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The six Grothendieck operations on o-minimal sheaves

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Abstract
In this paper we develop the formalism of the Grothendieck six operations on o-minimal sheaves. The Grothendieck formalism allows us to obtain o-minimal versions of: (i) derived projection formula; (ii) universal coefficient formula; (iii) derived base change formula; (iv) Künneth formula; (v) local and global Verdier duality.

Keywords o-Minimal structures · Proper direct image · Sheaves · Cohomology · Semi-algebraic · Globally sub-analytic

Mathematics Subject Classification 03C64 · 55N30

1 Introduction

The study of o-minimal structures [46] is the analytic part of model theory which deals with theories of ordered, hence topological, structures satisfying certain tameness properties. It generalizes piecewise linear geometry [46, Chapter 1, §7], semi-algebraic geometry [6] and globally sub-analytic geometry ([29], also called finitely sub-analytic in [45]) and it is claimed to be the formalization of Grothendieck's notion of tame topology (topologie modérée). See [46] and [48].

The most striking successes of this model-theoretic point of view of sub-analytic geometry include, on the one hand, an understanding of the behavior at infinity of certain important classes of sub-analytic sets as in Wilkie's [50] as pointed out by Bierstone and Milman [5],

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and on the other hand, the recent, somehow surprising, first unconditional proof of the André–Oort conjecture for mixed Shimura varieties expressible as products of curves by Pila [36] following previous work also using o-minimality by Pila and Zannier [37], Pila and Wilkie [38] and Peterzil and Starchenko [34].

The goal of this paper is to contribute further to the claim that o-minimality does indeed realize Grothendieck’s notion of topologie modérée by developing the formalism of the Grothendieck six operations on o-minimal sheaves, extending Delfs results for semi-algebraic sheaves [11] as well as Kashiwara and Schapira results for sub-analytic sheaves (defined and studied in [31] and successively in [42]) restricted to globally sub-analytic spaces (as we deal, for now, only with definable spaces and not locally definable spaces—the later more general case will be dealt with in a sequel to this paper).

In the semi-algebraic case nearly all of our results are completely new. Indeed, in [11, Section 8], Delfs constructs the semi-algebraic proper direct image functor and only proves two basic results about this functor (base change and commutativity with small inductive limits) and then conjectures in [11, Remark 8.11 ii)] that: “It seems that the results of this section suffice to prove a semi-algebraic analogue of Verdier duality (by the same proof as in the theory of locally compact spaces c.f [49]).”

In the globally sub-analytic case we introduce a new globally sub-analytic proper direct image functor which, unlike Kashiwara and Schapira [31] sub-analytic proper direct image functor, generalizes to arbitrary o-minimal structures including: (i) arbitrary real closed fields; (ii) the non-standard models of sub-analytic geometry [47]; (iii) the non-standard o-minimal structure which does not came from a standard one as in [26,33]. See Remark 3.14 for further details on this. Moreover, unlike [31,42], our definition is compatible with the restriction to open subsets (see Remark 3.15).

The Grothendieck formalism developed here allows us to obtain o-minimal versions of: (i) derived projection formula; (ii) derived base change formula; (iii) universal coefficient formula; (iv) Künneth formula; (v) local and the global Verdier duality. It also sets up the framework for: (a) defining new o-minimal homology theories $H_*(X, \mathbb{Q}) := H^*(a_{X!} \circ a_X^! \mathbb{Q})$ extending o-minimal singular homology or the o-minimal Borel-Moore homology $H_*^{\mathrm{BM}}(X, \mathbb{Q}) := H^*(a_{X*} \circ a_X^! \mathbb{Q})$; (b) the full development of o-minimal geometry including the theory of characteristic classes, Hirzebruch–Riemann–Roch formula and Atiyah–Singer theorem in the non-standard o-minimal context.

The results of the paper were recently used: (i) To settle Pillay’s conjecture for definably compact definable groups [27,40] in arbitrary o-minimal structures. See [17]. This conjecture is an o-minimal analogue of Hilbert’s fifth problem and it says roughly that a definably compact group has an infinitesimal normal subgroup such that the quotient equipped with a certain logic topology is a compact real Lie group of the same dimension. The solution to the conjecture is known to imply much information on the topology of definable groups as well as the structure of the definable subsets and, for example, explains how to obtain a compact real Lie group from an abelian variety over an arbitrary algebraically closed field of characteristic zero taking a quotient by a certain infinitesimal subgroup. (ii) In relation to the integral Hodge conjecture for real varieties by Benoist and Wittenberg [3,4].

We expect applications also in algebraic analysis. With our definition it is possible to treat the global sub-analytic sites in which are defined sheaves of functions with growth conditions up to infinity (e.g. tempered and Whitney $\mathcal{C}^\infty$ or holomorphic functions). This kind of objects are very important for applications as in [9] and in [10] (in which globally sub-analytic sheaves are hidden under the notion of ind-sheaves on a bordered space).

The structure of the paper is the following. In Sect. 2 we introduce the setting where we will work on, listing all the preliminary results that will be needed throughout the rest of the paper. In Sect. 3 we define the proper direct image operation on o-minimal sheaves and prove its fundamental properties. In Sect. 4 we obtain the local and the global Verdier duality and its consequences.

Finally we note that, for the readers convenience, at the beginning of each section, after we introduce the set up, we include an extended summary in which we compare the ideas involved in the proofs of the section with those present in the references cited in the bibliography.

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2 Preliminaries

In this paper we work in an arbitrary o-minimal structure $\mathbb{M} = (M, <, (c)_{c\in\mathcal{C}}, (f)_{f\in\mathcal{F}}, (R)_{R\in\mathcal{R}})$ with definable Skolem functions. We refer the reader to [46] for basic o-minimality.

We let Def be the category whose objects are definable spaces and whose morphisms are continuous definable maps between definable spaces—here and below "definable" always means definable in $\mathbb{M}$ with parameters. (See [46, Chapter 10, §1]). If $\mathbb{M}$ is a real closed field $(R, <, 0, 1, +, \cdot)$, then Def is the category whose objects are semi-algebraic spaces over R and whose morphisms are continuous semi-algebraic maps between such semi-algebraic spaces ([46, Chapter 2] and [11, Chapter I, Example 1.1]); if $\mathbb{M}$ is $\mathbb{R}_{\text{an}}=(\mathbb{R}, <, 0, 1, +, \cdot, (f)_{f\in\text{an}})$–the field of real numbers expanded by restricted analytic functions, then Def is the category whose objects are globally sub-analytic spaces together with continuous maps with globally sub-analytic graphs between such spaces [13]; if $\mathbb{M}$ is an ordered vector space $(V, <, 0, +, (d)_{d\in D})$ over an ordered division ring D, then Def is the category whose objects are the piecewise linear spaces in this vector space together with continuous piecewise linear maps between such spaces ([46, Chapter 1, §7]).

Objects of Def are equipped with a topology determined by the order topology on $(M, <)$. However, if $(M, <)$ is non-archimedean then infinite definable spaces are totally disconnected and not locally compact, so one studies definable spaces equipped with the o-minimal site and replaces topological notions (connected, normal, compact, proper) by their definable analogues (definably connected, definably normal, definably compact, definably proper). The o-minimal site [15] generalizes both the semi-algebraic site [11] and the sub-analytic site [31]. Given an object X of Def the o-minimal site X_{def} on X is the category $\text{Op}(X_{\text{def}})$ whose objects are open (in the topology of X mentioned above) definable subsets of X, the morphisms are the inclusions and the admissible covers $\text{Cov}(U)$ of $U \in \text{Op}(X_{\text{def}})$ are covers by open definable subsets of X with finite sub-covers.

As shown in [15, Proposition 3.2], if A is a commutative ring, then the category $\text{Mod}(A_{X_{\text{def}}})$ of sheaves of A-modules on X (relative to the o-minimal site) is isomorphic to the category $\text{Mod}(A_{\widetilde{X}})$ of sheaves of A-modules on a certain spectral topological space $\widetilde{X}$, the o-minimal spectrum of X, associated to X. The o-minimal spectrum $\widetilde{X}$ of a definable space X is the set of ultra-filters of definable subsets of X (also know in model theory as types on X) equipped with the topology generated by the subsets $\widetilde{U}$ with $U \in \text{Op}(X_{\text{def}})$. If $f : X \to Y$ is a morphism in Def, then one has a corresponding continuous map $\widetilde{f} : \widetilde{X} \to \widetilde{Y} : \alpha \mapsto \widetilde{f}(\alpha)$ where $f(\alpha)$ is the ultrafilter in $\widetilde{Y}$ determined by the collection $\{A : f^{-1}(A) \in \alpha\}$. (See [15, Definitions 2.2 and 2.18] or [8] and [39] where these notions were first introduced).

If $\mathbb{M}$ is a real closed field $(R, <, 0, 1, +, \cdot)$, and $V \subseteq R^n$ is an affine real algebraic variety over R, then $\widetilde{V}$ is homeomorphic to Sper $R[V]$, the real spectrum of the coordinate ring $R[V]$ of V ([6, Chapter 7, Section 7.2] or [11, Chapter I, Example 1.1]); and the isomorphism $\mathrm{Mod}(A_{V_{\mathrm{def}}}) \simeq \mathrm{Mod}(A_{\widetilde{V}})$ from [15] corresponds in this case to [11, Chapter 1, Proposition 1.4].

Below we denote by $\widetilde{\mathrm{Def}}$ the corresponding category of o-minimal spectra of definable spaces and continuous definable maps and

$$\mathrm{Def} \to \widetilde{\mathrm{Def}}$$

the functor just defined. Due to the isomorphism $\mathrm{Mod}(A_{X_{\mathrm{def}}}) \simeq \mathrm{Mod}(A_{\widetilde{X}})$ for every object X of Def we will work in this paper in $\widetilde{\mathrm{Def}}$.

2.1 Extended summary

In this section we will recall and prove some preliminary results that will be crucial for the rest of the paper. These preliminary results are necessary since the category $\widetilde{\mathrm{Def}}$ in which we will be working is rather different from the category Top of topological spaces.

An object of $\widetilde{\mathrm{Def}}$ is a T_0, quasi-compact and a spectral topological space, i.e., it has a basis of quasi-compact open subsets, closed under taking finite intersections and each irreducible closed subset is the closure of a unique point. Such objects are not T_1 (unless they are finite), namely not every point is closed, so they are not Hausdorff (unless they are finite). In particular, the usual (topological) definition of compact topological space cannot be used in $\widetilde{\mathrm{Def}}$. Similarly the usual (topological) definition of proper maps is of no use in $\widetilde{\mathrm{Def}}$.

Since $\widetilde{\mathrm{Def}}$ has cartesian squares (which one should point out are not cartesian squares in the category of topological spaces Top), in this section we introduce a category theory definition of these notions in $\widetilde{\mathrm{Def}}$ just like in semi-algebraic geometry [12, Section 9] (and also in algebraic geometry [23, Chapter II, Section 4] or [22, Chapter II, Section 5.4]) and point out all the properties needed later.

A fiber $\widetilde{f}^{-1}(\alpha)$ of a morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ in $\widetilde{\mathrm{Def}}$ is not in general an object of $\widetilde{\mathrm{Def}}$, but following [1, Lemma 3.1] in the affine case, such fiber is homeomorphic to an object $\widetilde{(f^{\mathbb{S}})^{-1}}(a)$ of $\widetilde{\mathrm{Def}(\mathbb{S})}$ where a is a realization of the type α and $\mathbb{S}$ is the prime model of the first-order theory of $\mathbb{M}$ over $\{a\} \cup M$. Here $\widetilde{\mathrm{Def}(\mathbb{S})}$ is the same as $\widetilde{\mathrm{Def}}$ but defined in the o-minimal structure $\mathbb{S}$. This phenomena is the model theoretic analogue of what happens in real-algebraic geometry ([11, Proposition 2.4], [44, Chapter II, 3.2]) (and also in algebraic geometry [23, Chapter II, Section 3]). Indeed, if $f : X \to Y$ is a morphism of real schemes over a real closed field R and $\alpha \in Y$, then $f^{-1}(\alpha)$ with the underlying topology is homeomorphic to the real scheme $X \times_Y \mathrm{Sper}\, k(\alpha)$ over the residue ordered field $k(\alpha)$ of α (recall that α is a prime cone). By [39] the real closure of $k(\alpha)$ is isomorphic to the prime model over R and a realization of α.

Due to the homeomorphism $\widetilde{f}^{-1}(\alpha) \simeq \widetilde{(f^{\mathbb{S}})^{-1}}(a)$, when $\alpha \in \widetilde{Y}$ is a closed point, we will be able to use the theory from [18, Section 3] of normal and constructible families of supports on the object $\widetilde{(f^{\mathbb{S}})^{-1}}(a)$ also on the fiber $\widetilde{f}^{-1}(\alpha)$ after we show that working in certain full subcategories $\widetilde{\mathbf{A}}$ of $\widetilde{\mathrm{Def}}$ the family c of complete supports on $\widetilde{(f^{\mathbb{S}})^{-1}}(a)$ is normal and constructible. Normal and constructible families of supports are the o-minimal analogue of the semi-algebraic and paracompactifying families of supports in semi-algebraic geometry [11, Chapter II, Section 1].

2.2 Morphisms proper in $\widetilde{\mathrm{Def}}$

Here we will introduce a category theory definition of the notions proper and complete in $\widetilde{\mathrm{Def}}$ just like in semi-algebraic geometry [12, Section 9] (and also in algebraic geometry [23, Chapter II, Section 4] or [22, Chapter II, Section 5.4]) and point out the properties needed later. Note that these notions and properties are noting but the corresponding notions and properties in Def, introduced and studied in [16], under the isomorphism $\mathrm{Def} \to \widetilde{\mathrm{Def}}$.

If X is an object of $\widetilde{\mathrm{Def}}$, then a subset $Z \subseteq X$ is called *constructible* if Z is also an object of $\widetilde{\mathrm{Def}}$.

Let $f : X \to Y$ be a morphism in $\widetilde{\mathrm{Def}}$. We say that:

- $f : X \to Y$ is *closed in* $\widetilde{\mathrm{Def}}$ if for every closed constructible subset A of X, its image $f(A)$ is a closed (constructible) subset of Y.
- $f : X \to Y$ is a *closed (resp. open) immersion* if $f : X \to f(X)$ is a homeomorphism and $f(X)$ is a closed (resp. open) subset of Y.

Since $\mathrm{Def} \to \widetilde{\mathrm{Def}}$ is an isomorphism of categories and cartesian squares exist in Def we have:

Fact 2.1 In the category $\widetilde{\mathrm{Def}}$ the *cartesian square* of any two morphisms $f : X \to Z$ and $g : Y \to Z$ in $\widetilde{\mathrm{Def}}$ exists and is given by a commutative diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "p_Y"] \arrow[d, "p_X"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

where the morphisms p_X and p_Y are known as projections. The Cartesian square satisfies the following universal property: for any other object Q of $\widetilde{\mathrm{Def}}$ and morphisms $q_X : Q \to X$ and $q_Y : Q \to Y$ of $\widetilde{\mathrm{Def}}$ for which the following diagram commutes,

$$\begin{tikzcd}
Q \arrow[rd, dashed, "u"] \arrow[rrd, bend left, "q_Y"] \arrow[rdd, bend right, "q_X"'] & & \\
& X \times_Z Y \arrow[r, "p_Y"] \arrow[d, "p_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

there exist a unique natural morphism $u : Q \to X \times_Z Y$ (called mediating morphism) making the whole diagram commute. As with all universal constructions, the cartesian square is unique up to a definable homeomorphism.

