“On the (Ab)Use of *Omega*?”

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Abstract

Several recent finance articles use the Omega measure (Keating and Shadwick, 2002), defined as a ratio of potential gains out of possible losses, for gauging the performance of funds or active strategies, in substitution of the traditional Sharpe ratio, with the arguments that return distributions are not Gaussian and volatility is not always the relevant risk metric. Other authors also use Omega for optimizing (non-linear) portfolios with important downside risk. However, we question in this article the relevance of such approaches. First, we show through a basic illustration that the Omega ratio is inconsistent with the Second-order Stochastic Dominance criterion. Furthermore, we observe that the trade-off between return and risk corresponding to the Omega measure, may be essentially influenced by the mean return. Next, we illustrate in static and dynamic frameworks that Omega-based optimal portfolios can be closely associated with classical optimization paradigms depending on the chosen threshold used in Omega. Finally, we present robustness checks on long-only asset and hedge fund databases, that confirm our results.

Keywords: Performance Measure, *Omega*, Return Distribution, Risk, Stochastic Dominance.
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1. Introduction

A relatively wide stream of the financial economics literature focuses on performance measurement with two main motivations: the introduction of measures capturing stylized facts of financial returns such as asymmetry or non-Gaussian densities, among many others; the use of performance measures in empirical studies dealing with managed portfolio evaluation or asset allocation. The interest in this research field started with the seminal contribution of Sharpe in 1966. An increasing number of studies appeared in the following decades (see the survey by Caporin et al., 2014). One of the most studied topics is the performance evaluation of active management, which probably represents the main element promoting a renewed interest in performance measurement. Among the most recent contributions, we can quote Cherny and Madan (2009), Capocci (2009), Darolles et al. (2009), Jha et al. (2009), Jiang and Zhu (2009), Stavetski (2009), Zakamouline and Koekebakker (2009), Darolles and Gouriéroux (2010), Glawischnig and Sommersguter-Reichmannn (2010), Billio et al. (2012), Cremers et al. (2013), Kapsos et al. (2014), Weng (2014), Billio et al. (2014) and Billio et al. (2015).

Performance evaluation has relevant implications from both a theoretical point of view, as it allows us to understand agent choices, and an empirical one, since practitioners are interested in ranking assets or managed portfolios according to a specific non-subjective *criterion*. As an example, financial advisors often rank mutual funds according to a specific performance measure. Moreover, when rankings produced by advisors are recognized as a reference by the investors, changes in the rankings might influence inflows and outflows (see Hendricks et al., 1993; Blake and Morey, 2000; Powell et al., 2002; Del Guercio and Tkac, 2008; Jagannathan et al., 2010).

A large number of performance measures have already been proposed and, as a consequence, related ranks can sensibly vary across different financial advisors, portfolio managers and investment institutions. The identification of the most appropriate performance measure depends on several elements. Among them, we cite the investors preferences and the properties or features of the analyzed asset/portfolio returns. Furthermore, the choice of the optimal performance measure, across a number of alternatives, also depends on the purpose of the analysis which might be, for instance, one of the following: an investment decision, the evaluation of managers abilities, the identification of management strategies and of their impact, either in terms of deviations from the benchmark or in terms of risk/return. Despite some known limitations, the Sharpe (1966) ratio is still considered as the reference performance measure. However, if we derive this ratio within a Markowitz framework, it shares the same drawbacks as the Mean-Variance model, where the representative investor is characterized by a quadratic utility function and/or the portfolio returns are assumed to be Gaussian.[1] Clearly, it is well established that financial returns are not Gaussian, also due to investment strategies based on derivatives with time-varying exposures and leverage effects. Furthermore, numerous theoretical articles and empirical studies show that it is unlikely that investors do not care about higher-order moments (*e.g.* Scott and Horwath, 1980; Golec and Tamarkin, 1998; Harvey and Siddique, 2000; Jondeau and Rockinger, 2006; Jurczenko and Maillet, 2006, and references herein).

As a consequence, both an incorrect assumption of Gaussianity and an under-estimation of the importance of higher-order moments for the investors, may lead to an underestimation of the portfolio total risk (see Fung and Hsieh, 2001; Lo, 2001; Mitchell and Pulvino, 2001; Kat and Brooks, 2002; Agarwal and Naik, 2004) and, consecutively, to biased investment rankings and financial decisions (a downward biased risk evaluation induces an upward biased Sharpe ratio). In this case, the Sharpe ratio does not completely reflect the attitude towards risk for all categories of investors. Moreover, the standard deviation equally weights positive and negative excess returns. In addition, it has been shown that volatility can be subject to manipulations (see Ingersoll et al., 2007). Using this quantity to evaluate the risk of assets with low liquidity can also be another issue (see Getmansky et al., 2004). Ultimately, the pertinence of using this performance measure relies heavily on the accuracy and stability of the first and second moment estimations (Merton, 1981; Engle and Bollerslev, 1986). Finally, the so-called "$\mu - \sigma$" Paradox illustrates that the Sharpe ratio is not consistent, in the sense of the Second-order Stochastic Dominance

[1] The Gaussianity assumption might be relaxed when considering, for instance, a Student-t density characterized by a specific degree of freedom.

criterion[2] as introduced by Hadar and Russell (1969), when return distributions are not Gaussian (see Weston and Copeland, 1998; Hodges, 1998).

Based on these critics, academics and practitioners proposed alternative *criteria* that can be adapted to more complex settings than the Gaussian one. Among the various proposed measures to remedy these aforementioned shortcomings, the Gain-Loss Ratio was introduced by Bernardo and Ledoit (2000) as a special case of the Grinblatt and Titman (1989) Positive Period Weighted Measure of performance; it was then generalized later on (with a variable threshold instead of the nil reference - see below) first by Keating and Shadwick (2002) and, secondly, by Farinelli and Tibiletti (2008), Caporin et al. (2014) and Bellini and Bernardino (2015) using variants of expected shortfall and expectiles. These performance measures were originally derived to fill the gap between model-based pricing and no-arbitrage pricing, and thus belong to the so-called "Good-Deal-Bound" literature (see Cochrane and Saa-Requejo, 2001; Cochrane, 2009; Biagini and Pinar, 2013). This literature mainly derives price bounds by precluding Stochastic Discount Factors which would create too attractive assets, with respect to their "Good-Deal" measure in the terminology of Cochrane (2009), and thereby inclining the no-arbitrage price bounds. Cochrane (2009) derives Good-Deal price bounds for option prices by restricting the Sharpe ratio. Černỳ (2003) generalizes this approach and made it well-applicable for skewed assets, whilst Cherny and Madan (2009) study the good properties that should exhibit any good performance measure. In this context, it appears that the Gain-Loss Ratio approach provides several significant advantages. A bounded ratio implies the absence of arbitrage opportunities. Another advantage, which might be the main reason for its success, results from its asset pricing model foundation. The Gain-Loss Ratio quantifies the attractiveness of an investment concerning a benchmark investor, *e.g.* a representative investor of a specified asset pricing model. Alongside the determination of price bounds on incomplete

[2]The following simple illustration shows that the Sharpe ratio does not even comply with the First-order Stochastic Dominance *criterion*: if we compare an investment A with potential outcomes such as {-1%;1%;5%} with probabilities {.1;.45;.45}, to an investment B with potential outcomes of {-1%;1%;3%} with (the same) probabilities {.1;.45;.45}, then the Sharpe ratio of B is higher than those of A (*i.e.* 1.30 *versus* 1.16) whilst, obviously, A dominates B according to the First-order Stochastic Dominance *criterion* – see other examples in Hodges (1998), Weston and Copeland (1998); see also Homm and Pigorsch (2012a) and (2012b) for further discussion.

markets, the Gain-Loss Ratio can also be used to compare asset pricing models or to measure the performance of funds in an economically meaningful way, *i.e.* concerning a benchmark Stochastic Discount Factor. In addition to the stated theoretical foundation and the preclusion of arbitrage opportunities, the Gain-Loss Ratio has many other desirable properties. It is not only very intuitive, but it also fulfills many requirements of a good performance measure, *e.g.* all conditions of an "acceptability index" as put forward by Cherny and Madan (2009).[3]

Nevertheless, some authors recently report a first notable drawback: in many standard models, *e.g.* the Black-Scholes model, the best Gain-Loss Ratio becomes infinity. Because of this shortcoming, Biagini and Pinar (2013) conclude that the "Gain-Loss Ratio is a poor performance measure..." (*sic*). One might argue that the Gain-Loss Ratio is only constructed for discrete probability space corresponding to real asset returns, but even in this case, the strong dependence on the specific discretization can be seen as a severe drawback. Voelzke (2015) recently attempted to solve this issue by proposing the so-called Substantial Gain-Loss Ratio, that allows us to work in continuous probability spaces without losing the positive properties of the Gain-Loss Ratio.

In recent years, different authors, however, widely used the extended version of the Gain-Loss Ratio, called the Omega performance measure as introduced by Keating and Shadwick (2002), when ordering investment portfolios according to a rational *criterion*.

The first set of authors consider Omega in the evaluation of active management strategies in contrast to the well-known Sharpe ratio, supporting their choice by the non-Gaussianity of returns and by the inappropriateness of volatility as a risk measure when strategies are non-linear (*e.g.* Eling and Schuhmacher, 2007; Annaert et al., 2009; Hamidi et al., 2009; Bertrand and Prigent, 2011; Ornelas et al., 2012; Zieling et al., 2014; Hamidi et al., 2014).

A second group of articles focuses on Omega as an objective function for portfolio optimization in order to introduce downside risk in the estimation of optimal portfolio

[3]Note, however, that Cherny and Madan (2009) use a non-standard definition of the Gain/loss ratio, with a difference in the numerator: they define the ratio as the expected return over the expected loss (for positive expected returns), which leads to the consistency of this ratio with the SSD criterion.

weights (*e.g.* Mausser et al., 2006; Farinelli et al., 2008, 2009; Kane et al., 2009; Gilli and Schumann, 2010; Hentati et al., 2010; Gilli et al., 2011). The maximization program in an Omega paradigm is indeed, *per* nature, a non-convex non-linear one. As formulated in Mausser et al. (2006), the investor's problem in finding the portfolio that optimizes Omega can be written, under some mild assumptions, as a linear one, after some variable changes, inspired by the works of Charnes and Cooper (1962) in the field of production optimization.[4]

Nevertheless, our main point in this article is that despite the fact that several authors in the literature claimed that the Gain-Loss Ratio (and its generalization: the Omega measure) "is consistent with second-order stochastic dominance *criterion*" (*sic*; *e.g.* Darsinos and Satchell, 2004, p.80; Biagini and Pinar, 2013, p. 229), it happens to be only valid for special cases; more precisely, this general statement is, unfortunately, untrue for the original general definitions of both the measures and *criterion* (see also Voelzke, 2015, for a generalized kernel pricing result).

Similar to Weston and Copeland (1998) and Hodges (1998), who illustrate the inconsistency of the Sharpe ratio when asset return densities do not belong to the same location-scale family (see Meyer, 1987, and Eling and Schuhmacher, 2011), we show in the present article that the Omega measure has some severe drawbacks. Of course, consistence with the stochastic dominance *criteria*, which is entailed by traditional rational agent behavior, is not a shared characteristic of all portfolio choice theories (for instance, both Rank Dependent Expected Utility and the Cumulative Prospect Theories[5] are not compliant), but this can be justified when optimal choices can be explained (based on a subjective transformation of objective probabilities and/or values for instance), and when they better reflect preferences of investors in empirical studies. This property of consistence with the stochastic dominance should, thus, not be overrated. However, in the case of Gain-Loss and Omega ratios, it might conduct investors to misleading and irrational rankings of risky assets, even, surprisingly, in an extremely simplified framework such as

[4]See Appendix A for details on this optimization approach.

[5]See Kahneman and Tversky (1979); Quiggin (1982); Tversky and Kahneman (1992); and also Zakamouline (2014), in the context of a piece-wise utility function with an extension of the Omega ratio proposed by Farinelli and Tibiletti (2008).

the Gaussian one. The main ideas that we present hereafter are: 1) the Omega measure is biased in some cases due to the importance of the mean return in its computation, and, 2) since gains can be compensated by losses in the computation of the Omega measure, two densities with (almost) the same means can share identical Omega measures, even when the risk related to one asset is obviously higher than the other one. In other words, the Omega ratio may, in some cases, overvalue performance and simply disregard risk, which is both counter-intuitive and far more severe than just a theoretical inconsistence with stochastic dominance *criteria*.

As shown below, these problems mainly come from the fact that Omega indeed does not comply, in general, with the Second-order Stochastic Dominance *criterion* (SSD, in short), as it is falsely stated in the literature. In the limit case, this implies indifference across assets with the same expected return but with different risk levels. In other words, we show that 1) on a theoretical basis, the Omega ratio is not compliant with the Second-order Stochastic Dominance *criterion*; 2) on an empirical basis, using Omega ratios can lead to obvious misleading financial choices and that 3) Omega and Sharpe approaches, whilst built on different concepts and measures, give very similar results when applied to real market data (long-only and hedge funds), which is in line with the empirical results by Eling and Schuhmacher (2007), showing very little differences between rankings based on various performance measures.

Finally, if the Omega measure 1) is not compliant with the main paradigm, 2) may sometimes lead to obvious irrational choices, 3) generally reaches the same conclusions than those based on the classical Sharpe ratio, we conclude that its use should always be cautious.

The article proceeds as follows. Section 2 introduces the Omega function in the general context of performance measurement and presents some of its properties. Section 3 demonstrates that the Omega measure is inconsistent with the Second-order Stochastic Dominance *criterion*. Then, we show that the trade-off of the Omega ratio – between return and risk – is mostly guided by the first moment of portfolio returns. In Section 4, we provide numerous illustrations, based on realistic simulations and several asset classes, showing incoherencies in fund analyses when considering various (static and dynamic) optimization settings. Section 5 presents some robustness checks based on different long-only asset and hedge fund databases. The last section concludes.[6]

2. Return Density-based Performance and the *Omega* Measures

The *Omega* measure belongs to a general class of performance measures based on features of the analyzed return density. Following the taxonomy proposed by Caporin et al. (2014), these measures can be represented with the following general notations:

$$PM_p = \mathcal{P}_p^+ (r) \times \left[\mathcal{P}_p^- (r)\right]^{-1}, \tag{1}$$

where PM_p is a performance measure of portfolio p, $\mathcal{P}_p^+ (.)$ and $\mathcal{P}_p^- (.)$ are two functions associated with the right and left parts of the support of the density of returns.

