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A nonlinear problem for the Laplace equation with a degenerating Robin condition

Paolo Musolino* | Gennady Mishuris

[1]Department of Mathematics, Aberystwyth University, Ceredigion SY23 3BZ, Wales, United Kingdom

Correspondence
*P. Musolino. Email: musolinopaolo@gmail.com

Summary

We investigate the behavior of the solutions of a mixed problem for the Laplace equation in a domain Ω. On a part of the boundary $\partial\Omega$ we consider a Neumann condition, whereas in another part we consider a nonlinear Robin condition which depends on a positive parameter δ in such a way that for $\delta = 0$ it degenerates into a Neumann condition. For δ small and positive, we prove that the boundary value problem has a solution $u(\delta, \cdot)$. We describe what happens to $u(\delta, \cdot)$ as $\delta \to 0$ by means of representation formulas in terms of real analytic maps. Then we confine ourselves to the linear case and we compute explicitly the power series expansion of the solution.

KEYWORDS:
boundary value problems for second-order elliptic equations, integral equations methods, singularly perturbed problem, Laplace operator, Robin problem, Neumann problem

1 | INTRODUCTION

In this paper, we study the asymptotic behavior of the solutions of a boundary value problem for the Laplace equation with a (nonlinear) Robin boundary condition which degenerates into a Neumann condition.

Boundary value problems with perturbed Robin or mixed conditions have been investigated by several authors. For example, Wendland, Stephan, and Hsiao[1] considered a family of Poincaré problems approximating a mixed boundary value problem for the Laplace equation in the plane. Kirsch[2] studied the convergence of the solution of the Helmholtz equation with boundary condition of the type $-\epsilon\frac{\partial u}{\partial \nu} + u = g$ to the solution with Dirichlet condition $u = g$ as $\epsilon \to 0$. Costabel and Dauge[3] studied a mixed Neumann-Robin problem for the Laplace operator, where the Robin condition contains a parameter ϵ so that it tends to a Dirichlet condition as $\epsilon \to 0$. An extension to nonlinear equation has been considered, for example, in Berestycki and Wei[4]. Degenerating nonlinear Robin conditions in the frame of homogenization problems have been studied by Gómez, Lobo, Pérez, and Sanchez-Palencia[5]. Singularly perturbed boundary conditions for the Maxwell equations have been analyzed for example in Ammari and Nédélec[6]. Moreover, Schmidt and Hiptmair[7] have exploited integral equation methods for singularly perturbed boundary conditions in the frame of transmission problems. Furthermore, an approach based on potential theory to prove the solvability of a small nonlinear perturbation of a homogeneous linear transmission problem can be found in Dalla Riva and Mishuris[8]. Concerning existence and uniqueness results for boundary value problems with nonlinear Robin conditions, we also mention, *e.g.*, Donato, Monsurrò, and Raimondi[9].

We note that the transmission problem for a composite domain with imperfect (nonnatural) conditions along the joint boundary is, in fact, a generalisation of the classical Robin problem. Such transmission conditions frequently appear in practical applications for various nonlinear multiphysics problems (*e.g.*, Mishuris, Miszuris, and Öchsner[10,11], and Mishuris[12]). Moreover, the imperfect transmission conditions allow one to perform numerical analysis of practical problems with thin interphases at low cost with sufficient accuracy (see Mishuris and Öchsner [13], Mishuris, Miszuris, and Öchsner [14], and Sonato, Piccolroaz, Miszuris, and Mishuris [15]).

In this paper, instead, we are interested in the case where the Robin condition degenerates into a Neumann condition.

In order to introduce the problem, we first define the geometric setting. We fix once for all a natural number

$$n \in \mathbb{N} \setminus \{0, 1\}\,.$$

Then we consider $\alpha \in]0,1[$ and two subsets Ω^i, Ω^o of $\mathbb{R}^n$ satisfying the following assumption:

$$\begin{gathered}\Omega^i \text{ and } \Omega^o \text{ are bounded open connected subsets of } \mathbb{R}^n \text{ of class } C^{1,\alpha}\\ \text{such that } \overline{\Omega^i} \subseteq \Omega^o \text{ and that } \mathbb{R}^n \setminus \overline{\Omega^i} \text{ and } \mathbb{R}^n \setminus \overline{\Omega^o} \text{ are connected.}\end{gathered}$$

For the definition of sets and functions of the Schauder class $C^{k,\alpha}$ ($k \in \mathbb{N}$) we refer, *e.g.*, to Gilbarg and Trudinger [16]. The letter 'i' stands for 'inner' and the letter 'o' stands for 'outer'. The symbol '$\overline{\cdot}$' denotes the closure. Then we introduce the domain Ω by setting

$$\Omega \equiv \Omega^o \setminus \overline{\Omega^i}\,.$$

We note that the boundary $\partial\Omega$ of Ω consists of the two connected components $\partial\Omega^o$ and $\partial\Omega^i$. Therefore, we can identify, for example, $C^{0,\alpha}(\partial\Omega)$ with the product $C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i)$. In order to define the boundary data, we fix two functions

$$g^o \in C^{0,\alpha}(\partial\Omega^o)\,, \qquad g^i \in C^{0,\alpha}(\partial\Omega^i)\,.$$

Then we take $\delta_0 > 0$ and a family $\{F_\delta\}_{\delta\in]0,\delta_0[}$ of functions from $\mathbb{R}$ to $\mathbb{R}$. Next, for each $\delta \in]0, \delta_0[$ we want to consider a nonlinear boundary value problem for the Laplace operator. Namely, we consider a Neumann condition on $\partial\Omega^o$ and a nonlinear Robin condition on $\partial\Omega^i$. Thus, for each $\delta \in]0, \delta_0[$ we consider the following boundary value problem:

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \Omega\,, \\ \frac{\partial}{\partial\nu_{\Omega^o}} u(x) = g^o(x) & \forall x \in \partial\Omega^o\,, \\ \frac{\partial}{\partial\nu_{\Omega^i}} u(x) = \delta F_\delta(u(x)) + g^i(x) & \forall x \in \partial\Omega^i\,, \end{cases} \tag{1}$$

where ν_{Ω^o} and ν_{Ω^i} denote the outward unit normal to $\partial\Omega^o$ and to $\partial\Omega^i$, respectively.

As a first step, under suitable assumptions, in this paper we show that for each δ positive and small enough, problem (1) has a solution, which we denote by $u(\delta, \cdot)$. Then we are interested in studying the behavior of $u(\delta, \cdot)$ as $\delta \to 0$ and thus we pose the following questions.

(i) Let x be a fixed point in Ω. What can be said of the map $\delta \mapsto u(\delta, x)$ when δ is close to 0 and positive?

(ii) What can be said of the map $\delta \mapsto \int_\Omega |\nabla u(\delta, x)|^2\, dx$ when δ is close to 0 and positive?

We also note that if in correspondence of the limiting value $\delta = 0$, we omit the term

$$\delta F_\delta(u(x))$$

in (1), then we obtain the Neumann problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \Omega\,, \\ \frac{\partial}{\partial\nu_{\Omega^o}} u(x) = g^o(x) & \forall x \in \partial\Omega^o\,, \\ \frac{\partial}{\partial\nu_{\Omega^i}} u(x) = g^i(x) & \forall x \in \partial\Omega^i\,. \end{cases} \tag{2}$$

On the other hand, by the Divergence Theorem and classical existence results for the Neumann problem, problem (2) has (at least) a solution if and only if

$$\int_{\partial\Omega^o} g^o\, d\sigma - \int_{\partial\Omega^i} g^i\, d\sigma = 0\,. \tag{3}$$

This means, in particular, that if (3) does not hold, then $u(\delta, \cdot)$ cannot converge to a solution of problem (2) as $\delta \to 0$.

In contrast with asymptotic expansion methods, in this paper, we answer the questions in (i), (ii) by representing the maps of (i), (ii) in terms of real analytic maps in Banach spaces and in terms of known functions of δ (for the definition and properties of real analytic maps, we refer to Deimling [17, p. 150]). We observe that if for example we know that the function in (i) equals for $\delta > 0$ a real analytic function defined in a whole neighborhood of $\delta = 0$, then we know that such a map can be expanded in power series for δ small.

Such an approach has been proposed by Lanza de Cristoforis [18] for the analysis of singularly perturbed problems in perforated domain as an alternative to asymptotic expansion methods (cf., *e.g.*, Maz'ya, Movchan, and Nieves [19] and Maz'ya, Nazarov, and Plamenevskij [20,21]). In particular, it has been exploited to analyze singularly perturbed (linear and nonlinear) Robin and mixed problems in domains with small holes (cf., e.g., Lanza de Cristoforis [22] and Dalla Riva and Musolino [23] for the Laplace equation and Dalla Riva and Lanza de Cristoforis [24,25] for the Lamé equations).

The paper is organized as follows. In Section 2, we consider some model problems in an annular domain where we can explicitly construct the solutions and discuss the behavior as δ tends to 0. In Section 3, we formulate our problem in terms of integral equations. In Section 4, we prove our main result, which answers our questions (i), (ii) above, and in Section 5 we discuss a local uniqueness property of the family of solutions. Finally, in Section 6 we make some comments on the linear case and compute the power series expansion of the solution.