The following is a very important remark that one should always have in mind:

Remark 2.2 The cartesian square in $\widetilde{\mathrm{Def}}$ of two morphisms $f : X \to Z$ and $g : Y \to Z$ in $\widetilde{\mathrm{Def}}$ is not the same as the cartesian square in Top of the two morphisms $f : X \to Z$ and $g : Y \to Z$ in Top. In particular, if pt denotes a fixed one point object of $\widetilde{\mathrm{Def}}$, then the *cartesian product*

$$\begin{tikzcd}
X \times_{\mathrm{pt}} Y \arrow[r, "p_Y"] \arrow[d, "p_X"] & Y \arrow[d, "a_Y"] \\
X \arrow[r, "a_X"] & \mathrm{pt}
\end{tikzcd}$$

in $\widetilde{\mathrm{Def}}$, also denoted by $X \times Y$, of two objects X and Y in $\widetilde{\mathrm{Def}}$ is not the same as usual the cartesian product (in Top) of the objects X and Y in Top.

Given a morphism $f : X \to Y$ in $\widetilde{\mathrm{Def}}$, the corresponding diagonal morphism is the unique morphism $\Delta : X \to X \times_Y X$ in $\widetilde{\mathrm{Def}}$ given by the universal property of cartesian squares:

$$\begin{tikzcd}
X \arrow[dr, dashed, "\Delta"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times_Y X \arrow[r, "p_X"] \arrow[d, "p_X"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y.
\end{tikzcd}$$

We say that:

- $f : X \to Y$ is *separated in* $\widetilde{\mathrm{Def}}$ if the corresponding diagonal morphism $\Delta : X \to X \times_Y X$ is a closed immersion.

We say that an object Z in $\widetilde{\mathrm{Def}}$ is *separated in* $\widetilde{\mathrm{Def}}$ if the morphism $Z \to \mathrm{pt}$ to a point is separated.

Let $f : X \to Y$ be a morphism in $\widetilde{\mathrm{Def}}$. We say that:

- $f : X \to Y$ is *universally closed in* $\widetilde{\mathrm{Def}}$ if for any morphism $g : Y' \to Y$ in $\widetilde{\mathrm{Def}}$ the morphism $f' : X' \to Y'$ in $\widetilde{\mathrm{Def}}$ obtained from the cartesian square

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

in $\widetilde{\mathrm{Def}}$ is closed in $\widetilde{\mathrm{Def}}$.

Definition 2.3 We say that a morphism $f : X \to Y$ in $\widetilde{\mathrm{Def}}$ is *proper in* $\widetilde{\mathrm{Def}}$ if $f : X \to Y$ is separated and universally closed in $\widetilde{\mathrm{Def}}$.

Definition 2.4 We say that an object Z of $\widetilde{\mathrm{Def}}$ is *complete in* $\widetilde{\mathrm{Def}}$ if the morphism $Z \to \mathrm{pt}$ is proper in $\widetilde{\mathrm{Def}}$.

Directly from the definitions (as in [22, Chapter II, Proposition 5.4.2 and Corollary 5.4.3], see also [12, Section 9]) one has the following. See [16, Proposition 3.7] for a detailed proof in Def which transfers to $\widetilde{\mathrm{Def}}$ due to the isomorphism Def $\to \widetilde{\mathrm{Def}}$:

Proposition 2.5 *In the category* $\widetilde{\mathrm{Def}}$ *the following hold:*

(1) *Closed immersions are proper in* $\widetilde{\mathrm{Def}}$.

(2) *A composition of two morphisms proper in* $\widetilde{\mathrm{Def}}$ *is proper in* $\widetilde{\mathrm{Def}}$.

(3) *Let*
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & Z \end{tikzcd}$$
be a morphism over Z *in* $\widetilde{\mathrm{Def}}$ *and* $Z' \to Z$ *a base extension in* $\widetilde{\mathrm{Def}}$. *If* $f : X \to Y$ *is proper in* $\widetilde{\mathrm{Def}}$, *then the corresponding base extension morphism* $f' : X \times_Z Z' \to Y \times_Z Z'$ *is proper in* $\widetilde{\mathrm{Def}}$.

(4) *Let*
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & Z \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X' \arrow[r, "f'"] \arrow[dr, "p'"'] & Y' \arrow[d, "q'"] \\ & Z \end{tikzcd}$$
be morphisms over Z *in* $\widetilde{\mathrm{Def}}$. *If* $f : X \to Y$ *and* $f' : X' \to Y'$ *are proper in* $\widetilde{\mathrm{Def}}$, *then the corresponding product morphism* $f \times f' : X \times_Z X' \to Y \times_Z Y'$ *is proper in* $\widetilde{\mathrm{Def}}$.

(5) *If* $f : X \to Y$ *and* $g : Y \to Z$ *are morphisms such that* $g \circ f$ *is proper in* $\widetilde{\mathrm{Def}}$, *then:*

   (i) f *is proper in* $\widetilde{\mathrm{Def}}$;
   (ii) *if* g *is separated in* $\widetilde{\mathrm{Def}}$ *and* f *is surjective, then* g *proper in* $\widetilde{\mathrm{Def}}$.

(6) *A morphism* $f : X \to Y$ *is proper in* $\widetilde{\mathrm{Def}}$ *if and only if* Y *can be covered by finitely many open definable subsets* V_i *such that* $f_| : f^{-1}(V_i) \to V_i$ *is proper in* $\widetilde{\mathrm{Def}}$.

From Proposition 2.5 we easily have:

Corollary 2.6 *Let* $f : X \to Y$ *be a morphism in* $\widetilde{\mathrm{Def}}$ *and* $Z \subseteq X$ *a complete object of* $\widetilde{\mathrm{Def}}$. *Then the following hold:*

(1) Z *is a closed (constructible) subset of* X.
(2) $f_{|Z} : Z \to Y$ *is proper in* $\widetilde{\mathrm{Def}}$.
(3) $f(Z) \subseteq Y$ *is (constructible) complete in* $\widetilde{\mathrm{Def}}$.
(4) *If* $f : X \to Y$ *is proper in* $\widetilde{\mathrm{Def}}$ *and* $C \subseteq Y$ *is a complete object of* $\widetilde{\mathrm{Def}}$, *then* $f^{-1}(C) \subseteq X$ *is (constructible) complete in* $\widetilde{\mathrm{Def}}$.

Below, if $\mathbf{C}$ is a subcategory of Def we denote by $\widetilde{\mathbf{C}}$ its image under Def $\to \widetilde{\mathrm{Def}}$. The following is a standard consequence of Proposition 2.5. See [16, Corollary 3.9] for a detailed proof in Def which transfers to $\widetilde{\mathrm{Def}}$ due to the isomorphism Def $\to \widetilde{\mathrm{Def}}$:

Corollary 2.7 *Let* $\mathbf{C}$ *be a full a subcategory of the category of definable spaces* Def *whose set of objects is:*

- *closed under taking locally closed definable subspaces of objects of* $\mathbf{C}$,
- *closed under taking cartesian products of objects of* $\mathbf{C}$,

Then the following are equivalent:

(1) *Every object* X *of* $\widetilde{\mathbf{C}}$ *is completable in* $\widetilde{\mathbf{C}}$ *i.e., there exists an object* X' *of* $\widetilde{\mathbf{C}}$ *which is complete in* $\widetilde{\mathrm{Def}}$ *together with an open immersion* $i : X \hookrightarrow X'$ *in* $\widetilde{\mathbf{C}}$ *with* $i(X)$ *dense in* X'. *Such* $i : X \hookrightarrow X'$ *is called a completion of* X *in* $\widetilde{\mathbf{C}}$.

(2) *Every morphism* $f : X \to Y$ *in* $\widetilde{\mathbf{C}}$ *is completable in* $\widetilde{\mathbf{C}}$ *i..e, there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "j"] & Y' \end{tikzcd}$$

of morphisms in $\widetilde{\mathbf{C}}$ *such that: (i)* $i : X \to X'$ *is a completion of* X *in* $\widetilde{\mathbf{C}}$*; (ii)* j *is a completion of* Y *in* $\widetilde{\mathbf{C}}$.

(3) *Every morphism* $f : X \to Y$ *in* $\widetilde{\mathbf{C}}$ *has a proper extension in* $\widetilde{\mathbf{C}}$ *i.e., there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & P \arrow[d, "\overline{f}"] \\ & Y \end{tikzcd}$$

of morphisms in $\widetilde{\mathbf{C}}$ *such that* ι *is a open immersion with* $\iota(X)$ *dense in* P *and* $\overline{f}$ *a proper in* $\widetilde{\mathrm{Def}}$.

Remark 2.8 Given a morphism $f : X \to Y$ in $\widetilde{\mathbf{C}}$, in Corollary 2.7 (3), we have $P = f'^{-1}(j(Y)) \subseteq X'$, an open subset of X', where $i : X \to X'$ is a completion of X in $\widetilde{\mathbf{C}}$, $j : Y \to Y'$ is a completion of Y in $\widetilde{\mathbf{C}}$ and f' is such that

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "j"] & Y' \end{tikzcd}$$

is a commutative diagram of morphisms in $\widetilde{\mathbf{C}}$. Moreover, $\overline{h} = j^{-1} \circ f'_{|P} : P \to Y$ where $j^{-1} : j(Y) \to Y$ is the inverse of $j : Y \to j(Y)$ and $\iota = i : X \to P \subseteq X'$.

In particular, if $\pi : X \times Y \to Y$ is a morphism of $\widetilde{\mathbf{C}}$, then in the commutative diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\iota"] \arrow[dr, "\pi"'] & P \arrow[d, "\overline{\pi}"] \\ & Y \end{tikzcd}$$

of morphisms in $\widetilde{\mathbf{C}}$ we have $P = X' \times Y$, $\iota = i \times \mathrm{id}$ (where $i : X \to X'$ is the completion of X in $\widetilde{\mathbf{C}}$) and $\overline{\pi} : P \to Y$ is the projection onto Y.

Finally we observe the following which was proved for Def and $\mathrm{Def}(\mathbb{S})$ in [16, Theorem 4.3], and transfers to $\widetilde{\mathrm{Def}}$ and $\widetilde{\mathrm{Def}(\mathbb{S})}$ due to the isomorphisms $\mathrm{Def} \to \widetilde{\mathrm{Def}}$ and $\mathrm{Def}(\mathbb{S}) \to \widetilde{\mathrm{Def}(\mathbb{S})}$:

Proposition 2.9 *Let* $\mathbb{S}$ *be an elementary extension of* $\mathbb{M}$. *Let* $f : X \to Y$ *a morphism in* $\widetilde{\mathrm{Def}}$. *Then the following are equivalent:*

(1) f *is separated (resp. proper) in* $\widetilde{\mathrm{Def}}$.

(2) *$f^{\mathbb{S}}$ is separated (resp. proper) in $\widetilde{\mathrm{Def}(\mathbb{S})}$.*

In particular, X is complete in $\widetilde{\mathrm{Def}}$ if and only if $X(\mathbb{S})$ is complete in $\widetilde{\mathrm{Def}(\mathbb{S})}$.

We end the section by describing the closed subsets of objects of $\widetilde{\mathrm{Def}}$ and also showing that a morphism in $\widetilde{\mathrm{Def}}$ is closed if and only if it closed in $\widetilde{\mathrm{Def}}$.

Remark 2.10 Let X be an object of $\widetilde{\mathrm{Def}}$ and $C = \bigcap_{i\in I} C_i \subseteq X$ with the C_i's constructible subsets. The following hold:

- C is quasi-compact subset of X (with the induced topology).
- A closed subset B of C is a quasi-compact subset of X. Indeed, $B = D \cap C$ with D a closed subset of X. So D is quasi-compact and B is also quasi-compact.
- A quasi-compact subset B of C is a quasi-compact subset of X.

Lemma 2.11 *Let $f : X \to Y$ be a morphism in $\widetilde{\mathrm{Def}}$. Then the following hold:*

1. *If $C = \bigcap_{i\in I} C_i \subseteq X$ with the C_i's constructible subsets, then $f(C) = \bigcap_{i\in I} f(C_i)$.*
2. *If $D = \bigcap_{j\in J} D_j \subseteq Y$ with the D_j's constructible subsets, then $f^{-1}(D) = \bigcap_{j\in J} f^{-1}(D_j)$.*
3. *$C \subseteq X$ is closed if and only if $C = \bigcap_{i\in I} C_i$ with each $C_i \subseteq X$ constructible and closed.*

Proof (1) Suppose that $C = \bigcap_{i\in I} C_i$ with the C_i's constructible. Then C is closed and hence compact in the Stone topology of X (the topology generated by the constructible subsets). By [15, Remark 2.19], $f : X \to Y$ is continuous with respect to the Stone topologies on X and Y, so $f(C)$ is compact in the Stone topology of X. Since Y is Hausdorff in the Stone topology, $f(C)$ is closed (in the Stone topology). Therefore,

$$f(C) = \bigcap\{E : f(C) \subseteq E \text{ and } E \text{ is constructible}\}.$$

If E is a constructible subset such that $f(C) \subseteq E$, then $C \subseteq f^{-1}(E)$ and by compactness, there is C_i such that $C \subseteq C_i \subseteq f^{-1}(E)$ and hence $f(C) \subseteq f(C_i) \subseteq E$. Therefore, $f(C) = \bigcap_{i\in I} f(C_i)$.

(2) Obvious.

(3) Suppose that C is closed. Then its complement is open and so a union of open constructible subsets. Then C is the intersection of closed constructible subsets [15, Remark 2.3]. The converse is clear.

□

Proposition 2.12 *Let $f : X \to Y$ be a morphism in $\widetilde{\mathrm{Def}}$. Then the following are equivalent:*

1. *f is closed.*
2. *f is closed in $\widetilde{\mathrm{Def}}$.*

Proof Assuming (1) clearly (2) is immediate. Assume (2). If $D \subseteq X$ be a closed constructible subset, then $f(D)$ is closed (and constructible by [15, Remark 2.19]); if $C \subseteq X$ is a general closed subset, we have $C = \bigcap_{i\in I} C_i$ with the C_i's closed and constructible (Lemma 2.11 (3)) and by Lemma 2.11 (1), $f(C) = \bigcap_{i\in I} f(C_i)$ with each $f(C_i)$ closed and constructible and hence, $f(C)$ is closed as required. □

2.3 Definably proper, definably compact and definably normal

Here we recall the relation between the notions of proper, complete and normal in $\widetilde{\mathrm{Def}}$ and notions definably proper, definably compact and definably normal.

Recall from [35] that if X is an object of Def and $C \subseteq X$ is a definable subset (i.e. it is also in Def), then we say that C is *definably compact* if for every morphism $\alpha : (a, b) \to C \subseteq X$ in Def, where $a < b$ are in $M \cup \{-\infty, +\infty\}$, the limits $\lim_{t\to a^+} \alpha(t)$ and $\lim_{t\to b^-} \alpha(t)$ exist in C.