In most cases, measures belonging to this class can be re-defined as ratios of two probability-weighted Power Expected Shortfalls (or Generalized Higher/Lower Partial Moments - see Caporin et al., 2014), which read:

$$\begin{aligned} PM_p = & \left[E_p \left(|\tau_1 - r|^{o_1} \left| r > \tau_3 \right.\right)\right]^{(k_1)^{-1}} \\ & \times \left\{\left[E_p \left(|\tau_2 - r|^{o_2} \left| r \leq \tau_4 \right.\right)\right]^{(k_2)^{-1}}\right\}^{-1} \times \psi_p \left[f_p (r)\right], \end{aligned} \tag{2}$$

where τ_1 is a threshold (a reserve return $\underline{r}$, a *Minimum* Acceptable Return – MAR, the null return, the risk-free rate r_f...) for computing gains; τ_2 is a similar threshold for calculating losses or risk; τ_3 and τ_4 are thresholds allowing the performance measure to focus more or less on the upper/lower part of the support density; constants o_1 and o_2 are intensification parameters monitoring the attitude of the investor toward gains and losses, while k_1 and k_2 are positive normalizing constants; finally $\psi_p \left[f_p (r)\right]$ is a factor depending on the density of the portfolio returns, $f_p (r)$, where r denotes the portfolio returns.

Note that some thresholds might be defined based on quantiles (*i.e.* Value-at-Risk,

[6] A general Appendix and a Web Appendix (available on demand) also include complementary results, details on simulation schemes used in the *corpus* of the text, proofs and some extra robustness results.

denoted VaR), such as $\tau_3 = VaR_{-r_p,a_1}$ and $\tau_4 = VaR_{r_p,a_2}$ with a_1 and a_2 being confidence levels.

Equation (2) thus highlights that measures belonging to this class capture features of the return density that are going beyond the first two moments.[7] The Keating and Shadwick (2002) *Omega* measure, following Bernardo and Ledoit (2000), corresponds to the case where $\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau$ and $o_1 = o_2 = k_1 = k_2 = 1$. We have thus a unique threshold and all intensification and normalizing constants are fixed at 1. *Omega* indeed writes:[8]

$$\begin{aligned}\Omega_p(\tau) = & \left(GHPM_{r_p,\tau,\tau,1}\right) \times \left(GLPM_{r_p,\tau,\tau,1}\right)^{-1} \\ = & E_p\left[r-\tau\,|r>\tau\right] \times \left\{E_p\left[\tau-r\,|r\leq\tau\right]\right\}^{-1} \times \left[\frac{1-F_p(\tau)}{F_p(\tau)}\right],\end{aligned} \tag{3}$$

where $GHPM_{r_p,\tau,\tau,1}$, $GLPM_{r_p,\tau,\tau,1}$ and $E\left[.\,|.\right]$ are, respectively, the Higher/Lower Partial Moments and the conditional expectation operator, and $F_p(.)$ is the Cumulative Density Function of the return r.

We see here that the *Omega* ratio separately considers, in a simple and intuitive way, favorable and unfavorable potential excess returns with respect to a threshold that has to be given (arbitrary). More precisely, as noted by Kazemi et al. (2004) and Bertrand and Prigent (2011), the *Omega* measure is equal to the ratio of the expectations (under the historical probability $\mathbb{P}$) of a call option to a put option written on the risky reference asset with a strike price corresponding to the threshold. The main advantage of the *Omega* measure is that it incorporates, in some ways, some features of the return distribution, such as moments, including skewness and *kurtosis*. A ranking is theoretically always possible, whatever the threshold, in contrast to the Sharpe ratio where the threshold is fixed and equal to the riskless return. Furthermore, it displays some properties such as (see Kazemi et al., 2004; Bertrand and Prigent, 2011):

- for any portfolio p (with a symmetric return distribution), $\Omega_p(\tau) = 1$ when $\tau =$

[7] See Caporin et al. (2014) for further discussions on this class of measures and for a list of derived measures.

[8] Several alternative expressions of the *Omega* measure exist in the literature (*Cf.* Appendix C).

$E_p\,[r]$;

- for any portfolio p (and for all distributions), $\Omega_p\,(.)$ is a monotonous decreasing function in $\tau \in \mathbb{R}$;

- for any couple of portfolios $p = \{A, B\}$, $\Omega_A\,(.) = \Omega_B\,(.)$ for all $\tau \in \mathbb{R}$, if and only if $F_A\,(\cdot) = F_B\,(\cdot)$, where functions $F_p\,(.)$ is the Cumulative Density Function of the returns on a portfolio p;

- for any portfolio p and if there exists one risk-free asset p_0 with return r_{p_0}, then $\Omega_p\,(.) < \Omega_{p_0}\,(.)$ for all $\tau \leq r_{p_0}$, with $\Omega_{p_0}\,(.) = +\infty$.

In this setting, the threshold τ must be exogenously specified as it may vary according to investment objectives and individual preferences. As mentioned by Unser (2000), we are often only interested in the evaluation of "risky" outcomes that reflect the attitude towards downside risk. Usually, their values are smaller than a given target, which is, for example, the riskless asset or the rate of a financial index (benchmark). Downside risk measures have been examined, for instance, in Ebert (2005), and are linked to the measures introduced by Fishburn (1977 and 1984). For the background literature on risk measures and their applications to finance and insurance, we refer to Kaas et al. (2004), Denuit et al. (2006) and Prigent (2007).

The first purpose of this article is to highlight a severe drawback that characterizes the *Omega* measure. We refer, here, to the possibility that the use of *Omega* leads investors to create risky asset rankings which are misleading and not compatible with a rational behavior. Such an event might be realized in simplified frameworks, such as the Gaussian one, but also under less stringent hypotheses on the features of risky asset return densities. In order to introduce the *Omega* "curse", we will use hereafter the definition of consistency in the sense of the classical Stochastic Dominance[9] that says that a risk

[9] *Cf.* Quirk and Saposnik (1962), Hadar and Russell (1969), Hadar and Russell (1971), Hanoch and Levy (1969), Rothschild and Stiglitz (1970) for the traditional Stochastic Dominance *criterion*. See also De Giorgi (2005), Levy (1992) and Levy (1998) for surveys on Stochastic Dominance (*Cf.* Levy (1998) for definitions). See also Leshno and Levy (2002) and Tzeng et al. (2013) for other types of stochastic dominance such as the Almost Stochastic Dominance.

measure denoted ρ, is consistent with the Stochastic Dominance *criterion*[10] if and only if $A \succ_{SD} B$ implies $A \prec_{\rho} B$ and $A =_{SD} B$ implies $A =_{\rho} B$.

Let us come back now to performance measurement (where the higher the measure, the better the investment). Considering *Omega* as an investment decision *criterion*, we will demonstrate below that this performance measure is inconsistent with the Second-order Stochastic Dominance *criterion*, so that:[11] on the one hand, we have $A \preceq_{\Omega} B$ (*i.e.* $\Omega_A(\tau) \leq \Omega_B(\tau)$ for some thresholds τ) and, on the other hand, we have $A \succ_{SSD} B$. More precisely, to illustrate this apparent opposition as a counter-example of the consistency of the Omega *criterion*, we will first show in the following section, posing $E_A[r] = E_B[r]$ for the sake of simplicity, that we can have both $\Omega_A(\tau) = \Omega_B(\tau)$ for some thresholds τ, and $A \succ_{SSD} B$, implying here as the SSD requires that fund B be objectively riskier than fund A.

3. Inconsistency of the *Omega* when Ranking Funds

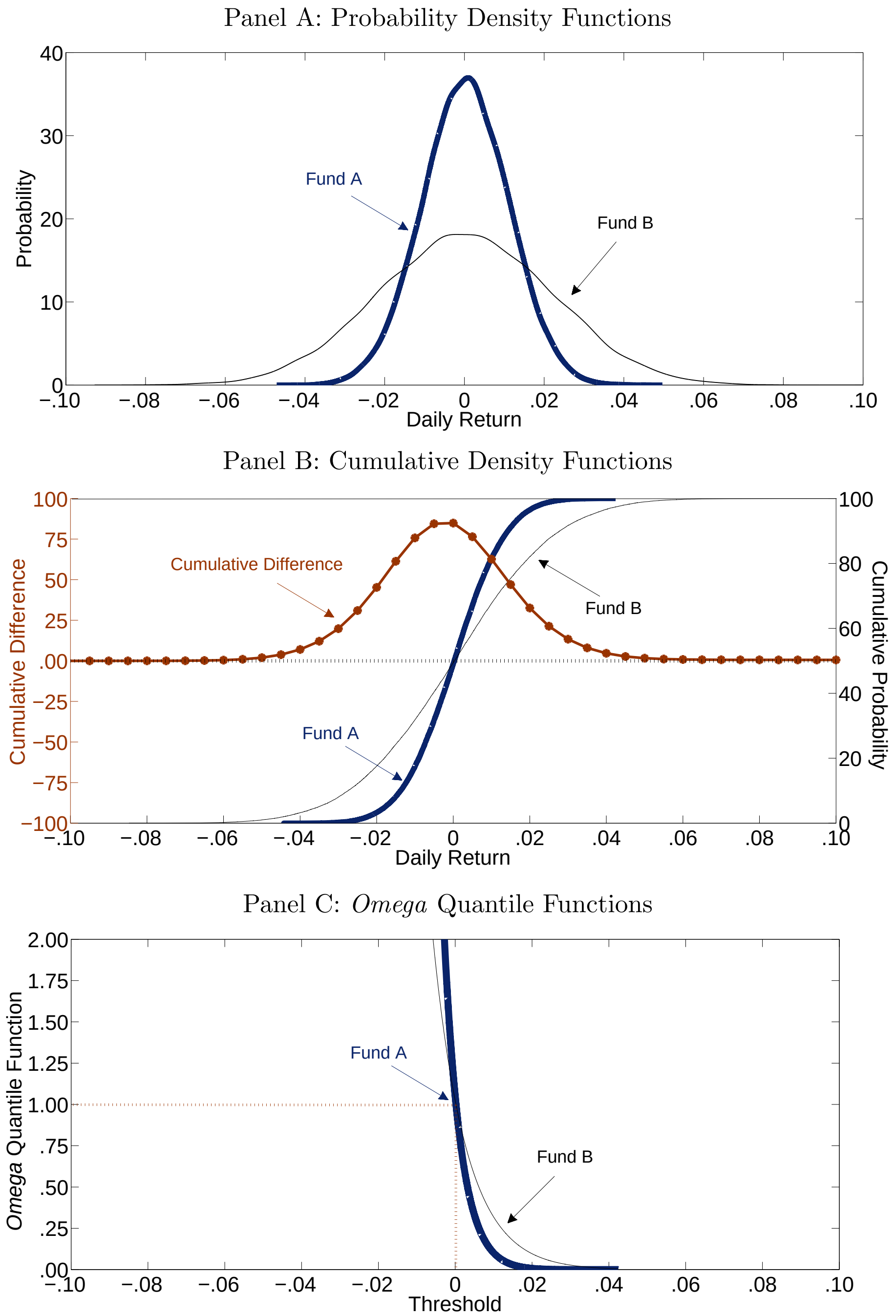
Let us first grasp the intuition and consider the daily returns of fund A as illustrated in Figure 1, where (Panel A) represents the Probability Density Functions (PDF in short) of returns on fund A and fund B in the Gaussian paradigm (for the sake of simplicity in this example). Both funds have exactly the same average daily return, but the return distribution of fund B has twice the volatility of that of fund A. Figure 1 (Panel B) also presents the Cumulative Distribution Function (CDF), the Cumulative Difference in CDF, as well as the explicit function of the *Omega* quantile function[12] (Panel C) for both funds.

[10] In a general context, we use $A \succ_X B$ (respectively, $\approx, \succeq, \prec, \preceq$) when A is strictly preferred to B (respectively, A is equivalent to B, A is preferred or equivalent to B, B is strictly preferred to A, B is preferred or equivalent to A) according to a *criterion* X. *Cf.* Levy (1998) for more details.

[11] See Footnote 10 for the notations.

[12] The *Omega* Quantile Function highlights the relation existing between the *Omega* measure and the density assumed for the returns. Appendix C describes how the *Omega* measure can be expressed as a function of the return density, thus recovering the so-called *Omega* Quantile Function. The *formula* is provided for a generic density. Specific cases can be obtained by integrating the results reported in Appendix C with those of Malo (2005).

Figure 1: Gaussian Density Functions, Stochastic Dominance *Criteria* and *Omega* for Funds A and B (when $\tau = E_A(r) = E_B(r)$)



The first Figure (Panel A) shows the Probability Density Functions of fund A (bold line) and fund B (thin line) daily returns. Both fund returns follow Gaussian densities, such as the first two moments of the fund A distribution being calibrated on those of the daily Dow Jones Index over the period 1$^{\text{st}}$ January, 1900 to the 1$^{\text{st}}$ March, 2013 (data source: *Datastream*). The second Figure (Panel B) pictures the Cumulative Density Functions of fund A (bold line) and fund B (thin line) returns on the (right) y-axis and the cumulative difference (line with markers) in CDF for the two funds on the (left) y-axis. The third Figure (Panel C) represents the *Omega* quantile function corresponding to several thresholds.

If we now study the Stochastic Dominance *criterion*, Figure 1 (Panel B) shows that fund A does not dominate fund B according to the FSD *criterion* – since the two CDFs cross – but fund B is dominated by fund A according to the SSD *criterion*, because the cumulative difference in CDF does not change sign (*Cf.* Appendix B).

In this first simple illustration, we clearly see that fund A should be preferred by any rational investor with a concave utility function, since the choice of fund B goes with a higher risk for the same expected return.[13] However, we note that the two assets have both an *Omega* equal to 1.00 when the threshold is zero (Panel C). Furthermore, the *Omega* quantile functions reported in Panel C show that, depending on the threshold, fund A is, or is not, preferred to fund B.[14]

We have to admit that these facts are rather counter-intuitive and that *Omega* deserves some *extra* cautionary interest.

Figure 1 illustrates that the *Omega* measure is inconsistent with the SSD *criterion* in a simplified Gaussian framework. However, we can also provide the same qualitative conclusion within a more complex setting with asymmetric and leptokurtic (log-normal) densities. For instance, if we consider fund A and fund B with equal mean returns and a threshold below the means, the *Omega* measure will still be inconsistent with the SSD *criterion*.[15] Indeed, if we compute the *Omega* ratio for two series with some peculiar features, we can show that the asset ordering is inconsistent with the SSD *criterion* as presented in the following proposition:[16]

[13]Complementary illustrations are available on demand with different hypotheses on densities and related moments (*Cf.* Web Appendix C).

[14]Even if it is difficult to establish results in a general multivariate framework (multiple comparisons between funds/assets), we report some complementary results corresponding to various ranking orders depending on moments and threshold in Web Appendix E.