2 | MODEL PROBLEMS

In order to illustrate some aspects of the problem under investigation, in this section, we consider the set

$$\Omega \equiv \mathbb{B}_n(0,1) \setminus \overline{\mathbb{B}_n(0,1/2)}\,,$$

i.e., we take $\Omega^o \equiv \mathbb{B}_n(0,1)$ and $\Omega^i \equiv \mathbb{B}_n(0,1/2)$, where, for $r > 0$, the symbol $\mathbb{B}_n(0,r)$ denotes the open ball in $\mathbb{R}^n$ of center 0 and radius r.

2.1 | A linear problem

We begin with a linear problem and to do so we take $a, b \in \mathbb{R}$. Then for each $\delta \in]0,+\infty[$ we consider the problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{B}_n(0,1) \setminus \overline{\mathbb{B}_n(0,1/2)}\,, \\ \frac{\partial}{\partial \nu_{\mathbb{B}_n(0,1)}} u(x) = a & \forall x \in \partial\mathbb{B}_n(0,1)\,, \\ \frac{\partial}{\partial \nu_{\mathbb{B}_n(0,1/2)}} u(x) = \delta u(x) + b & \forall x \in \partial\mathbb{B}_n(0,1/2)\,. \end{cases} \tag{4}$$

As is well known, for each $\delta \in]0,+\infty[$ problem (4) has a unique solution in $C^{1,\alpha}(\overline{\Omega})$ and we denote it by u_δ. On the other hand, if instead we put $\delta = 0$ in (4) we obtain

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{B}_n(0,1) \setminus \overline{\mathbb{B}_n(0,1/2)}\,, \\ \frac{\partial}{\partial \nu_{\mathbb{B}_n(0,1)}} u(x) = a & \forall x \in \partial\mathbb{B}_n(0,1)\,, \\ \frac{\partial}{\partial \nu_{\mathbb{B}_n(0,1/2)}} u(x) = b & \forall x \in \partial\mathbb{B}_n(0,1/2)\,. \end{cases} \tag{5}$$

The solvability of problem (5) is subject to a compatibility condition on the Neumann data on $\partial\mathbb{B}_n(0,1)$ and on $\partial\mathbb{B}_n(0,1/2)$. More precisely, problem (5) has a solution if and only if

$$a \int\limits_{\partial\mathbb{B}_n(0,1)} d\sigma - b \int\limits_{\partial\mathbb{B}_n(0,1/2)} d\sigma = 0\,,$$

i.e., if and only if

$$a s_n - b\frac{1}{2^{n-1}} s_n = 0\,, \tag{6}$$

where s_n denotes the $(n-1)$-dimensional measure of $\partial\mathbb{B}_n(0,1)$. Condition (6) can be rewritten as

$$a = \frac{b}{2^{n-1}}\,. \tag{7}$$

In particular, if $a = \frac{b}{2^{n-1}}$ then the Neumann problem (5) has a one-dimensional space of solutions; if instead $a \neq \frac{b}{2^{n-1}}$, problem (5) does not have any solution.

This implies that in general the solution u_δ of problem (4) cannot converge to a solution of (5) as $\delta \to 0$, if the compatibility condition (7) does not hold. Therefore, we wish to understand the behavior of u_δ as $\delta \to 0$ and we do so by constructing explicitly u_δ.

In order to construct the solution u_δ we consider separately case $n = 2$ and $n \geq 3$.

If $n = 2$ we look for the function u_δ in the form

$$u_\delta(x) \equiv A_\delta \log|x| + B_\delta \qquad \forall x \in \overline{\Omega},$$

with A_δ and B_δ to be set so that the boundary conditions of problem (4) are satisfied.

We first note that

$$\nabla u_\delta(x) = A_\delta \frac{x}{|x|^2},$$

and that accordingly

$$\frac{\partial u_\delta(x)}{\partial \nu_{\mathbb{B}_2(0,1)}} = \frac{x}{|x|} \cdot A_\delta \frac{x}{|x|^2} = A_\delta \qquad \forall x \in \partial\mathbb{B}_2(0,1),$$

which implies that we must have

$$A_\delta = a$$

in order to fulfil the Neumann condition on $\partial\mathbb{B}_n(0,1)$. On the other hand, as far as the Robin condition on $\partial\mathbb{B}_2(0,1/2)$ is concerned, we must find B_δ such that

$$\frac{x}{|x|} \cdot a \frac{x}{|x|^2} = \delta(a \log|x| + B_\delta) + b \qquad \forall x \in \partial\mathbb{B}_2(0,1/2).$$

Then a straightforward computation implies that we must have

$$B_\delta = \frac{1}{\delta}(2a - b) + a \log 2.$$

As a consequence, if $n = 2$ we have

$$u_\delta(x) \equiv a \log|x| + a \log 2 + \frac{1}{\delta}(2a - b) \qquad \forall x \in \overline{\Omega}. \tag{8}$$

Then we turn to consider the case of dimension $n \geq 3$ and we look for a solution of problem (4) in the form

$$u_\delta(x) \equiv A_\delta \frac{1}{(2-n)|x|^{n-2}} + B_\delta \qquad \forall x \in \overline{\Omega},$$

with A_δ and B_δ to be set so that the boundary conditions of problem (4) are satisfied. By arguing as above, one deduces that

$$A_\delta = a, \qquad B_\delta = \frac{1}{\delta}(2^{n-1}a - b) + a\frac{2^{n-2}}{n-2},$$

and thus

$$u_\delta(x) \equiv a \frac{1}{(2-n)|x|^{n-2}} + a\frac{2^{n-2}}{n-2} + \frac{1}{\delta}(2^{n-1}a - b) \qquad \forall x \in \overline{\Omega}. \tag{9}$$

Thus, by looking at (8) and (9), we note that if condition (7) does not hold, then

$$\lim_{\delta \to 0} \|u_\delta\|_\infty = +\infty.$$

Comparing (8) and (9) one can write the solutions in a uniform manner:

$$u_\delta(x) = u^{(0)}(x) + \frac{1}{\delta} u^{(1)}(x), \tag{10}$$

where

$$u^{(0)}(x) \equiv \begin{cases} a \log|x| + a \log 2 & \text{if } n = 2, \\ a\frac{1}{(2-n)|x|^{n-2}} + a\frac{2^{n-2}}{n-2} & \text{if } n \geq 3, \end{cases}, \quad u^{(1)}(x) \equiv 2^{n-1}a - b, \qquad \forall x \in \overline{\Omega},$$

and both functions $u^{(0)}, u^{(1)} \in L_\infty(\Omega)$. In particular, we note that $u^{(0)}$ is the unique solution of (5) such that

$$\int_{\partial\mathbb{B}_n(0,1/2)} u^{(0)}\, d\sigma = 0.$$

On the other hand, if (7) holds, we have $u^{(1)} \equiv 0$ and

$$u_\delta(x) \equiv u^{(0)}(x),$$

for all $\delta \in]0, +\infty[$, and u_δ is also a solution to problem (5).

2.2 A nonlinear problem

In this section we analyze a nonlinear problem and, for the sake of simplicity, we confine to the case of dimension $n = 2$.

For each $\delta \in]0, +\infty[$ we consider the problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{B}_2(0,1) \setminus \overline{\mathbb{B}_2(0,1/2)}\,, \\ \frac{\partial}{\partial \nu_{\mathbb{B}_2(0,1)}} u(x) = a & \forall x \in \partial\mathbb{B}_2(0,1)\,, \\ \frac{\partial}{\partial \nu_{\mathbb{B}_2(0,1/2)}} u(x) = \delta^2 u(x) + \delta^2 u^2(x) + b & \forall x \in \partial\mathbb{B}_2(0,1/2)\,. \end{cases} \tag{11}$$

Now we note that we can collect δ in the right hand side of the third equation in (11) and thus we can write the Robin condition as

$$\frac{\partial}{\partial \nu_{\mathbb{B}_2(0,1/2)}} u(x) = \delta\Big(\delta u(x) + \delta u^2(x)\Big) + b \qquad \forall x \in \partial\mathbb{B}_2(0,1/2)\,.$$

If for each $\delta \in]0, +\infty[$ we introduce the function

$$F_\delta(\tau) = \delta\tau + \delta\tau^2 \qquad \forall \tau \in \mathbb{R}\,,$$

we can rewrite problem (11) as follows:

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{B}_2(0,1) \setminus \overline{\mathbb{B}_2(0,1/2)}\,, \\ \frac{\partial}{\partial \nu_{\mathbb{B}_2(0,1)}} u(x) = a & \forall x \in \partial\mathbb{B}_2(0,1)\,, \\ \frac{\partial}{\partial \nu_{\mathbb{B}_2(0,1/2)}} u(x) = \delta F_\delta(u(x)) + b & \forall x \in \partial\mathbb{B}_2(0,1/2)\,. \end{cases} \tag{12}$$

Then, again we look for a solution u_δ in the form

$$u_\delta(x) \equiv A_\delta \log|x| + B_\delta \qquad \forall x \in \overline{\Omega}\,,$$

with A_δ and B_δ to be set so that the boundary conditions of problem (12) are satisfied.