For definable subsets $X \subseteq M^n$ with their induced topology we have [35, Theorem 2.1]:

Fact 2.13 *A definable subset $X \subseteq M^n$ is definably compact if and only if it is closed and bounded in M^n*

Recall also that a morphism $f : X \to Y$ in Def is called *definably proper* if for every definably compact definable subset K of Y its inverse image $f^{-1}(K)$ is a definably compact definable subset of X. (See [46, Chapter 6, Section 4]).

From the definitions we see that:

Remark 2.14 An object X of Def is definably compact if and only if the morphism $X \to \mathrm{pt}$ in Def is definably proper.

Recall that we have separated, proper or complete in Def if and only if under the isomorphim $\mathrm{Def} \to \widetilde{\mathrm{Def}}$ we have separated, proper or complete in $\widetilde{\mathrm{Def}}$.

From the way cartesian squares are defined in Def we easily obtain the following:

Remark 2.15 Let $f : X \to Y$ be a morphism in Def. Then the following are equivalent:

(1) $f : X \to Y$ is separated in Def.
(2) The fibers $f^{-1}(y)$ of f are Hausdorff (with the induced topology).

In particular, X is separated in Def if and only if X is Hausdorff.

Under our assumption that $\mathbb{M}$ has definable Skolem functions we have (see [16, Theorem 3.15 and Corollary 3.17]):

Fact 2.16 *Let X and Y be objects of* Def *such that X and Y are Hausdorff and Y locally definably compact (i.e., every $y \in Y$ has a definably compact neighborhood). Let $f : X \to Y$ be a morphim in* Def. *Then the following are equivalent:*

(1) *f is proper in* Def.
(2) *f is definably proper.*

In particular, X is Hausdorff and definably compact if and only if X is complete in Def.

Recall that an object X of Def is *definably normal* if one of the following equivalent conditions holds:

(1) for every disjoint closed definable subsets Z_1 and Z_2 of X there are disjoint open definable subsets U_1 and U_2 of X such that $Z_i \subseteq U_i$ for $i = 1, 2$.
(2) for every $S \subseteq X$ closed definable and $W \subseteq X$ open definable such that $S \subseteq W$, there is an open definable subsets U of X such that $S \subseteq U$ and $\overline{U} \subseteq W$.

Under our assumption that $\mathbb{M}$ has definable Skolem functions we have (see [16, Theorem 2.11]):

Fact 2.17 *If X is an object of* Def *which is Hausdorff and definably compact (i.e complete in* Def*), then X is definably normal.*

The relationship between definably normal (in Def) and normal in $\widetilde{\mathrm{Def}}$ (as spectral topological spaces) is given by the following. See [15, Proposition 2.12 and Theorem 2.13].

Fact 2.18 *If X is an object of* Def, *then the following are equivalent:*

(1) *$\widetilde{X}$ is normal. In fact, if F and G are two disjoint closed subsets of $\widetilde{X}$ then there exist two disjoint constructible open subsets U and V of $\widetilde{X}$ such that $F \subseteq U$ and $G \subseteq V$.*
(2) *Every point $\alpha \in \widetilde{X}$ has a unique closed specialization.*
(3) *X is definably normal.*

Definable normality gives the shrinking lemma (see [46, Chapter 6, (3.6)] in Def or [15, Proposition 2.17] in $\widetilde{\mathrm{Def}}$):

Fact 2.19 *(The shrinking lemma) Let X be an object of* Def *which is definably normal. If $\{U_i : i = 1, \dots, n\}$ is a covering of X by open definable subsets, then there are definable open subsets V_i and definable closed subsets C_i of X ($1 \leq i \leq n$) such that $V_i \subseteq C_i \subseteq U_i$ and $\{V_i : i = 1, \dots, n\}$ is a covering of X.*

We end this section with the following result:[1]

Theorem 2.20 *Let Z and K be objects of* Def *with Z definably normal and K Hausdorff and definably compact (i.e. complete in* Def*). Then $Z \times K$ is definably normal.*

This theorem is a consequence of the following proposition, the affine case of the theorem, together with the shrinking lemma and Facts 2.17 and 2.18.

Proposition 2.21 *Let $Z \subseteq M^n$ be an affine definably normal, definable subset and let $K \subseteq M^k$ be a closed and bounded definable subset. Then $Z \times K \subseteq M^{n+k}$ is definably normal. In particular, every closed and bounded definable subset of M^k is definably normal.*

Proof of Theorem 2.20 By Fact 2.17 K is also definably normal. So by the shrinking lemma there are $Z_1, \dots, Z_l \subseteq Z$ closed affine definable subsets such that $Z = \bigcup_i Z_i$ and there are $K_1, \dots, K_n \subseteq K$ closed affine definable subsets such that $K = \bigcup_j K_j$. Now $Z \times K = \bigcup_{i,j} Z_i \times K_j$ with each $Z_i \times K_j$ a closed definable subset of $Z \times K$ and each K_j definably compact. So each $Z_i \times K_j$ is definably normal by Fact 2.13 and Proposition 2.21. Now going to $\widetilde{\mathrm{Def}}$ but omitting the tilde, if $\alpha \in Z \times K$, then $\alpha \in Z_i \times K_j$ for some i and j, and so, by Fact 2.18, α has a unique closed specialization ρ in $Z_i \times K_j$. Since $Z_i \times K_j$ is closed in $Z \times K$, every specialization of α in $Z \times K$ is in $Z_i \times K_j$ [15, Proposition 2.7] and ρ is a closed specialization of α in $Z \times K$. So ρ is the unique closed specialization of α in $Z \times K$. By Fact 2.18, the tilde of $Z \times K$ in $\widetilde{\mathrm{Def}}$ is normal and $Z \times K$ (in Def) is definably normal. □

Proposition 2.21 will be obtained after a couple of lemmas showing special cases of it. Below we will use the usual notation

$$\Gamma(h)_B := \{(x, y) \in B \times M : h(x) = y\},$$
$$(f, g)_B := \{(x, y) \in B \times M : f(x) < y < g(x)\}$$

---

[1] In topology, Borsuk asked in 1937 if the product of a normal space and the unit interval is normal and the question was only solved, negatively, without any set theoretic conditions beyond the axiom of choice, in 1971 by Rudin [43]. We thank the referee for pointing this to us. So Theorem 2.20 is yet another manifestation of tameness of o-minimal structures.

and

$$[f, g]_B := \{(x, y) \in B \times M : f(x) \leq y \leq g(x)\}$$

where $h, f, g : B \to M$ are functions.

The following is obtained from the definition of cells [46, Chapter 3, §2]:

Remark 2.22 Let $C \subseteq M^n$ be a d-dimensional cell. Then by the definition of cells, C is a $(i_1, \ldots, i_n)$-cell for some unique sequence $(i_1, \ldots, i_n)$ of 0's and 1's. Moreover, if $\lambda(1) < \cdots < \lambda(d)$ are the indices $\lambda \in \{1, \ldots, n\}$ for which $i_\lambda = 1$ and

$$p_{\lambda(1),\ldots,\lambda(d)} : M^n \to M^d : (x_1, \ldots, x_n) \mapsto (x_{\lambda(1)}, \ldots, x_{\lambda(d)})$$

is the projection, then $C' := p_{\lambda(1),\ldots,\lambda(d)}(C)$ is an open d-dimensional cell in M^d and the restriction $p_C := p_{\lambda(1),\ldots,\lambda(d)|C} : C \to C'$ is a definable homeomorphism [46, Chapter 3, (2.7)].

Let $\tau(1) < \cdots < \tau(n-d)$ be the indices $\tau \in \{1, \ldots, n\}$ for which $i_\tau = 0$. For each such τ, by the definition of cells, there is a definable continuous function $h_\tau : \pi_{\tau-1}(C) \subseteq M^{\tau-1} \to M$ where, for each $k = 1, \ldots, n$, $\pi_k : M^n \to M^k$ is the projection onto the first k-coordinates. Moreover we have $\pi_\tau(C) = \{(x, h_\tau(x)) : x \in \pi_{\tau-1}(C)\}$.

Let $f = (f_1, \ldots, f_{n-d}) : C' \to M^{n-d}$ be the definable continuous map where for each $l = 1, \ldots, n-d$ we set $f_l = h_{\tau(l)} \circ \pi_{\tau(l)-1} \circ p_C^{-1}$. Let $\sigma : M^n \to M^n : (x_1, \ldots, x_n) \mapsto (x_{\lambda(1)}, \ldots, x_{\lambda(d)}, x_{\tau(1)}, \ldots, x_{\tau(n-d)})$. Then we clearly have

$$\sigma(C) = \left\{(x, f(x)) : x \in C'\right\}.$$

Now let $h : C \to M$ be any continuous definable map. Then the definable set $W = (p_{\lambda(1),\ldots,\lambda(d)})^{-1}(C')$ is an open definable neighborhood of C in M^n and $g : W \to M$ given by $g(z) = h(\sigma^{-1}(p_{\lambda(1),\ldots,\lambda(d)}(z), f(p_{\lambda(1),\ldots,\lambda(d)}(z))))$ is a continuous definable map such that $g_{|C} = h$.

Lemma 2.23 *Let $Z \subseteq M^n$ be a definably normal definable subset. Let $S \subseteq Z \times [a, b]$ be a closed definable subset and $W \subseteq Z \times [a, b]$ an open definable subset. Then for every closed definable subset $F \subseteq \pi(S)$ such that $S \cap \pi^{-1}(F) \subseteq W$ there is an open definable neighborhood O of F in Z such that $O \subseteq \overline{O} \cap Z \subset \pi(W)$ and $S \cap \pi^{-1}(\overline{O} \cap Z) \subseteq W$.*

Proof Let $\pi : Z \times [a, b] \to Z$ be the projection and let $\pi' : Z \times [a, b] \to [a, b]$ be the other projection. Since $[a, b]$ is definably compact, it is complete in Def (Fact 2.16). In particular, the projection $\pi : Z \times [a, b] \to Z$ is closed in Def.

Let $W^c = (Z \times [a, b]) \setminus W$. If $S \subseteq W$ then since $\pi(S) \subseteq \pi(W)$ is closed in Z and Z is definably normal, there is an open definable neighborhood O of $\pi(S) \supseteq F$ in Z such that $O \subseteq \overline{O} \cap Z \subset \pi(W)$ and so $S \cap \pi^{-1}(\overline{O} \cap Z) = S \subseteq W$. So we may suppose that $S \cap W^c \neq \emptyset$.

For $z \in Z$ let

$$D(z) = \{((d_1^-, d_1^+), \ldots, (d_n^-, d_n^+) \in M^{2n} : z \in \Pi_{i=1}^n (d_i^-, d_i^+) \cap Z\}$$

and for $d \in D(z)$ let

$$U(z, d) = \Pi_{i=1}^n (d_i^-, d_i^+) \cap Z.$$

Then $\{U(z, d)\}_{d \in D(z)}$ is a uniformly definable system of fundamental open neighborhoods of z in Z. Moreover, since the relation $d \preceq d'$ on $D(z)$ given by $U(z, d) \subseteq U(z, d')$ is a definable downwards directed order on $D(z)$, by [25, Lemma 4.2.18] (or [24, Lemma 2.19]), there is a definable type β on $D(z)$ such that for every $d \in D(z)$ we have $\{d' \in D(z) : d' \preceq d\} \in \beta$.[2]

Suppose that $z \in F$ and for all $d \in D(z)$, we have $(S \cap \pi^{-1}(U(z,d))) \cap W^c \neq \emptyset$. Then by definable Skolem functions, there is a definable map

$$h : D(z) \to S \cap W^c \subseteq Z \times [a,b]$$

such that for every $d \in D(z)$ we have $h(d) \in (S \cap \pi^{-1}(U(z,d))) \cap W^c$.

Let α be the definable type on $S \cap W^c$ determined by the collection $\{A \subseteq S \cap W^c : h^{-1}(A) \in \beta\}$. Let α_1 be the definable type on $\pi(S)$ determined by the collection $\{A \subseteq \pi(S) : \pi^{-1}(A) \in \alpha\}$ and let α_2 be the definable type on $[a,b]$ determined by the collection $\{A \subseteq [a,b] : \pi'^{-1}(A) \in \alpha\}$.

We have that z is the limit of α_1 i.e., for every open definable subset V of Z such that $z \in V$ we have $V \in \alpha_1$. Indeed, given any such V there is d' such that $U(z,d') \subseteq V$ and

$$\begin{aligned} h^{-1}(\pi^{-1}(V)) &\supseteq h^{-1}(\pi^{-1}(\pi(S) \cap U(z,d'))) \\ &\supseteq h^{-1}((S \cap \pi^{-1}(U(z,d'))) \cap W^c) \\ &\supseteq \{d'' \in D(z) : d'' \preceq d'\}. \end{aligned}$$

On the other hand, since α_2 is a definable type on the closed and bounded definable set $[a,b]$, it has a limit, say $c \in [a,b]$ [16, Fact 5.1]. It follows that $(z,c) \in Z \times [a,b]$ is the limit of α. Since $S \cap W^c$ is closed and α is a definable type on $S \cap W^c$, its limit (z,c) is in $S \cap W^c$. But then $(z,c) \in S \cap \pi^{-1}(z) \subseteq W^c$ which is a contradiction.

So for each $z \in F$ there is $d \in D(z)$ such that $S \cap \pi^{-1}(U(z,d)) \subseteq W$. By definable Skolem functions there is a definable map $\epsilon : F \to M^{2n}$ such that for each $z \in F$ we have $\epsilon(z) \in D(z)$ and $S \cap \pi^{-1}(U(z,\epsilon(z))) \subseteq W$. Then

$$U(F,\epsilon) = \bigcup_{z \in F} U(z,\epsilon(z))$$

is an open definable neighborhood of F in Z such that

$$S \cap \pi^{-1}(U(F,\epsilon)) = \bigcup_{z \in F} S \cap \pi^{-1}(U(z,\epsilon(z))) \subseteq W.$$

Since $U(F,\epsilon) \cap \pi(W)$ is an open definable neighborhood of F in Z, F is closed in Z and Z is definably normal, there is an open definable neighborhood O of F in Z such that $O \subseteq \overline{O} \cap Z \subseteq U(F,\epsilon) \cap \pi(W) \subset \pi(W)$ and $S \cap \pi^{-1}(\overline{O} \cap Z) \subseteq W$. □

Lemma 2.24 *Let $Z \subseteq M^n$ be a definably normal definable subset. Let S and T be disjoint, closed definable subsets of $Z \times [a,b]$ such that $\pi(S) = \pi(T)$. Then there are disjoint open definable subsets U and V of $Z \times [a,b]$ such that $S \subseteq V$ and $T \subseteq U$.*

Proof Since $S \cap T = \emptyset$ and by o-minimality [46, Chapter 3, (3.6)] there is N such that for each $z \in Z$ the definable sets $S \cap \pi^{-1}(z)$ and $T \cap \pi^{-1}(z)$ are each a union of at most N points and intervals (in $\pi^{-1}(z)$), by definable Skolem functions, there are definable maps $f_l^-, f_l^+, g_l^-, g_l^+ : Z \to M$ for $l = 1, \ldots, N$, such that

$$f_1^- < f_1^+ < g_1^- < g_1^+ < f_2^- < f_2^+ < \ldots$$

and

---

[2] Recall that a type β on X (i.e. an ultrafilter of definable subsets of X) is a *definable type on X* if and only if for every uniformly definable family $\{Y_t\}_{t \in T}$ of definable subsets of X, the set $\{t \in T : Y_t \in \beta\}$ is a definable set.

