

BFC-theorems for higher commutator subgroups

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Abstract

A BFC-group is a group in which all conjugacy classes are finite with bounded size. In 1954 B. H. Neumann discovered that if G is a BFC-group then the derived group G' is finite. Let $w = w(x_1, \dots, x_n)$ be a multilinear commutator. We study groups in which the conjugacy classes containing w -values are finite of bounded order. Let G be a group and let $w(G)$ be the verbal subgroup of G generated by all w -values. We prove that if $|x^G| \leq m$ for every w -value x , then the derived subgroup of $w(G)$ is finite of order bounded by a function of m and n . If $|x^{w(G)}| \leq m$ for every w -value x , then $[w(w(G)), w(G)]$ is finite of order bounded by a function of m and n .

1 Introduction

Given a group G and an element $x \in G$, we write x^G for the conjugacy class containing x . Of course, if the number of elements in x^G is finite, we have $|x^G| = [G : C_G(x)]$. A group is said to be a BFC-group if its conjugacy classes are finite and of bounded size. One of the most famous of B. H. Neumann's theorems says that in a BFC-group the commutator subgroup G' is finite [6]. It follows that if $|x^G| \leq m$ for each $x \in G$, then the order of G' is bounded by a number depending only on m . A first explicit bound for the order of G' was found by J. Wiegold [11], and the best known was obtained in [5] (see also [7] and [9]).

The recent article [3] deals with groups G in which conjugacy classes containing commutators are bounded. By a commutator we mean any element $x \in G$ which can be written in the form

$$x = [x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$$

for suitable $x_1, x_2 \in G$. The results obtained in [3] can be summarized as follows.

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Theorem 1.1. *Let m be a positive integer and G a group. If $|x^G| \leq m$ for any commutator x , then $|G''|$ is finite and m -bounded. If $|x^{G'}| \leq m$ for any commutator x , then $|\gamma_3(G')|$ is finite and m -bounded.*

Here G'' denotes the second commutator subgroup of G and $\gamma_3(G')$ denotes the third term of the lower central series of G' . Throughout the article we use the expression “ (a, b, \dots) -bounded” to mean that a quantity is finite and bounded by a certain number depending only on the parameters a, b, \dots .

Comparing Neumann’s theorem with Theorem 1.1 one cannot help but wondering whether these results are parts of some more general phenomenon. The purpose of the present article is to address this question.

Given a group-word $w = w(x_1, \dots, x_k)$, the verbal subgroup $w(G)$ of a group G determined by w is the subgroup generated by the set G_w consisting of all values $w(g_1, \dots, g_k)$, where g_1, \dots, g_k are elements of G . In particular, we will work with multilinear commutator words. These are words which are obtained by nesting commutators, but using always different variables. More formally, the word $w(x) = x$ in one variable is a multilinear commutator; if u and v are multilinear commutators involving different variables then the word $w = [u, v]$ is a multilinear commutator, and all multilinear commutators are obtained in this way. Note that in the literature the multilinear commutators are sometimes called outer commutator words. In the recent article by A. Shalev [10] they are called general commutator words. Examples of multilinear commutators include the familiar lower central words $\gamma_k(x_1, \dots, x_k) = [x_1, \dots, x_k]$ and derived words δ_k , on 2^k variables, defined recursively by

$$\delta_0 = x_1, \quad \delta_k = [\delta_{k-1}(x_1, \dots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \dots, x_{2^k})].$$

Of course, $\gamma_k(G)$ is the k -th term of the lower central series of G while $\delta_k(G) = G^{(k)}$ is the k -th term of the derived series.

We establish the following results.

Theorem 1.2. *Let $w = w(x_1, \dots, x_n)$ be a multilinear commutator, and let G be a group such that $|x^G| \leq m$ for every w -value x in G . Then the commutator subgroup of $w(G)$ has finite (m, n) -bounded order.*

Theorem 1.3. *Let $w = w(x_1, \dots, x_n)$ be a multilinear commutator, and let G be a group such that $|x^{w(G)}| \leq m$ for every w -value x in G . Then $[w(w(G)), w(G)]$ has finite (m, n) -bounded order.*

It is easy to see that the theorem of Neumann can be obtained from each of the above theorems by considering the case where $w(x) = x$ while Theorem 1.1 follows by taking $w(x_1, x_2) = [x_1, x_2]$.