As we have seen, to ensure the validity of the Neumann condition on $\partial\mathbb{B}_2(0,1)$ we must have

$$A_\delta = a\,.$$

On the other hand, in order to satisfy the Robin condition, we have to find B_δ such that

$$2a = \delta F_\delta(-a\log 2 + B_\delta) + b\,.$$

Motivated by the linear case, we find it convenient to replace B_δ by $\tilde{B}_\delta/\delta + a\log 2$. In other words we look for a solution u_δ in the form

$$u_\delta(x) \equiv A\log|x| + \frac{\tilde{B}_\delta}{\delta} + a\log 2 \qquad \forall x \in \overline{\Omega}\,,$$

with $\tilde{B}_\delta$ such that

$$2a = \delta F_\delta\Big(\frac{1}{\delta}\tilde{B}_\delta\Big) + b\,. \tag{13}$$

Then we note that if we set

$$\tilde{F}(\xi,\delta) = \delta\xi + \xi^2 \qquad \forall (\xi,\delta) \in \mathbb{R}^2\,,$$

we have

$$\delta F_\delta\Big(\frac{1}{\delta}\xi\Big) = \tilde{F}(\xi,\delta) \qquad \forall \xi \in \mathbb{R}\,. \tag{14}$$

As a consequence, we can rewrite equation (13) as

$$2a = \tilde{F}\Big(\tilde{B}_\delta, \delta\Big) + b\,. \tag{15}$$

For general $\tilde{F}$, under suitable assumptions, one can try to resolve equation (15) by means of the Implicit Function Theorem. On the other hand, for our specific case, for each $\delta \in]0, +\infty[$ one has that the solutions ζ in $\mathbb{C}$ of equation

$$2a = \tilde{F}\Big(\zeta, \delta\Big) + b$$

are delivered by

$$\frac{-\delta + z_0}{2}\,, \qquad \frac{-\delta - z_0}{2}\,.$$

where

$$z_0^2 = \delta^2 - 4(b - 2a)\,.$$

Thus, if we look for solutions $\tilde{B}_\delta \in \mathbb{R}$ of equation (15) for δ positive and close to 0, we may have one, two, or no solutions to (15) depending on the sign of

$$-4(b-2a)\,.$$

Therefore, for δ small and positive we may have one, two, or no solutions to the nonlinear problem (11). In particular, a crucial role for the solvability of problem (11) is played by the function $\tilde{F}$ which ensures the validity of equation (14).

2.2.1 | A family of nonlinear problems

In order to play with the structure of the nonlinear boundary condition, for each $\delta \in]0,+\infty[$, we consider the family of problems

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{B}_2(0,1) \setminus \overline{\mathbb{B}_2(0,1/2)}\,, \\ \frac{\partial}{\partial \nu_{\mathbb{B}_2(0,1)}} u(x) = a & \forall x \in \partial\mathbb{B}_2(0,1)\,, \\ \frac{\partial}{\partial \nu_{\mathbb{B}_2(0,1/2)}} u(x) = \delta u(x)(1 + c\delta^{\gamma_1} u^{\gamma_2}(x)) + b & \forall x \in \partial\mathbb{B}_2(0,1/2)\,, \end{cases} \tag{16}$$

where $c \in \mathbb{R}$ and $\gamma_1, \gamma_2 \in \mathbb{N}$. Note that such type of boundary conditions is crucially important for practical applications. For example, in metallurgy and metal forming processes, the typical boundary condition involves $\gamma_2 = 4$ where the respective term corresponds to the heat exchange due to the radiation at high temperature (see Golitsyna[26], Letavin and Mishuris[27], and Letavin and Shestakov[28]).

Now we note that we can rewrite the Robin condition as

$$\frac{\partial}{\partial \nu_{\mathbb{B}_2(0,1/2)}} u(x) = \delta\Big(u(x) + c\delta^{\gamma_1} u^{\gamma_2+1}(x)\Big) + b \qquad \forall x \in \partial\mathbb{B}_2(0,1/2)\,.$$

As above, for each $\delta \in]0,+\infty[$ we introduce the function

$$F_\delta(\tau) = \tau + c\delta^{\gamma_1}\tau^{\gamma_2+1} \qquad \forall \tau \in \mathbb{R}\,.$$

Then, we can rewrite problem (16) as follows:

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{B}_2(0,1) \setminus \overline{\mathbb{B}_2(0,1/2)}\,, \\ \frac{\partial}{\partial \nu_{\mathbb{B}_2(0,1)}} u(x) = a & \forall x \in \partial\mathbb{B}_2(0,1)\,, \\ \frac{\partial}{\partial \nu_{\mathbb{B}_2(0,1/2)}} u(x) = \delta F_\delta(u(x)) + b & \forall x \in \partial\mathbb{B}_2(0,1/2)\,. \end{cases} \tag{17}$$

Again, we look for a solution u_δ in the form

$$u_\delta(x) \equiv A_\delta \log|x| + a\log 2 + \frac{1}{\delta}\tilde{B}_\delta \qquad \forall x \in \overline{\Omega}\,,$$

with A_δ and $\tilde{B}_\delta$ to be set so that the boundary conditions of problem (17) are satisfied.

As we have seen, we must have

$$A_\delta = a\,,$$

and, in order to satisfy the Robin condition, we have to find $\tilde{B}_\delta$ such that

$$2a = \delta F_\delta\Big(\frac{1}{\delta}\tilde{B}_\delta\Big) + b\,. \tag{18}$$

Then we note that if we set

$$\tilde{F}(\xi,\omega) = \xi + c\omega\xi^{\gamma_2+1} \qquad \forall (\xi,\omega) \in \mathbb{R}^2\,,$$

we have

$$\delta F_\delta\Big(\frac{1}{\delta}\xi\Big) = \tilde{F}(\xi, \delta^{\gamma_1-\gamma_2}) \qquad \forall \xi \in \mathbb{R}\,, \delta \in]0,+\infty[\,.$$

Since we want to pass to the limit in $\tilde{F}(\xi, \delta^{\gamma_1-\gamma_2})$ as $\delta \to 0$, we find it convenient to assume that

$$\gamma_1 \geq \gamma_2\,.$$

As a consequence, we rewrite equation (18) as

$$2a = \tilde{F}\Big(\tilde{B}_\delta, \delta^{\gamma_1-\gamma_2}\Big) + b\,. \tag{19}$$

We try to resolve equation (19) around $\delta = 0$ by means of the Implicit Function Theorem. We treat separately the case $\gamma_1 = \gamma_2$ and the case $\gamma_1 > \gamma_2$. If $\gamma_1 > \gamma_2$, then there exists a unique $\tilde{B}_0$ such that

$$2a - b = \tilde{F}\left(\tilde{B}_0, 0\right),$$

i.e.,

$$\tilde{B}_0 = 2a - b\,.$$

Then by applying the Implicit Function Theorem around the pair $(2a-b, 0)$, one can prove that there exist a small neighborhood $]2a - b - \epsilon_1, 2a - b + \epsilon_1[\times] - \epsilon_2, \epsilon_2[$ of $(2a - b, 0)$ and a function

$$]-\epsilon_2, \epsilon_2[\ni \omega \mapsto \tilde{B}(\omega) \in]2a - b - \epsilon_1, 2a - b + \epsilon_1[$$

such that

$$\tilde{B}(0) = 2a - b\,, \qquad 2a - b = \tilde{F}\left(\tilde{B}(\omega), \omega\right) \qquad \forall \omega \in]-\epsilon_2, \epsilon_2[\,.$$

Therefore, there exists $\delta_1 \in]0, +\infty[$ small enough, such that

$$2a - b = \tilde{F}\left(\tilde{B}(\delta^{\gamma_1-\gamma_2}), \delta^{\gamma_1-\gamma_2}\right) \qquad \forall \delta \in]0, \delta_1[\,,$$

and thus we can take

$$\tilde{B}_\delta = \tilde{B}(\delta^{\gamma_1-\gamma_2}) \qquad \forall \delta \in]0, \delta_1[\,.$$

Accordingly,

$$u_\delta(x) = a \log|x| + a\log 2 + \frac{\tilde{B}(\delta^{\gamma_1-\gamma_2})}{\delta} \qquad \forall x \in \overline{\Omega}\,, \qquad \forall \delta \in]0, \delta_1[\,.$$

Now we turn to consider the case $\gamma_1 = \gamma_2$ and we note that

$$\tilde{F}(\xi, \delta^{\gamma_1-\gamma_1}) = \tilde{F}(\xi, 1) = \xi + c\xi^{\gamma_1+1} \qquad \forall (\xi, \delta) \in \mathbb{R}^2\,.$$

As a consequence, there are $\gamma_1 + 1$ complex solutions ζ to the equation

$$2a - b = \zeta + c\zeta^{\gamma_1+1}\,. \tag{20}$$

Then if we denote by $\{\tilde{B}_j\}_{j=1}^k$ the set of (distinct) real solutions to equation (20), for each of them we can construct the corresponding function, and thus we can define a family of solutions $\{u_{j,\delta}\}_{\delta\in]0,+\infty[}$ to problem (16), by setting

$$u_{j,\delta}(x) \equiv a\log|x| + a\log 2 + \frac{1}{\delta}\tilde{B}_j \qquad \forall x \in \overline{\Omega}\,, \qquad \forall \delta \in]0, +\infty[\,,$$

for each $j \in \{1, \dots, k\}$. Note that this can be presented in the form:

$$u_{j,\delta}(x) = u^{(0)}(x) + \frac{1}{\delta}u^{(j)}(x)\,, \tag{21}$$

thus the non-uniqueness is related to the second term of this representation only. Moreover, it makes sense also to underline that the first term in the solutions for the linear (10) and nonlinear (21) problems coincides.

3 | AN INTEGRAL EQUATION FORMULATION OF THE BOUNDARY VALUE PROBLEM

In order to analyze problem (1) for δ close to 0, we exploit classical potential theory, which allows to obtain an integral equation formulation of (1). To do so, we need to introduce some notation.