- $\bigcup_l (f_l^-, f_l^+)_Z$ contains S;
- $\bigcup_l (g_l^-, g_l^+)_Z$ contains T;
- $\bigcup_l (f_l^-, f_l^+)_Z \cap T = \emptyset$;
- $\bigcup_l (g_l^-, g_l^+)_Z \cap S = \emptyset$.

Note that by construction, for each $z \in \pi(S)$ the definable set $(f_l^-, f_l^+)_Z \cap (S \cap \pi^{-1}(z))$ is either empty, a point or a closed interval (in $\pi^{-1}(z)$). Similarly, for each $z \in \pi(T)$ the definable set $(g_l^-, g_l^+)_Z \cap (T \cap \pi^{-1}(z))$ is either empty, a point or a closed interval (in $\pi^{-1}(z)$).

Let $D = \pi(S) = \pi(T)$. We prove the result by induction on $\dim D$. Suppose that $\dim D = 0$, so $D = \{d_0, \ldots, d_k\} \subseteq Z$. For $i = 0, \ldots, k$ and $l = 1, 2, \ldots, N$, let $s_{l,i}^- = f_l^-(d_i)$, $s_{l,i}^+ = f_l^+(d_i)$, $t_{l,i}^- = g_l^-(d_i)$, $t_{l,i}^+ = g_l^+(d_i)$. Then

$$s_{1,i}^- < s_{1,i}^+ < t_{1,i}^- < t_{1,i}^+ < s_{2,i}^- < s_{2,i}^+ < \ldots$$

and

- $\bigcup_l (s_{l,i}^-, s_{l,i}^+)_{\{d_i\}}$ contains $S \cap \pi^{-1}(d_i)$;
- $\bigcup_l (t_{l,i}^-, t_{l,i}^+)_{\{d_i\}}$ contains $T \cap \pi^{-1}(d_i)$;
- $\bigcup_l (s_{l,i}^-, s_{l,i}^+)_{\{d_i\}} \cap T = \emptyset$;
- $\bigcup_l (t_{l,i}^-, t_{l,i}^+)_{\{d_i\}} \cap S = \emptyset$.

Because Z is definably normal, for each $i = 0, \ldots, k$, let B_i be an open definable neighborhood of d_i in Z, such that $B_i \cap B_{i'} = \emptyset$ for $i \neq i'$. Let $\sigma_{l,i}^-, \sigma_{l,i}^+, \tau_{l,i}^-, \tau_{l,i}^+ : B_i \to M$ be the constant definable maps with values $s_{l,i}^-, s_{l,i}^+, t_{l,i}^-$ and $t_{l,i}^+$ respectively. Then

$$\sigma_{1,i}^- < \sigma_{1,i}^+ < \tau_{1,i}^- < \tau_{1,i}^+ < \sigma_{2,i}^- < \sigma_{2,i}^+ < \ldots.$$

Moreover, if

$$V_i = \bigcup_l (\sigma_{l,i}^-, \sigma_{l,i}^+)_{B_i} \cap Z \times [a, b]$$

and

$$U_i = \bigcup_l (\tau_{l,i}^-, \tau_{l,i}^+)_{B_i} \cap Z \times [a, b],$$

then V_i and U_i are open definable subsets of $Z \times [a, b]$ such that

$$S \cap \pi^{-1}(\{d_i\}) \subseteq V_i,$$
$$T \cap \pi^{-1}(\{d_i\}) \subseteq U_i$$

and $U_i \cap V_{i'} = \emptyset$ for every i and i'. Therefore, if $V = \bigcup_i V_i$ and $U = \bigcup_i U_i$, then V and U are open definable subsets of $Z \times [a, b]$ such that $S \subseteq V$, $T \subseteq U$ and $V \cap U = \emptyset$.

Suppose that $\dim D = k$ and the result holds for every pair (S', T') of disjoint, closed definable subsets of $Z \times [a, b]$ such that $\pi(S') = \pi(T')$ and this pojection has dimension smaller than k.

Take a cell decomposition of D such that on each cell the definable maps $f_l^-, f_l^+, g_l^-, g_l^+$ for $l = 1, 2, \ldots$ are all continuous. Let $C = C_0 \sqcup \ldots \sqcup C_k$ be the union of the cells C_i with $\dim C_i = \dim D$ and let $E = D \backslash C$. Then C and E are definable and $\dim E < \dim D$ [46, Chapter 3, (2.11)]. Since, $\dim \overline{E} \cap Z = \dim E$ [46, Chapter 4, (1.8)], by the induction hypothesis, there are V'' and U'' open definable subsets of $Z \times [a, b]$ such that

$$S \cap \pi^{-1}(\overline{E} \cap Z) \subseteq V'',$$
$$T \cap \pi^{-1}(\overline{E} \cap Z) \subseteq U''$$

and $V'' \cap U'' = \emptyset$.

Now the definable maps $f^-_{l|C_i}, f^+_{l|C_i}, g^-_{l|C_i}, g^+_{l|C_i}$ are all continuous and

$$f^-_{1|C_i} < f^+_{1|C_i} < g^-_{l|C_i} < g^+_{l|C_i} < f^-_{2|C_i} < f^+_{2|C_i} < \dots.$$

By the last paragraph of Remark 2.22 we may extend them to continuous definable maps $\sigma^-_{l,i}, \sigma^+_{l,i}, \tau^-_{l,i}, \tau^+_{l,i} : B_i \to M$ for $l = 1, 2, \dots$ such that

$$\sigma^-_{1,i} < \sigma^+_{1,i} < \tau^-_{1,i} < \tau^+_{1,i} < \sigma^-_{2,i} < \sigma^+_{2,i} < \dots,$$

where B_i is an open definable neighborhood of C_i in Z. Moreover, if

$$V'_i = \bigcup_l (\sigma^-_{l,i}, \sigma^+_{l,i})_{B_i} \cap Z \times [a, b]$$

and

$$U'_i = \bigcup_l (\tau^-_{l,i}, \tau^+_{l,i})_{B_i} \cap Z \times [a, b],$$

then V'_i and U'_i are open definable subsets of $Z \times [a, b]$ such that

$$S \cap \pi^{-1}(C_i) \subseteq V'_i,$$
$$T \cap \pi^{-1}(C_i) \subseteq U'_i$$

and $U'_i \cap V'_i = \emptyset$ for each i. However we might have $U'_i \cap V'' \neq \emptyset$ or $V'_i \cap U'' \neq \emptyset$ or $U'_i \cap V'_{i'} \neq \emptyset$ for $i \neq i'$. We will first modify U'_i, V'_i, U'' and V'' so that $U'_i \cap V'' = \emptyset$ and $V'_i \cap U'' = \emptyset$.

Since $\pi(V'') \cap \pi(U'')$ is an open definable neighborhood of $\overline{E} \cap Z$ in Z and Z is definably normal, by Lemma 2.23, there is an open definable neighborhood O'' of $\overline{E} \cap Z$ in Z such that $O'' \subseteq \overline{O''} \cap Z \subseteq \pi(V'') \cap \pi(U'')$,

$$S \cap \pi^{-1}(\overline{E} \cap Z) \subseteq S \cap \pi^{-1}(\overline{O''} \cap Z) \subseteq V''$$

and

$$T \cap \pi^{-1}(\overline{E} \cap Z) \subseteq T \cap \pi^{-1}(\overline{O''} \cap Z) \subseteq U''.$$

In particular, we also have $S \cap \pi^{-1}(\overline{O''} \cap Z) \cap \overline{U''} = \emptyset$ and $T \cap \pi^{-1}(\overline{O''} \cap Z) \cap \overline{V''} = \emptyset$.

Let

$$U' = \pi^{-1}(O'') \cap U''$$

and

$$V' = \pi^{-1}(O'') \cap V''.$$

Then U' and V' are open definable subsets of $Z \times [a, b]$ such that $V' \cap U' = \emptyset$,

$$S \cap \pi^{-1}(\overline{E} \cap Z) \subseteq S \cap \pi^{-1}(O'') \subseteq V'$$

and

$$T \cap \pi^{-1}(\overline{E} \cap Z) \subseteq T \cap \pi^{-1}(O'') \subseteq U'.$$

For each $i = 0, \dots, k$, let

$$V_i = V_i' \backslash (\pi^{-1}(\overline{O''} \cap Z) \cap \overline{U''})$$

and

$$U_i = U_i' \backslash (\pi^{-1}(\overline{O''} \cap Z) \cap \overline{V''}).$$

Then V_i and U_i are open definable subsets of $Z \times [a, b]$ such that

$$S \cap \pi^{-1}(C_i) \subseteq V_i$$

and

$$T \cap \pi^{-1}(C_i) \subseteq U_i.$$

Moreover, $U_i \cap V_i = \emptyset$, $U_i \cap V' = \emptyset$ and $V_i \cap U' = \emptyset$ for each i. We would also like to have $U_i \cap V_{i'} = \emptyset$ for all i, i'. For that we will need to shrink the U_i's and V_i's without destroying what we already achieved.

For each $i = 0, \dots, k$, let $D_i = C_i \backslash O''$. Then D_i is a closed definable subset of Z. In fact, let $z \in Z$ such that every open definable neighborhood of z in Z intersects D_i. Then $z \in D$ and so either $z \in E$ or $z \in C_{i'}$ for some i'. It cannot be the case that $z \in E$ for then O'' is an open definable neighborhood of z in Z which does not intersect D_i. Now since $C_{i'}$ is relatively open in D we have $i' = i$ and $z \in D_i$.

Since the D_i's are closed definable subsets of Z, two by two disjoint and $B_i \backslash (\overline{E} \cap Z)$ is an open definable neighborhood of D_i in Z, because Z is definably normal, for each i there is an open definable neighborhood O_i of D_i in Z such that $O_i \subseteq \overline{O_i} \cap Z \subseteq B_i \backslash (\overline{E} \cap Z)$ and $O_i \cap O_{i'} = \emptyset$ for $i \neq i'$.

For each i let

$$\mathcal{V}_i = V_i \cap \pi^{-1}(O_i)$$

and

$$\mathcal{U}_i = U_i \cap \pi^{-1}(O_i).$$

Then $\mathcal{V}_i$ and $\mathcal{U}_i$ are open definable subsets of $Z \times [a, b]$ such that

$$S \cap \pi^{-1}(D_i) \subseteq \mathcal{V}_i$$

and

$$T \cap \pi^{-1}(D_i) \subseteq \mathcal{U}_i$$

Moreover, $\mathcal{U}_i \cap V' = \emptyset$ and $\mathcal{V}_i \cap U' = \emptyset$ and $\mathcal{U}_i \cap \mathcal{V}_{i'} = \emptyset$ for all i, i'.

Finally, let

$$V = \left(\bigcup_{i=1}^{k} \mathcal{V}_i \right) \cup V'$$

and

$$U = \left(\bigcup_{i=1}^{k} \mathcal{U}_i\right) \cup U'.$$

Then V and U are open definable subsets of $Z \times [a, b]$ and since $C_i = D_i \sqcup (C_i \cap O'')$, we have that $S \subseteq V$, $T \subseteq U$ and $U \cap V = \emptyset$. □

Proof of Proposition 2.21 We have $K \subseteq [a, b]^k$. So if $Z \times [a, b]^k$ is definably normal, then $Z \times K$ is definably normal as well being a closed definable subset. By induction we can easily reduce to $k = 1$, so we just need to show that $Z \times [a, b]$ is definably normal, i.e., given disjoint, closed definable subsets S and T in $Z \times [a, b]$, we need to find disjoint definable open neighborhoods $V \supseteq S$ and $U \supseteq T$ in $Z \times [a, b]$.

If $\pi(S) \cap \pi(T) = \emptyset$, since Z is definably normal and $\pi : Z \times [a, b] \to Z$ is closed in Def, there are disjoint, definable open neighborhoods $V' \supseteq \pi(S)$ and $U' \supseteq \pi(T)$ in Z. But then $V = \pi^{-1}(V') \supseteq S$ and $U = \pi^{-1}(U') \supseteq T$ are disjoint, definable open neighborhoods in $Z \times [a, b]$.

So suppose that $D = \pi(S) \cap \pi(T) \neq \emptyset$. Then by Lemma 2.24 there are disjoint definable open neighborhoods $V'' \supseteq S \cap \pi^{-1}(D)$ and $U'' \supseteq T \cap \pi^{-1}(D)$ in $Z \times [a, b]$. Since $\pi(V'') \cap \pi(U'')$ is an open definable neighborhood of D in Z and Z is definably normal, by Lemma 2.23, there is an open definable neighborhood O'' of D in Z such that $O'' \subseteq \overline{O''} \cap Z \subseteq \pi(V'') \cap \pi(U'')$,

$$S \cap \pi^{-1}(D) \subseteq S \cap \pi^{-1}(\overline{O''} \cap Z) \subseteq V''$$

and

$$T \cap \pi^{-1}(D) \subseteq T \cap \pi^{-1}(\overline{O''} \cap Z) \subseteq U''.$$

In particular, we also have $S \cap \pi^{-1}(\overline{O''} \cap Z) \cap \overline{U''} = \emptyset$ and $T \cap \pi^{-1}(\overline{O''} \cap Z) \cap \overline{V''} = \emptyset$.

Let

$$U' = \pi^{-1}(O'') \cap U''$$

and

$$V' = \pi^{-1}(O'') \cap V''.$$

Then U' and V' are open definable subsets of $Z \times [a, b]$ such that $V' \cap U' = \emptyset$,

$$S \cap \pi^{-1}(D) \subseteq S \cap \pi^{-1}(O'') \subseteq V'$$

and

$$T \cap \pi^{-1}(D) \subseteq T \cap \pi^{-1}(O'') \subseteq U'.$$

The definable sets $\pi(S)\setminus O''$ and $\pi(T)\setminus O''$ are disjoint closed definable subsets of Z. Therefore, since $Z\setminus D$ is an open definable neighborhood of them in Z and Z is definably normal, there are disjoint open definable neighborhoods $W_S \supseteq \pi(S)\setminus O''$ and $W_T \supseteq \pi(T)\setminus O''$ in Z such that $W_S \subseteq \overline{W_S} \cap Z \subseteq Z\setminus D$ and $W_T \subseteq \overline{W_T} \cap Z \subseteq Z\setminus D$.

Let

$$\mathcal{V} = \pi^{-1}(W_S)\setminus(\pi^{-1}(\overline{O''} \cap Z) \cap \overline{U''})$$

and

$$\mathcal{U} = \pi^{-1}(W_T)\setminus(\pi^{-1}(\overline{O''} \cap Z) \cap \overline{V''}).$$

Then $\mathcal{V}$ and $\mathcal{U}$ are open definable subsets of $Z \times [a, b]$ such that

$$S \cap \pi^{-1}(\pi(S)\setminus O'') \subseteq \mathcal{V}$$

and

$$T \cap \pi^{-1}(\pi(T)\setminus O'') \subseteq \mathcal{U}$$

Moreover, $\mathcal{U} \cap V' = \emptyset$, $\mathcal{V} \cap U' = \emptyset$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

Finally, let

$$V = \mathcal{V} \cup V'$$

and

$$U = \mathcal{U} \cup U'.$$

Then V and U are open definable subsets of $Z \times [a, b]$ and since $\pi(S) = (\pi(S)\setminus O'') \sqcup (\pi(S) \cap O'')$ and $\pi(T) = (\pi(T)\setminus O'') \sqcup (\pi(T) \cap O'')$ we have that $S \subseteq V$, $T \subseteq U$ and $U \cap V = \emptyset$. □

2.4 Fibers in $\widetilde{\mathrm{Def}}$

A fiber $f^{-1}(\alpha)$ of a morphism $f : X \to Y$ in $\widetilde{\mathrm{Def}}$ is not in general an object of $\widetilde{\mathrm{Def}}$, but by a model theoretic trick such fibers can each be seen as objects of $\widetilde{\mathrm{Def}(\mathbb{S})}$ for some elementary extension $\mathbb{S}$ of $\mathbb{M}$. This follows from our assumption that $\mathbb{M}$ has definable Skolem functions.