A number of further results of similar nature can be deduced from rather general Theorem 4.1 which states that for any subgroup K of G such that $w(G) \leq K$ and $|a^K| \leq m$ for each $a \in G_w$, the order of $[w(K), w(G)]$ is (m, w) -bounded (see Section 4 for details). For example, let $w = [[x_1, x_2], [x_3, x_4]]$ be the metabelian word. If $|x^G| \leq m$ for each w -value in G , then $G^{(3)}$ has finite m -bounded order. If $|x^{G'}| \leq m$ for each w -value in G , then $\gamma_3(G^{(2)})$ has finite m -bounded order.

The proofs of Theorem 1.2 and Theorem 1.3 are based on rather specific combinatorial techniques which were developed in [4, 1, 2] for handling multilinear commutator words. It seems unlikely that results of similar nature for arbitrary words hold.

2 Preliminary results

Throughout this section, w will be a fixed word.

Let G be a group generated by a set X such that $X = X^{-1}$. Given an element $g \in G$, we write $l_X(g)$ for the minimal number l with the property that g can be written as a product of l elements of X . Clearly, $l_X(g) = 0$ if and only if $g = 1$. We call $l_X(g)$ the length of g with respect to X . The following result is Lemma 2.1 in [3].

Lemma 2.1. *Let H be a group generated by a set $X = X^{-1}$ and let K be a subgroup of finite index m in H . Then each coset Kb contains an element g such that $l_X(g) \leq m - 1$.*

In the sequel the above lemma will be used in the situation where $H = w(G)$ and $X = G_w \cup G_w^{-1}$ is the set of w -values and their inverses in G . Therefore we will write $l(g)$ to denote the smallest number such that the element $g \in w(G)$ can be written as a product of as many w -values or their inverses.

Recall that if G is a group, $a \in G$ and H is a subgroup of G , then $[H, a]$ denotes the subgroup of G generated by all commutators of the form $[h, a]$, where $h \in H$. It is well-known that $[H, a]$ is normalized by a and H .

The following result is analogous to Lemma 2.3 in [3].

Lemma 2.2. *Let w be a word and let G be a group such that for every $x \in G_w$ the centralizer $C_{w(G)}(x)$ of x in $w(G)$ has finite index at most m in $w(G)$. Then $[w(G), x]$ has m -bounded order for every $x \in G_w$.*

Proof. Let $W = w(G)$. Since $C_W(x)$ has index at most m in W , by Lemma 2.1 we can choose elements y_1, \dots, y_m such that $l(y_i) \leq m - 1$ and $[W, x]$ is generated by the commutators $[y_i, x]$. For each $i = 1, \dots, m$ write $y_i = y_{i1} \cdots y_{im-1}$, where $y_{ij} \in G_w \cup G_w^{-1}$. The standard commutator identities show that $[y_i, x]$ can be written as a product of conjugates in W of the commutators $[y_{ij}, x]$. Let h_1, \dots, h_s be the conjugates in W of elements from the set $\{x, y_{ij}; 1 \leq i, j \leq m\}$. Since $C_W(h)$ has finite index at most m in W for each $h \in G_w$, it follows that s is m -bounded. Let $T = \langle h_1, \dots, h_s \rangle$. It is clear that $[W, x] \leq T'$ and so it is sufficient to show that T' has finite m -bounded order. As $|W : C_W(h_i)| \leq m$ for every i , it follows that the center $Z(T)$ has index at most m^s in T . Thus, Schur's theorem [8, 10.1.4] tells us that T' has finite m -bounded order, as required. \square

Lemma 2.3. *Let w be a word, let G be a group and let K be a subgroup of G containing $w(G)$ such that $|K : C_K(x)| \leq m$ for every $x \in G_w$. Then $[w(G), x]^K$ has m -bounded order for every $x \in G_w$.*

Proof. Choose $x \in G_w$ and let $W = w(G)$. By Lemma 2.2, $[W, x]$ has m -bounded order. Observe that $[W, x]$ has at most m conjugates in K and the conjugates normalize

each other. Thus, $[W, x]^K$ is a product of at most m subgroups that normalize each other and have m -bounded order. The lemma follows. \square

The following lemma can be seen as a development related to Lemma 2.4 in [3] and Lemma 4.5 in [11]. It plays a central role in our arguments.