Let S_n be the function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_n(x) \equiv \begin{cases} \frac{1}{s_n}\log|x| & \forall x \in \mathbb{R}^n \setminus \{0\}, \text{ if } n = 2\,, \\ \frac{1}{(2-n)s_n}|x|^{2-n} & \forall x \in \mathbb{R}^n \setminus \{0\}, \text{ if } n > 2\,. \end{cases}$$

S_n is well-known to be a fundamental solution of the Laplace operator.

We now introduce the single layer potential. If $\mu \in C^0(\partial\Omega)$, we set

$$v[\partial\Omega, \mu](x) \equiv \int_{\partial\Omega} S_n(x - y)\mu(y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n\,,$$

where $d\sigma$ denotes the area element of a manifold imbedded in $\mathbb{R}^n$. As is well-known, if $\mu \in C^0(\partial\Omega)$, then $v[\partial\Omega,\mu]$ is continuous in $\mathbb{R}^n$. Moreover, if $\mu \in C^{0,\alpha}(\partial\Omega)$, then the function $v^+[\partial\Omega,\mu] \equiv v[\partial\Omega,\mu]_{|\overline{\Omega}}$ belongs to $C^{1,\alpha}(\overline{\Omega})$, and the function $v^-[\partial\Omega,\mu] \equiv v[\partial\Omega,\mu]_{|\mathbb{R}^n\setminus\Omega}$ belongs to $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega)$. Then we set

$$w_*[\partial\Omega,\mu](x) \equiv \int_{\partial\Omega} \nu_\Omega(x)\cdot\nabla S_n(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega\,,$$

where ν_Ω denotes the outward unit normal to $\partial\Omega$. If $\mu \in C^{0,\alpha}(\partial\Omega)$, the function $w_*[\partial\Omega,\mu]$ belongs to $C^{0,\alpha}(\partial\Omega)$ and we have

$$\frac{\partial}{\partial\nu_\Omega}v^\pm[\partial\Omega,\mu] = \mp\frac{1}{2}\mu + w_*[\partial\Omega,\mu] \qquad \text{on } \partial\Omega\ .$$

Then we have the technical Lemma 1 below on the representation of harmonic functions as the sum of a single layer potential with a density with zero integral mean and a constant. Therefore, we find it convenient to set

$$C^{0,\alpha}(\partial\Omega)_0 \equiv \left\{ f \in C^{0,\alpha}(\partial\Omega) : \int_{\partial\Omega} f\,d\sigma = 0 \right\}.$$

The proof of Lemma 1 can be deduced by classical potential theory (cf. Folland[29, Ch. 3]).

Lemma 1. Let $u \in C^{1,\alpha}(\overline{\Omega})$ be such that $\Delta u = 0$ in Ω. Then there exists a unique pair $(\mu,c) \in C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ such that

$$u = v^+[\partial\Omega,\mu] + c \qquad \text{in } \overline{\Omega}\,.$$

By exploiting Lemma 1, we can establish a correspondence between the solutions of problem (1) and those of a (nonlinear) system of integral equations.

Proposition 1. Let $\delta \in]0,\delta_0[$. Then the map from the set of pairs (μ,ξ) of $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ such that

$$\begin{cases} -\frac{1}{2}\mu(x) + w_*[\partial\Omega,\mu](x) = g^o(x) & \forall x \in \partial\Omega^o\,, \\ \frac{1}{2}\mu(x) - w_*[\partial\Omega,\mu](x) = \delta F_\delta\Big(v[\partial\Omega,\mu](x) + \frac{\xi}{\delta}\Big) + g^i(x) & \forall x \in \partial\Omega^i\,, \end{cases} \tag{22}$$

to the set of those functions $u \in C^{1,\alpha}(\overline{\Omega})$ which solve problem (1), which takes a pair (μ,ξ) to the function

$$v^+[\partial\Omega,\mu] + \frac{\xi}{\delta} \tag{23}$$

is a bijection.

Proof. If $(\mu,\xi) \in C^{0,\alpha}(\partial\Omega) \times \mathbb{R}$ then we know that $v^+[\partial\Omega,\mu] + \frac{\xi}{\delta}$ belongs to $C^{1,\alpha}(\overline{\Omega})$ and is harmonic in Ω. Moreover, if (μ,ξ) satisfies system (22), then the jump formulas for the normal derivative of the single layer potential imply the validity of the boundary condition in problem (1). Hence, the function in (23) solves problem (1).

Conversely, if $u \in C^{1,\alpha}(\overline{\Omega})$ satisfies problem (1), then the representation Lemma 1 for harmonic functions in terms of single layer potentials plus constants ensures that there exists a unique pair $(\mu,\xi) \in C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ such that $u = v^+[\partial\Omega,\mu] + \frac{\xi}{\delta}$. Then the jump formulas for the normal derivative of a single layer potential and the boundary condition in (1) imply that the system of integral equations of (22) is satisfied. Hence, the map of the statement is a bijection. □

Now that the correspondence between the solutions of boundary value problem (1) and those of the system of integral equations (22) is established, we wish to study the behavior of the solutions to system (22) as $\delta \to 0$. Then we note that we can write

$$\delta F_\delta\Big(v[\partial\Omega,\mu](x) + \frac{\xi}{\delta}\Big) = \delta F_\delta\Big(\frac{1}{\delta}(\delta v[\partial\Omega,\mu](x) + \xi)\Big)\,.$$

Therefore, to analyze the second equation in (22) for δ small, we need to make some other assumptions on the structure of the family of functions

$$\mathbb{R} \ni \tau \mapsto \delta F_\delta\Big(\frac{1}{\delta}\tau\Big) \qquad \text{as } \delta \to 0\,.$$

So we assume that

$$\begin{aligned} &\text{there exist } \delta_1 \in]0,\delta_0[,\ m \in \mathbb{N}, \text{ a real analytic function } \tilde{F} \text{ from } \mathbb{R}^{m+1} \text{ to } \mathbb{R}, \\ &\text{a function } \omega(\cdot) \text{ from }]0,\delta_1[\text{ to } \mathbb{R}^m \text{ such that } \omega_0 \equiv \lim_{\delta\to 0}\omega(\delta) \in \mathbb{R}^m \text{ and that} \\ &\delta F_\delta\Big(\frac{1}{\delta}\tau\Big) = \tilde{F}(\tau,\omega(\delta)) \text{ for all } (\tau,\delta) \in \mathbb{R}\times]0,\delta_1[. \end{aligned} \tag{24}$$

Thus, under the additional assumption (24), if we let δ tend to 0 in (22), we obtain the following limiting system of integral equations

$$\begin{cases} -\frac{1}{2}\mu(x) + w_*[\partial\Omega, \mu](x) = g^o(x) & \forall x \in \partial\Omega^o\,, \\ \frac{1}{2}\mu(x) - w_*[\partial\Omega, \mu](x) = \tilde{F}(\xi, \omega_0) + g^i(x) & \forall x \in \partial\Omega^i\,. \end{cases} \tag{25}$$

Then, as a preliminary step in the analysis of the system of integral equations (22) for δ close to 0, in the following lemma we study the limiting system (25).

Lemma 2. Let assumption (24) hold. Assume that $\tilde{\xi} \in \mathbb{R}$ is such that

$$\tilde{F}(\tilde{\xi}, \omega_0) = \frac{1}{|\partial\Omega^i|_{n-1}}\left(\int_{\partial\Omega^o} g^o\, d\sigma - \int_{\partial\Omega^i} g^i\, d\sigma\right), \tag{26}$$

where $|\partial\Omega^i|_{n-1}$ denotes the $(n-1)$-dimensional measure of $\partial\Omega^i$. Then there exists a unique $\tilde{\mu} \in C^{0,\alpha}(\partial\Omega)_0$ such that

$$\begin{cases} -\frac{1}{2}\tilde{\mu}(x) + w_*[\partial\Omega, \tilde{\mu}](x) = g^o(x) & \forall x \in \partial\Omega^o\,, \\ \frac{1}{2}\tilde{\mu}(x) - w_*[\partial\Omega, \tilde{\mu}](x) = \tilde{F}(\tilde{\xi}, \omega_0) + g^i(x) & \forall x \in \partial\Omega^i\,. \end{cases}$$

Proof. We note that if $\tilde{\xi}$ is as in (26), then the function $\tilde{g}$ defined as

$$\tilde{g}(x) \equiv \begin{cases} g^o(x) & \forall x \in \partial\Omega^o\,, \\ -\left(\tilde{F}(\tilde{\xi}, \omega_0) + g^i(x)\right) & \forall x \in \partial\Omega^i\,, \end{cases}$$

belongs to $C^{0,\alpha}(\partial\Omega)_0$. Then by classical potential theory (cf. Folland[29, Ch. 3]), there exists a unique $\tilde{\mu} \in C^{0,\alpha}(\partial\Omega)_0$ such that

$$-\frac{1}{2}\tilde{\mu}(x) + w_*[\partial\Omega, \tilde{\mu}](x) = \tilde{g}(x) \qquad \forall x \in \partial\Omega\,,$$

and the validity of the statement follows. □

In view of Proposition 1 and under assumption (24), in order to study the solutions of (22), we find it convenient to introduce the map $\Lambda \equiv (\Lambda^o, \Lambda^i)$ from $\mathbb{R}^{m+1} \times C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ to $C^{0,\alpha}(\partial\Omega)$ defined by setting

$$\begin{aligned} &\Lambda^o[\delta, \omega, \mu, \xi] \equiv -\tfrac{1}{2}\mu(x) + w_*[\partial\Omega, \mu](x) - g^o(x) && \forall x \in \partial\Omega^o\,, \\ &\Lambda^i[\delta, \omega, \mu, \xi] \equiv \tfrac{1}{2}\mu(x) - w_*[\partial\Omega, \mu](x) - \tilde{F}\left(\delta v[\partial\Omega, \mu](x) + \xi, \omega\right) - g^i(x) && \forall x \in \partial\Omega^i\,, \end{aligned}$$

for all $(\delta, \omega, \mu, \xi) \in \mathbb{R}^{m+1} \times C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$.