Before we proceed we recall the basic model theoretic consequences of our assumption on $\mathbb{M}$ that will be required later.

By [41, Theorem 5.1]:

Fact 2.25 *Since $\mathbb{M}$ has definable Skolem functions, then for any parameter v in some model of the first-order theory of $\mathbb{M}$, there is a prime model $\mathbb{S}$ of the first-order theory of $\mathbb{M}$ over $M \cup \{v\}$ such that, for every $s \in S$, there is a definable set D and a definable map $f : D \to M$ (all definable in $\mathbb{M}$), such that $v \in D(\mathbb{S})$ and $s = f^{\mathbb{S}}(v)$.*

By Fact 2.25 we have:

Fact 2.26 *Since $\mathbb{M}$ has definable Skolem functions, if $\mathbb{S}$ is a prime model of the first-order theory of $\mathbb{M}$ over $M \cup \{v\}$ where v is a parameter in some model of the first-order theory of $\mathbb{M}$, then if $u \in S$ and $\mathbb{K}$ is a prime model of the theory of $\mathbb{M}$ over $M \cup \{u\}$, then we have either $\mathbb{K} = \mathbb{M}$ or $\mathbb{K} = \mathbb{S}$.*

Indeed, consider the model theoretic algebraic closure operator $\mathrm{acl}_{\mathbb{M}}(\bullet)$ in $\mathbb{S}$ given by $a \in \mathrm{acl}_{\mathbb{M}}(C)$ with $C \subseteq S$ if and only if there is $c \in C$ and a definable set D and a definable map $f : D \to M$ (all definable in $\mathbb{M}$) such that $c \in D(\mathbb{S})$ and $f^{\mathbb{S}}(c) = a$. By [41, Theorem 4.1], this model theoretic algebraic closure operator satisfies the exchange property: if $a \in \mathrm{acl}_{\mathbb{M}}(Cb)\setminus\mathrm{acl}_{\mathbb{M}}(C)$, then $b \in \mathrm{acl}_{\mathbb{M}}(Ca)$. Thus, by Fact 2.25, if $u \in S$ and $\mathbb{K}$ is a prime model of the theory of $\mathbb{M}$ over $M \cup \{u\}$, then we have either $\mathbb{K} = \mathbb{M}$ or $\mathbb{K} = \mathbb{S}$.

Let $\mathbb{S}$ be an elementary extension of $\mathbb{M}$. For each object $\widetilde{X}$ of $\widetilde{\mathrm{Def}}$ we have a restriction map $r : \widetilde{X(\mathbb{S})} \to \widetilde{X}$ such that given an ultrafilter $\alpha \in \widetilde{X(\mathbb{S})}$, $r(\alpha)$ is the ultrafilter in $\widetilde{X}$

determined by the collection $\{A : A(\mathbb{S}) \in \alpha\}$. This is a continuous surjective map which is neither open nor closed.

The following result was proved in [1, Lemma 3.1] in the affine case - for continuous definable maps between definable sets, under the assumption that $\mathbb{M}$ is an o-minimal expansion of an ordered group. However, the only thing needed from $\mathbb{M}$ is that it has definable Skolem functions. In fact all that is required is Fact 2.25. A special case of this result, when the map is a projection $X \times [a, b] \to X$, was proved before in [15, Claim 4.5].

Lemma 2.27 *Let $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ be a morphism in $\widetilde{\mathrm{Def}}$, $\alpha \in \widetilde{Y}$, $a \models \alpha$ a realization of α and $\mathbb{S}$ a prime model of the first-order theory of $\mathbb{M}$ over $\{a\} \cup M$. Then there is a homeomorphism*

$$r_{|} : \widetilde{(f^{\mathbb{S}})^{-1}(a)} \to \widetilde{f}^{-1}(\alpha)$$

induced by the restriction $r : \widetilde{X(\mathbb{S})} \to \widetilde{X}$.

Proof Since $(f^{-1}(B))(\mathbb{S}) = (f^{\mathbb{S}})^{-1}(B(\mathbb{S}))$ for every definable subset $B \subseteq Y$, we have a commutative diagram

$$\begin{tikzcd}
\widetilde{(f^{\mathbb{S}})^{-1}(a)} \arrow[r, hook] \arrow[d, "r_{|}"] & \widetilde{X(\mathbb{S})} \arrow[r, "\widetilde{f^{\mathbb{S}}}"] \arrow[d, "r"] & \widetilde{Y(\mathbb{S})} \arrow[d, "r"] \\
\widetilde{f}^{-1}(\alpha) \arrow[r, hook] & \widetilde{X} \arrow[r, "\widetilde{f}"] & \widetilde{Y}
\end{tikzcd}$$

and so $r_{|} : \widetilde{(f^{\mathbb{S}})^{-1}(a)} \to \widetilde{f}^{-1}(\alpha)$ is well defined.

We now have to show that $r_{|} : \widetilde{(f^{\mathbb{S}})^{-1}(a)} \to \widetilde{f}^{-1}(\alpha)$ is a continuous and open bijection. Let Y_i be a definable chart of Y such that $\alpha \in \widetilde{Y_i}$ and let $Z_i = f^{-1}(Y_i)$. Since $\widetilde{((f_{|Z_i})^{\mathbb{S}})^{-1}(a)} = \widetilde{(f^{\mathbb{S}})^{-1}(a)}$ and $\widetilde{(f_{|Z_i})}^{-1}(\alpha) = \widetilde{f}^{-1}(\alpha)$, the map we are interested on is the same as the restriction in the following commutative diagram

$$\begin{tikzcd}
\widetilde{((f_{|Z_i})^{\mathbb{S}})^{-1}(a)} \arrow[r, hook] \arrow[d, "r_{|}"] & \widetilde{Z_i(\mathbb{S})} \arrow[r, "\widetilde{f^{\mathbb{S}}}"] \arrow[d, "r"] & \widetilde{Y_i(\mathbb{S})} \arrow[d, "r"] \\
\widetilde{f_{|Z_i}}^{-1}(\alpha) \arrow[r, hook] & \widetilde{Z_i} \arrow[r, "\widetilde{f}"] & \widetilde{Y_i}.
\end{tikzcd}$$

Let X_j be a definable chart of X. Since $\widetilde{((f_{|Z_i \cap X_j})^{\mathbb{S}})^{-1}(a)} = \widetilde{((f_{|Z_i})^{\mathbb{S}})^{-1}(a)} \cap \widetilde{X_j(\mathbb{S})}$ and $\widetilde{(f_{|Z_i \cap X_j})}^{-1}(\alpha) = \widetilde{f_{|Z_i}}^{-1}(\alpha) \cap \widetilde{X_j}$, then the restriction in the following commutative diagram

$$\begin{tikzcd}
\widetilde{((f_{|Z_i \cap X_j})^{\mathbb{S}})^{-1}(a)} \arrow[r, hook] \arrow[d, "r_{|}"] & \widetilde{(Z_i \cap X_j)(\mathbb{S})} \arrow[r, "\widetilde{f^{\mathbb{S}}}"] \arrow[d, "r"] & \widetilde{Y_i(\mathbb{S})} \arrow[d, "r"] \\
\widetilde{f_{|Z_i \cap X_j}}^{-1}(\alpha) \arrow[r, hook] & \widetilde{Z_i \cap X_j} \arrow[r, "\widetilde{f}"] & \widetilde{Y_i}
\end{tikzcd}$$

is a continuous and open bijection by [1, Lemma 3.1]. Therefore, since the $\widetilde{X_j}$'s (resp. $\widetilde{X_j(\mathbb{S})}$) cover $\widetilde{f_{|Z_i}}^{-1}(\alpha)$ (resp. $\widetilde{((f_{|Z_i})^{\mathbb{S}})^{-1}(a)}$), it follows that $r_{|} : \widetilde{(f^{\mathbb{S}})^{-1}(a)} \to \widetilde{f}^{-1}(\alpha)$ is a continuous and open surjection.

Let $\beta_1, \beta_2 \in \widetilde{(f^{\mathbb{S}})^{-1}}(a) = \widetilde{((f_{|Z_i})^{\mathbb{S}})^{-1}}(a)$ be such that $r_|(\beta_1) = r_|(\beta_2) \in \widetilde{f}^{-1}(\alpha) = \widetilde{(f_{|Z_i})}^{-1}(\alpha)$. Let X_j be a definable chart of X such that $r_|(\beta_1) = r_|(\beta_2) \in \widetilde{X_j}$. Then $r_|(\beta_1) = r_|(\beta_2) \in \widetilde{f_{|Z_i}}^{-1}(\alpha) \cap \widetilde{X_j} = \widetilde{(f_{|Z_i \cap X_j})}^{-1}(\alpha)$. On the other hand, we also have $\beta_1, \beta_2 \in \widetilde{X_j(\mathbb{S})}$ and so $\beta_1, \beta_2 \in \widetilde{((f_{|Z_i})^{\mathbb{S}})^{-1}}(a) \cap \widetilde{X_j(\mathbb{S})} = \widetilde{((f_{|Z_i \cap X_j})^{\mathbb{S}})^{-1}}(a)$. Hence, by the above, $\beta_1 = \beta_2$. This proves that $r_| : \widetilde{(f^{\mathbb{S}})^{-1}}(a) \to \widetilde{f}^{-1}(\alpha)$ is also an injection as required. □

2.5 On families of supports on fibers in $\widetilde{\mathrm{Def}}$

In the paper [18] normal and constructible families of supports on objects of $\widetilde{\mathrm{Def}}$ played a fundamental role. Here we introduce the notion of a normal and constructible family of supports Φ on a fiber $f^{-1}(\alpha)$ of a morphism $f : X \to Y$ in $\widetilde{\mathrm{Def}}$.

First recall the following definitions from [18, page 1267].

Definition 2.28 Let X be an object in $\widetilde{\mathrm{Def}}$. A family Φ of closed subsets of X is a *family of supports on X* if:

- every closed subset of a member of Φ is in Φ;
- Φ is closed under finite unions.

A family Φ of supports on X is said to be *constructible* if:

- every member of Φ is contained in a member of Φ which is constructible.

A family Φ of supports on X is said to be *normal* if:

- every member of Φ is normal;
- for each member S of Φ, if U is an open neighborhood of S in X, then there exists a closed constructible neighborhood of S in U which is a member of Φ.

Remark 2.29 Let X be an object in $\widetilde{\mathrm{Def}}$ and let Φ be a constructible family of supports on X. If $C \in \Phi$ and V is an open neighborhood of C in X, then there is a constructible closed subset B of X and a constructible open subset U of X such that $C \subseteq B \subseteq U \subseteq V$ and $B \in \Phi$. When Φ is also normal, we may take B to closed and constructible neighborhood of C in V.

In fact, since C is quasi-compact we find $U \subseteq V$. Since, by Lemma 2.11, $C = \bigcap_{i \in I} C_i \subseteq U$ with C_i's closed constructible subsets of X, we have $X \backslash U \subseteq \bigcup_{i \in I} X \backslash C_i$. Since $X \backslash U$ is quasi-compact, being closed, there is $I_0 \subseteq I$ finite such that $\bigcap_{i \in I_0} C_i \subseteq U$. On the other hand, since Φ is constructible, there is $D \in \Phi$ constructible such that $C \subseteq D$. Take $B = D \cap \bigcap_{i \in I_0} C_i$.

Working with the previous definition in $\widetilde{\mathrm{Def}}(\mathbb{S})$ we can use the homeomorphism $r_| : (f^{\mathbb{S}})^{-1}(a) \to f^{-1}(\alpha)$ of Lemma 2.27 to define the notion of a normal and constructible family of supports on the fibers $f^{-1}(\alpha)$:

Definition 2.30 Let $f : X \to Y$ be a morphism in $\widetilde{\mathrm{Def}}$, $\alpha \in Y$, $a \models \alpha$ a realization of α and $\mathbb{S}$ a prime model of the first-order theory of $\mathbb{M}$ over $\{a\} \cup M$.

- A family of supports Ψ on the quasi-compact space $f^{-1}(\alpha)$ is *constructible* if its inverse image $(r_|)^{-1}\Psi$ is a constructible family of supports on the object $(f^{\mathbb{S}})^{-1}(a)$ of $\widetilde{\mathrm{Def}}(\mathbb{S})$.

- A family of supports Ψ on the quasi-compact space $f^{-1}(\alpha)$ is *normal* if its inverse image $(r_{|})^{-1}\Psi$ is a normal family of supports on the object $(f^{\mathbb{S}})^{-1}(a)$ of $\widetilde{\mathrm{Def}}(\mathbb{S})$.

If there is no risk of confusion, and since $r_{|}$ is a homeomorphism, we also use Ψ to denote the inverse image of Ψ by $r_{|}$.

We also say that a subset Z of $f^{-1}(\alpha)$ is *constructible* if its inverse image $(r_{|})^{-1}(Z)$ is a constructible subset of the object $(f^{\mathbb{S}})^{-1}(a)$ of $\widetilde{\mathrm{Def}(\mathbb{S})}$. Of course, a subset Z of $f^{-1}(\alpha)$ is quasi-compact if its inverse image $(r_{|})^{-1}(Z)$ is a quasi-compact subset of the object $(f^{\mathbb{S}})^{-1}(a)$ of $\widetilde{\mathrm{Def}(\mathbb{S})}$.

Let X be an object of $\widetilde{\mathrm{Def}}$. Then:

- The *family of complete supports on* X, denoted c, is the constructible family of supports on X of all closed subsets A of X with $A \subseteq Z$ for some constructible complete in $\widetilde{\mathrm{Def}}$ subset Z of X.
- The *family of complete supports on* $X(\mathbb{S})$, also denoted c, is the constructible family of supports on $X(\mathbb{S})$ of all closed subsets A of $X(\mathbb{S})$ with $A \subseteq Z$ for some constructible complete in $\widetilde{\mathrm{Def}(\mathbb{S})}$ subset Z of $X(\mathbb{S})$.

We can now introduce one of the main definitions of the paper:

Definition 2.31 Let $f : X \to Y$ be a morphism in $\widetilde{\mathrm{Def}}$, $\alpha \in Y$, $a \models \alpha$ a realization of α and $\mathbb{S}$ a prime model of the first-order theory of $\mathbb{M}$ over $\{a\} \cup M$.

- The *family of complete supports on* $(f^{\mathbb{S}})^{-1}(a)$ (an object of $\widetilde{\mathrm{Def}(\mathbb{S})}$), denoted c, is the constructible family of supports on $(f^{\mathbb{S}})^{-1}(a)$ of all closed subsets A of $(f^{\mathbb{S}})^{-1}(a)$ with $A \subseteq Z$ for some constructible complete in $\widetilde{\mathrm{Def}(\mathbb{S})}$ subset Z of $(f^{\mathbb{S}})^{-1}(a)$.
- The *family of complete supports on* $f^{-1}(\alpha)$, denoted c, is the (constructible) family of supports on $f^{-1}(\alpha)$ whose inverse image by $r_{|}$ is the constructible family of complete supports on $(f^{\mathbb{S}})^{-1}(a)$.