Lemma 2.4. *Let m, j be positive integers and $w = w(x_1, \dots, x_n)$ a word. Let G be a group and K be a subgroup of G containing $w(G)$ such that $|K : C_K(x)| \leq m$ for every $x \in G_w$. Assume that M is a normal subgroup of K of index j . Choose $k_i \in K$ for $i = 1, \dots, n$. Then there exist elements $\tilde{k}_i \in k_i M$, for $i = 1, \dots, n$, and a normal subgroup \tilde{M} of M of finite (j, m) -bounded index, such that the order of $[w(G), w(\tilde{k}_1 \tilde{M}, \dots, \tilde{k}_n \tilde{M})]^K$ is finite and m -bounded.*

Proof. Let $W = w(G)$. Consider the set

$$S = \{w(k_1 u_1, \dots, k_n u_n) \mid u_1, \dots, u_n \in M\}.$$

Choose in S an element $a = w(\tilde{k}_1, \dots, \tilde{k}_n)$ such that the number of conjugates of a in W is maximal among the elements of S , that is $|a^W| \geq |g^W|$ for all $g \in S$.

By Lemma 2.1 we can choose $b_1, \dots, b_r \in W$ such that $l(b_i) \leq m - 1$ and $a^W = \{a^{b_i} \mid i = 1, \dots, r\}$ where $a^{b_i} \neq a^{b_j}$ if $i \neq j$. Set $\tilde{M} = M \cap (C_K(\langle b_1, \dots, b_r \rangle))_K$ (i.e. \tilde{M} is the intersection of M and all K -conjugates of $C_K(\langle b_1, \dots, b_r \rangle)$). Since $l(b_i) \leq m - 1$ and $C_K(x)$ has index at most m in K for each $x \in G_w$, the subgroup $C_K(\langle b_1, \dots, b_r \rangle)$ has m -bounded index in K , and so \tilde{M} has (j, m) -bounded index.

Consider the element $w(\tilde{k}_1 v_1, \dots, \tilde{k}_n v_n) \in S$ where each $v_i \in \tilde{M}$, for $i = 1, \dots, n$. We have

$$w(\tilde{k}_1 v_1, \dots, \tilde{k}_n v_n) = va,$$

for some $v \in \tilde{M} \leq C_K(b_1, \dots, b_r)$. It follows that $(va)^{b_i} = va^{b_i}$ for each $i = 1, \dots, r$. Therefore the elements va^{b_i} form the conjugacy class $(va)^W$ because they are all different and their number is the allowed maximum. So, for an arbitrary element $h \in W$ there exists $b \in \{b_1, \dots, b_r\}$ such that $(va)^h = va^b$ and hence $v^h a^h = va^b$. Therefore $[h, v] = v^{-h} v = a^h a^{-b}$ and so $[h, v]^a = a^{-1} a^h a^{-b} a = [a, h][b, a] \in [W, a]$. Thus $[W, v]^a \leq [W, a]$ and

$$[W, va] = [W, a][W, v]^a \leq [W, a].$$

Therefore $[W, w(\tilde{k}_1 \tilde{M}, \dots, \tilde{k}_n \tilde{M})] \leq [W, a]$. Lemma 2.3 states that $[W, a]^K$ has m -bounded order. The result follows. \square

3 Combinatorics of multilinear commutators

We will need some machinery concerning combinatorics of commutators, so we now recall some notation from the paper [2].

Throughout this section $w = w(x_1, \dots, x_n)$ will be a multilinear commutator. If A_1, \dots, A_n are subsets of a group G , we write $w(A_1, \dots, A_n)$ for the subgroup generated by the w -values $w(a_1, \dots, a_n)$ with $a_i \in A_i$.