In the following proposition, we investigate the solutions of the system of integral equations (22), by applying the Implicit Function Theorem to Λ, under suitable assumptions on the partial derivative $\partial_\tau \tilde{F}(\tilde{\xi}, \omega_0)$ of the function $(\tau, \omega) \mapsto \tilde{F}(\tau, \omega)$ with respect to the variable τ computed at the point $(\tilde{\xi}, \omega_0)$.

Proposition 2. Let assumption (24) hold. Let $(\tilde{\mu}, \tilde{\xi})$ be as in Lemma 2. Assume that

$$\partial_\tau \tilde{F}(\tilde{\xi}, \omega_0) \neq 0\,. \tag{27}$$

Then there exist $\delta_2 \in]0, \delta_1[$, an open neighborhood $\mathcal{U}$ of ω_0 in $\mathbb{R}^m$, an open neighborhood $\mathcal{V}$ of $(\tilde{\mu}, \tilde{\xi})$ in $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$, and a real analytic map (M, Ξ) from $]-\delta_2, \delta_2[\times \mathcal{U}$ to $\mathcal{V}$ such that

$$\omega(\delta) \in \mathcal{U} \qquad \forall \delta \in]0, \delta_2[\,,$$

and such that the set of zeros of Λ in $]-\delta_2, \delta_2[\times \mathcal{U} \times \mathcal{V}$ coincides with the graph of (M, Ξ). In particular,

$$(M[0, \omega_0], \Xi[0, \omega_0]) = (\tilde{\mu}, \tilde{\xi})\,.$$

Proof. We first note that by classical potential theory (cf. Miranda[30], Lanza de Cristoforis and Rossi[31, Thm. 3.1]), by assumption (24), and by analyticity results for the composition operator (cf. Böhme and Tomi[32, p. 10], Henry[33, p. 29], Valent[34, Thm. 5.2, p. 44]), we conclude that Λ is real analytic. Then we note that the partial differential $\partial_{(\mu,\xi)}\Lambda[0, \omega_0, \tilde{\mu}, \tilde{\xi}]$ of Λ at $(0, \omega_0, \tilde{\mu}, \tilde{\xi})$ with respect to the variable (μ, ξ) is delivered by

$$\begin{aligned} &\partial_{(\mu,\xi)}\Lambda^o[0, \omega_0, \tilde{\mu}, \tilde{\xi}](\overline{\mu}, \overline{\xi})(x) \equiv -\tfrac{1}{2}\overline{\mu}(x) + w_*[\partial\Omega, \overline{\mu}](x) && \forall x \in \partial\Omega^o\,, \\ &\partial_{(\mu,\xi)}\Lambda^i[0, \omega_0, \tilde{\mu}, \tilde{\xi}](\overline{\mu}, \overline{\xi})(x) \equiv \tfrac{1}{2}\overline{\mu}(x) - w_*[\partial\Omega, \overline{\mu}](x) - \partial_\tau \tilde{F}(\tilde{\xi}, \omega_0)\overline{\xi} && \forall x \in \partial\Omega^i\,, \end{aligned}$$

for all $(\overline{\mu}, \overline{\xi}) \in C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$. Then by assumption (27) and by classical potential theory (cf. Folland[29, Ch. 3]), we deduce that $\partial_{(\mu,\xi)}\Lambda[0, \omega_0, \tilde{\mu}, \tilde{\xi}]$ is a homeomorphism from $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ onto $C^{0,\alpha}(\partial\Omega)$. Then, by the Implicit Function Theorem for real analytic maps in Banach spaces (cf., *e.g.*, Deimling[17, Theorem 15.3]), we deduce the validity of the statement. □

Now that we have converted problem (1) into an equivalent system of integral equations for which we have exhibited a real analytic family of solutions, we are ready to introduce the family of solutions to (1).

Definition 1. Let the assumptions of Proposition 2 hold. Then we set

$$u(\delta, x) \equiv v^+[\Omega, M[\delta, \omega(\delta)]](x) + \frac{\Xi[\delta, \omega(\delta)]}{\delta} \qquad \forall x \in \overline{\Omega},$$

for all $\delta \in]0, \delta_2[$.

By Propositions 1 and 2 and by Definition 1, we deduce that for each $\delta \in]0, \delta_2[$ the function $u(\delta, \cdot) \in C^{1,\alpha}(\overline{\Omega})$ is a solution to problem (1).

4 | A FUNCTIONAL ANALYTIC REPRESENTATION THEOREM FOR THE FAMILY OF SOLUTIONS

In the following theorem, we exploit the analyticity result of Proposition 2 concerning the solutions of the system of integral equations (22) in order to prove representation formulas for $u(\delta, \cdot)$ and its energy integral in terms of real analytic maps and thus to answer to questions (i), (ii) of the Introduction.

Theorem 1. Let the assumptions of Proposition 2 hold. Then the following statements hold.

(i) There exists a real analytic map U from $]-\delta_2, \delta_2[\times\mathcal{U}$ to $C^{1,\alpha}(\overline{\Omega})$ such that

$$u(\delta, x) = U[\delta, \omega(\delta)](x) + \frac{\Xi[\delta, \omega(\delta)]}{\delta} \qquad \forall x \in \overline{\Omega}, \tag{28}$$

for all $\delta \in]0, \delta_2[$. Moreover, $U[0, \omega_0]$ is a solution of the Neumann problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \Omega, \\ \frac{\partial}{\partial \nu_{\Omega^o}} u(x) = g^o(x) & \forall x \in \partial\Omega^o, \\ \frac{\partial}{\partial \nu_{\Omega^i}} u(x) = \frac{1}{|\partial\Omega^i|_{n-1}} \left(\int_{\partial\Omega^o} g^o \, d\sigma - \int_{\partial\Omega^i} g^i \, d\sigma \right) + g^i(x) & \forall x \in \partial\Omega^i, \end{cases} \tag{29}$$

and

$$\Xi[0, \omega_0] = \tilde{\xi}.$$

(ii) There exists a real analytic map E from $]-\delta_2, \delta_2[\times\mathcal{U}$ to $\mathbb{R}$ such that

$$\int_{\Omega} |\nabla u(\delta, x)|^2 \, dx = E[\delta, \omega(\delta)], \tag{30}$$

for all $\delta \in]0, \delta_2[$. Moreover,

$$E[0, \omega_0] = \int_{\Omega} |\nabla \tilde{u}(x)|^2 \, dx,$$

where $\tilde{u}$ is any solution of the Neumann problem (29).

Proof. We first prove statement (i). We set

$$U[\delta, \omega](x) \equiv v^+[\partial\Omega, M[\delta, \omega]](x) \qquad \forall x \in \overline{\Omega},$$

for all $(\delta, \omega) \in]-\delta_2, \delta_2[\times\mathcal{U}$. Then by Proposition 2 and by classical mapping properties of layer potentials (cf. Miranda[30], Lanza de Cristoforis and Rossi[31, Thm. 3.1]), we conclude that U is real analytic. Since

$$\tilde{F}(\tilde{\xi}, \omega_0) = \frac{1}{|\partial\Omega^i|_{n-1}} \left(\int_{\partial\Omega^o} g^o \, d\sigma - \int_{\partial\Omega^i} g^i \, d\sigma \right),$$

and $M[0,\omega_0]=\tilde{\mu}$, then

$$\begin{cases} \frac{\partial}{\partial \nu_{\Omega^o}} v^+[\partial\Omega,\tilde{\mu}](x) = -\frac{1}{2}\tilde{\mu}(x) + w_*[\partial\Omega,\tilde{\mu}](x) = g^o(x) & \forall x \in \partial\Omega^o\,, \\ \frac{\partial}{\partial \nu_{\Omega^i}} v^+[\partial\Omega,\tilde{\mu}](x) = \frac{1}{2}\tilde{\mu}(x) - w_*[\partial\Omega,\tilde{\mu}](x) = \frac{1}{|\partial\Omega^i|_{n-1}}\left(\int_{\partial\Omega^o} g^o\,d\sigma - \int_{\partial\Omega^i} g^i\,d\sigma\right) + g^i(x) & \forall x \in \partial\Omega^i\,. \end{cases}$$

As a consequence, $v^+[\partial\Omega,\tilde{\mu}]$ solves problem (29). Then we deduce the validity of statement (i) (see also Proposition 2).

