map*

$$\begin{array}{ccc} \mathcal{M}_g & \longrightarrow & \mathcal{A}_g \\ C & \longmapsto & (JC, \Theta_C). \end{array}$$

By the Torelli theorem (see e.g. [And58]), this map is injective on coarse moduli spaces, in the sense that the Jacobian determines the curve C up to isomorphism. If one takes the stack structure into account, one notices that the general abelian variety has an automorphism $x \leftrightarrow -x$ of order 2, whereas the automorphism group of a general curve of genus 3 is trivial. So the Torelli map is a degree 2 map onto its image for $g \geq 3$. In genus 2 all curves have an involution, namely, the hyperelliptic involution, and the Torelli map is injective as a map of stack.

Of course, one would like to be able to extend the Torelli map to a map from the Deligne–Mumford compactification $\overline{\mathcal{M}}_g$ to some toroidal compactification of \mathcal{A}_g . Indeed, it has been known for a long time that the Torelli map extends to a map $\overline{\mathcal{M}}_g \rightarrow \mathcal{A}_g^{\text{Vor}}$ [Nam76a, Nam76b], which is known to be compatible with the modular interpretation of the two moduli spaces [Ale04]. The combinatorial conditions necessary to ensure that the Torelli map extends to a given stable curve are studied by Alexeev and Brunyate in [AB11]. Using their more refined criterion, they could prove that the Torelli map does not extend to a regular map to $\mathcal{A}_g^{\text{centr}}$

for $g \geq 9$ and that the image of $\mathcal{M}_g \rightarrow \mathcal{A}_g^{\text{Vor}}$ is contained in a common open subset of $\mathcal{A}_g^{\text{Vor}}$ and $\mathcal{A}_g^{\text{perf}}$, i.e. to the matroidal locus, so that the extended Torelli map can also be viewed as a map $\overline{\mathcal{M}}_g \rightarrow \mathcal{A}_g^{\text{Vor}}$.

The relationship between curves and abelian varieties can be pushed further by working with Prym varieties of covers of curves. Here, the classical case is that of étale double covers.

Definition 6. An *admissible double cover* is a finite surjective morphism $\pi : \tilde{C} \rightarrow C$ such that

- C is a stable curve;
- the genus of \tilde{C} satisfies $p_a(\tilde{C}) = 2p_a(C) - 1$;
- the restriction $\pi : \pi^{-1}(D) \rightarrow D$ to the preimage of any irreducible component $D \subset C$ has degree 2;
- the fixed points of the involution $i : \tilde{C} \rightarrow \tilde{C}$ given by sheet interchange are nodes of \tilde{C} , and i does not interchange the two branches of a node.

The moduli stack $\overline{\mathcal{R}}_g$ of admissible double covers of a genus g curve is a proper Deligne–Mumford stack. The natural map $\overline{\mathcal{R}}_g \rightarrow \overline{\mathcal{M}}_g$ that realizes it as a finite cover of degree $2^{2g} - 1$ of the moduli space of curves, branched over the divisor $\Delta_0 \subset \overline{\mathcal{M}}_g$ whose general element is an irreducible nodal curve. The open substack in which the curve C is smooth is denoted by \mathcal{R}_g .

A smooth admissible double cover $\pi : \tilde{C} \rightarrow C$ with $p_a(C) = g + 1$ has an associated principally polarized abelian variety of dimension g , known as the *Prym variety* $\text{Prym}(\tilde{C}/C)$ of π . Following Mumford’s work [Mum74], one can construct it by looking at the norm map

$$\begin{aligned} \text{Nm} : J(\tilde{C}) &\longrightarrow J(C) \\ [D] &\longmapsto [\pi_*(D)] \end{aligned}$$

and setting $\text{Prym}(\tilde{C}/C)$ to be the connected component of the identity of the kernel of Nm . After translation, the Prym variety can also be thought of as a component of the preimage of the canonical divisor K_C under the norm map $\text{Pic}^{2g-2}(\tilde{C}) \rightarrow \text{Pic}^{2g-2}(C)$, namely the component $\text{Nm}^{-1}(K_C)^{\text{even}}$ corresponding to line bundles whose space of sections has even dimension. Then the locus

$$\Xi_C = \{L \in \text{Nm}^{-1}(K_C)^{\text{even}} \mid h^0(\tilde{C}, L) \geq 2\}$$

is the divisor defining the desired principal polarization. In this way one obtains the *Prym map* $\mathcal{R}_{g+1} \rightarrow \mathcal{A}_g$.

Let us observe that this construction makes sense also when C is a stable curve. In this case, one has to consider J_C and $J_{\tilde{C}}$ as the connected component of the identity in the Picard group. If the curve C only has separating nodes, i.e. if the dual graph of C is a tree, then J_C is an abelian variety, but in general J_C is just a semiabelian variety, namely the extension of the Jacobian of its normalization by $H^0(\Gamma(C), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^*$. The construction of the norm map carries on in this settings and one can define $P(\tilde{C}/C)$ exactly as in the smooth case.

Analogously to the case of the Torelli map, one would like to be able to extend the Prym map to a map $\overline{\mathcal{R}}_{g+1} \rightarrow \mathcal{A}_g^{\Sigma}$ for some toroidal compactification \mathcal{A}_g^{Σ} of \mathcal{A}_g . However, it turns out that the Prym map cannot be extended to any of the best known toroidal compactifications. The conditions for the existence of a regular extension of the Prym map to an admissible double cover $\tilde{C} \rightarrow C$ were studied in

[ABH02] for the second Voronoi compactification and in [CMLGH15] for the perfect cone and the central cone decomposition.