Let I be a subset of $\{1, \dots, n\}$. Suppose that we have a family A_{i_1}, \dots, A_{i_s} of subsets of G with indices running over I and another family B_{l_1}, \dots, B_{l_t} of subsets with indices running over $\{1, \dots, n\} \setminus I$. We write

$$w_I(A_i; B_l)$$

for $w(X_1, \dots, X_n)$, where $X_k = A_k$ if $k \in I$, and $X_k = B_k$ otherwise. On the other hand, whenever $a_i \in A_i$ for $i \in I$ and $b_l \in B_l$ for $l \in \{1, \dots, n\} \setminus I$, the symbol $w_I(a_i; b_l)$ stands for the element $w(x_1, \dots, x_n)$, where $x_k = a_k$ if $k \in I$, and $x_k = b_k$ otherwise. Sometimes, for the sake of shortness, we will omit the indices and we will simply write

$$w_I(A; B),$$

for $w(X_1, \dots, X_n)$, where $X_k = A$ if $k \in I$, and $X_k = B$ otherwise.

The next lemma is Lemma 2.4 in [2].

Lemma 3.1. *Let $w = w(x_1, \dots, x_n)$ be a multilinear commutator. Assume that H is a normal subgroup of a group G . Let $g_1, \dots, g_n \in G$, $h \in H$ and fix $s \in \{1, \dots, n\}$. Then there exist $y_j \in g_j^H$, for $j = 1, \dots, n$, such that*

$$w_{\{s\}}(g_s h; g_l) = w(y_1, \dots, y_n) w_{\{s\}}(h; g_l).$$

The following corollaries are special cases of Lemma 2.5 and Lemma 4.1 of [2], respectively. For the reader's convenience, we include the proofs.

Corollary 3.2. *Let $w = w(x_1, \dots, x_n)$ be a multilinear commutator. Let K be a group and M a normal subgroup of K . Assume that*

$$w(k_1 M, \dots, k_n M) = 1$$

for some elements $k_i \in K$. Let I be a proper subset of $\{1, \dots, n\}$. Then

$$w_I(k_i M; M) = 1.$$

Proof. It is sufficient to prove that the result holds when $|I| = n - 1$, and then repeatedly apply this case, with $k_i = 1$ if needed. Let $\{s\} = \{1, \dots, n\} \setminus I$. Clearly $w_I(k_i M; M) = w_{\{s\}}(M; k_l M)$, so we have to prove that $w_{\{s\}}(M; k_l M) = 1$.

Choose an element $w_{\{s\}}(m_s; k_l m_l) \in w_{\{s\}}(M; k_l M)$ and consider the element $w_{\{s\}}(k_s m_s; k_l m_l)$. By Lemma 3.1,

$$w_{\{s\}}(k_s m_s; k_l m_l) = w(y_1, \dots, y_n) w_{\{s\}}(m_s; k_l m_l), \quad (3.1)$$

where $y_l \in (k_l m_l)^M \subseteq k_l M$, for $j \neq s$, and $y_s \in k_s^M \subseteq k_s M$. Both $w_{\{s\}}(k_s m_s; k_l m_l)$ and $w(y_1, \dots, y_n)$ are trivial by assumption, since they belong to $w(k_1 M, \dots, k_n M) = 1$. By (3.1), we conclude that $w_{\{s\}}(m_s; k_l m_l) = 1$ as well, and the result follows. \square

Corollary 3.3. *Let $w = w(x_1, \dots, x_n)$ be a multilinear commutator and let I be a subset of $\{1, \dots, n\}$. Let M be a normal subgroup of a group K . Assume that*

$$w_J(K; M) = 1 \quad \text{for every } J \subsetneq I.$$

Suppose we are given elements $k_i \in K$ with $i \in I$ and elements $m_j \in M$ with $j \in \{1, \dots, n\}$. Then we have

$$w_I(k_i m_i; m_l) = w_I(k_i; m_l).$$

Proof. Fix an index $s \in I$ and let $J = I \setminus \{s\}$. Then we can write

$$w_I(k_i m_i; m_l) = w_{\{s\}}(k_s m_s; c_l)$$

where $c_l = k_l m_l$ if $l \in J$, and $c_l = m_l$ if $l \notin I$. Moreover we can write $k_s m_s = h k_s$ for some $h \in M$. By Lemma 3.1,