[15]Several complementary tests using various densities lead to the same qualitative results. More interestingly, if we hypothesize that the mean return of fund A is superior to that of fund B by 1.00%, which is higher than the threshold, *Omegas* of the two funds will be equal, despite a trade-off of 1 against 10 in terms of expected return *versus* volatility. Complementary results are available upon request (*Cf.* Web Appendix C).

[16]These statements are in coherence with Darsinos and Satchell (2004) and Kazemi et al. (2004), since we can show that, under some crucial conditions on the threshold, the traditional second-order stochastic dominance, such that $A \succeq_{SSD} B$, implies a Omega (and Sharpe-Omega) dominance so that $A \succeq_{\Omega} B$ (and $A \succeq_{S-\Omega} B$), see Web Appendix B, C and F for some comments.

Proposition 1. *Inconsistency of the* Omega *Ranking*

Consider returns on two assets A and B, with non-degenerated densities, with asset A being preferred to asset B according to the Second-order Stochastic Dominance criterion *(denoted $A \succ_{SSD} B$). We define the following quantity:*

$$\Delta(\tau) = \frac{\int_{-\infty}^{\tau} [F_B(r) - F_A(r)]\ dr}{\int_{-\infty}^{+\infty} [F_B(r) - F_A(r)]\ dr}$$

A necessary condition for having Omega *inconsistent with the Second-order Stochastic Dominance* criterion *(SSD) is that $\Delta(\tau) > 1$.*

Under this condition, Omega *is inconsistent with the Second-order Stochastic Dominance if:*

$$\Omega_B(\tau) < \frac{\Delta(\tau) - 1}{\Delta(\tau)}.$$

Proof: see Appendix D.

Through Proposition 1, we show that the *Omega* measure is inconsistent with the SSD *criterion* if $\Delta(\tau) > 1$. Despite the Proposition provides a condition based only on the *Omega* of asset B, the information on asset A density is constrained by the existence of Stochastic Dominance and is included in the quantity $\Delta(\tau)$. In addition, we here observe that the choice of funds according to *Omega* is directly dependent on the threshold, which leads us to present the following corollary.

Corollary 1. *Inconsistency of the* Omega *Ranking under Symmetry of Densities and Equality of Means*

If the two assets of Proposition 1 have returns with a density of the same family, with identical means $E_A(r) = E_B(r) = \mu$, but different volatilities such as $\sigma_A(r) < \sigma_B(r)$, then, the Second-order Stochastic Dominance criterion *implies that the ranking in terms of* Omega *of the two assets will be:*

1. *if $\tau = \mu$, $A \approx_{\Omega} B$;*

2. *if* $\tau > \mu$, $A \prec_{\Omega} B$,

3. *if* $\tau < \mu$, $A \succ_{\Omega} B$;

and thus Omega *is inconsistent with the Second-order Stochastic Dominance* criterion.[17]
Proof: see Appendix D.

Note that the Corollary is coherent with the Proposition as under the assumptions of the Corollary, $\Delta(\tau)$ diverges to plus infinity and the condition of Proposition 1 becomes $\Omega_B(\tau) < 1$. The latter holds when $\tau < \mu$.

We indeed show that the *Omega* function is not consistent with the SSD *criterion* in general. In fact, we can derive conditions (sufficient and, in some specific cases, also necessary) for having asset rankings inconsistent with the SSD *criterion*. In the peculiar case of two funds with equal means and symmetric distributions, we can obtain any ranking in terms of *Omega* depending on the threshold (below, equal or higher than the mean return), as briefly mentioned by Bertrand and Prigent (2011).[18]

There is a very counter-intuitive result, since we notice here that the center of the distribution, where returns are close to 0 (*i.e.* when nothing happens in the market), has a significant impact on the *Omega* measure. If we compare two *Omega* ratios for the same portfolio, and move the threshold of the second *Omega* slightly to the right (respectively, to the left) of the mean return of the underlying distribution, this will transform relative gains (respectively, losses) into relative losses (respectively, gains), and thus inverse the portfolio ordering. This drawback may have severe consequences in terms of investment choices when comparing several portfolios.

As a result of Corollary 1, if one investor makes decisions based on *Omega*, he will prefer the safest asset for thresholds below the mean, and the riskiest asset for thresholds above the mean. This *phenomenon* is associated with choices that are driven by the

[17] However, having in mind that τ is a "*Minimum* Acceptable Return" (MAR in short), cases 1 and 2 may appear unrealistic when comparing two portfolios. Probably only funds p with an $\Omega_p(\tau)$ that is much greater than 1 (first part of a ranking) and a μ_p that is much greater than τ would be considered (focusing on large losses). Nevertheless, these cases are frequent when all assets are considered in an optimization program (see the next section).

[18] *Cf.* Bertrand and Prigent (2011), Proposition 1 in section 2.3 on page 1,815.

probability of obtaining positive outcomes with high thresholds. In fact, an agent prefers the asset that has the highest probability of being above the threshold, without giving much importance to the risk associated with his choice. On the contrary, if the threshold is below the mean, the choice made by the agent will focus more on risk.

We point out that our result on the inconsistency of *Omega* with the SSD *criterion* refines the findings of Darsinos and Satchell (2004) and Kazemi et al. (2004).[19]

We can thus derive here some preliminary conclusions. First, the *Omega* ratio is inconsistent in the sense of the SSD. Secondly, the threshold should be linked to the agent's preferences since it determines the ranking of the funds and reflects her investment choice. In other words, the *Omega* is not a "universal" performance measure. More specifically, when studying the case of two symmetric densities (simple illustration of Gaussian laws) such as $E[r_A] = E[r_B] = \mu$, we may face an irrational ordering when the chosen threshold is high. We can also show that even in a more complex setting based on two asymmetric and leptokurtic (log-normal) densities (with some uncertainty), the results remain similar.[20]

This latter fact leads us to a more general study into the trade-off between expected return and risk when *Omega* is driving allocation and investment choices. Even if the *Omega* ratio was truly compatible with the SSD *criterion* in most cases (but not all as shown earlier), what really matters is the price of exchange between performance and risk that the *Omega* function values.

Based on the simulation scheme used in Ingersoll et al. (2007), Figure 2 below represents the *Iso-Omega* curves for various thresholds[21] τ, set to .00%, 5.00%, 7.00% and 10.00% (see Keating and Shadwick, 2002; Kane et al., 2009; Hentati et al., 2010; Bertrand and Prigent, 2011; Gilli et al., 2011).

For a given threshold, an *Iso-Omega* curve corresponds to identical *Omega* levels

[19] See Web Appendix B in which we prove that our results are valid within their framework.

[20] Another counter-intuitive example with a log-normal (asymmetric) density is available upon request (*Cf.* Web Appendix C).

[21] See Web Appendix A for a full sketch of the algorithm. For the sake of realism, we herein restrict our next analysis to the various values of thresholds we found in the literature. However, results with other values (very low or very high) are in line with those presented below.

for various portfolios characterized by different mean returns and volatilities.[22] For the sake of simplicity, let us compare a fund A with a fund B characterized by the same volatilities. We can then distinguish three main cases. In the first one, if the funds A and B are located on the same *Iso-Omega* curve, this will imply that they have identical *Omega*-rankings. In the second case, if fund A is above fund B (*ceteris paribus*), this means that *Omega* will prefer fund A. Finally, in the third case, we assume that fund A is below fund B (*ceteris paribus*), then the *Omega* ratio will prefer fund B.

In Figure 2, we note that when the threshold is equal to zero, the trade-off is very close to 1, as for the Sharpe ratio, since we require 100 basis points of *extra* over-performance for the same amount of over-volatility to reverse the fund rankings obtained with the *Omega* measure (slope of the *iso-Omega* curve being equal to 1 for the threshold on this sample). For a threshold equal to 10.00%, we only require 100 basis points of *extra* over-performance for 400 basis points of over-volatility. This last case can be associated with the behavior of a greedy agent. Finally, we see here that the lower the threshold, the closer the rankings between the Sharpe ratio and the *Omega criterion*.

However, we have seen previously that the threshold should not be too high; otherwise the conclusion drawn from the *Omega* measure ranking might be just... wrong.

Let us investigate more precisely the risk of being irrational when choosing to adopt the *Omega* measure in comparing two funds. In the following, the first one, named again fund A (run by an informed portfolio manager), has a higher mean return than the second one (an uninformed manager), called fund B, but with a higher volatility (due to the noisy signal received by the informed manager). Table 1 displays the frequency at which the Sharpe ratio is higher for fund A than for fund B (group of S) and the frequency at which the *Omega* measure concludes the opposite (third to fifth groups of columns), when setting the thresholds equal to 10.00%, 5.00% and .00%. Lines in the table distinguish several cases corresponding to an increase in the volatility of portfolio returns, obtained

[22]The trade-off between risk and return observed in Figure 2, when drawing the *Iso-Omega* curves, is based on the simulation scheme used in Ingersoll et al. (2007). Please see the source of Figure 2 and Web Appendix A for more details on the sketch of the algorithm. Other complementary tests on various databases show that these trade-offs may vary according to the statistical properties of the underlying return series.

Figure 2: *Iso-Omega* Curves displaying the Quantity of Over-volatility required for a given Over-performance for reversing the *Omega* Ranking

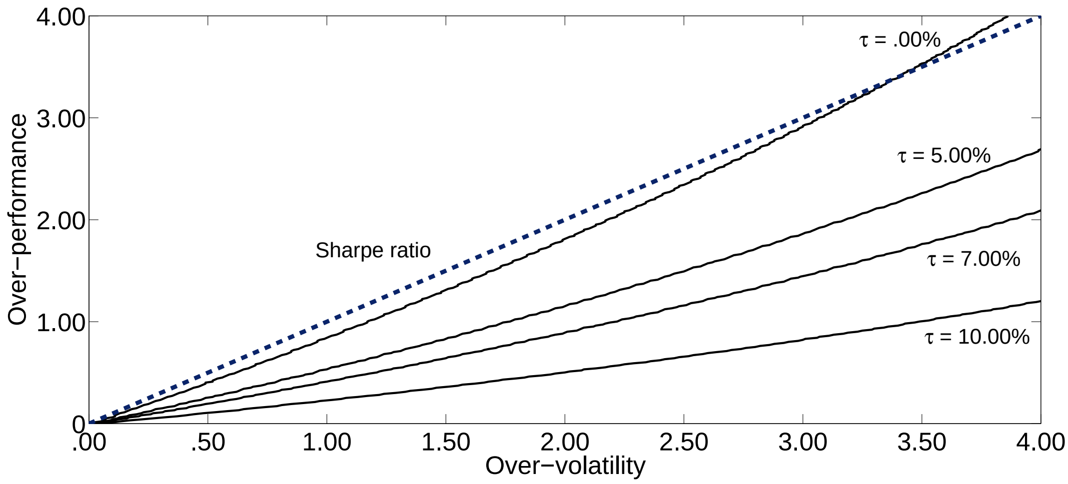


Illustration of *Iso-Omega* curves, *i.e.* over-performances (y-axis) *versus* over-volatilities (x-axis) yielding the same ranking according to the *Omega*; both are expressed in percentage terms. The ranking curves (solid lines) are computed for the *Omega* measure (*Cf.* Keating and Shadwick, 2002) when the threshold is equal to .00%, 5.00%, 7.00% and 10.00%. Each *Iso*-curve represents the amount of over-performance of a portfolio compared to another one, for a given over-volatility, required to reverse the *Omega* ranking between these two funds. The dashed bold line is the *Iso*-Sharpe (1966) ratio curve assuming that a unit of *extra* over-volatility – for a unit of a given over-performance – is required to inverse the ranking between fund A and fund B.

by multiplying the probability density functions by a distortion coefficient m varying from 1 to 2 with successive steps equal to .20 (third to eighth rows), without altering the associated mean returns.[23]

In Table 1, we observe in the second column that the Sharpe ratio naturally decreases when the studied portfolio total risk increases, for a same amount of mean return. If we now focus on the results given by the *Omega* measure, we can see that the value of the threshold - equal to 10.00%, 5.00% and .00% (third to fifth groups of columns), has a real impact on the final ranking. Obviously, the intuition tells us that the Sharpe and the *Omega* ratios are equal most of the time.

[23]Still following the simulation scheme proposed in Ingersoll et al. (2007), our results are based on a random selection of 10,000 pairs of simulated portfolios (one uninformed portfolio return series against one informed manager series for all comparisons).

Table 1: Contradiction in Choices of Investments using various *Criteria*

	$S_B \leq S_A$			Cases when $A \succ_E B$, $A \succ_{Sharpe} B$ and $A \prec_\Omega B$								
				$\Omega_p\,(\tau = .10)$			$\Omega_p\,(\tau = .05)$			$\Omega_p\,(\tau = .00)$		
Dist. Coef. \ Risk-free Rate	10.00%	5.00%	.00%	10.00%	5.00%	.00%	10.00%	5.00%	.00%	10.00%	5.00%	.00%
$m = 1.00$	48.27%	47.47%	46.58%	.11%	.19%	.24%	1.90%	.21%	.13%	3.84%	2.09%	.30%
$m = 1.20$	44.99%	42.29%	39.54%	.19%	.09%	.12%	6.10%	.31%	.04%	12.18%	6.48%	.38%
$m = 1.40$	42.14%	37.92%	33.79%	.19%	.02%	.03%	10.01%	.31%	.04%	19.86%	10.94%	.51%
$m = 1.60$	39.43%	33.88%	28.54%	.23%	.00%	.00%	14.03%	.39%	.00%	27.68%	15.85%	.61%
$m = 1.80$	37.11%	30.62%	24.50%	.19%	.00%	.00%	17.48%	.33%	.00%	34.00%	19.99%	.56%
$m = 2.00$	35.21%	27.97%	21.35%	.17%	.00%	.00%	20.56%	.32%	.00%	39.28%	23.58%	.55%

This table displays the frequency to which the Sharpe ratio is higher for fund A than for fund B (group of S) and the frequency to which the *Omega* measure concludes the opposite (third to fifth groups of columns) according to several thresholds: 10.00%, 5.00% and .00%. Following the simulation scheme developed in Ingersoll et al. (2007), portfolio returns are defined as: $\tilde{r}_p = \exp\left\{\left[\mu_m + \alpha_p - .5\left(\sigma_m^2 + \upsilon_p^2\right)\right]\Delta t + \left(\sigma_m\tilde{\varepsilon} + \upsilon_p\tilde{\eta}\right)\sqrt{\Delta t}\right\} - 1$, where μ_m is the market portfolio return, α_p is the *extra*-performance generated by the manager, σ_m is the market portfolio total risk, υ_p corresponds to the residual portfolio specific risk, Δt is the data frequency, $\tilde{\varepsilon}$ and $\tilde{\eta}$ are Gaussian random variables. We use here four different profiles of investors. We characterize the first one by setting $\alpha_p = .00\%$ and $\upsilon_p = .20\%$. The three other managers are defined as $\alpha_p = 1.00\%$ and the residual portfolio specific risks are respectively equal to $\upsilon_p = .20\%$, $\upsilon_p = 2.00\%$ and $\upsilon_p = 20.00\%$. Then, we randomly choose two portfolios among all these profiles and order them according to their mean returns. Finally, we compute the associated Sharpe ratios and *Omega* measures for each threshold and determine how often these measures conclude as the ordering given by their mean returns. A supplementary check confirms that the risk *premia* for all portfolios are positive, which proves consistency between rankings with the Sharpe ratios. Our simulations are based on the comparison of 10,000 pairs of portfolios, with a 5-year return history, for each distortion coefficient m varying from 1 to 2 with a step equal to .20. The market hypotheses are: risk free rate 5.00% *per* year, market *premium* 12.00%, market standard deviation 20.00%.