We now consider statement (ii). By the Divergence Theorem and standard properties of harmonic functions and their normal derivatives, we have

$$\begin{aligned} \int_\Omega |\nabla u(\delta,x)|^2\,dx &= \int_{\partial\Omega}\left(v^+[\partial\Omega, M[\delta,\omega(\delta)]] + \frac{\Xi[\delta,\omega(\delta)]}{\delta}\right)\frac{\partial v^+[\partial\Omega, M[\delta,\omega(\delta)]]}{\partial\nu_\Omega}\,d\sigma\,, \\ &= \int_{\partial\Omega} v^+[\partial\Omega, M[\delta,\omega(\delta)]]\frac{\partial v^+[\partial\Omega, M[\delta,\omega(\delta)]]}{\partial\nu_\Omega}\,d\sigma\,, \end{aligned}$$

for all $\delta\in]0,\delta_2[$. Thus, we find natural to set

$$E[\delta,\omega] = \int_{\partial\Omega} v^+[\partial\Omega, M[\delta,\omega]]\frac{\partial v^+[\partial\Omega, M[\delta,\omega]]}{\partial\nu_\Omega}\,d\sigma \qquad \forall(\delta,\omega)\in]-\delta_2,\delta_2[\times\mathcal{U}\,.$$

By Proposition 2, by mapping properties of layer potentials, and by standard calculus in Schauder spaces, we deduce the real analyticity of E from $]-\delta_2,\delta_2[\times\mathcal{U}$ to $\mathbb{R}$. Since

$$\begin{aligned} E[0,\omega_0] &= \int_{\partial\Omega} v^+[\partial\Omega,\tilde{\mu}]\frac{\partial v^+[\partial\Omega,\tilde{\mu}]}{\partial\nu_\Omega}\,d\sigma \\ &= \int_\Omega |\nabla v^+[\partial\Omega,\tilde{\mu}]|^2\,dx\,, \end{aligned}$$

and $v^+[\partial\Omega,\tilde{\mu}]$ is a solution of problem (29), we deduce the validity of statement (ii). □

Remark 1. We observe that Theorem 1 implies that the quantities in the left-hand sides of (28) and of (30) can be represented as convergent power series of $(\delta,\omega(\delta)-\omega_0)$.

5 | LOCAL UNIQUENESS OF THE FAMILY OF SOLUTIONS

We now show by means of the following theorem that the family $\{u(\delta,\cdot)\}_{\delta\in]0,\delta_2[}$ is locally essentially unique (cf. Lanza de Cristoforis[22, Thm. 4.1 (iii)]).

Theorem 2. Let the assumptions of Proposition 2 hold. If $\{d_j\}_{j\in\mathbb{N}}$ is a sequence of $]0,\delta_0[$ converging to 0 and if $\{u_j\}_{j\in\mathbb{N}}$ is a sequence of functions such that

$$\begin{aligned} & u_j \in C^{1,\alpha}(\overline{\Omega})\,, \\ & u_j \text{ solves (1) for } \delta = d_j\,, \\ & \lim_{j\to\infty} d_j u_{j|\partial\Omega^i} = \tilde{\xi} \qquad \text{in } C^{0,\alpha}(\partial\Omega^i)\,, \end{aligned} \tag{31}$$

then there exists $j_0\in\mathbb{N}$ such that $u_j(\cdot)=u(d_j,\cdot)$ for all $j\geq j_0$.

Proof. Since u_j solves problem (1), Proposition 1 ensures that for each $j\in\mathbb{N}$ there exists a pair $(\mu_j,\xi_j)\in C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R}$ such that

$$u_j = v^+[\partial\Omega,\mu_j] + \frac{\xi_j}{d_j} \qquad \text{in } \overline{\Omega}\,. \tag{32}$$

We now rewrite equation

$$\Lambda[\delta,\omega,\mu,\xi]=0$$

in the following form

$$
\begin{cases}
-\frac{1}{2}\mu(x) + w_*[\partial\Omega,\mu](x) = g^o(x) \quad \forall x \in \partial\Omega^o\,, \\
\frac{1}{2}\mu(x) - w_*[\partial\Omega,\mu](x) - \partial_\tau \tilde{F}(\tilde{\xi},\omega_0)\Big(\delta v[\partial\Omega,\mu](x)+\xi\Big) \\
\qquad = -\partial_\tau \tilde{F}(\tilde{\xi},\omega_0)\Big(\delta v[\partial\Omega,\mu](x)+\xi\Big) + \tilde{F}\Big(\delta v[\partial\Omega,\mu](x)+\xi,\omega\Big) + g^i(x) \quad \forall x \in \partial\Omega^i\,,
\end{cases}
\tag{33}
$$

for all $(\delta,\omega,\mu,\xi) \in \mathbb{R}^{m+1}\times C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R}$. Next we denote by $N[\delta,\omega,\mu,\xi] = (N^o[\delta,\omega,\mu,\xi], N^i[\delta,\omega,\mu,\xi])$ and by $B[\delta,\omega,\mu,\xi] = (B^o[\delta,\omega,\mu,\xi], B^i[\delta,\omega,\mu,\xi])$ the left and right hand side of such an equality, respectively. By standard properties of single layer potentials, we conclude that N is real analytic (cf. Miranda[30], Lanza de Cristoforis and Rossi[31, Thm. 3.1]).

Next we note that $N[\delta,\omega,\cdot,\cdot]$ is linear for all fixed $(\delta,\omega)\in\mathbb{R}^{m+1}$. Accordingly, the map from $\mathbb{R}^{m+1}$ to $\mathcal{L}(C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R}, C^{0,\alpha}(\partial\Omega))$ which takes (δ,ω) to $N[\delta,\omega,\cdot,\cdot]$ is real analytic. Here $\mathcal{L}(C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R}, C^{0,\alpha}(\partial\Omega))$ denotes the Banach space of linear and continuous operators from $C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R}$ to $C^{0,\alpha}(\partial\Omega)$. We also note that

$$
N[0,\omega_0,\cdot,\cdot] = \partial_{(\mu,\xi)}\Lambda[0,\omega_0,\tilde{\mu},\tilde{\xi}](\cdot,\cdot)\,,
$$

and that accordingly, $N[0,\omega_0,\cdot,\cdot]$ is a linear homeomorphism (see the proof of Proposition 2). Since the set of linear homeomorphisms is open in the set of linear and continuous operators, and since the map which takes a linear invertible operator to its inverse is real analytic (cf. *e.g.*, Hille and Phillips[35, Thms. 4.3.2 and 4.3.4]), there exists an open neighborhood $\mathcal{W}$ of $(0,\omega_0)$ in $]-\delta_2,\delta_2[\times\mathcal{U}$ such that the map which takes (δ,ω) to $N[\delta,\omega,\cdot,\cdot]^{(-1)}$ is real analytic from $\mathcal{W}$ to $\mathcal{L}(C^{0,\alpha}(\partial\Omega), C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R})$. Clearly, there exists $j_1\in\mathbb{N}$ such that

$$
(d_j,\omega(d_j))\in\mathcal{W} \qquad \forall j\geq j_1\,.
$$

Since $\Lambda[d_j,\omega(d_j),\mu_j,\xi_j]=0$, the invertibility of $N[\delta,\omega,\cdot,\cdot]$ and equality (33) guarantee that

$$
(\mu_j,\xi_j) = N[d_j,\omega(d_j),\cdot,\cdot]^{(-1)}[B[d_j,\omega(d_j),\mu_j,\xi_j]] \qquad \forall j\geq j_1\,.
$$

By (32), we have

$$
d_j u_j = d_j v^+[\partial\Omega,\mu_j]+\xi_j\,,
$$

for all $j\geq j_1$. Accordingly,

$$
\begin{aligned}
-\partial_\tau\tilde{F}(\tilde{\xi},\omega_0)\Big(d_j v[\partial\Omega,\mu_j](x)+\xi_j\Big) + \tilde{F}\Big(d_j v[\partial\Omega,\mu_j](x)+\xi_j,\omega(d_j)\Big)+g^i(x)\\
= -\partial_\tau\tilde{F}(\tilde{\xi},\omega_0)\Big(d_j u_j(x)\Big) + \tilde{F}\Big(d_j u_j(x),\omega(d_j)\Big)+g^i(x) \qquad \forall x\in\partial\Omega^i\,,
\end{aligned}
$$

for all $j\geq j_1$. Then by assumptions (24) and (31) and by analyticity results for the composition operator (cf. Böhme and Tomi[32, p. 10], Henry[33, p. 29], Valent[34, Thm. 5.2, p. 44]), we have

$$
\lim_{j\to\infty} -\partial_\tau\tilde{F}(\tilde{\xi},\omega_0)\Big(d_j u_{j|\partial\Omega^i}\Big) + \tilde{F}\Big(d_j u_{j|\partial\Omega^i},\omega(d_j)\Big)+g^i = -\partial_\tau\tilde{F}(\tilde{\xi},\omega_0)\tilde{\xi} + \tilde{F}\Big(\tilde{\xi},\omega_0\Big)+g^i\,,
\tag{34}
$$

in $C^{0,\alpha}(\partial\Omega^i)$. The analyticity of the map which takes (δ,ω) to $N[\delta,\omega,\cdot,\cdot]^{(-1)}$ implies that

$$
\lim_{j\to\infty} N[d_j,\omega(d_j),\cdot,\cdot]^{(-1)} = N[0,\omega_0,\cdot,\cdot]^{(-1)} \qquad \text{in } \mathcal{L}(C^{0,\alpha}(\partial\Omega), C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R})\,.
\tag{35}
$$

Since the evaluation map from $\mathcal{L}(C^{0,\alpha}(\partial\Omega), C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R})\times C^{0,\alpha}(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R}$, which takes a pair (A,v) to $A[v]$ is bilinear and continuous, the limiting relations (34) and (35) imply that

$$
\lim_{j\to\infty}(\mu_j,\xi_j) = N[0,\omega_0,\cdot,\cdot]^{(-1)}[g^o, -\partial_\tau\tilde{F}(\tilde{\xi},\omega_0)\tilde{\xi}+\tilde{F}\Big(\tilde{\xi},\omega_0\Big)+g^i]
\tag{36}
$$

in $C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R}$. Since $\Lambda[0,\omega_0,\tilde{\mu},\tilde{\xi}]=0$, the right hand side of (36) equals $(\tilde{\mu},\tilde{\xi})$. Hence,

$$
\lim_{j\to\infty}(d_j,\omega(d_j),\mu_j,\xi_j) = (0,\omega_0,\tilde{\mu},\tilde{\xi}) \qquad \text{in } \mathbb{R}^{m+1}\times C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R}\,.
$$

Then Proposition 2 implies that there exists $j_0\in\mathbb{N}$ such that

$$
\xi_j = \Xi[d_j,\omega(d_j)]\,, \qquad \mu_j = M[d_j,\omega(d_j)] \qquad \forall j\geq j_0\,.
$$

Accordingly, $u_j = u(d_j,\cdot)$ for $j\geq j_0$ (see Definition 1). □

6 | REMARKS ON THE LINEAR CASE

In this section, we wish to make further considerations on the linear case. In particular, we plan to compute asymptotic expansions of the solutions as the parameter δ tends to 0.