$$w_{\{s\}}(h k_s; c_l) = w(y_1, \dots, y_n) w_{\{s\}}(k_s; c_l), \quad (3.2)$$

where $y_s \in h^K \subseteq M$ and $y_l \in m_l^K \subseteq M$ if $l \notin I$. In particular

$$w(y_1, \dots, y_n) \in w_J(K; M).$$

and thus $w(y_1, \dots, y_n) = 1$, since $w_J(K; M) = 1$ by assumption. From (3.2) we deduce that

$$w_{\{s\}}(k_s m_s; c_l) = w_{\{s\}}(h k_s; c_l) = w_{\{s\}}(k_s; c_l).$$

By repeating the argument for every $s \in I$, we get the desired conclusion. \square

4 Proof of the main theorems

Our main theorems are both consequences of the following result, which will be proved in this section.

Theorem 4.1. *Let m be a positive integer and $w = w(x_1, \dots, x_n)$ a multilinear commutator word. Suppose that G is a group having a subgroup K such that $W = w(G) \leq K$ and $|K : C_K(a)| \leq m$ for every $a \in G_w$. Then $[w(K), W]$ has (m, n) -bounded order.*

For the reader's convenience, the most technical part of our argument is isolated in the following proposition.

Proposition 4.2. *Let $I \subseteq \{1, \dots, n\}$ with $|I| = s$. Under the hypotheses of Theorem 4.1, assume that there exist a normal subgroup N of K of finite order r and a normal subgroup M of K of finite index j such that*

$$[W, w_J(K; M)] \leq N \quad \text{for every } J \subsetneq I.$$

Then there exist a finite normal subgroup N_I of K of (r, j, m, s) -bounded order with $N \leq N_I$ and a normal subgroup M_I of K of (j, m, s) -bounded index with $M_I \leq M$ such that

$$[W, w_I(K; M_I)] \leq N_I.$$

Proof. Consider a set C of coset representatives of M in K , and let Ω be the set of n -tuples $\underline{c} = (c_1, \dots, c_n)$ where $c_r \in C$ if $r \in I$ and $c_r = 1$ otherwise. Notice that the order of Ω is j^s . For any n -tuple $\underline{c} = (c_1, \dots, c_n) \in \Omega$, by Lemma 2.4, there exist elements $d_i \in c_i M$, with $i = 1, \dots, n$ and a normal subgroup $M_{\underline{c}}$ of (j, m) -bounded index in K such that the order of

$$[W, w(d_1 M_{\underline{c}}, \dots, d_n M_{\underline{c}})]^K$$

is m -bounded. Let

$$\begin{aligned} M_I &= M \cap \left(\bigcap_{\underline{c} \in \Omega} M_{\underline{c}} \right), \\ N_I &= N \prod_{\underline{c} \in \Omega} [W, w(d_1 M_{\underline{c}}, \dots, d_n M_{\underline{c}})]^K. \end{aligned}$$

As $|\Omega| = j^s$, it follows that M_I has (j, m, s) -bounded index in K and the order of N is (r, j, m, s) -bounded.

Let Z/N_I be the center of WN_I/N_I in the quotient group K/N_I and let $\bar{K} = K/Z$. The image of a subgroup U of K in \bar{K} will be denoted by \bar{U} , and similarly for the image of an element.

Let us consider an arbitrary element of the type $w_I(k_i, h_l) \in w_I(K; M_I)$, with $k_i \in K$ and $h_l \in M_I$. Consider the n -tuple $\underline{c} = (c_1, \dots, c_n) \in \Omega$ defined by $k_i \in c_i M$ if $i \in I$ and $c_i = 1$ otherwise. Let d_1, \dots, d_n the elements as above, corresponding to the n -tuple \underline{c} . Then