One of the main reasons that can explain the rate of false answers is the value of the threshold. But we also note some significant differences between the *Omega* and the Sharpe ratios (even when the threshold is low), mainly because *Omega* is focusing too much on the difference of expected returns, disregarding in some sense the implied risks. Another potential explanation is in the relative value of the mean returns of the compared funds (above and/or below the threshold); in some limited cases, the *Omega* ratio will be biased, just as the Sharpe ratio, when considering portfolios with negative mean performances (see Israëlsen, 2005).

In a simple setting (a Gaussian case for a non-multiple comparison of two funds), we can thus say that the choice of the level of the threshold is of interest: if too "high", some aberrant results may appear (see Proposition 1) and if too "low" (compared to mean returns of investments considered), the risk of error is accentuated (see results in Table 1). A second rule of thumb is thus to choose a not too high or too low threshold.

We have seen previously that we may face, under some circumstances, misleading financial advice when using *Omega* to rank investments when the value of the threshold is too high or too low. But what is the typical behavior of portfolios optimized according to this *criterion* when the threshold is in a reasonable band? The answer to this question is the objective of the next section.

4. *Omega* as an Optimization *Criterion*?

We studied the inconsistency of the *Omega* ratio (*Cf.* Proposition 1) through simple illustrations based on realistic simulations comparing two investment portfolios. We have shown that using *Omega* as a ranking *criterion* may be misleading in some cases since it is inconsistent with the SSD *criterion*. We would like now to study the behavior of *Omega* in a static and dynamic optimization framework.

We also ran some robustness checks, comparing the performance of portfolios optimized according to the *Omega criterion* and other classical paradigms, in both static and dynamic ways. For the sake of robustness, we have dealt with some empirical illustrations of properties based on, first, realistic simulations and, secondly, on three different market databases used in the literature on portfolio optimization (namely Hentati et al., 2010; Darolles et al., 2009; DeMiguel et al., 2009).

Since the *Omega* ratio can be inconsistent with the SSD *criterion* in some cases, we show as a natural consequence, through Figure 3, that using this measure as an optimization objective function can also lead to some mismatches, even in the simple Gaussian framework.[24] More precisely, we illustrate a typical case that can happen in a very common situation when we optimize a portfolio according to *Omega*. First, we simulate two Net Asset Value series (once again using the traditional Gaussian *hypothesis* on returns), corresponding to distinct risky assets, respectively characterized by annual mean returns (volatilities) equal to 9.96% (6.44%) for fund A and 10.44% (12.29%) for fund B. We consider the return of the riskless asset at 2.00% *per* year without loss of generality. The threshold used for computing the *Omega* portfolio is set to 10.00%. In a static way, we then optimize portfolios according to the *maximum* Sharpe ratio, the *maximum Omega* and the *minimum* volatility *criteria*.

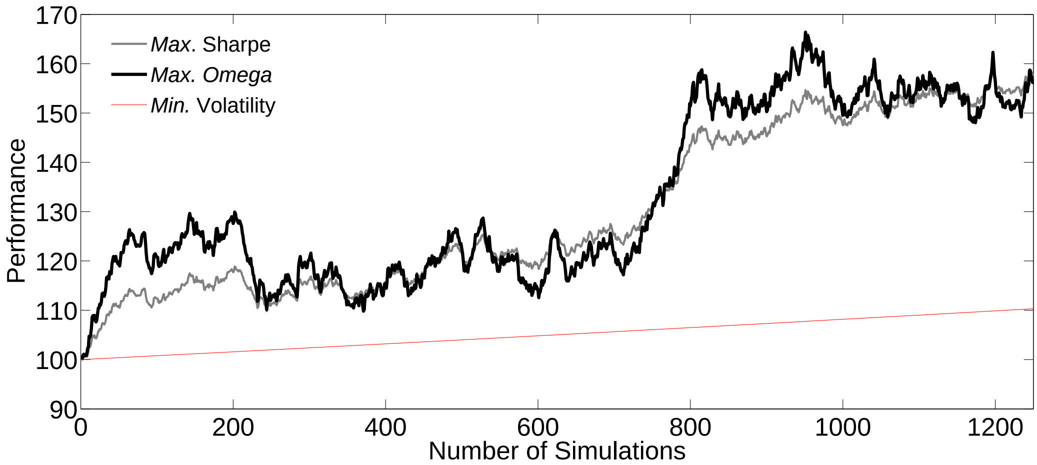
We clearly see in Figure 3 that the *Omega*-based optimal portfolio is misleading compared to the other ones (namely, *maximum* Sharpe and *minimum* volatility). Indeed, it is completely invested in Fund B that over-performs Fund A by only .48% but with a much higher volatility (12.29% *versus* 6.44%).[25] Then, we can say that the *Omega* measure may not, always, allow us to rationally choose the best investment.

Nevertheless, real financial series are known to be non-Gaussian. We thus want to highlight some peculiarities of *Omega*-based strategies on a real dataset. For the sake of comparison, we refer in the following illustrations to the dataset used in Hentati et al. (2010) and based on the same methodology. This database is composed of five indexes, expressed in USD with a monthly frequency, representing the most common asset classes often used in asset allocation optimization problems, namely: the HFRX Global Index for hedge funds, the UBS GC Index for convertibles, the JPM GBI Index for bonds, the

[24]Figure 3 is just one mere illustration among thousands of tests, that exhibits the *phenomenon*. Our results are based on the simulation of 10,000 pairs of Gaussian distributions. We also obtain the same conclusion when using a log-normal law for simulating the NAV of fund A and fund B (results are available upon request).

[25]Complementary results are available upon request (*Cf.* Web Appendix C).

Figure 3: An Illustration of a Misleading Choice of *Omega* (Static Analysis)



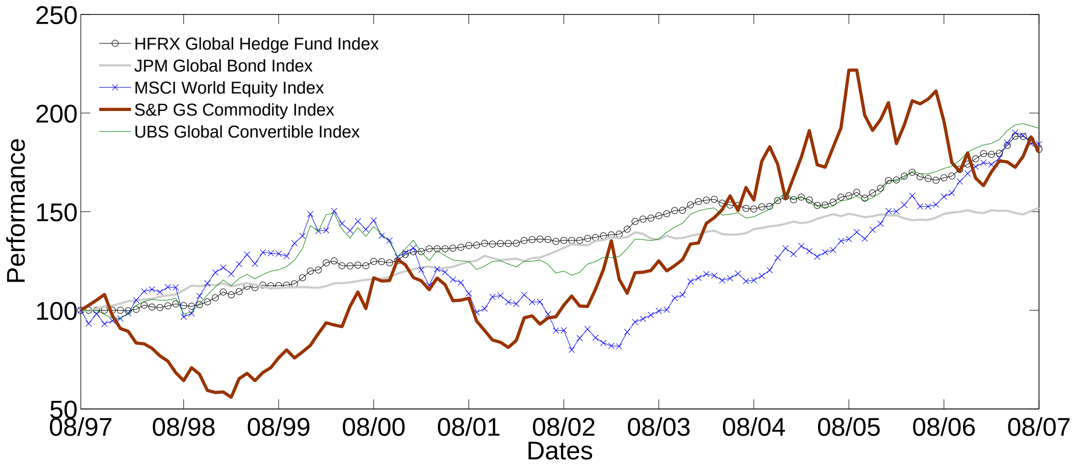
The figure illustrates over a 5-year period three static portfolios optimized according to the following paradigms: *maximum* Sharpe ratio (light line), *maximum Omega* ratio (bold line) and *minimum* volatility (thin line). The x-axis represents the number of simulated observations and the y-axis is the cumulative performance of optimal portfolios. Fund *A* and Fund *B* return densities are simulated according to Gaussian laws and are characterized, respectively, by annual mean returns (volatilities) equal to 9.96% (6.44%) and 10.44% (12.29%). We consider the return of the risk-free asset at 2.00% *per* year (fixed), without loss of generality (complementary results are available upon request - *Cf.* Web Appendix C).

MSCI World Free Equity Index for equities and the S&P GSCI Index for commodities.

In Figure 4, we represent the realized performance of the five indexes from 31st August, 1997 to 31st August, 2007. A brief analysis shows that only the commodity market suffered strongly from the Asian and the Russian crises. Indeed, the S&P GS Commodity Index fell by 40% between October 1997 and December 1998. In contrast, despite heterogeneous correlation levels between the studied markets, the successive financial turbulences that occurred in 2000 (Internet Bubble), 2001 (terrorist attack and Junk Bonds) and 2002 (Brazilian crisis) had a more global impact since the convertible, commodity and equity indexes, dropped between 20.00% and 40.00% within this time period. In Figure 5, we present and compare our results obtained when using the *Omega* ratio and other common paradigms as optimization *criteria*.

We start our study within a static framework using a classical optimization algorithm over the five assets used by Hentati et al. (2010). In Figure 5, we represent five (long-only) optimal portfolios, each corresponding to a specific direction of an in-sample optimization. The first two maximize (only) mean return and mean return under a volatility constraint.

Figure 4: Performance of the Five Indexes representing the Studied Asset Classes



Data sources: Source: *Bloomberg* and *Datastream*; monthly data in USD from 31^{st} August, 1997 to 31^{st} August, 2007. On the y-axis cumulative monthly returns in base 100. The figure shows the performance of the five studied indexes, namely, the HFRX Global Hedge Fund Index (thin line with circular markers), the UBS Global Convertible Index (thin light line), the JPM Global Bond Index (bold light line), the MSCI World Equity Index (thin line with cross markers) and the S&P GS Commodity Index (bold dark line).

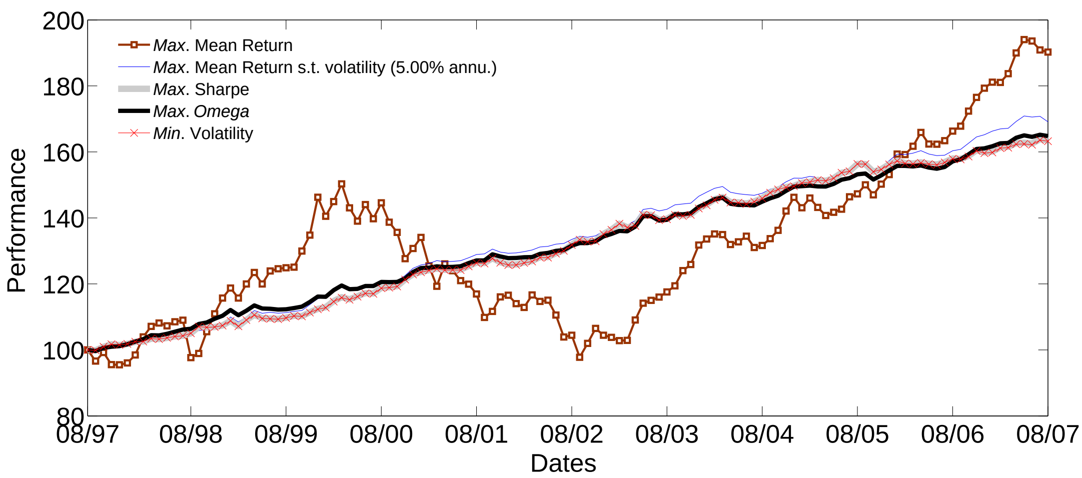
The next two paradigms consist of maximizing the Sharpe (1966) and *Omega* (2002) ratios. Finally, the last considered portfolio minimizes volatility. The composition of these static portfolios is represented in Table 2.

We observe that the first one-period optimized portfolio, corresponding to the *maximum* mean return *criterion*, displays the highest performance, but also the strongest volatility since it totally disregards risk (or losses) by definition. In contrast, the other four static optimal portfolios have very similar profiles and smoothed performances in the sample. Then, the use of the *Omega* (2002) ratio as an optimization *criterion* does not present any major improvement compared to other classical directions (using this database).

In order to complement the previous static analysis presented in Figure 5, we show in Table 2 the one-period optimal allocation for each portfolio (reported in the first column) over the five studied indexes (second to sixth columns).

Here, we read results that are perfectly in line with the given optimization paradigms and our previous illustration. The first optimal portfolio, corresponding to the *maximum*

Figure 5: Performance of Five Portfolios optimized according to the *Omega Criterion* and other Classical Paradigms (Static Analysis)



Data sources: *Bloomberg* and *Datastream*; monthly data in USD from 31st August, 1997 to 31st August, 2007. On the y-axis cumulative monthly returns in base 100. The figure presents a static analysis of the performance of portfolios optimized according to the following *criteria*: *maximum* mean return (line with square markers), *maximum* mean return under a volatility constraint (thin light line), *maximum* Sharpe ratio (bold light line), *maximum Omega* measure (bold dark line) and *minimum* volatility (thin line with cross markers). These five optimal portfolios are composed of the HFRX Global Hedge Fund Index, the UBS Global Convertible Index, the JPM Global Bond Index, the MSCI World Equity Index and the S&P GS Commodity Index (*Cf.* Figure 4).

mean return *criterion*, naturally overweighs in a static framework the indexes displaying the highest final performance (see Figure 4 and 5). The one based on the *Omega* ratio is very slightly more diversified but it is fundamentally almost identical to the three other remaining optimal portfolios that disregard the equity and convertible bond indexes.

In Figure 6, we now propose an illustration of a dynamic out-of-sample analysis of the performance of the five portfolios previously studied. The rolling window for estimations is based on two years and we assume a monthly rebalancing, without specific constraints, during the period of time that ran from 31st August, 1999 to 31st August, 2007.