2.6 The full subcategories $\widetilde{\mathbf{A}}$ of $\widetilde{\mathrm{Def}}$

Here we introduce the full subcategories $\widetilde{\mathbf{A}}$ of $\widetilde{\mathrm{Def}}$ on which we develop the Grothendieck formalism of the six operations on o-minimal sheaves. We explain exactly which properties of objects and morphisms of $\widetilde{\mathbf{A}}$ are used later and give the main examples of such subcategories.

The full subcategories $\widetilde{\mathbf{A}}$ of $\widetilde{\mathrm{Def}}$ on which we must work are such that:

(A0) cartesian products (in $\widetilde{\mathrm{Def}}$) of objects of $\widetilde{\mathbf{A}}$ are objects of $\widetilde{\mathbf{A}}$ and locally closed constructible subsets of objects of $\widetilde{\mathbf{A}}$ are objects of $\widetilde{\mathbf{A}}$;
(A1) in every object of $\widetilde{\mathbf{A}}$ every open constructible subset is a finite union of open and normal constructible subsets;
(A2) every object of $\widetilde{\mathbf{A}}$ has a completion in $\widetilde{\mathbf{A}}$.

For some of the results about the proper direct image we will required that the morphisms $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ in $\widetilde{\mathbf{A}}$ involved satisfy the following:

(A3) if $u \in Y$, then for every elementary extension $\mathbb{S}$ of $\mathbb{M}$ and every $F \in \mathrm{Mod}(A_{X_{\mathrm{def}}})$ we have an isomorphism

$$H_c^*(f^{-1}(u); F_{|f^{-1}(u)}) \simeq H_c^*((f^{\mathbb{S}})^{-1}(u); F(\mathbb{S})_{|(f^{\mathbb{S}})^{-1}(u)})$$

where $\widetilde{F}(\mathbb{S}) = r^{-1}\widetilde{F}$, $r : \widetilde{X(\mathbb{S})} \to \widetilde{X}$ is the restriction and H_c^* is the cohomology with definably compact supports [18, Example 2.10 and Definition 2.12].

Examples of categories $\widetilde{\mathbf{A}}$ and morphisms in such categories satisfying (A3) will be given below. Before let us compare our conditions (A0)–(A3) with those used by Delfs in the semi-algebraic case [11]:

Remark 2.32 In the semi-algebraic setting instead of (A1) the following stronger version is used, e.g. in the proof of [11, Chapter II, Section 8, Proposition 8.2 and Corollaries 8.3 and 8.4]:

(A1)* in every object of $\widetilde{\mathbf{A}}$ every open constructible subset is normal.

The reason is that, in that setting [11, Chapter II, Section 8], one works with the tilde of locally complete semi-algebraic spaces which are in particular regular, and regular semi-algebraic spaces are affine and hence semi-algebraically normal. If we were only interested in working with the tilde of regular definable spaces in o-minimal expansions of real closed fields, then for the same reason [46, Chapter 10, (1.8) and Chapter 6 (3.5)] the stronger assumption (A1)* would also hold.

In our application of the results of this paper to the solution of Pillay's conjecture for definably compact groups in arbitrary o-minimal structures [17] we have to work with the tilde of locally closed definable subsets of definable manifolds defined in cartesian products of definable group-intervals. In that setting (A1) always holds [17, Corollary 3.15] but if these definable group-intervals are orthogonal (in a model theoretic sense) to each other then (A1)* fails [17, Example 3.16].

The only consequence of (A1) which will be used throughout the paper is the following result. Thus this result could be a replacement for (A1).

Before we prove the result let us define an object X of $\widetilde{\mathrm{Def}}$ to be *affine* if it is the tilde of an affine object of Def. Recall that an affine definable space is a definable space which is definably homeomorphic to a definable set equipped with its induced topology.

Proposition 2.33 *Let X be an object of $\widetilde{\mathbf{A}}$. If $\alpha \in X$, then there is an open, affine, normal constructible subset U of X such that $\alpha \in U$ and α is closed in U.*

Proof The specializations of a point α in an object Y of $\widetilde{\mathrm{Def}}$ form finite chains by [15, Lemma 2.11] and if the object is normal, α has a unique closed specialization ρ (Fact 2.18) which must be the minimum of any chain of specialization of α.

Since X has a finite cover by affine, open, constructible subsets, we may assume that X is affine. By (A1) let U be an open, normal constructible open subset of X such that $\alpha \in U$. Let ρ be the unique closed specialization of α in U. If α is closed in U we are done, otherwise, $U \setminus \{\rho\}$ is open (in U and so also in X), and so, by (A1), there is an open, normal constructible open subset V of $U \setminus \{\rho\}$ (and so of X) such that $\alpha \in V$. Since every specialization of α in V is also a specialization of α in U and the maximum length of chains of specializations of α in V is smaller than the maximum length of chains of specializations of α in U, repeating the process finitely many times we get the result. □

Remark 2.34 In the semi-algebraic setting condition (A2) holds for the tildes of locally complete semi-algebraic spaces, and similarly it holds for the tildes of locally definably compact definable spaces in o-minimal expansions of real closed fields.

In arbitrary o-minimal structures locally definably compact does not imply the existence of completions in Def. For example, in $\mathbb{M} = (\mathbb{R}, <, 0, -, +, (q)_{q \in \mathbb{Q}})$, if $X = (0, +\infty)$, then

X is locally definably compact but X has no completion in Def. Indeed, let $i : X \to Z$ be a completion of X in Def. Then because Z is complete in Def, it is Hausdorff and definably compact (Fact 2.16, $\mathbb{M}$ has definable Skolem functions by [46, Chapter 6 (1.2)]). Therefore the morphism $i : (0, +\infty) \to Z$ has the limit $\lim_{t \to +\infty} i(t)$, say a, in Z. Since Z is definably normal (Fact 2.17), by shrinking lemma, a has an affine definably compact neighborhood. So we obtain a morphism $\iota : [d, +\infty) \to [-b, b]$ in Def which is injective and piecewise linear by [46, Chapter 1 (7.8)]. This is absurd.

An analogue of (A2) is required even in algebraic geometry since when one has to define the proper direct image functor of a separated morphism of finite type of schemes one has to use Nagata's theorem on the existence of proper extensions of such morphisms (i.e., existence of completions as in (A2)).

Remark 2.35 If X is an object of $\widetilde{\mathbf{A}}$ and $i : X \to X'$ is a completion of X in $\widetilde{\mathbf{A}}$, then X' is normal. Indeed, under the inverse of the isomorphism $\mathrm{Def} \to \widetilde{\mathrm{Def}}$, we have that X' is complete in Def, therefore, by Fact 2.16, X' in Def is Hausdorff and definably compact, by 2.17, X' in Def is definably normal and by Fact 2.18 X' in $\widetilde{\mathrm{Def}}$ is normal.

The first consequence of (A2) that we require is:

Proposition 2.36 *Let X be an object of $\widetilde{\mathrm{Def}}$ with a completion in $\widetilde{\mathrm{Def}}$. Then the family c of complete supports on X is a normal and constructible family of supports on X.*

In particular, if $f : X \to Y$ is a morphism in $\widetilde{\mathbf{A}}$ and $\alpha \in Y$, then the family c of complete supports on the fiber $f^{-1}(\alpha)$ is a normal and constructible family of supports on $f^{-1}(\alpha)$.

Proof Let $i : X \to X'$ be a completion of X in $\widetilde{\mathrm{Def}}$. Let $C \in c$ and let $V \subseteq X$ be an open neighborhood of C in X. By Remark 2.29 we may assume that C and V are constructible. Then $i_{|C} : C \to X'$ is proper in $\widetilde{\mathrm{Def}}$ (Corollary 2.6 (2)) and so $i(C)$ is closed constructible in X'. Since $i(V)$ is an open constructible neighborhood of $i(C)$ in X' and X' is normal (Remark 2.35), by shrinking lemma and Corollary 2.6 (1) and (3) applied to $i^{-1} : i(X) \to X$, there is a complete constructible neighborhood D of C in X such that $D \subseteq V$.

Now let $f : X \to Y$ be a morphism in $\widetilde{\mathbf{A}}$ and $\alpha \in Y$. Let $a \models \alpha$ be a realization of α and $\mathbb{S}$ a prime model of the first-order theory of $\mathbb{M}$ over $\{a\} \cup M$. By (A2) let $i : X \to X'$ be a completion of X in $\widetilde{\mathbf{A}}$. By Proposition 2.9 $i^{\mathbb{S}} : X(\mathbb{S}) \to X'(\mathbb{S})$ is a completion of $X(\mathbb{S})$ in $\widetilde{\mathrm{Def}(\mathbb{S})}$.

Since $\mathbb{S}$ has definable Skolem functions, by the first paragraph in $\widetilde{\mathrm{Def}(\mathbb{S})}$, we see that the family c of complete supports on $X(\mathbb{S})$ is a normal and constructible family of supports on $X(\mathbb{S})$. Since $(f^{\mathbb{S}})^{-1}(a)$ is closed constructible subset of $X(\mathbb{S})$ it follows that the family c of complete supports on $(f^{\mathbb{S}})^{-1}(a)$ is a normal and constructible family of supports on $(f^{\mathbb{S}})^{-1}(a)$. Therefore, by Definition 2.31, the family c of complete supports on the fiber $f^{-1}(\alpha)$ is a normal and constructible family of supports on $f^{-1}(\alpha)$. □

In the semi-algebraic setting (A1)* is used in combination with (A2) to verify the results [11, Chapter II, Section 8, Corollaries 8.3 and 8.4]. Here, with slightly more work, namely Theorem 2.20, from the weaker (A1) we are able to obtain the following suitable replacement of those results:

Proposition 2.37 *Let $f : X \to Y$ be a morphism in $\widetilde{\mathbf{A}}$ and let $\alpha \in Y$ be closed. If K is a constructible element of the family c of complete supports on the fiber $f^{-1}(\alpha)$, then there exists a constructible and complete neighborhood B of K in X.*

Proof First we observe the we may assume that Y is affine and normal. Indeed, by Proposition 2.33, there is an open, affine, normal constructible subset U of Y such that $\alpha \in U$ and α is closed in U. Let $V = f^{-1}(U)$ so that $f^{-1}(\alpha) = (f_{|V})^{-1}(\alpha)$. If B is a constructible and complete neighborhood of $K \subseteq (f_{|V})^{-1}(\alpha)$ in V then B is still a constructible and complete neighborhood of $K \subseteq f^{-1}(\alpha)$ in X.

To proceed note that in Def the morphism $f : X \to Y$ is the composition of the graph embedding $X \to X \times Y : x \mapsto (x, f(x))$ and the projection $X \times Y \to Y$. Since the graph embedding is a closed immersion (f is continuous and both X and Y (in Def) are Hausdorff, for example by (A1)), it follows that it is proper in Def (Proposition 2.5 (1)). Thus it is enough to show the result for a projection $\pi : X \times Y \to Y$ in $\widetilde{\mathbf{A}}$. In this case, by (A2) and Corollary 2.7 (3), we have a commutative diagram

$$\begin{tikzcd}
X \times Y \arrow[r, "\iota"] \arrow[dr, "\pi"'] & P \arrow[d, "\overline{\pi}"] \\
& Y
\end{tikzcd}$$

of morphisms in $\widetilde{\mathbf{A}}$ such that ι is an open immersion and $\overline{\pi}$ is proper in $\widetilde{\mathrm{Def}}$. Moreover, by Remark 2.8, $P = X' \times Y$, $\iota = i \times \mathrm{id}$ with $i : X \to X'$ a completion of X in $\widetilde{\mathbf{A}}$ and $\overline{\pi} : X' \times Y \to Y$ is the projection onto Y.

Since X' is complete in $\widetilde{\mathrm{Def}}$, under the inverse of the isomorphism $\mathrm{Def} \to \widetilde{\mathrm{Def}}$, we have that X' is complete in Def, and so, by Fact 2.16, X' in Def is Hausdorff and definably compact. By Fact 2.18 Y in Def is definably normal. Therefore, by Theorem 2.20, $P = X' \times Y$ in Def is definably normal, and so by Fact 2.18, $P = X' \times Y$ in $\widetilde{\mathrm{Def}}$ is normal.

Let $a \models \alpha$ be a realization of α and $\mathbb{S}$ a prime model of the first-order theory of $\mathbb{M}$ over $\{a\} \cup M$. Let L be a constructible and complete in $\widetilde{\mathrm{Def}(\mathbb{S})}$ subset of $(\pi^{\mathbb{S}})^{-1}(a)$ such that $K = r_{|}(L)$.

By Corollary 2.6 (3) in $\widetilde{\mathrm{Def}(\mathbb{S})}$, we have that $\iota^{\mathbb{S}}(L)$ is constructible and complete in $\widetilde{\mathrm{Def}(\mathbb{S})}$ subset of $(\overline{\pi}^{\mathbb{S}})^{-1}(a) \subseteq P(\mathbb{S})$. In particular, by Corollary 2.6 (1) in $\widetilde{\mathrm{Def}(\mathbb{S})}$, $\iota^{\mathbb{S}}(L)$ is constructible and closed subset of $(\overline{\pi}^{\mathbb{S}})^{-1}(a)$.

From the commutative diagram,

$$\begin{tikzcd}
(\pi^{\mathbb{S}})^{-1}(a) \arrow[r, "r_{|}"] \arrow[d, "\iota^{\mathbb{S}}"'] & \pi^{-1}(\alpha) \arrow[d, "\iota_{|}"] \\
(\overline{\pi}^{\mathbb{S}})^{-1}(a) \arrow[r, "r_{|}"] & \overline{\pi}^{-1}(\alpha)
\end{tikzcd}$$

and the fact that the $r_{|}$ are homeomorphisms (Lemma 2.27) it follows that $\iota_{|}(K) = r_{|}(\iota^{\mathbb{S}}(L))$ is a constructible element of the family c of complete supports on the fiber $\overline{\pi}^{-1}(\alpha)$. In particular, by the above $\iota_{|}(K)$ is a closed constructible subset of $\overline{\pi}^{-1}(\alpha)$ which is closed in P (α is closed in Y). Hence $\iota_{|}(K)$ is closed in P and moreover $\iota_{|}(K) \cap (P \backslash \iota(X \times Y)) = \emptyset$.

Since P is normal, by the shrinking lemma, we can find disjoint constructible open neighborhoods U and V of $\iota_{|}(K)$ and $P \backslash \iota(X \times Y)$ respectively in P. Set $C = P \backslash V$. Then $\iota_{|}(K) \subset C \subset \iota(X \times Y) \subseteq P$ with C constructible and complete in $\widetilde{\mathrm{Def}}$. By Proposition 2.36 we can assume that C is a constructible complete in $\widetilde{\mathrm{Def}}$ neighborhood of $\iota_{|}(K)$ in $\iota(X \times Y)$.