$$[W, w(d_1 M_I, \dots, d_n M_I)] \leq N_I,$$

that is

$$\overline{w(d_1 M_I, \dots, d_n M_I)} = 1,$$

in the quotient group $\bar{K} = K/Z$. By Corollary 3.2, we deduce that

$$\overline{w_I(d_i M_I; M_I)} = 1. \quad (4.1)$$

Moreover, as $c_i M = d_i M$, we have that $k_i = d_i v_i$ for some $v_i \in M$. It also follows from our assumptions that

$$\overline{w_J(K; M)} = 1$$

for every proper subset J of I . Thus we can apply Corollary 3.3 and we obtain that

$$w_I(\bar{k}_i; \bar{h}_l) = w_I(\bar{d}_i \bar{v}_i; \bar{h}_l) = w_I(\bar{d}_i; \bar{h}_l) = 1,$$

where in the last equality we have used (4.1). Since $w_I(k_i, h_l)$ was an arbitrary generator of $w_I(K; M_I)$, it follows that

$$\overline{w_I(K; M_I)} = 1,$$

that is

$$[W, w_I(K; M_I)] \leq N_I,$$

as desired. \square

Now the proof of Theorem 4.1 is an easy induction.

Proof of Theorem 4.1. We will prove that for every $s = 0, \dots, n$ there exist a finite normal subgroup N_s of K of (m, n) -bounded order and a normal subgroup M_s of K of (m, n) -bounded index such that

$$[W, w_I(K; M_s)] \leq N_s$$

for every subset I of $\{1, \dots, n\}$ with $|I| \leq s$. Once this is done, the theorem will follow taking $s = n$.

Assume that $s = 0$. We apply Lemma 2.4 with $M = K$ and $k_i = 1$ for every $i = 1, \dots, n$. Thus there exist $a_1, \dots, a_n \in K$ and a normal subgroup of m -bounded index M_0 of K , such that the order of

$$N_0 = [W, w(a_1 M_0, \dots, a_n M_0)]^K$$

is m -bounded.

Let Z/N_0 be the center of WN_0/N_0 in the quotient group K/N_0 and let $\bar{K} = K/Z$. We have that

$$\overline{w(a_1 M_0, \dots, a_n M_0)} = 1,$$

so it follows from Corollary 3.2 that

$$\overline{w(M_0, \dots, M_0)} = 1,$$

that is, $[W, w(M_0)] \leq N_0$. This proves the case $s = 0$.

Now assume $s \geq 1$. Choose $I \subseteq \{1, \dots, n\}$ with $|I| = s$. By induction, the hypotheses of Proposition 4.2 are satisfied with $N = N_{s-1}$ and $M = M_{s-1}$, so there exist a finite normal subgroup N_I of K of (m, n) -bounded order with $N_{s-1} \leq N_I$ and a normal subgroup M_I of K of (m, n) -bounded index with $M_I \leq M_{s-1}$ such that

$$[W, w_I(K; M_I)] \leq N_I.$$

Let

$$M_s = \bigcap_{|I|=s} M_I, \quad N_s = \prod_{|I|=s} N_I,$$

where the intersection (resp. the product) ranges over all subsets I of $\{1, \dots, n\}$ of size s .

As there is a n -bounded number of choices for I , it follows that N_s (resp. M_s) has (m, n) -bounded order (resp. (m, n) -bounded index in K). Note that $M_s \leq M_{s-1}$ and $N_{s-1} \leq N_s$. Therefore

$$[W, w_I(K; M_s)] \leq N_s$$

for every $I \subseteq \{1, \dots, n\}$ with $|I| \leq s$. This completes the induction step and the proof of the theorem. \square

Proof of Theorem 1.2. Let $w = w(x_1, \dots, x_n)$ be a multilinear commutator, and let G be a group such that $|x^G| \leq m$ for every w -value x in G . Apply Theorem 4.1 with $K = G$. It follows that $[w(G), w(G)]$ has (m, n) -bounded order, as desired. \square

Proof of Theorem 1.3. Let $w = w(x_1, \dots, x_n)$ be a multilinear commutator, and let G be a group such that $|x^{w(G)}| \leq m$ for every w -value x in G . Apply Theorem 4.1 with $K = w(G)$. It follows that $[w(w(G)), w(G)]$ has (m, n) -bounded order, as desired. \square

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