This figure shows that not only does the dynamic portfolio maximizing (only) the mean return displays (almost) the highest terminal performance, but also the largest volatility compared to the other strategies. In contrast, the less risky optimal portfolio is the one minimizing volatility that presents the lowest but smoothed performance. If we now focus on the portfolio maximizing the *Omega* ratio, we observe that it underperforms

Table 2: Composition of the Five Optimal Portfolios according to the *Omega Criterion* and other Classical Paradigms (Static Analysis)

	Optimal Weights				
Optimization Paradigms	HFRX Hedge Fund Global	JPM Global Bond Index	MSCI World Equity Index	S&P GS Com. Index	UBS Global Conv. Index
Max. Mean Return	.00%	.00%	.00%	.00%	100.00%
Max. Mean s.t. Vol. (5.00%)	54.20%	43.90%	.00%	1.90%	.00%
Max. Sharpe	28.37%	66.45%	.00%	5.18%	.00%
Max. Omega ($\tau = .00\%$)	32.40%	60.15%	1.99%	.68%	4.77%
Min. Volatility	27.20%	67.47%	.00%	5.33%	.00%

Data sources: *Bloomberg* and *Datastream*; monthly data in USD from 31^{st} August, 1997 to 31^{st} August, 2007. The table shows the one-period optimal weights of portfolios corresponding to the following directions: *maximum* mean return, *maximum* mean return under a volatility constraint, *maximum* Sharpe, *maximum Omega* and *minimum* volatility; and for each studied index, namely, the HFRX Global Hedge Fund Index, the JPM Global Bond Index, the MSCI World Equity Index, the S&P GS Commodity Index and the UBS Global Convertible Index.

almost all the other ones, and more precisely, those based on the *maximum* mean return, the *maximum* Sharpe ratio and the *maximum* mean return under a volatility constraint.

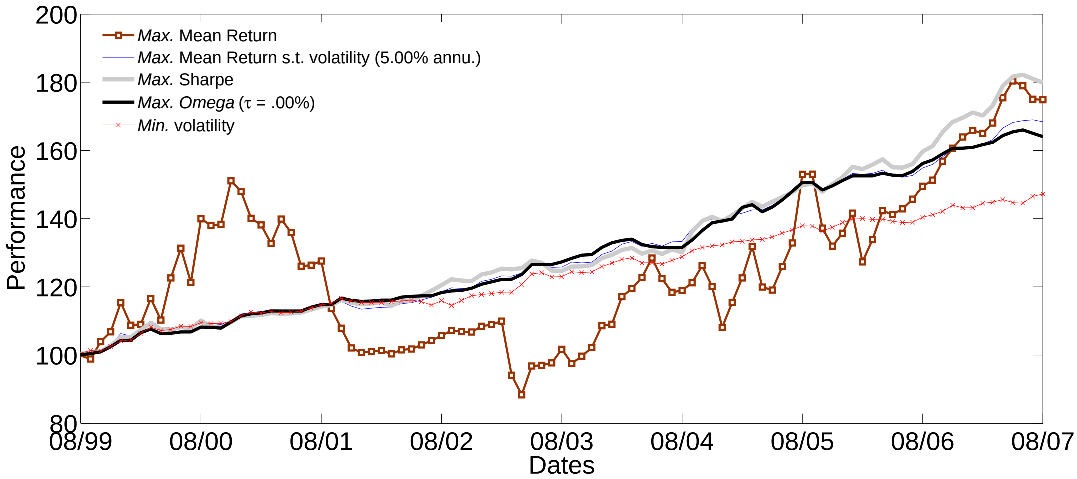
We also run some robustness checks,[26] varying the inputs, in a static and a dynamic framework, when optimizing portfolios according to the Omega ratio. First, we use the same sub-sample of hedge funds adopted in Darolles et al. (2009), we calculate the Sharpe ratios of several strategies based on the methodology and the datasets used in DeMiguel et al. (2009). Theses checks confirm our previous results: thus, Omega-based optimal portfolios, respectively, for a low and a high threshold, are very close to those obtained in minimum volatility and maximum mean return optimization paradigms. In other words, it happens in these tests that the use of the Omega ratio does not present here a true added value.

We now focus our study on the use of the *Omega* ratio as an optimization *criterion*. Similar to our previous analysis, we thus present, first, a static framework, in which we examine the evolution of the performance of optimal portfolios using the *Omega* optimization *criterion* according to several thresholds, respectively, equal to .00%, 5.00%, 7.00% and 10.00%.

As shown in Figure 7, the higher the value of the threshold, the riskier the profile

[26] For space reasons, all robustness checks have been reported in Web Appendix D.

Figure 6: Performance of Five Portfolios optimized according to the *Omega Criterion* and other Classical Paradigms (Dynamic Analysis)

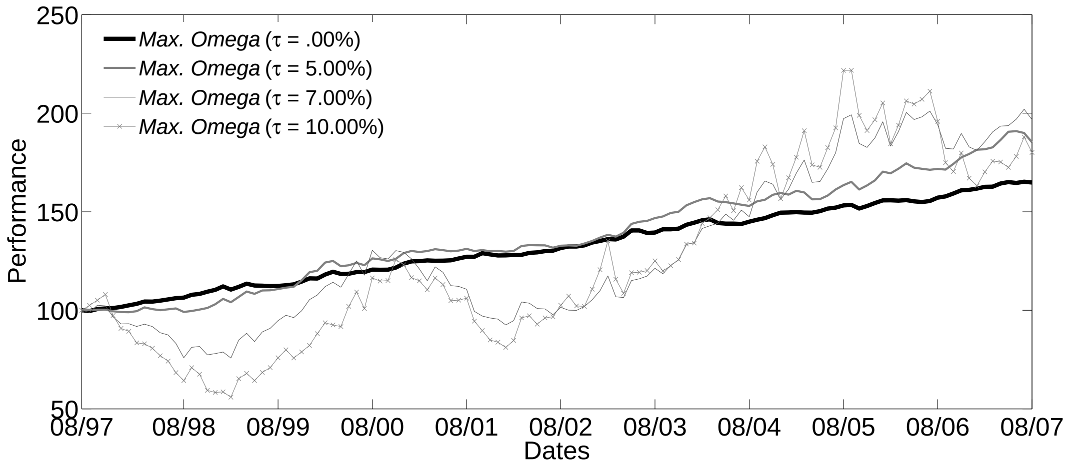


Data sources: *Bloomberg* and *Datastream*; monthly data in USD from 31^{st} August, 1997 to 31^{st} August, 2007. On the y-axis cumulative monthly returns in base 100. The figure presents a dynamic analysis of the performance of portfolios optimized according to the following *criteria*: *maximum* mean return (line with square markers), *maximum* mean return under a volatility constraint (thin light line), *maximum* Sharpe ratio (bold light line), *maximum Omega* measure (bold dark line) and *minimum* volatility (thin line with cross markers). We consider a two-year in-sample window and a monthly rebalancing for the projection period. These five optimal portfolios are initially composed of the HFRX Global Hedge Fund Index, the UBS Global Convertible Index, the JPM Global Bond Index, the MSCI World Equity Index and the S&P GS Commodity Index (*Cf.* Figure 4).

of the optimal portfolio. For instance, the one considering a null threshold presents the lowest but the most stable performance with a very low risk, whereas the optimal portfolio based on a threshold set to 10.00% displays a slightly better performance, but for a much more important risk level. However, we can wonder about the (rational) *maximum* value of the threshold since performances of static optimal portfolios based on rates equal to 5.00%, 7.00% and 10.00% are almost identical.

In Figure 8, we now compare the performance of four dynamic portfolios optimized according to the same setting of thresholds used in the previous analysis. As expected, this figure shows that the dynamic portfolio based on a threshold equal to 10.00% clearly overperforms the optimal portfolio for which the threshold is set to .00%. This last observation raises an important question of the usefulness and justification in optimization problems of the *Omega*, and confirms our intuition about its sensitivity to the value of the threshold. dedicated to some robustness checks.

Figure 7: Performance of Four Portfolios optimized according to the *Omega* Ratio when varying the Threshold (Static Analysis)



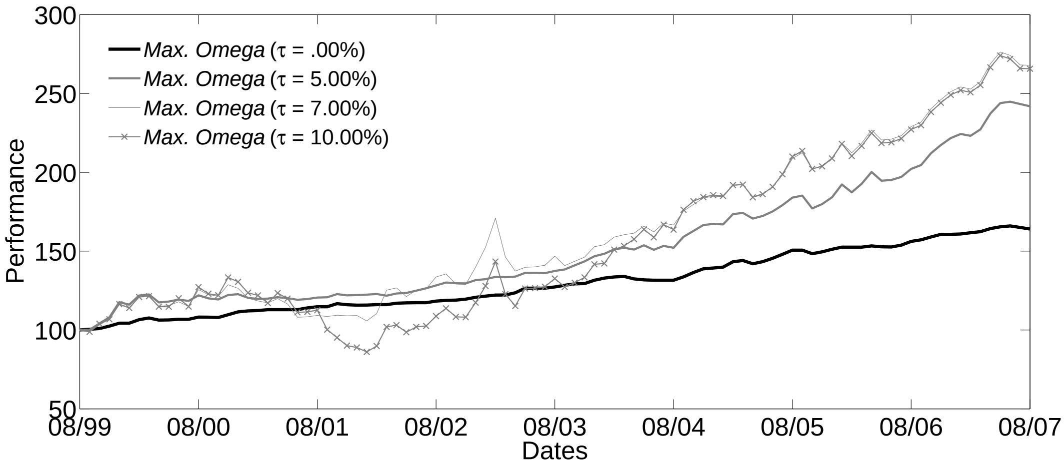
Data sources: *Bloomberg* and *Datastream*; monthly data in USD from 31st August, 1997 to 31st August, 2007. On the y-axis cumulative monthly returns in base 100. This figure presents a static analysis of the performance of portfolios optimized according to the *Omega criteria* when varying the thresholds, respectively, equal to .00% (bold dark line), 5.00% (bold light line), 7.00% (thin light line) and 10.00% (thin light line with cross markers). These four optimal portfolios are initially composed of the HFRX Global Hedge Fund Index, the UBS Global Convertible Index, the JPM Global Bond Index, the MSCI World Equity Index and the S&P GS Commodity Index (*Cf.* Figure 4).

5. Conclusion

We recently observed that an increasing number of significant finance articles refer to the *Omega* measure for evaluating the performance of funds or of active strategies (*e.g.* Eling and Schuhmacher, 2007; Farinelli and Tibiletti, 2008; Annaert et al., 2009; Bertrand and Prigent, 2011; Zieling et al., 2014; Kapsos et al., 2014; Hamidi et al., 2014), since both the return distributions may not be Gaussian and the volatility may not be the relevant risk measure in such a case. The *Omega* measure is also used in some non-linear portfolio optimization problems when returns are characterized by severe downside risks (*e.g.* Mausser et al., 2006; Kane et al., 2009; Gilli et al., 2011).

Our article started with a first simple intuition, which was that the *Omega* measure might excessively privilege the performance regardless of the implied risk and thus exhibits inconsistency with the Stochastic Dominance *criteria*. The second intuition was on the role of the threshold that we should define for computing *Omega*, which might be very crucial, as already pointed out by Bertrand and Prigent (2011). Based on these two

Figure 8: Performance of Four Portfolios optimized according to the *Omega* Ratio when varying the Threshold (Dynamic Analysis)



Data sources: *Bloomberg* and *Datastream*; monthly data in USD from 31st August, 1997 to 31st August, 2007. On the y-axis cumulative monthly returns in base 100. This figure presents a dynamic analysis of the performance of portfolios optimized according to the *Omega criteria* when varying the thresholds, respectively, equal to .00% (bold dark line), 5.00% (bold light line), 7.00% (thin light line) and 10.00% (thin light line with cross markers). We consider a two-year in-sample window and a monthly rebalancing for the projection period. These four optimal portfolios are initially composed of the HFRX Global Hedge Fund Index, the UBS Global Convertible Index, the JPM Global Bond Index, the MSCI World Equity Index and the S&P GS Commodity Index (*Cf.* Figure 4).

assumptions, we have studied more precisely the *Omega* ratio under two main aspects.

The first one is the relevance of using *Omega* as a portfolio ranking *criterion*. First, we show through a simple realistic illustration that the *Omega* measure can be equal for two portfolios, even when the total risk of the first fund is twice that of the second one. Secondly, we observe that for positive thresholds close to .00%, the trade-offs return/risk of the *Omega* and the Sharpe ratios are both almost equal to 1.00. This trade-off is inferior to 1.00 (for instance, equal to .25) when using high thresholds in the computation of the *Omega* measure (for example, 10.00%). The lower the threshold, the closer the trade-offs using the Sharpe ratio and the *Omega criterion*. We have thus demonstrated that portfolio rankings obtained according to this performance measure are seriously influenced by the chosen threshold, and more precisely by the difference between this threshold and the mean return of studied underlying return distributions. Choices of investments guided by the *Omega* measure may thus yield different investment decisions, depending on the threshold, that should not be too high or too low.

The second interest is the use of the *Omega* ratio within an optimization framework. In a simple static Gaussian optimization setting, we first show that the *Omega* ratio can lead to clear under-optimal solutions in some realistic cases (mainly when the given threshold is higher than the mean return of return distributions of some considered risky assets). Our next results are based on three different market databases used in publications, respectively, by Hentati et al. (2010), Darolles et al. (2009) and DeMiguel et al. (2009). The first dataset represents five main asset classes, the second database regroups 18 hedge funds and the third one corresponds to six sub-markets as classified by Kenneth French. Using these three datasets, we provide illustrations in which *Omega*-based optimal portfolios are sensibly very similar to those obtained according to more classical *criteria* (namely the *maximum* Sharpe ratio and the *minimum* volatility). Furthermore, we show in our study that an *Omega*-based optimal portfolio is similar to a *maximum* mean return-based optimal portfolio for a high threshold, and to a *minimum* volatility-based optimal portfolio for a low threshold. This confirms the intuition that a low threshold corresponds to risk-averse agents (when the risk-free rate is the preferred asset in the limit case) and a high threshold represents greedy investors (with a strong preference for performance).

We conclude that the *Omega* ratio can entail misleading financial decisions and lead us to question the use of this measure in any of the traditional finance applications it was designed for; we finally have some doubts about its accuracy, stability and overall usefulness.[27]

Finally, the *Omega* ratio belongs to a general class of performance measurement based on features of return densities. *Omega* is, indeed, a simplified version of some other measures when varying the parameters in the general function proposed by Caporin et al. (2014). For instance, the Sharpe-*Omega* ratio (Kazemi et al., 2004) is reported in some studies[28] to be equal to $\Omega_p(\tau) - 1$; thus, it would also be interesting to challenge such measures. In the same vein, we should question other related measures such as, for instance, the Sortino-Meer-Plantinga (1999) Upside-Potential ratio, the Gemmill-Hwang-Salmon

[27]See Hübner (2007) for a general discussion on the precision and stability of performance measures.
[28]See also Appendix C for a comparison between the *Omega* and the Sharpe-*Omega* ratios.