10. CURVE MODELS OF ABELIAN VARIETIES OF DIMENSION $g \leq 5$

The moduli space of principally polarized abelian varieties is in most instances very well behaved. Nevertheless, the richness of the geometry of the moduli space of curves makes it easier to study. The best situation occurs when there is a dominant map from a moduli space of curves to \mathcal{A}_g . Then one can combine the geometry of curves and abelian varieties to obtain very useful additional information.

The classical example of this is the Torelli map. In view of the Torelli theorem, the Torelli map is birational on the coarse moduli spaces whenever $\dim \mathcal{M}_g = \dim \mathcal{A}_g$ holds, that is, for $g \leq 3$. In particular, one can deduce the rationality of A_2 and A_3 directly from the rationality of M_2 and M_3 . However, for $g \geq 4$ the general ppav of dimension g is no longer a Jacobian.

The situation for the Prym map is more complicated. For instance, the Prym map is known to be *generically injective* on the coarse moduli spaces for $g \geq 7$ by [FS82]. However, it is shown in [Don92] that the Prym map is never injective. For instance, its restriction to the locus of hyperelliptic curves has positive-dimensional fibers.

If one looks at the dimension of the spaces, one has $\dim(\mathcal{R}_{g+1}) \geq \dim(\mathcal{A}_g)$ for $g \leq 5$. Therefore, one expects it to be a dominant map in this range, and indeed a general ppav of dimension $g \leq 5$ is a Prym variety, as already proved by Wirtinger in [Wir95]. For instance, this implies that in this range the coarse moduli space R_{g+1} is unirational and this can be used to prove that A_g is unirational as well. This approach works for R_6 and A_5 [Don84] as well as for R_5 and A_4 [Cle83, ILGS09]. As we already explained, A_g is of general type for $g \geq 7$. The birational type of A_6 is an open problem.

11. CURVE MODELS OF ABELIAN 6-FOLDS

The locus of Prym varieties has codimension 3 in \mathcal{A}_6 , so that the Prym map cannot be used to control the geometry of \mathcal{A}_6 and answer questions about its birational type. However, there is now a construction available that allows to describe the general abelian 6-fold in terms of curve geometry. Namely, in [ADF⁺15], it is proved that the generalized Prym–Tyurin–Kanev map from the space of E_6 -covers of the projective line to \mathcal{A}_6 is dominant. The E_6 -covers are a special class of degree 27 covers of \mathbb{P}^1 with monodromy group equal to the Weyl group of the E_6 lattice and 24 branch points. In particular, the Hurwitz space parametrizing them has dimension equal to $\dim \overline{\mathcal{M}}_{0,24} = 21$. Special examples of E_6 -covers arise quite naturally if one considers a general pencil of cubic surfaces. Specifically, let $X \subset \mathbb{P}^4$ be a smooth cubic threefold and $\{S_t \mid t \in \mathbb{P}^1\}$ a general pencil of hyperplane sections of X . Such a pencil contains 24 nodal surfaces. Then

$$C = \{(L, t) \in \mathbb{G}(1, \mathbb{P}^4) \times \mathbb{P}^1 \mid L \subset S_t\} \longrightarrow \mathbb{P}^1$$

is a degree 27 cover ramified at 24 points. From the ramification pattern one deduce that C is a curve of genus 46. However, the expected dimension of the space of pencil of cubic surfaces is $16 < 21$, so that it is clear that the general E_6 -cover does not come from a pencil of cubics.

Nevertheless, most features of pencil of cubics extend to general E_6 -covers. Let us recall that $W(E_6)$ is the monodromy group for family of lines on smooth cubic surfaces. Hence, requiring that the monodromy group of the cover $C \rightarrow \mathbb{P}^1$ is $W(E_6)$ implies that one can identify consistently the fiber of $C \rightarrow \mathbb{P}^1$ with the 27 lines on a cubic surface. This allows to define a correspondence $D \subset C \times C$ by

$$D = \{((L_1, t), (L_2, t)) \in C \times C \mid \text{the lines } L_1 \text{ and } L_2 \text{ are incident to each other}\}.$$

The correspondence D defines a map $JC \rightarrow JC$, simply by mapping the class of a point $(L, t) \in C$ to the sum of the classes of all $(M, t) \in C$ with M incident to L . Then Kanev [Kan87] proved that the locus in JC where D acts as multiplication by -5 is a ppav of dimension 6. The main result of [ADF⁺15] is that the map from the Hurwitz scheme parametrizing E_6 -covers to A_6 defined by this construction is dominant. As this Hurwitz scheme is a finite cover of $\overline{\mathcal{M}}_{0,24}/S_{24}$, it has dimension 21, hence the Prym–Tyurin–Kanev map is also generically injective.

The idea of the proof is again based on techniques from toroidal geometry. Namely, one can consider the extension of the Prym–Tyurin–Kanev map to E_6 -covers $C \rightarrow R$ obtained by letting the pair (\mathbb{P}^1, B) consisting of \mathbb{P}^1 and the branch locus B of the cover degenerate to a maximally degenerate 24-pointed stable rational curve (R, B') in $\overline{\mathcal{M}}_{0,24}/S_{24}$. Locally, the extension of the Prym–Tyurin–Kanev map to such a degenerate cover gives rise to a toroidal map. In particular, this allows to compute explicitly the rank of the map by analyzing the appropriate fan.

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