We first note that the results of Section 4 apply to the linear case. In particular, in case

$$F_\delta(\tau) = \tau \qquad \forall(\tau,\delta) \in \mathbb{R}\times]0,+\infty[\,,$$

problem (1) reduces to the following linear problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \Omega\,, \\ \frac{\partial}{\partial \nu_{\Omega^o}} u(x) = g^o(x) & \forall x \in \partial\Omega^o\,, \\ \frac{\partial}{\partial \nu_{\Omega^i}} u(x) = \delta u(x) + g^i(x) & \forall x \in \partial\Omega^i\,. \end{cases} \tag{37}$$

For each $\delta \in]0,+\infty[$, we know that problem (37) has a unique solution in $C^{1,\alpha}(\overline{\Omega})$ and we denote it by $u[\delta]$.

Clearly,

$$\delta F_\delta(\frac{1}{\delta}\tau) = \tau \qquad \forall(\tau,\delta) \in \mathbb{R}\times]0,+\infty[\,,$$

and thus we can take for example

$$\omega(\delta) = 0 \qquad \forall \delta \in]0,+\infty[\,,$$

and

$$\tilde{F}(\tau,\omega) = \tau \qquad \forall(\tau,\omega) \in \mathbb{R}^2\,.$$

In particular,

$$\omega_0 = 0\,, \qquad \tilde{\xi} = \frac{1}{|\partial\Omega^i|_{n-1}} \left(\int_{\partial\Omega^o} g^o\, d\sigma - \int_{\partial\Omega^i} g^i\, d\sigma \right),$$

and

$$\partial_\tau \tilde{F}(\tilde{\xi},\omega_0) = 1 \neq 0\,.$$

Therefore, the results of Sections 3 and 4 apply to the present case. More precisely, by simplifying the arguments of Propositions 1 and 2, we deduce the validity of the following proposition.

Proposition 3. Let $\Lambda_\# \equiv (\Lambda^o_\#, \Lambda^i_\#)$ be the map from $\mathbb{R} \times C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ to $C^{0,\alpha}(\partial\Omega)$ defined by setting

$$\begin{aligned} \Lambda^o_\#[\delta,\mu,\xi] &\equiv -\tfrac{1}{2}\mu(x) + w_*[\partial\Omega,\mu](x) - g^o(x) && \forall x \in \partial\Omega^o\,, \\ \Lambda^i_\#[\delta,\mu,\xi] &\equiv \tfrac{1}{2}\mu(x) - w_*[\partial\Omega,\mu](x) - \delta v[\partial\Omega,\mu](x) - \xi - g^i(x) && \forall x \in \partial\Omega^i\,, \end{aligned}$$

for all $(\delta,\mu,\xi) \in \mathbb{R} \times C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$. Let $(\tilde{\mu}_\#, \tilde{\xi}_\#)$ be the unique solution in $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ of

$$\begin{cases} -\frac{1}{2}\mu(x) + w_*[\partial\Omega,\mu](x) = g^o(x) & \forall x \in \partial\Omega^o\,, \\ \frac{1}{2}\mu(x) - w_*[\partial\Omega,\mu](x) = \xi + g^i(x) & \forall x \in \partial\Omega^i\,. \end{cases}$$

Then there exist $\delta_0 \in]0,+\infty[$, an open neighborhood $\mathcal{V}$ of $(\tilde{\mu}_\#, \tilde{\xi}_\#)$ in $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$, and a real analytic map $(M_\#, \Xi_\#)$ from $]-\delta_0,\delta_0[$ to $\mathcal{V}$ such that the set of zeros of $\Lambda_\#$ in $]-\delta_0,\delta_0[\times\mathcal{V}$ coincides with the graph of $(M_\#,\Xi_\#)$. In particular,

$$(M_\#[0], \Xi_\#[0]) = (\tilde{\mu}_\#, \tilde{\xi}_\#)\,. \tag{38}$$

Moreover,

$$u[\delta](x) \equiv v^+[\Omega, M_\#[\delta]](x) + \frac{\Xi_\#[\delta]}{\delta} \qquad \forall x \in \overline{\Omega}\,,$$

for all $\delta \in]0,\delta_0[$.

Then we can follow the lines of the proof of Theorem 1 and obtain the following real analytic representation result for $u[\delta]$ and its energy integral.

Theorem 3. Let $\Xi_\#$ be as in Proposition 3. Then the following statements hold.

(i) There exists a real analytic map $U_\#$ from $]-\delta_0,\delta_0[$ to the space $C^{1,\alpha}(\overline{\Omega})$ such that

$$u[\delta](x) = U_\#[\delta](x) + \frac{\Xi_\#[\delta]}{\delta} \qquad \forall x \in \overline{\Omega}\,,$$

for all $\delta \in]0, \delta_0[$. Moreover, $U_\#[0]$ is a solution of the Neumann problem

$$
\begin{cases}
\Delta u(x) = 0 & \forall x \in \Omega\,,\\
\frac{\partial}{\partial \nu_{\Omega^o}} u(x) = g^o(x) & \forall x \in \partial\Omega^o\,,\\
\frac{\partial}{\partial \nu_{\Omega^i}} u(x) = \frac{1}{|\partial\Omega^i|_{n-1}}\left(\int_{\partial\Omega^o} g^o\, d\sigma - \int_{\partial\Omega^i} g^i\, d\sigma \right) + g^i(x) & \forall x \in \partial\Omega^i\,,
\end{cases}
\tag{39}
$$

and

$$
\Xi_\#[0] = \frac{1}{|\partial\Omega^i|_{n-1}}\left(\int_{\partial\Omega^o} g^o\, d\sigma - \int_{\partial\Omega^i} g^i\, d\sigma \right).
$$

(ii) There exists a real analytic map $E_\#$ from $]-\delta_0, \delta_0[$ to $\mathbb{R}$ such that

$$
\int_\Omega |\nabla u(\delta, x)|^2\, dx = E_\#[\delta]\,,
$$

for all $\delta \in]0, \delta_0[$. Moreover,

$$
E_\#[0] = \int_\Omega |\nabla \tilde{u}(x)|^2\, dx\,,
$$

where $\tilde{u}$ is any solution of the Neumann problem (39).

6.1 | Asymptotic expansion of $u[\delta]$

By Theorem 4 (i), we know that there exist a sequence of functions $\{u_{\#,k}\}_{k\in\mathbb{N}} \subseteq C^{1,\alpha}(\overline{\Omega})$ and a sequence of real numbers $\{\xi_{\#,k}\}_{k\in\mathbb{N}}$ such that

$$
u[\delta](x) = \sum_{k=0}^{+\infty} u_{\#,k}(x)\delta^k + \sum_{k=0}^{+\infty} \xi_{\#,k}\delta^{k-1} \qquad \forall x \in \overline{\Omega}\,,
\tag{40}
$$

where the series are uniformly convergent for δ in a neighborhood of 0. As for the model problem (10), we note that we can rewrite equation (40) in the form

$$
u[\delta](x) = u^{(0)}(x) = \frac{1}{\delta} u^{(1)}_\delta(x) \qquad \forall x \in \overline{\Omega}\,,
$$

where in this case in general $u^{(1)}_\delta$ depends on δ.

In order to construct the sequences $\{u_{\#,k}\}_{k\in\mathbb{N}} \subseteq C^{1,\alpha}(\overline{\Omega})$ and $\{\xi_{\#,k}\}_{k\in\mathbb{N}}$, we wish to exploit the integral equation formulation of problem (37) and the approach of Dalla Riva, Musolino, and Rogosin[36].