(2006) Loss-Averse Performance measures and the Farinelli-Tibiletti (2008) measure (*Cf.* Caporin et al., 2014), in order to see if they share, or not, the same drawbacks.

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Appendix A. Optimization of *Omega*

Given the growing popularity of the *Omega* ratio, building a portfolio that maximizes *Omega* has become both a topical and practical problem. Maximizing *Omega* can be difficult since the resulting optimization problem is not convex. The problem of finding a portfolio that maximizes the *Omega* measure has been considered by Avouyi-Dovi et al. (2004), who use a global optimization approach, and Passow (2005), who considers a parametric approach using the four-parameter family of Johnson distributions. Briec et al. (2004) develop an implicit enumeration strategy for solving non-convex optimization problems to obtain closed-form solutions. Briec and Kerstens (2006) refine this methodology by showing how a non-linear problem can be transposed into a more familiar linear programming problem.

Based on the work of Charnes and Cooper (1962), and in the same vein as Briec and Kerstens (2006), Mausser et al. (2006) propose a non-parametric model[29], in which the investor's problem of finding the portfolio that optimizes the *Omega* measure is written as:

$$\Omega_p^*(\tau) = \max_{(w_i,u_t,d_t)\in[0,1]\times\mathbb{R}_+^2} \frac{\sum_{t=1}^{T} P_{p,t}^u(r_t)\times u_t}{\sum_{t=1}^{T} P_{p,t}^d(r_t)\times d_t}, \tag{A.1}$$

with:

$$\begin{cases} \sum_{i=1}^{N} r_{i,t}w_i - u_t + d_t = \tau, \\ \sum_{i=1}^{N} w_i = 1, \end{cases}$$

and:

$$\begin{cases} Aw \leq b, \\ u_t, d_t, w_i \geq 0, \\ u_t d_t = 0, \end{cases}$$

where u_t is the excess return of the portfolio p above the threshold τ and $P_{p,t}^u(r_t)$ is the corresponding probability, d_t is the excess return of the portfolio p below the threshold τ and $P_{p,t}^d(r_t)$ is the corresponding probability, w_i is the weight assigned to the asset i, $r_{i,t}$ is the return at time t of the asset i.

However, Mausser et al. (2006) show that this non-convex non-linear program is, under some conditions, equivalent to a linear program through a transformation of variables, if we drop the complementarity constraint $u_t d_t = 0$ – see also Charnes and Cooper (1962).

If we define:

$$\zeta = \left[\sum_{t=1}^{T} P_{p,t}^d(r_t)\, d_t\right]^{-1}, \tag{A.2}$$

we note that if $\Omega_p^*(\tau) \in (0,+\infty)$ then $\zeta > 0$ is finite.

[29] A parametric approach for optimizing *Omega* discussed in Passow (2005) can be an alternative to the approach proposed by Mausser et al. (2006).

The transformed variables are defined as:

$$\begin{cases} \tilde{w}_i = w_i \times \zeta, \\ \tilde{u}_t = u_t \times \zeta, \\ \tilde{d}_t = d_t \times \zeta. \end{cases} \tag{A.3}$$

Since $\zeta > 0$, we observe that the non-negativity of the transformed variables is equivalent to that of the original variables. Additionally, it is possible to return to the original variables using the inverse transformation such as:

$$\begin{cases} w_i = \tilde{w}_i \times (\zeta)^{-1}, \\ u_t = \tilde{u}_t \times (\zeta)^{-1}, \\ d_t = \tilde{d}_t \times (\zeta)^{-1}. \end{cases} \tag{A.4}$$

Substituting the transformed variables into the problem (A.2) and dropping the constraint $u_t d_t = 0$ give the following optimization problem:

$$\tilde{\Omega}_p(\tau) = \max_{(\tilde{w}_i, \tilde{u}_t, \tilde{d}_t, \zeta) \in [0,1] \times \mathbb{R}_+^3} \sum_{t=1}^{T} P_{p,t}^u(r_t)\, \tilde{u}_t, \tag{A.5}$$

with:

$$\begin{cases} \sum_{i=1}^{N} r_{i,t} \tilde{w}_i - \tilde{u}_t + \tilde{d}_t - \tau \times \zeta = 0, \\ \sum_{i=1}^{N} \tilde{w}_i - \zeta = 0, \\ \sum_{t=1}^{T} P_{p,t}^d(r_t)\, \tilde{d}_t = 1, \end{cases}$$

and:

$$\begin{cases} A\tilde{w} - b\zeta \leq 0, \\ \tilde{u}_t, \tilde{d}_t, \tilde{w}_i \geq 0, \end{cases}$$

which is a linear program in the variables ζ, $\tilde{w}_i$, $\tilde{u}_t$, $\tilde{d}_t$.

It is not automatically assumed that the optimal solution of equation (A.5) gives an optimal solution to (A.1) by reversing the transformation (A.3), because the complementarity constraint $u_t d_t = 0$ has been dropped.

Note, however, that when the optimal *Omega* is less than one (which is unrealistic), the linear program (A.5) will not reach an *optimum* since it will produce a solution with an optimal value that violates the complementarity constraints $u_t d_t = 0$ and is therefore financially meaningless.

Appendix B. On the First-order and Second-order Stochastic Dominance *Criteria*

Several studies, such as Darsinos and Satchell (2004), refer to the traditional (First and Second-order) Stochastic Dominance definitions[30] to show that *Omega* is a consistent measure.

In the same vein, we will use the following notations. Let us consider two portfolios A and B, r is a random variable that corresponds to returns on a portfolio p with $p = \{A, B\}$, with $r \in [a, b]$, where (a, b) are two constants that define the support[31] of r such as $(a, b) = \{\min(r), \max(r)\}$, $f_p(r)$ is the Probability Density Function, $F_p(r)$ denotes the Cumulative Density Function, $u(.)$ is a utility function belonging to the set U, of increasing, concave and continuous real valued functions, $u'(.)$ is its first derivative, and $1\mathrm{I}_{\{.\}}$ is the Heaviside function.

Without loss of generality, we only consider here the non-trivial case,[32] such as $f_A(r) \neq f_B(r)$ for at least one $r \in [a, b]$.

Proposition B1. *First-order Stochastic Dominance* Criterion

A portfolio A is preferred to a portfolio B according to the First-order Stochastic Dominance criterion*, i.e.:*

$$A \succ_{FSD} B, \tag{B.1}$$

if and only if:

$$F_A(r) \leq F_B(r) \text{ for all } r \in [a, b], \tag{B.2}$$

with at least a return $r \in [a, b]$ for which the strict inequality is true, and where $F_p(.)$ corresponds to the Cumulative Density Functions of returns r on the portfolios $p = \{A, B\}$, *which entails that:*

$$E_A[u(1 + r)] \geq E_B[u(1 + r)], \tag{B.3}$$

with at least a function $u(.)$ for which the strict inequality is true, and where $u(.)$ is an increasing and continuous utility function, $E_p[u(.)]$ is the expected utility and r corresponds to the portfolio returns.

Proof of Proposition B1

- Regarding the sufficient part of Proposition B1. [*i.e.* Equation (B.3) is sufficient for Equation (B.1)].

[30] *Cf.* Weston and Copeland (1998) and Huang and Litzenberger (1988) for the traditional definitions of the First and Second-order Stochastic Dominance *criteria*.

[31] So that $F_p(a) = 0$ and $F_p(b) = 1$, where $F_p(.)$ is the CDF of returns r on portfolios p, herein with $p = \{A, B\}$.

[32] In fact, the Second-order Stochastic Dominance *criterion* may lead to irrational investment choices. According to the traditional definition of the Stochastic Second-order Dominance *criterion*, the trivial case where $F_A(.) = F_B(.)$ is not excluded. When $\int_a^\tau F_B(r) - F_A(r)dr \geq 0$ for all $\tau \in [a, b]$, we can face the following counter-intuitive situation in which (with the previous notations): 1) $E_A(r) = E_B(r)$; 2) $\sigma_A < \sigma_B$; 3) $A \succeq_{SSD} B$; 4) $\int_a^\tau F_B(r) - F_A(r)dr \geq 0$ for all $\tau \in [a, b]$ (see Figure 1 for an illustration of this case when *Omega* is the decision *criterion.*); while, obviously, A should be strictly preferred to B. In the strict version of the SSD *criterion*, with a strict inequality (in the non-trivial case where $F_A(r) \neq F_B(r)$), we have, in this case, $A \succ_{SSD} B$, since $\int_a^\tau F_B(r) - F_A(r)dr \geq 0$ for all $\tau \in [a, b]$ and $\int_a^\tau F_B(r) - F_A(r)dr > 0$ for some $\tau \in [a, b]$.

Let us start by assuming that the returns r on the portfolios A and B belong to the interval $[a, b]$ where a and b are some constants (corresponding to some low and high returns defining the support of the returns).

First, we suppose that, $\forall r \in [a, b]$:

$$F_A(r) \leq F_B(r), \tag{B.4}$$

with a strict inequality for some r.

Secondly, let us consider a rational continuous and increasing utility function, denoted $u(.)$, representing a non-satiable individual's preferences and assume, without loss of generality, that this individual has an initial wealth equal to 1.00. If the individual invests in portfolio A, his expected utility is $E_A[u(1+r)]$ and similarly for B, where r denotes the returns. We want to show that (B.1) implies (B.3), or similarly:

$$\int_a^b u(1+r)\, dF_A(r) \geq \int_a^b u(1+r)\, dF_B(r). \tag{B.5}$$

If we now integrate by parts Equation (B.5), we have (with the previous notations):

$$\begin{aligned} &\int_a^z u(1+r)\, d[F_A(r) - F_B(r)] \\ &= u(1+r)[F_A(r) - F_B(r)]_a^z - \int_a^z [F_A(r) - F_B(r)]\, du(1+r). \end{aligned} \tag{B.6}$$

Since $F_A(b) = F_B(b) = 1$, we get:

$$\int_a^b u(1+r)\, d[F_A(r) - F_B(r)] = -\int_a^b [F_A(r) - F_B(r)]\, du(1+r). \tag{B.7}$$

However, following the *hypothesis* that $F_A(r) - F_B(r) \leq 0$ for all $r \in [a, b]$ and $F_A(r) - F_B(r) < 0$ for some r, and $u(.)$ is strictly increasing, the sign of Equation (B.7) is positive. Thus, Equation (B.4) is a sufficient condition for Equation (B.3).■

- Regarding the necessary part of Proposition B1. [*i.e.* Equation (B.3) is necessary for Equation (B.1)].

Let us suppose that $A \succ_{FSD} B$, $F_A(r) \geq F_B(r)$ for all $r \in [a, b]$. Since $F_A(.)$ and $F_B(.)$ are increasing and right continuous, there must be an interval $[c, d] \subset [a, b]$ so that, $\forall r \in [c, d]$:

$$F_A(r) > F_B(r). \tag{B.8}$$

If we now define a continuous and increasing utility function, with standard properties, that increases only on $[1+c, 1+d]$, the sign of Equation (B.7) would be negative, which contradicts the *hypothesis* that $A \succ_{FSD} B$.

First, let us define the simplest utility function $u(.)$, as:

$$u(1+\tau) = \int_a^\tau 1\mathrm{I}_{\{1+c \le x \le 1+d\}}(1+r)\,dr, \tag{B.9}$$

where $1\mathrm{I}_{\{.\}}$ is the Heaviside function.

Recalling that $u(.)$ is increasing and continuous, we have:

$$u^{'}(1+\tau) \;=\; \begin{cases} 1 & \text{if } (1+\tau) \in [1+c, 1+d], \\ 0 & \text{elsewhere.} \end{cases} \tag{B.10}$$

Equation (B.10) corresponds to the marginal utility that is equal to 1 on $[1+c, 1+d]$ and 0 elsewhere. Secondly, using Equation (B.7) and Equation (B.8), we obtain:

$$\int_a^b u(1+r)\,d\left[F_A(r) - F_B(r)\right] = -\int_a^b \left[F_A(r) - F_B(r)\right] du(1+r). \tag{B.11}$$

However, results in Equation (B.11) contradict $A \succ_{FSD} B$.

Thus, we must have, $\forall r \in [a, b]$:

$$F_A(r) \le F_B(r), \tag{B.12}$$

with a strict inequality for some r.

Then, Equation (B.3) is a sufficient and necessary condition for Equation (B.1).■

Proposition B.2. *Second-order Stochastic Dominance* Criterion

A portfolio A is preferred to a portfolio B according to the Second-order Stochastic Dominance criterion*, i.e. (with the previous notations):*

$$A \succ_{SSD} B, \tag{B.13}$$

if and only if:

$$\int_a^\tau \left[F_B(r) - F_A(r)\right]\; dr \ge 0 \text{ for all } \tau \in [a, b], \tag{B.14}$$

with at least a return $\tau \in [a, b]$ for which the strict inequality is true, which implies that:

$$E_A\left[u(1+r)\right] \ge E_B\left[u(1+r)\right], \tag{B.15}$$

with at least a function $u(.)$ for which the strict inequality is true.

Proof of Proposition B2

- Regarding the sufficient part of Proposition B2. [*i.e.* Equation (B.15) is sufficient for Equation (B.13)].

Let us first define the continuous function $S(.)$ as being:

$$
S(\tau) := \begin{cases} \int\limits_a^\tau [F_A(r) - F_B(r)] \ dr \leq 0 & \text{for all } \tau \in [a,b], \\ \int\limits_a^\tau [F_B(r) - F_A(r)] \ dr < 0 & \text{for a least one } \tau \in [a,b], \end{cases} \tag{B.16}
$$

If we integrate by parts Equation (B.7), we have:

$$
-\int\limits_a^\tau [F_A(r) - F_B(r)] \, du(1+r) = -\int\limits_a^\tau u'(1+r) \, dS(r). \tag{B.17}
$$

where $u'(.)$ is the first derivative of the utility function $u(.)$.