Since $\iota : X \times Y \to \iota(X \times Y)$ is a homeomorphism $B = \iota^{-1}(C)$ is a constructible and complete neighborhood of K in $X \times Y$. □

Remark 2.38 In the semi-algebraic setting (A3) is proved for all tildes of semi-algebraic morphisms between the tildes of locally complete semi-algebraic spaces [11, Chapter II, Theorem 6.10]. A similar proof works in o-minimal expansions of real closed fields as it is based on the definable/semi-algebraic triangulation theorems.

Condition (A3) is required below for the base change formula (Proposition 3.33), derived base change formula (Theorem 3.37), the Künneth formula (Theorem 3.38) and dual base change formula (Proposition 4.6).

In each case where (A3) is used that fact will be explicitly mentioned. We also note that although (A3) looks rather technical and in some specific cases may follow from simpler assumptions, we preferred to state exactly what is used in the proofs since in some situations we only need to apply the consequences of (A3) to very specific morphisms and not all morphisms.

We have:

Example 2.39 Categories $\widetilde{\mathbf{A}}$ satisfying also (A3) include the tildes of:

(i) Regular, locally definably compact definable spaces in o-minimal expansions of real closed fields. ((A1) is by [46, Chapter 10, Theorem (1.8)] and [46, Chapter 6, Lemma (3.5)]; (A2) and (A3) by [19, Fact 4.7 and Corollary 4.8].)

(ii) Hausdorff locally definably compact definable spaces in o-minimal expansions of ordered groups with completions in Def. ((A1) by [46, Chapter 6, Lemma (3.5)]; (A2) by assumption and (A3) by [19, Theorem 4.5].)

(iii) Locally closed definable subsets of cartesian products of a given definably compact definable group in an arbitrary o-minimal structure. (A1) by [17, Corollary 3.15]; (A2) follows since: (i) definably compact groups are definably normal ([20, Corollary 2.3] or [16, Theorem 2.11]) and (ii) a locally closed definable subset of a cartesian product of a given definably compact definable group has a definable completion, namely its closure; (A3) by [19, Theorem 1.1].

In these examples, (A3) is obtained in [19] after extending an invariance result for closed and bounded definable sets in o-minimal expansions of ordered groups from [2].

Finally we note that in topology we do not have the problem of fibers described above and what we need from (A0), (A1) and (A2) holds on Hausdorff locally compact topological spaces.

3 Proper direct image

In this section we will develop the theory of proper direct image in full subcategories $\widetilde{\mathbf{A}}$ of $\widetilde{\mathrm{Def}}$ introduced in Sect. 2.6.

Notation: For the rest of the paper we let A be a commutative ring with unit. If X is a topological space we denote by $\mathrm{Mod}(A_X)$ the category of sheaves of A-modules on X and we call its objects A-sheaves on X.

Since $\widetilde{\mathrm{Def}}$ is a subcategory of Top, if X is an object of $\widetilde{\mathrm{Def}}$ then we have the classical operations

$$\mathcal{H}om_{A_X}(\bullet,\bullet),\ \ \bullet\otimes_{A_X}\bullet,\ \ f_*,\ \ f^{-1},\ \ (\bullet)_Z,\ \ \Gamma_Z(X;\bullet),\ \ \Gamma(X;\bullet)$$

on A-sheaves on X, where $Z\subseteq X$ is a locally closed subset. Below we may use freely these operations and refer the reader to [30, Chapter II, Sections 2.1–2.4] for the details on sheaves on topological spaces and on the properties of these basic operations.

Later in this section we may also use the derived versions of many of the properties relating the above operations and we refer to reader to [30, Chapter II, Section 2.6] for details. The reader can also see the classical references [7], [21] and [28] for similar details on sheaves on topological spaces.

Convention: For the rest of the paper we will work in the category $\widetilde{\mathrm{Def}}$ (resp. $\widetilde{\mathbf{A}}$ or $\widetilde{\mathrm{Def}(\mathbb{S})}$) and omit the tilde on objects and morphisms. We will say "proper" (resp. "separated" or "complete") instead of "proper in $\widetilde{\mathrm{Def}}$" (resp. "separated in $\widetilde{\mathrm{Def}}$" or "complete in $\widetilde{\mathrm{Def}}$").

We will assume that all morphism and objects that we consider in $\widetilde{\mathrm{Def}}$ are separated. This is the case for morphisms and objects in $\widetilde{\mathbf{A}}$ (by (A2) all objects of $\widetilde{\mathbf{A}}$ are tildes of open definable subsets of objects complete in Def (i.e. Hausdorff and definably compact) and so are Hausdorff).

If X is an object of $\widetilde{\mathrm{Def}}$ (resp. $\widetilde{\mathbf{A}}$), then $\mathrm{Op}(X)$ denotes the category of open subsets of X with inclusions and $\mathrm{Op}^{\mathrm{cons}}(X)$ is the full sub-category of constructible open subsets of X.

3.1 Extended summary

Since our definition of the o-minimal proper direct image functor $f_{\underset{\cdot}{\wr}}$ associated to a morphism $f : X \to Y$ in $\widetilde{\mathrm{Def}}$ is similar to the definition of the topological proper direct image functor $f_!$, we only replace proper in Top by proper in $\widetilde{\mathrm{Def}}$, we follow closely the proofs in topology [30, Chapter II, Sections 2.5 and 2.6] but we have to deal with: (i) the fact that the fibers $f^{-1}(\alpha)$ are often not objects of $\widetilde{\mathrm{Def}}$; (ii) the fact that cartesian squares in $\widetilde{\mathrm{Def}}$ are not cartesian squares in Top; (iii) the fact that objects of $\widetilde{\mathrm{Def}}$ are not locally compact topological spaces.

The most technical result is the Fiber formula (Corollary 3.12) which is a consequence of the Relative fiber formula (Proposition 3.8) in whose proof we use heavily the consequences of our assumptions (A0), (A1) and (A2) discussed in the Sect. 2.6. Our methods are different from those in semi-algebraic case: the fiber formula there [11, Chapter II, Corollary 8.8] is obtained as a consequence of a Base change formula [11, Chapter II, Theorem 8.7] and no semi-algebraic analogue of the Relative fiber formula is proved by Delfs.

We then proceed with the theory of f-soft sheaves which gives us the $f_{\underset{\cdot}{\wr}}$-injective objects (Proposition 3.26) and later the bound on the cohomological dimension of $f_{\underset{\cdot}{\wr}}$ (Theorem 3.39). The f-soft sheaves are those whose restriction to fibers $f^{-1}(\alpha)$ are c-soft. We show our results about f-soft sheaves after we remark (Remark 3.18) that by our assumptions (A0), (A1) and (A2) we can always reduce to the case where α is a closed point in which case c is a normal and constructible family of supports on $f^{-1}(\alpha)$ (Proposition 2.36) and the results follow, via $\widetilde{(f^{\mathbb{S}})}^{-1}(a) \simeq f^{-1}(\alpha)$, from similar results already proved in [18, Section 3]. This theory has not been considered in the semi-algebraic case.

The Projection formula (Proposition 3.28) is obtained in a standard way but the Base change formula (Proposition 3.33) requires assumption (A3) and, after we prove a technical lemma (Lemma 3.32), we use (A3) more or less as the corresponding fact [11, Chapter I, Theorem 6.10] is used in the prove of the semi-algebraic Base change formula [11, Chapter II, Theorem 8.7]. Using the f-soft resolutions we obtain in a standard way the Derived projection formula (Theorem 3.35) and the Derived base change formula (Theorem 3.37) after we prove a technical lemma (Lemma 3.36) requiring assumption (A3). This part was also not considered in the semi-algebraic case.

The complication with the Base change formula and the Derived base change formula and the need to use (A3) arises from the fact that in a cartesian square

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

in $\widetilde{\mathrm{Def}}$, if $\gamma \in Y'$, then we may have $(\widetilde{f'^{\mathbb{S}}})^{-1}(a) \simeq f'^{-1}(\gamma)$ and $(\widetilde{f^{\mathbb{K}}})^{-1}(b) \simeq f^{-1}(g(\gamma))$ with $\mathbb{K} = \mathbb{M}$ and $\mathbb{S} \neq \mathbb{M}$. So after we identify $(f'^{\mathbb{S}})^{-1}(a)$ with $(f^{\mathbb{S}})^{-1}(b)$ via $g'^{\mathbb{S}}$ we need to know that $H^*_c(f^{-1}(b); F_{|f^{-1}(b)}) \simeq H^*_c((f^{\mathbb{S}})^{-1}(b); F(\mathbb{S})_{|(f^{\mathbb{S}})^{-1}(b)})$.

3.2 Proper direct image

Here we define the proper direct image functor and prove some of its basic properties.

Definition 3.1 Let $f : X \to Y$ be a morphism in $\widetilde{\mathrm{Def}}$ and let $F \in \mathrm{Mod}(A_X)$. The *proper direct image* is the subsheaf of f_*F defined by setting for $U \in \mathrm{Op}^{\mathrm{cons}}(Y)$, an open constructible subset of Y,

$$\Gamma(U; f_!F) = \varinjlim_{Z} \Gamma_Z(f^{-1}(U); F),$$

where Z ranges through the family of closed constructible subsets of $f^{-1}(U)$ such that $f_{|Z} : Z \to U$ is proper.

From the definition we have:

Remark 3.2 The functor $f_!$ is clearly left exact; if $f : X \to Y$ is proper, then $f_! = f_*$; if $i : Z \to Y$ is the inclusion of a locally closed subset, then $i_!$ is the extension by zero functor (i.e. $(i_!F)_\alpha \simeq F_\alpha$ or 0 according to $\alpha \in Z$ or $\alpha \notin Z$) and $(\bullet)_Z = i_! \circ i^{-1}(\bullet)$. Compare with [30, Chapter II, Proposition 2.5.4].

Remark 3.3 If we consider the morphism $a_X : X \to \{\mathrm{pt}\}$ to a point in $\widetilde{\mathrm{Def}}$, then we have

$$(a_{X!}F)_{\mathrm{pt}} \simeq \Gamma(\mathrm{pt}; a_{X!}F) = \Gamma_c(X; F).$$

Proposition 3.4 *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\widetilde{\mathrm{Def}}$. Then $(g \circ f)_! \simeq g_! \circ f_!$.*

Proof Indeed, if $V \in \mathrm{Op}^{\mathrm{cons}}(Z)$, then a section of $\Gamma(V; g_! \circ f_!F)$ is represented by $s \in \Gamma(f^{-1}(g^{-1}(V)); F)$ such that $\mathrm{supp}(s) \subset f^{-1}(S) \cap Z$, where Z, S are closed constructible subsets of $f^{-1}(g^{-1}(V))$ and $g^{-1}(V)$ respectively and such that $f_{|Z} : Z \to g^{-1}(V)$, $g_{|S} : S \to V$ are proper. The set $Z \cap f^{-1}(S)$ is closed constructible and the restriction $(g \circ f) : Z \cap f^{-1}(S) \to V$ is proper (Proposition 2.5 (1) and (2)). Conversely if $V \in \mathrm{Op}^{\mathrm{cons}}(Z)$, then a section of $\Gamma(V; (g \circ f)_!F)$ is represented by $s \in \Gamma_Z(f^{-1}(g^{-1}(V)); F)$ where Z is closed constructible subset of $f^{-1}(g^{-1}(V))$ such that $(g \circ f)_{|Z} : Z \to V$ is proper. But then $f_{|Z} : Z \to g^{-1}(V)$ is proper and $g_{|f(Z)} : f(Z) \to V$ is proper (Proposition 2.5 (5)). □

In particular, we have:

Remark 3.5 For $f : X \to Y$ a morphism in $\widetilde{\mathrm{Def}}$, since $a_X = a_Y \circ f$, we have $\Gamma_c(Y; f_{!!}F) \simeq \Gamma_c(X; F)$.

Proposition 3.6 *Let $f : X \to Y$ be a morphism in $\widetilde{\mathrm{Def}}$. The functor $f_{!!}$ commutes with filtrant inductive limits.*

Proof Let $(F_i)_{i \in I}$ be a filtrant inductive system in $\mathrm{Mod}(A_X)$. Since open constructible sets are quasi-compact, then for any $V \in \mathrm{Op}^{\mathrm{cons}}(X)$ and any closed constructible subset S of V the functors $\Gamma(V; \bullet)$ and $\Gamma(V \backslash S; \bullet)$ commute with filtrant $\varinjlim$ [18, Remark 2.7]. Hence $\Gamma_S(V; \bullet)$ commutes with filtrant $\varinjlim$. Therefore we have

$$\begin{aligned}
\Gamma(U; \varinjlim_i f_{!!}F_i) &\simeq \varinjlim_i \Gamma(U; f_{!!}F_i) \\
&\simeq \varinjlim_{i,Z} \Gamma_Z(f^{-1}(U); F_i) \\
&\simeq \varinjlim_Z \Gamma_Z(f^{-1}(U); \varinjlim_i F_i) \\
&\simeq \Gamma(U; f_{!!}\varinjlim_i F_i),
\end{aligned}$$

where Z ranges through the family of closed constructible subsets of $f^{-1}(U)$ such that $f_{|Z} : Z \to U$ is proper. □

The following lemma follows immediately from the definition of proper direct image and we leave the details to the reader

Lemma 3.7 *Let $f : X \to Y$ be a morphism in $\widetilde{\mathrm{Def}}$ and W an open subset of Y. Consider the commutative diagram*

$$\begin{tikzcd}
f^{-1}(W) \arrow[r, hook, "j"] \arrow[d, "f_|"] & X \arrow[d, "f"] \\
W \arrow[r, hook, "i"] & Y.
\end{tikzcd}$$

If $F \in \mathrm{Mod}(A_X)$, then $(f_{!!}F)_{|W} \simeq (f_|)_{!!}(F_{|f^{-1}(W)})$.

Proposition 3.8 (Relative fiber formula) *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\widetilde{\mathbf{A}}$, $\beta \in Z$ and let F be a sheaf in $\mathrm{Mod}(A_X)$. If $K \in c$ in $g^{-1}(\beta)$, then $\Gamma(K; f_{!!}F) \simeq \Gamma_c(f^{-1}(K); F)$.*

Proof First we show that we may assume without loss of generality that β is a closed point in Z.

By Proposition 2.33, there exists $Z' \in \mathrm{Op}^{\mathrm{cons}}(Z)$ with $\beta \in Z'$ such that β is closed in Z'. Let $Y' = g^{-1}(Z')$ and $X' = f^{-1}(Y')$. Let also $f' = f_{|X'}$ and $g' = g_{|Y'}$. By Lemma 3.7 we have $(f_{!!}F)_{|K} = ((f_{!!}F)_{|Y'})_{|K} \simeq (f'_{!!}F_{|X'})_{|K}$, and since $f'^{-1}(K) = f^{-1}(K)$, we have also $F_{|f^{-1}(K)} = F_{|f'^{-1}(K)}$. Thus after replacing Z (resp., Y and X) by Z' (resp. Y' and X') and f (resp., g) by f' (resp., g'), we may assume without loss of generality that β is a closed point of Z.