Now we observe that the real analyticity result of Proposition 3 implies that there exists $\delta_1 \in]0, \delta_0[$ small enough such that we can expand $M_\#[\delta]$ and $\Xi_\#[\delta]$ into power series of δ, *i.e.*,

$$
M_\#[\delta] = \sum_{k=0}^{+\infty} \frac{\mu_{\#,k}}{k!}\delta^k\,, \qquad \Xi_\#[\delta] = \sum_{k=0}^{+\infty} \frac{\xi_{\#,k}}{k!}\delta^k\,,
\tag{41}
$$

for some $\{\mu_{\#,k}\}_{k\in\mathbb{N}}$, $\{\xi_{\#,k}\}_{k\in\mathbb{N}}$ and for all $\delta \in]-\delta_1, \delta_1[$. Moreover,

$$
\mu_{\#,k} = \left(\partial^k_\delta M_\#[\delta]\right)_{|\delta=0}\,, \qquad \xi_{\#,k} = \left(\partial^k_\delta \Xi_\#[\delta]\right)_{|\delta=0}\,,
$$

for all $k \in \mathbb{N}$. Therefore, in order to obtain a power series expansion for $u[\delta]$ for δ close to 0, we want to exploit the expansion of $(M_\#[\delta], \Xi_\#[\delta])$. Since the coefficients of the expansions in (41) are given by the derivatives with respect to δ of $M_\#[\delta]$ and $\Xi_\#[\delta]$, we would like to obtain some equations identifying $\left(\partial^k_\delta M_\#[\delta]\right)_{|\delta=0}$ and $\left(\partial^k_\delta \Xi_\#[\delta]\right)_{|\delta=0}$. The plan is to obtain such equations by deriving with respect to δ equality (38), which then leads to

$$
\partial^k_\delta \Big(\Lambda_\#[\delta, M_\#[\delta], \Xi_\#[\delta]] \Big) = 0 \qquad \forall \delta \in]-\delta_1, \delta_1[\,, \qquad \forall k \in \mathbb{N}\,.
\tag{42}
$$

Then, as Proposition 4 below shows, by taking $\delta = 0$ in (42), we will obtain integral equations identifying $\left(\partial^k_\delta M_\#[\delta]\right)_{|\delta=0}$ and $\left(\partial^k_\delta \Xi_\#[\delta]\right)_{|\delta=0}$.

Proposition 4. Let δ_0, $M_\#[\cdot]$, and $\Xi_\#[\cdot]$ be as in Proposition 3. Then there exist $\delta_1 \in]0, \delta_0[$ and a sequence of functions $\{\mu_{\#,k}\}_{k\in\mathbb{N}} \subseteq C^{0,\alpha}(\partial\Omega)_0$ and a sequence of real numbers $\{\xi_{\#,k}\}_{k\in\mathbb{N}}$ such that

$$M_\#[\delta] = \sum_{k=0}^{+\infty} \frac{\mu_{\#,k}}{k!}\delta^k \qquad \text{and} \qquad \Xi_\#[\delta] = \sum_{k=0}^{+\infty} \frac{\xi_{\#,k}}{k!}\delta^k \qquad \forall \delta \in]-\delta_1, \delta_1[\,, \tag{43}$$

where the two series converge uniformly for $\delta \in]-\delta_1, \delta_1[$. Moreover, the following statements hold.

(i) The pair $(\mu_{\#,0}, \xi_{\#,0})$ is the unique solution in $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ of the following system of integral equations

$$\begin{cases} -\frac{1}{2}\mu_{\#,0}(x) + w_*[\partial\Omega, \mu_{\#,0}](x) = g^o(x) & \forall x \in \partial\Omega^o\,, \\ \frac{1}{2}\mu_{\#,0}(x) - w_*[\partial\Omega, \mu_{\#,0}](x) = \xi_{\#,0} + g^i(x) & \forall x \in \partial\Omega^i\,. \end{cases}$$

Moreover,

$$\xi_{\#,0} = \frac{1}{|\partial\Omega^i|_{n-1}}\left(\int_{\partial\Omega^o} g^o\, d\sigma - \int_{\partial\Omega^i} g^i\, d\sigma\right).$$

(ii) For all $k \in \mathbb{N}\setminus\{0\}$ the pair $(\mu_{\#,k}, \xi_{\#,k})$ is the unique solution in $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ of the following system of integral equations

$$\begin{cases} -\frac{1}{2}\mu_{\#,k}(x) + w_*[\partial\Omega, \mu_{\#,k}](x) = 0 & \forall x \in \partial\Omega^o\,, \\ \frac{1}{2}\mu_{\#,k}(x) - w_*[\partial\Omega, \mu_{\#,k}](x) = kv[\partial\Omega, \mu_{\#,k-1}](x) + \xi_{\#,k} & \forall x \in \partial\Omega^i\,. \end{cases} \tag{44}$$

Moreover,

$$\xi_{\#,k} = \frac{1}{|\partial\Omega^i|_{n-1}}\left(-\int_{\partial\Omega^i} kv[\partial\Omega, \mu_{\#,k-1}]\, d\sigma\right). \tag{45}$$

Proof. We first note that Proposition 3 implies the existence of δ_1 and of sequences $\{\mu_{\#,k}\}_{k\in\mathbb{N}}$ and $\{\xi_{\#,k}\}_{k\in\mathbb{N}}$ such that (43) holds. Moreover, Proposition 3 immediately implies the validity of statement (i). Then observe that $\Lambda_\#\left[\delta, M_\#[\delta], \Xi_\#[\delta]\right] = 0$ for all $\delta \in]-\delta_0, \delta_0[$. Accordingly, the map which takes δ to $\Lambda_\#\left[\delta, M_\#[\delta], \Xi_\#[\delta]\right]$ has derivatives which are equal to zero, *i.e.*, $\partial_\delta^k(\Lambda_\#\left[\delta, M_\#[\delta], \Xi_\#[\delta]\right]) = 0$ for all $\delta \in]-\delta_0, \delta_0[$ and all $k \in \mathbb{N}\setminus\{0\}$. Then a straightforward calculation shows that

$$\partial_\delta^k\left(\Lambda_\#^o\left[\delta, M_\#[\delta], \Xi_\#[\delta]\right]\right)(x) = -\frac{1}{2}\partial_\delta^k M_\#[\delta](x) + w_*\left[\partial\Omega, \partial_\delta^k M_\#[\delta]\right](x) = 0 \qquad \forall x \in \partial\Omega^o\,, \tag{46}$$

$$\begin{aligned} \partial_\delta^k\left(\Lambda_\#^i\left[\delta, M_\#[\delta], \Xi_\#[\delta]\right]\right)(x) &= \frac{1}{2}\partial_\delta^k M_\#[\delta](x) - w_*\left[\partial\Omega, \partial_\delta^k M_\#[\delta]\right](x) \\ &- \sum_{j=0}^{k}\binom{k}{j}\left(\partial_\delta^{k-j}\delta\right)v\left[\partial\Omega, \partial_\delta^j M_\#[\delta]\right](x) - \partial_\delta^k \Xi_\#[\delta] = 0 \qquad \forall x \in \partial\Omega^i\,, \end{aligned} \tag{47}$$

for all $\delta \in]-\delta_0, \delta_0[$ and all $k \in \mathbb{N}\setminus\{0\}$. Then, one verifies that system (46)-(47) with $\delta = 0$ can be rewritten as system (44) for all $k \in \mathbb{N}\setminus\{0\}$. Hence, classical potential ensures that the solution $(\mu_{\#,k}, \xi_{\#,k}) \in C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ of system (44) exists and is unique. Then, by integrating, one deduces the validity of (45). The proof is now complete. □

Finally, by Propositions 3 and 4, Theorem 4 and standard calculus in Banach spaces, one deduces the validity of the following.

Theorem 4. Let δ_1, $\{u_{\#,k}\}_{k\in\mathbb{N}}$, and $\{\xi_{\#,k}\}_{k\in\mathbb{N}}$ be as in Proposition 4. Let

$$u_{\#,k}(x) \equiv v^+[\partial\Omega, \mu_{\#,k}](x) \qquad \forall x \in \overline{\Omega}\,, \quad \forall k \in \mathbb{N}\,.$$

Then there exists $\delta_2 \in]0, \delta_1[$ such that

$$u[\delta](x) = \sum_{k=0}^{+\infty} u_{\#,k}(x)\delta^k + \sum_{k=0}^{+\infty} \xi_{\#,k}\delta^{k-1} \qquad \forall x \in \overline{\Omega}\,, \quad \forall \delta \in]0, \delta_2[$$

where the series are uniformly convergent for δ in $]-\delta_2, \delta_2[$.

Then by Proposition 4 and Theorem 4, we can deduce a representation formula similar to the one of the solution of the model problem (10) (where $u^{(1)}$ is replaced by the δ-dependent function $u_\delta^{(1)}$).

Corollary 1. Let the assumptions of Theorem 4 hold. Let

$$u^{(0)}(x) \equiv u_{\#,0}(x) - \xi_{\#,1} \qquad \forall x \in \overline{\Omega}\,,$$

and

$$u^{(1)}_{\delta}(x) \equiv \xi_{\#,0} + \sum_{k=2}^{+\infty} \xi_{\#,k}\delta^k + \sum_{k=1}^{+\infty} u_{\#,k}(x)\delta^{k+1} \qquad \forall x \in \overline{\Omega}\,, \quad \forall \delta \in]0, \delta_2[\,.$$

Then

$$u[\delta](x) = u^{(0)}(x) + \frac{1}{\delta}u^{(1)}_{\delta}(x) \qquad \forall x \in \overline{\Omega}\,, \quad \forall \delta \in]0, \delta_2[\,,$$

and $u^{(0)}$ is the unique solution in $C^{1,\alpha}(\overline{\Omega})$ of problem (39) such that

$$\int_{\partial\Omega^i} u^{(0)}\,d\sigma = 0\,.$$

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