From Equation (B.9) and Equation (B.16), and assuming that function $u'(.)$ is decreasing, we can rewrite Equation (B.17) as follows:

$$
\int\limits_a^\tau u(1+x) \, d[F_A(x) - F_B(x)] = \int\limits_a^r S(x) \, du'(1+x). \tag{B.18}
$$

From the Equation (B.17), stating that $\int_a^\tau [F_A(r) - F_B(r)] \ dr \geq 0$ for all $\tau \in [a,b]$, and $\int\limits_a^\tau [F_B(r) - F_A(r)] \ dr > 0$ for at least one $\tau \in [a,b]$, we can write that:

$$
\begin{cases} \int\limits_a^\tau S(r) \, du'(1+r) \geq 0 \text{ for all } \ \tau \in [a,b], \\ \int\limits_a^\tau S(r) \, du'(1+r) > 0 \text{ for at least one } \tau \in [a,b].. \end{cases} \tag{B.19}
$$

Thus, Equation (B.15) is a sufficient condition for Equation (B.13).■

- Regarding the necessary part of Proposition B2. [*i.e.* Equation (B.13) is necessary for Equation (B.15)].

Let us assume that, first, $A \succ_{SSD} B$ and, secondly, that function $S(.)$ is not negative. From the continuity of $S(.)$, it then follows that there must exist an interval $[c,d]$, with $c \neq d$ and $\forall \tau \in [c,d]$, such as:

$$
S(\tau) > 0. \tag{B.20}
$$

If we can find a concave utility function whose first derivative is continuous, except possibly on a countable set, and strictly decreases only on $[1+c, 1+d]$, then the inequality in Equation (B.19) will not be valid anymore and the *hypothesis* that $A \succ_{SSD} B$ will be contradicted.

Let us define, $\forall (\tau, r) \in [a,b]^2$:

$$
u(1+\tau) = \int\limits_a^\tau \int\limits_y^b 1\!\mathrm{l}_{\{1+c \leq x \leq 1+d\}} (1+r) \, dr d\tau. \tag{B.21}
$$

Since the utility function $u(.)$ is continuously differentiable and concave, we have (with the previous notations):

$$u'(1+\tau) = \int_{\tau}^{b} 1\!\mathrm{l}_{\{1+c \leq x \leq 1+d\}} (1+r)\, dr. \tag{B.22}$$

We then obtain:

$$\int_{a}^{\tau} u(1+\tau)\, d\left[F_A(r) - F_B(r)\right] = \int_{a}^{\tau} S(r)\, du'(1+r)\,. \tag{B.23}$$

Equation (B.23), being negative, contradicts the *hypothesis* that $A \succ_{SSD} B$, and thus, Equation (B.13) is a necessary condition for having Equation (B.15). ■

Appendix C. Complementary Results on the *Omega* measure

Some authors (*e.g.* Mausser et al., 2006; Farinelli and Tibiletti, 2008; Bertrand and Prigent, 2011) write the *Omega* ratio as:

$$\begin{aligned}\Omega_p(\tau) &= E_p[\max(r-\tau,0)] \times \{E_p[\min(r-\tau,0)]\}^{-1} \\ &= E_p[(r-\tau)^+] \times \{E_p[(r-\tau)^-]\}^{-1},\end{aligned} \tag{C.1}$$

where $E_p[(.)^+] = \max(.,.)$ and $E_p[(.)^-] = \min(.,.)$.

They use the traditional option payoff notation (*Cf.* Kazemi et al., 2004, Eq. 11 page 24), which is different in the presentation from the original version proposed in Keating and Shadwick (2002), who define the *Omega* measure by means of integrals, and also different from the notation we adopted in Section 2. We provide here detailed equations showing the link between alternative representations of the $\Omega_p(\tau)$ function.

Keating and Shadwick (2002) define *Omega* by means of integrals based on the CDF of the portfolio returns. Following their contribution, *Omega* is given as:

$$\Omega_p(\tau) = \frac{\int_{\tau}^{+\infty} [1 - F_p(r)]\, dr}{\int_{-\infty}^{\tau} F_p(r)\, dr}, \tag{C.2}$$

where $F_p(r)$ is the CDF of the portfolio returns.

Focus now on the numerator. By integrating in parts, we can easily obtain the following equalities:

$$\begin{aligned}\int_{\tau}^{+\infty} [1 - F_p(r)]\, dr &= r[1 - F_p(r)]\,|_{\tau}^{+\infty} + \int_{\tau}^{+\infty} r f_p(r)\, dr \\ &= -\tau[1 - F(\tau)] + E_p[r|r > \tau][1 - F_p(\tau)] \\ &= E_p[r - \tau|r > \tau][1 - F_p(\tau)] \\ &= E_p[max(r - \tau, 0)].\end{aligned} \tag{C.3}$$

We can proceed in a similar way on the denominator, obtaining a second set of equalities:

$$\begin{aligned}\int_{-\infty}^{\tau} F_p(r)\, dr &= rF_p(r)\,|_{-\infty}^{\tau} - \int_{-\infty}^{\tau} r f_p(r)\, dr \\ &= \tau F_p(\tau) - E_p[r|r \leq \tau] F_p(\tau) \\ &= E_p[\tau - r|r \leq \tau] F_p(\tau) \\ &= E_p[max(\tau - r, 0)].\end{aligned} \tag{C.4}$$

Therefore, combining the previous equalities, we can obtain the following alternative equivalent representations of the *Omega* function:

$$
\begin{aligned}
\Omega_p(\tau) &= \frac{\int\limits_{\tau}^{+\infty}[1-F_p(r)]\,dr}{\int\limits_{-\infty}^{\tau}F_p(r)\,dr} \\
&= \frac{E_p[r-\tau|r>\tau]\,[1-F_p(\tau)]}{E_p[\tau-r|r\leq\tau]\,F_p(\tau)} \\
&= \frac{E_p[max(r-\tau,0)]}{E_p[max(\tau-r,0)]}.
\end{aligned}
\tag{C.5}
$$

The first equation corresponds to the original *formula* in Keating and Shadwick (2002), the second equation is the one we adopted in Section 2, while the last equation is inspired by the option theory and is referred to, among others, by Mausser et al. (2006), Farinelli and Tibiletti (2008), and Bertrand and Prigent (2011).

Alternatively, following Gouriéroux and Liu (2012), we can express *Omega* thanks to a quantile function, defined as: $Q_p(u) = F_p^{-1}(u)$. From the previous equalities, we have:

$$
\begin{aligned}
E_p(r|\ r\leq\tau) &= \frac{1}{F_p(\tau)}\int\limits_{-\infty}^{\tau} r f_p(r)\,dr \\
&= \frac{1}{F_p(\tau)}\int\limits_{0}^{F_p(\tau)} Q_p(u)\,du,
\end{aligned}
\tag{C.6}
$$

and we can redefine *Omega* as follows:

$$
\Omega_p(\tau) = \frac{[1-F_p(\tau)]}{F_p(\tau)}\frac{\int_{F_p(\tau)}^{+\infty}[Q_p(u)-\tau]\,du}{\int_{-\infty}^{F_p(\tau)}[\tau-Q_p(u)]\,du}. \tag{C.7}
$$

In addition, once a specific distribution is chosen, a more detailed or analytical representation is, in general, available. A related representation for the *Omega* can be derived from the work of Malo (2005). For instance, we can assume that the portfolio net asset value, denoted X_t, follows a mean reverting process such as:

$$
dX_t = \theta(\mu - X_t)\,dt + \sigma_p(X_t)\,dW_t. \tag{C.8}
$$

where $\theta > 0$ measures the speed of mean reversion, μ is the long run mean to which the process tends to revert, $\sigma_p(.)$ is the diffusion function and W_t denotes the Wiener process.

Under the conditions in Malo (2005), the following results hold:

$$
E_p[r|r\leq\tau] = \mu - \frac{\sigma_p^2(\tau)}{2\theta}\frac{f_p(\tau)}{F_p(\tau)}, \tag{C.9}
$$

and:

$$
E_p[r|r>\tau] = \mu + \frac{\sigma_p^2(\tau)}{2\theta}\frac{f_p(\tau)}{[1-F_p(\tau)]}. \tag{C.10}
$$

Now focus on the *Omega* measure, which can be written, following previously reported equivalences, such as:

$$\Omega_p(\tau) = \frac{-\tau\left[1 - F_p(\tau)\right] + E_p\left[r|r > \tau\right]\left[1 - F_p(\tau)\right]}{\tau F_p(\tau) - E_p\left[r|r \le \tau\right] F_p(\tau)}. \tag{C.11}$$

Using the results obtained in Equation (C.9) and Equation (C.10), we thus have:

$$\begin{aligned}
\Omega_p(\tau) &= \frac{-\tau\left[1 - F_p(\tau)\right] + \left\{\mu + \left[\sigma_p^2(\tau)/2\theta\right] \times \left[f_p(\tau)/1 - F_p(\tau)\right]\right\}\left[1 - F_p(\tau)\right]}{\tau F_p(\tau) - \left\{\mu - \left[\sigma_p^2(\tau)/2\theta\right] \times \left[f_p(\tau)/1 - F_p(\tau)\right]\right\} F_p(\tau)} \\
&= \frac{(\mu - \tau)\left[1 - F_p(\tau)\right] + \left[\sigma_p^2(\tau)/2\theta\right] f_p(\tau)}{(\tau - \mu) F_p(\tau) + \left[\sigma_p^2(\tau)/2\theta\right] f_p(\tau)}.
\end{aligned} \tag{C.12}$$

The last *formula* is the generic *Omega Quantile Function*. Once a specific density is chosen, following the examples in Malo (2005), explicit/specific *Omega Quantile Functions* can be easily recovered (see below Table C.1 for some examples with the most usual densities used in Finance). For instance, in the case of Gaussian innovations with mean μ and variance σ^2, we have $\sigma_p^2(\tau) = 2\theta\sigma^2$, thus obtaining:

$$\Omega_p(\tau) = \frac{(\mu - \tau)\left[1 - F_p(\tau)\right] + \sigma^2 f_p(\tau)}{(\tau - \mu) F_p(\tau) + \sigma^2 f_p(\tau)}. \tag{C.13}$$

If we further impose $\mu = 0$ and $\sigma^2 = 1$, we get:

$$\Omega_p(\tau) = \frac{-\tau\left[1 - F_p(\tau)\right] + f_p(\tau)}{\tau F_p(\tau) + f_p(\tau)}. \tag{C.14}$$

Finally, we can derive a relation between *Omega* and the Sharpe-*Omega* ratio (Kazemi et al., 2004). Now focusing on the expectation of a random variable r, the following relations hold when considering deviations from a given threshold:

$$\begin{aligned}
E_p\left[r - \tau\right] &= \int_{-\infty}^{+\infty} (r - \tau) f_p(r)\, dr \\
&= \int_{-\infty}^{\tau} (r - \tau) f_p(r)\, dr + \int_{\tau}^{+\infty} (r - \tau) f_p(r)\, dr \\
&= E_p\left[(r - \tau)\,|r \le \tau\right] F_p(r) + E_p\left[(r - \tau)\,|r > \tau\right]\left[1 - F_p(r)\right] \\
&= E_p\left[(r - \tau)\,|r > \tau\right]\left[1 - F_p(r)\right] - E_p\left[(\tau - r)\,|r \le \tau\right] F_p(r) \\
&= E_p\left[max\,(r - \tau, 0)\right] - E_p\left[max\,(\tau - r, 0)\right].
\end{aligned} \tag{C.15}$$

Therefore:

$$E_p\left[max\,(r - \tau, 0)\right] = E_p\left[max\,(\tau - r, 0)\right] + E_p\left[r - \tau\right], \tag{C.16}$$

which leads to:

$$
\begin{aligned}
\Omega_p(\tau) &= \frac{E_p\left[max\left(\tau-r,0\right)\right]+E_p\left[r-\tau\right]}{E_p\left[max\left(\tau-r,0\right)\right]} \\
&= \frac{E_p\left[r-\tau\right]}{E_p\left[max\left(\tau-r,0\right)\right]}+1, \\
&= \text{S-}\Omega_p+1. \qquad \text{(C.17)}
\end{aligned}
$$

where S-Ω_p is the Sharpe-*Omega* ratio (Kazemi et al., 2004).

The *Omega* function can thus be recast into a quantity that can be interpreted in a way similar to the Sharpe-*Omega* ratio, with a risk indicator which is a weighted expected shortfall with respect to a given threshold.

The Sharpe-*Omega* measure thus shares with *Omega* similar drawbacks (same ranks for funds for both measures). Indeed, the riskiest asset will be better ranked by the Sharpe-*Omega* ratio for two given assets with different risks and equal expectations on return deviations from a common threshold.

Table C.1: *Omega* Quantile Functions for Some of the Most Common Distributions

Name Support	Density Function $f_p(r)$	CDF $F_p(r)$	*Omega* Quantile Function $\Omega_p(\tau)$
Normal $(-\infty,+\infty)$	$\left(\sigma\sqrt{2\pi}\right)^{-1} \times exp\left[-\frac{1}{2}(r^*)^2\right]$	$\left(\sigma\sqrt{2\pi}\right)^{-1} \times \int_{-\infty}^{r} exp\left[-\frac{1}{2}(x^*)^2\right]dx$	$\left\{(\mu-\tau)[1-F_p(\tau)]+\sigma^2 f_p(\tau)\right\} \times \left\{(\tau-\mu)F_p(\tau)+\sigma^2 f_p(\tau)\right\}^{-1}$
Student $(-\infty,+\infty)$	$\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sigma\sqrt{\nu\pi}}\left[1+\frac{1}{\nu}(r^*)^2\right]^{\frac{-(\nu+1)}{2}}$	$\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sigma\sqrt{\nu\pi}}\int_{-\infty}^{r}\left[1+\frac{1}{\nu}(x^*)^2\right]^{\frac{-(\nu+1)}{2}}dx$	$\left\{(\mu-\tau)[1-F_p(\tau)]+\sigma^2\left(\frac{\nu+(\tau^*)^2}{\nu-1}\right)f_p(\tau)\right\} \times \left\{(\tau-\mu)F_p(\tau)+\sigma^2\left(\frac{\nu+(\tau^*)^2}{\nu-1}\right)f_p(\tau)\right\}^{-1}$
Logistic $(-\infty,+\infty)$	$\frac{exp(-r^*)}{\sigma[1+exp(-r^*)]^2}$	$\int_{-\infty}^{r}\frac{exp(-x^*)}{\sigma[1+exp(-x^*)]^2}dx$	$\left\{(\mu-\tau)[1-F_p(\tau)]+\sigma^2\varpi f_p(\tau)\right\} \times \left\{(\tau-\mu)F_p(\tau)+\sigma^2\varpi f_p(\tau)\right\}^{-1}$
Extreme Value $(-\infty,+\infty)$	$\frac{1}{\sigma}exp\left(-r^*-e^{-r^*}\right)$	$\int_{-\infty}^{r} exp(-x^*)\,dx$	$\left\{(\mu-\tau)[1-F_p(\tau)]+\sigma^2\kappa f_p(\tau)\right\} \times \left\{(\tau-\mu)F_p(\tau)+\sigma^2\kappa f_p(\tau)\right\}^{-1}$

Source: This table shows *Omega* quantile functions expressed for some of the most common distributions (See Aitchison and Brown, 1963; Malo, 2005; McNeil et al., 2005). For the sake of simplicity, we use the following notations: $r^* = (r-\mu)/\sigma$, $\tau^* = (\tau-\mu)/\sigma$, $\varpi = \left(e^{\tau^*}+e^{-\tau^*}+2\right)log\left(1+e^{\tau^*}\right)-\tau^*\left(1+e^{\tau^*}\right)$, and $\kappa = e^{\tau^*}\left\{\gamma-\tau^*+exp\left[exp(-\tau^*)\right]Ei\left[-exp(-\tau^*)\right]\right\}$ where γ is the Euler constant and Ei is the exponential integral function. Computations by the authors.

Appendix D. Proofs of Proposition 1 and Corollary 1

Proposition 1. *Inconsistency of the* Omega *Ranking*

Consider returns on two assets A and B, with non-degenerated densities, with asset A being preferred to asset B according to the Second-order Stochastic Dominance criterion *(denoted $A \succ_{SSD} B$). We define the following quantity:*

$$\Delta(\tau) = \frac{\int_{-\infty}^{\tau} [F_B(r) - F_A(r)]\ dr}{\int_{-\infty}^{+\infty} [F_B(r) - F_A(r)]\ dr}$$

A necessary condition for having Omega *inconsistent with the Second-order Stochastic Dominance* criterion *(SSD) is that $\Delta(\tau) > 1$.*
Under this condition, Omega *is inconsisten with the Second-order Stochastic Dominance if*

$$\Omega_B(\tau) < \frac{\Delta(\tau) - 1}{\Delta(\tau)}.$$

Proof of Proposition 1.

The Second-order Stochastic Dominance of A with respect to B implies that:

$$\int_{-\infty}^{\tau} [F_B(r) - F_A(r)]\ dr \geq 0, \tag{D.1}$$

where τ is a given threshold and the equality is strict for at least one value of τ.

We search conditions giving:

$$\Omega_B(\tau) > \Omega_A(\tau), \tag{D.2}$$

thus pointing at a preference ordering which is reversed to that obtained by Second-order Stochastic Dominance.

First, we note that

$$\begin{cases} \Omega_A(\tau) = \int_{\tau}^{+\infty} [1 - F_A(r)]\, dr \times \left[\int_{-\infty}^{\tau} F_A(r)\, dr \right]^{-1}, \\ \Omega_B(\tau) = \int_{\tau}^{+\infty} [1 - F_B(r)]\, dr \times \left[\int_{-\infty}^{\tau} F_B(r)\, dr \right]^{-1}, \end{cases}$$

and that both Omega measures are, by definition, positive.

To simplify notations, we redefine the previous ratios as:

$$\Omega_p(\tau) = \frac{i_p(\tau)}{I_p(\tau)}, \tag{D.3}$$

where $i_p(\tau) = \int_{\tau}^{+\infty} [1 - F_p(r)]\, dr$ and $I_p(\tau) = \int_{-\infty}^{\tau} F_p(r)\, dr$ with $p = \{A, B\}$.

Using Equations (D.2) and (D.3), we have:

$$
\begin{aligned}
&\Omega_B(\tau) - \Omega_A(\tau) > 0 \\
&\Leftrightarrow \frac{i_B(\tau)}{I_B(\tau)} - \frac{i_A(\tau)}{I_A(\tau)} > 0 \\
&\Leftrightarrow i_B(\tau)I_A(\tau) - i_A(\tau)I_B(\tau) > 0 \\
&\Leftrightarrow i_B(\tau)I_A(\tau) - i_A(\tau)I_B(\tau) + i_B(\tau)I_B(\tau) - i_B(\tau)I_B(\tau) > 0 \\
&\Leftrightarrow i_B(\tau)\left[I_A(\tau) - I_B(\tau)\right] - I_B(\tau)\left[i_A(\tau) - i_B(\tau)\right] > 0.
\end{aligned} \tag{D.4}
$$

If we focus on the last term of Equation (D.4), we obtain the following equalities:

$$
\begin{aligned}
i_A(\tau) - i_B(\tau) &= \int_{\tau}^{+\infty} \left[1 - F_A(r)\right] dr - \int_{\tau}^{+\infty} \left[1 - F_B(r)\right] dr \\
&= \int_{\tau}^{+\infty} \left\{\left[1 - F_A(r)\right] - \left[1 - F_B(r)\right]\right\} dr \\
&= -\int_{\tau}^{+\infty} \left[F_A(r) - F_B(r)\right] dr.
\end{aligned} \tag{D.5}
$$

From Equation (D.1), we thus have:

$$
\lim_{\tau \to +\infty} \left\{ \int_{-\infty}^{\tau} \left[F_A(r) - F_B(r)\right] \right\} dr = \int_{-\infty}^{+\infty} \left[F_A(r) - F_B(r)\right] dr \leq 0. \tag{D.6}
$$

Equation (D.6) is identical to Equation (D.1) when $\tau = +\infty$. When $\tau = +\infty$, however, $I_A^{+\infty} - I_B^{+\infty} = E\left[B\right] - E\left[A\right]$.[33] Further, we stress that if the Second Order Stochastic dominance is strict for each value of τ, the quantity in D.6 is negative, being the opposite of equation D.1 for the threshold τ diverging toward $+\infty$, while if the dominance is not strict it can be negative, but can also be null.

We can rewrite the indefinite integral as follows:

$$
\int_{-\infty}^{+\infty} \left[F_A(r) - F_B(r)\right] dr = \int_{-\infty}^{\tau} \left[F_A(r) - F_B(r)\right] dr + \int_{\tau}^{+\infty} \left[F_A(r) - F_B(r)\right] dr. \tag{D.7}
$$

From this relation, we can write that:

$$
\begin{aligned}
-\int_{\tau}^{+\infty} \left[F_A(r) - F_B(r)\right] dr &= \int_{-\infty}^{\tau} \left[F_A(r) - F_B(r)\right] dr - \int_{-\infty}^{+\infty} \left[F_A(r) - F_B(r)\right] dr \\
&= \left[I_A(\tau) - I_B(\tau)\right] - \left(E\left[B\right] - E\left[A\right]\right).
\end{aligned} \tag{D.8}
$$

[33]The term $I_p^{+\infty}$ corresponds to the limit of $I_p(\tau)$ when τ tends to infinity, *i.e.* $I_p^{+\infty} = \lim_{\tau \to +\infty} \left[\int_{-\infty}^{\tau} F_p\left(r\right) dr\right]$, with $p = \{A, B\}$. Moreover $\int_{-\infty}^{+\infty} \left[F_A(r) - F_B(r)\right] dr = E\left[B\right] - E\left[A\right]$.

Substituting Equation (D.8) into Equation (D.4) leads us to the following result:

$$\begin{aligned}
& i_B(\tau)\left[I_A(\tau) - I_B(\tau)\right] - I_B(\tau)\left[i_A(\tau) - i_B(\tau)\right] > 0 \\
& \Leftrightarrow i_B(\tau)\left[I_A(\tau) - I_B(\tau)\right] - I_B(\tau)\left\{\left[I_A(\tau) - I_B(\tau)\right] - \left(E\left[B\right] - E\left[A\right]\right)\right\} > 0 \\
& \Leftrightarrow \left[i_B(\tau) - I_B(\tau)\right] \times \left[I_A(\tau) - I_B(\tau)\right] + I_B(\tau)\left(E\left[B\right] - E\left[A\right]\right) > 0 \qquad .
\end{aligned} \tag{D.9}$$

Then, taking in mind that the last term in parentheses is negative, we have

$$\left[i_B(\tau) - I_B(\tau)\right] \times \frac{I_A(\tau) - I_B(\tau)}{E\left[B\right] - E\left[A\right]} + I_B(\tau) < 0 \quad . \tag{D.10}$$

We redefine the ratio as

$$\Delta\left(\tau\right) = \frac{I_A(\tau) - I_B(\tau)}{E\left[B\right] - E\left[A\right]} \quad . \tag{D.11}$$

and note that, being a ratio of negative quantities it must be positive. Furthermore, it is related to the Stochastic Dominance being, by definition, equivalent to

$$\Delta\left(\tau\right) = \frac{I_B(\tau) - I_A(\tau)}{E\left[A\right] - E\left[B\right]} \quad , \tag{D.12}$$

where the numerator measures the condition for stochastic dominance at the threshold level τ whether the denominator measures the condition for τ diverging to infinity. Note that the denominator might be zero when SSD is not strict, leading to a divergence of $\Delta\left(\tau\right)$ to infinity.

If we assume the Stochastic Dominance is strict for the limiting case, then $\Delta\left(\tau\right)$ always exist, and it assumes values on $\{0\} \cup \mathbb{R}^{+}$.

Therefore, a condition for having B Omega-preferred to A comes from a relation of the form

$$\begin{aligned}
& \left[i_B(\tau) - I_B(\tau)\right] \times \Delta\left(\tau\right) + I_B(\tau) < 0, \\
& \Leftrightarrow i_B(\tau)\Delta\left(\tau\right) - I_B(\tau)\Delta\left(\tau\right) + I_B(\tau) < 0 \\
& \Leftrightarrow \left[1 - \Delta\left(\tau\right)\right] I_B(\tau) + i_B(\tau)\Delta\left(\tau\right) < 0 \\
& \Leftrightarrow \left[1 - \Delta\left(\tau\right)\right] + \frac{i_B(\tau)}{I_B(\tau)}\Delta\left(\tau\right) < 0 \\
& \Leftrightarrow \left[1 - \Delta\left(\tau\right)\right] + \Omega_B\left(\tau\right)\Delta\left(\tau\right) < 0 \\
& \Leftrightarrow \Omega_B\left(\tau\right) < \frac{\Delta\left(\tau\right) - 1}{\Delta\left(\tau\right)}.
\end{aligned} \tag{D.13}$$

We distinguish between three cases of $\Delta\left(\tau\right)$

- $\Delta\left(\tau\right) = 0$. In this case $\Omega_B(\tau)$ cannot be greater than $\Omega_A(\tau)$ as, by simplifying (D.10) we have $I_B\left(\tau\right) < 0$, which is impossible;

- $0 < \Delta\left(\tau\right) \leq 1$. This gives from (D.13) that to have B Omega-preferred to A, we must have $\Omega_B(\tau) < 0$, which is not possible as the indicator is positive by construction;

- $\Delta\left(\tau\right) > 1$. This is the only case in which the preference ordering is reversed with respect to that of the Second Order Stochastic dominance, leading to the condition reported in Proposition 1.

Note that, when $\Delta(\tau)$ diverges to plus infinity, the condition of Proposition 1 converges to $\Omega_B(\tau) < 1$.

Corollary 1. *Inconsistency of the* Omega *Ranking under Symmetry of Densities and Equality of Means*

If the two assets of Proposition 1 have returns with a density of the same family, with identical means $E_A(r) = E_B(r) = \mu$, but different volatilities such as $\sigma_A(r) < \sigma_B(r)$, then, the Second-order Stochastic Dominance criterion *implies that the ranking in terms of* Omega *of the two assets will be:*

1. *if $\tau = \mu$, $A \approx_\Omega B$;*
2. *if $\tau > \mu$, $A \prec_\Omega B$,*
3. *if $\tau < \mu$, $A \succ_\Omega B$;*

and thus Omega *is inconsistent with the Second-order Stochastic Dominance* criterion (SSD in cases 1 and 2).

Proof of Corollary 1.

We are in the first case of Proposition 1 and the proof is thus straightforward.

However, a more direct relation between thresholds and preference can be now obtained. The proof corresponds to that of Proposition 1 up to Equation (D.4) of the previous proof. By the SSD properties, the symmetry of $F_A(r)$ and $F_B(r)$, and the equalities of the means, we have the indefinite integral as:

$$\begin{aligned} \int_{-\infty}^{+\infty} [F_B(r) - F_A(r)]\, dr = & \int_{-\infty}^{\tau} [F_B(r) - F_A(r)]\, dr + \int_{\tau}^{+\infty} [F_B(r) - F_A(r)]\, dr \\ = & 0. \end{aligned} \tag{D.14}$$

We can express the second term as:

$$\int_{\tau}^{+\infty} [F_B(r) - F_A(r)]\, dr = -\int_{\tau}^{+\infty} [1 - F_B(r) - 1 - F_A(r)]\, dr. \tag{D.15}$$

Thus:

$$\begin{aligned} I_B(\tau) - I_A(\tau) - [i_B(\tau) - i_A(\tau)] = 0 \\ I_B(\tau) - I_A(\tau) = i_B(\tau) - i_A(\tau), \end{aligned} \tag{D.16}$$

and by replacing Equation (D.16) into Equation (D.13), we have:

$$\begin{aligned} & i_A(\tau)[I_B(\tau) - I_A(\tau)] - I_A(\tau)[I_B(\tau) - I_A(\tau)] > 0 \\ \Leftrightarrow & [i_A(\tau) - I_A(\tau)]\,[I_B(\tau) - I_A(\tau)] > 0. \end{aligned} \tag{D.17}$$

The second term in Equation (D.17) derives from the SSD, and the symmetry of the density and the equality of the means, leads us to write:

$$[I_B(\tau) - I_A(\tau)] > 0, \tag{D.18}$$

with:

$$\lim_{\tau \to +\infty} \left[I_B(\tau) - I_A(\tau) \right] = 0. \tag{D.19}$$

For the first term in Equation (D.17), *i.e.* $i_A(\tau) - I_A(\tau)$, we have by symmetry:

$$\begin{cases} \int\limits_{-\infty}^{\tau} F_A(r)dr = \int\limits_{\tau}^{+\infty} \left[1 - F_A(r)\right] dr, \;\text{ if }\; \tau = \mu, \\ \text{and} \\ \int\limits_{-\infty}^{\tau} F_A(r)dr < \int\limits_{\tau}^{+\infty} \left[1 - F_A(r)\right] dr, \;\text{ if }\; \tau < \mu. \end{cases} \tag{D.20}$$

In conclusion, when we consider two symmetric distributions and equal mean returns, $\Omega_A(\tau) > \Omega_B(\tau)$ is true if and only if $\tau < \mu$.

The result is coherent with Proposition 1, as under the condition of the Corollary, $\Delta(\tau)$ diverges to plus infinity, and to obtain that B is preferred to A on the basis on an *Omega* criterion, we need the threshold τ to be above the mean μ. ■