Let $b \models \beta$ be a realization of β and $\mathbb{S}$ a prime model of the first-order theory of $\mathbb{M}$ over $\{b\} \cup M$. Then we have the following commutative diagram

$$\begin{tikzcd}
(f^{\mathbb{S}})^{-1}((g^{\mathbb{S}})^{-1}(b)) \arrow[r, "r_{|}"] \arrow[d, "(f_{|})^{\mathbb{S}}"'] & f^{-1}(g^{-1}(\beta)) \arrow[d, "f_{|}"] \\
(g^{\mathbb{S}})^{-1}(b) \arrow[r, "r_{|}"] & g^{-1}(\beta)
\end{tikzcd}$$

given by Lemma 2.27 (where we have omitted the tildes and used the fact that $(g \circ f)^{\mathbb{S}} = (g^{\mathbb{S}}) \circ (f^{\mathbb{S}})$).

By (A0), (A2) and Corollary 2.7 (2), we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
Y \arrow[r, "i_Y"] \arrow[d, "g"'] & Y' \arrow[d, "g'"] \\
Z \arrow[r, "i_Z"] & Z'
\end{tikzcd}$$

of completions in $\widetilde{\mathbf{A}}$.

Let $\Phi_f = \{A : A \subseteq X$ is closed, $A \subseteq B$ for some closed constructible subset B of X such that $f_{|B} : B \to Y$ is proper in $\widetilde{\mathrm{Def}}\,\}$. Then:

Claim 3.9

$$\Phi_f \cap f^{-1}(K) = c_{|f^{-1}(K)}.$$

In particular, if $Y = Z$, $g = \mathrm{id}$ *and* $\alpha \in Y$ *is closed, then for* c *the family of complete supports of* $f^{-1}(\alpha)$, *we have*

$$c = \Phi_f \cap f^{-1}(\alpha).$$

Let $Z \in \Phi_f$. We may assume that Z is a closed constructible subset of X such that $f_{|Z} : Z \to Y$ is proper in $\widetilde{\mathrm{Def}}$. Then, $f^{\mathbb{S}}_{|Z(\mathbb{S})} : Z(\mathbb{S}) \to Y(\mathbb{S})$ is proper in $\widetilde{\mathrm{Def}}(\mathbb{S})$ (Proposition 2.9). Since $K \in c$ in $g^{-1}(\beta)$, by definition, there is a $C \subseteq (g^{\mathbb{S}})^{-1}(b)$ constructible and complete in $(g^{\mathbb{S}})^{-1}(b)$ such that $K \subseteq r_{|}(C)$. By Corollary 2.6 (4), $(f^{\mathbb{S}})^{-1}(C) \cap Z(\mathbb{S})$ is constructible and complete in $(f^{\mathbb{S}})^{-1}((g^{\mathbb{S}})^{-1}(b))$. Therefore, $r_{|}((f^{\mathbb{S}})^{-1}(C) \cap Z(\mathbb{S})) = Z \cap f^{-1}(r_{|}(C))$ is constructible and complete in $f^{-1}(g^{-1}(\beta))$. Since $Z \cap f^{-1}(K)$ is a closed subset of $Z \cap f^{-1}(r_{|}(C))$ we have that $Z \cap f^{-1}(K) \in c$ in $f^{-1}(g^{-1}(\beta))$.

Conversely, let $Z \in c_{|f^{-1}(K)}$, then there is C a constructible complete subset of $f^{-1}(g^{-1}(\beta))$ such that $Z \subseteq C \subseteq f^{-1}(K)$. Since β is closed in Z, it follows from Proposition 2.37 applied to $g \circ f$ that there is a $B \subseteq X$ constructible and complete in X such that $C \subseteq B$. Hence $f_{|B}$ is proper (Corollary 2.6 (2)). So $C \subseteq B \cap f^{-1}(K) \in \Phi_f \cap f^{-1}(K)$. □

We proceed with the proof of the Proposition. We have

$$\Gamma(K; f_{!}F) \simeq \varinjlim_{U} \Gamma(U; f_{!}F) \simeq \varinjlim_{U,Z} \Gamma_Z(f^{-1}(U); F),$$

where U ranges through the family of open constructible neighborhoods of K and Z is closed constructible in $f^{-1}(U)$ and such that $f_{|Z} : Z \to U$ is proper.

By Claim 3.9 we can conclude the proof after the following two steps.

Claim 3.10 *We have an isomorphism*

$$\varinjlim_{U,Z}\Gamma_Z(f^{-1}(U);F)\simeq\varinjlim_{U}\Gamma_{\Phi_f\cap f^{-1}(U)}(f^{-1}(U);F),$$

where U ranges through the family of open constructible neighborhoods of K and Z is a closed constructible subset of $f^{-1}(U)$ and such that $f_{|Z}:Z\to U$ is proper.

Let $s\in\Gamma(f^{-1}(U);F)$ whose support is contained in Z a closed constructible subset of $f^{-1}(U)$ such that $f_{|Z}:Z\to U$ is proper. Since $K\in c$ in $g^{-1}(\beta)$ and β is closed in Z, it follows from Proposition 2.37 applied to g that there is $B\subseteq Y$ a constructible and complete in Y neighborhood of K in Y. Since K is closed in Y ($g^{-1}(\beta)$ is closed in Y), it is closed in B. Since $i_{Y|B}:B\to Y'$ is proper (Corollary 2.6 (2)), $i_Y(K)=i_{Y|B}(K)$ is a closed subset of $i_Y(B)$ (Proposition 2.12). Since $i_Y(B)$ is a closed constructible subset of Y' (Corollary 2.6 (1)), it follows that $i_Y(K)$ is a closed subset of Y'. Since Y' is normal (Remark 2.35), there exists an open constructible neighborhood V of K such that $\overline{V}\subset U$. Then $f_{|Z\cap f^{-1}(\overline{V})}:Z\cap f^{-1}(\overline{V})\to\overline{V}$ is proper and composing with the inclusion $\overline{V}\hookrightarrow Y$, the morphism $f_{|Z\cap f^{-1}(\overline{V})}:Z\cap f^{-1}(\overline{V})\to Y$ is proper (Proposition 2.5 (1) and (2)). Hence $Z\cap f^{-1}(\overline{V})\in\Phi_f$. The support of the restriction of s to $f^{-1}(V)$ is contained in $Z\cap f^{-1}(V)=Z\cap f^{-1}(\overline{V})\cap f^{-1}(V)\in\Phi_f\cap f^{-1}(V)$ and the result follows. □

Claim 3.11 *We have an isomorphism*

$$\varinjlim_{U}\Gamma_{\Phi_f\cap f^{-1}(U)}(f^{-1}(U);F)\simeq\Gamma_{\Phi_f\cap f^{-1}(K)}(f^{-1}(K);F),$$

where U ranges through the family of open constructible neighborhoods of K.

Let $s\in\Gamma(f^{-1}(K);F)$ and let $Z\in\Phi_f$ closed constructible subset of $f^{-1}(U)$ containing its support. The section s is represented by a section $t\in\Gamma(W;F)$ for some open constructible neighborhood W of $f^{-1}(K)$. Since $Z\in\Phi_f$ and $K\in c$ in $g^{-1}(\beta)$, we have $Z\cap f^{-1}(K)\in c$ in $f^{-1}(g^{-1}(\beta))$ (Claim 3.9). Since β is closed in Z, it follows from Proposition 2.37 applied to $g\circ f$ that there is $B\subseteq X$ a constructible and complete in X neighborhood of $Z\cap f^{-1}(K)$ in X. Since $Z\cap f^{-1}(K)$ is closed in X ($f^{-1}(g^{-1}(\beta))$ is closed in X), it is closed in B. Since $i_{X|B}:B\to X'$ is proper (Corollary 2.6 (2)), $i_X(Z\cap f^{-1}(K))=i_{X|B}(Z\cap f^{-1}(K))$ is a closed subset of $i_X(B)$ (Proposition 2.12). Since $i_X(B)$ is a closed constructible subset of X' (Corollary 2.6 (1)), it follows that $i_X(Z\cap f^{-1}(K))$ is a closed subset of X' .

Since X' is normal (Remark 2.35) there exists an open constructible neighborhood V of $Z\cap f^{-1}(K)$ with $\overline{V}\subset W$. We have that $f'_{|\overline{V}}$ is proper, so $f_{|\overline{V}}$ is proper (Proposition 2.5 (5) (i)) and $\overline{V}\in\Phi_f$. Since $f'^{-1}(K)=f^{-1}(K)$, we have $f'^{-1}(K)\cap((\overline{V}\cap\operatorname{supp}(t))\setminus V)=\emptyset$ and so $K\cap f'((\overline{V}\cap\operatorname{supp}(t))\setminus V)=\emptyset$. Since K and $f'((\overline{V}\cap\operatorname{supp}(t))\setminus V)$ are closed subsets of Y' (recall that $f'_{|\overline{V}}$ is closed) and Y' is normal, by shrinking lemma, there exists an open constructible subset U of Y with $K\subseteq U$ such that $f^{-1}(U)\cap\overline{V}\cap\operatorname{supp}(t)\subset V$. Let us define $\widetilde{t}\in\Gamma(f^{-1}(U);F)$ by

$$\begin{aligned}\widetilde{t}_{|f^{-1}(U)\setminus(\overline{V}\cap\operatorname{supp}(t))}&=0,\\ \widetilde{t}_{|f^{-1}(U)\cap V}&=t_{|f^{-1}(U)\cap V}.\end{aligned}$$

By construction $\widetilde{t}_{|f^{-1}(K)}=s$. Note that $\operatorname{supp}(\widetilde{t})$ is contained in $f^{-1}(U)\cap\overline{V}$ and $\overline{V}\in\Phi_f$. □

Setting $Y=Z$ and $g=\mathrm{id}$ we obtain

Corollary 3.12 (Fiber formula) *Let* $f : X \to Y$ *be a morphism in* $\widetilde{\mathbf{A}}$ *and let* F *be a sheaf in* $\mathrm{Mod}(A_X)$. *Let* $\alpha \in Y$. *Then* $(f_{\wr}F)_\alpha \simeq \Gamma_c(f^{-1}(\alpha); F)$.

We end the section comparing the o-minimal proper direct image functor $f_{\wr}$ with Verdier's [49] topological proper direct image functor $f_!$ and Kashiwara and Schapira [31] sub-analytic proper direct image functor $f_{!!}$.

To make the comparison more clear we will use the isomorphism $\mathrm{Mod}(A_{\widetilde{X}}) \simeq \mathrm{Mod}(A_{X_{\mathrm{def}}})$ and the fact that $f : X \to Y$ is proper in Def if and only if $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is proper in $\widetilde{\mathrm{Def}}$ to introduce $f_{\wr} : \mathrm{Mod}(A_{X_{\mathrm{def}}}) \to \mathrm{Mod}(A_{Y_{\mathrm{def}}})$ as $\widetilde{f}_{\wr} : \mathrm{Mod}(A_{\widetilde{X}}) \to \mathrm{Mod}(A_{\widetilde{Y}})$. Recall that given an object X of Def the o-minimal site X_{def} on X is the category $\mathrm{Op}(X_{\mathrm{def}})$ whose objects are open (in the topology of X mentioned above) definable subsets of X, the morphisms are the inclusions and the admissible covers $\mathrm{Cov}(U)$ of $U \in \mathrm{Op}(X_{\mathrm{def}})$ are covers by open definable subsets of X with finite sub-covers.

We now explain the relationship between the o-minimal proper direct image functor and Verdier's [49] proper direct image functor.

Remark 3.13 Consider the category Def associated to an o-minimal expansion $\mathbb{M} = (\mathbb{R}, <, (c)_{c\in\mathcal{C}}, (f)_{f\in\mathcal{F}}, (R)_{R\in\mathcal{R}})$ of the ordered set of real numbers.

For a continuous map $f : X \to Y$ between locally compact topological spaces Verdier's proper direct image functor $f_!$ is given by: for $G \in \mathrm{Mod}(A_X)$ and $U \in \mathrm{Op}(Y)$ we have

$$\Gamma(U; f_!G) = \varinjlim_{Z} \Gamma_Z(f^{-1}(U); G),$$

where Z ranges through the family of closed subsets of $f^{-1}(U)$ such that $f_{|Z} : Z \to U$ is proper (in Top).

If X is an object of Def then X is also a topological space (with the usual topology generated by open definable subsets) and we have the natural morphism of sites $\rho : X \to X_{\mathrm{def}}$ induced by the inclusion $\mathrm{Op}(X_{\mathrm{def}}) \subset \mathrm{Op}(X)$.

Since if $f_{|Z} : Z \to Y$ is proper in Def then $f_{|Z} : Z \to Y$ is proper in Top [16, Theorem 4.11], we have $\Gamma(U; f_{\wr} \circ \rho_* F) \subseteq \Gamma(U; \rho_* \circ f_! F)$ for $U \in \mathrm{Op}(X_{\mathrm{def}})$. However, in general, we have $\rho_* \circ f_! \neq f_{\wr} \circ \rho_*$.

For example, suppose that $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $f : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the first coordinate and $U = (0, +\infty)$. Let $\phi : (0, +\infty) \to (0, +\infty)$ be given by $\phi(t) = \frac{1}{t}$ (resp. $\phi(t) = \frac{1}{\ln(t)}$) if $\mathbb{M}$ is semi-bounded [14] (resp. polynomially bounded [48]). Set $S = \{(x, y) \in (0, +\infty) \times \mathbb{R} : y = \phi(x))\}$ and consider the sheaf A_S. Let $s \in \Gamma(f^{-1}(U); A_S)$. Then $f : \mathrm{supp}(s) \to U$ is proper in Top (it is a homeomorphism into its image), so $s \in \Gamma(U; f_! A_S) = \Gamma(U; \rho_* \circ f_! A_S)$, but $s \notin \Gamma(U; f_{\wr} \circ \rho_* A_S)$ since there is no closed definable subset Z of $f^{-1}(U)$ such that $f_{|Z} : Z \to U$ is proper in Def and $S \subseteq Z$.

Now we explain the relationship between the o-minimal proper direct image functor and Kashiwara–Schapira's [31] sub-analytic proper direct image functor.

Remark 3.14 Consider the category Def associated to the o-minimal structure $\mathbb{R}_{\mathrm{an}} = (\mathbb{R}, <, 0, 1, +, \cdot, (f)_{f\in\mathrm{an}})$—the field of real numbers expanded by restricted globally analytic functions [48]. As explained in [48], in this case, Def is the category of globally sub-analytic spaces with continuous maps with globally sub-analytic graphs.

If X is a real analytic manifold, then we can equip X with its sub-analytic site, denoted X_{sa}, where the objects are open sub-analytic subsets $\mathrm{Op}(X_{sa})$ and $S \subset \mathrm{Op}(X_{sa}) \cap \mathrm{Op}(U)$ is covering of $U \in \mathrm{Op}(X_{sa})$ if for any compact subset K of X there is a finite subset $S_0 \subseteq S$

such that $K \cap \bigcup_{V \in S_0} V = K \cap U$, see [31]. In this context we have a sub-analytic proper direct image functor $f_{!!}$ given by: for $F \in \mathrm{Mod}(A_{X_{sa}})$ and $U \in \mathrm{Op}(Y_{sa})$ we have

$$\Gamma(U; f_{!!}F) = \varinjlim_{K} \Gamma_{K \cap f^{-1}(U)}(f^{-1}(U); F),$$

where K ranges through the family of compact sub-analytic subsets of X. We recall that this is a direct construction of Kashiwara and Schapira [31] sub-analytic proper direct image functor originally constructed as a special case of a more general construction within the theory of ind-sheaves (the direct construction for subanalytic sheaves was performed in [42]).

If X is also globally sub-analytic (i.e. a definable subset in the o-minimal structure $\mathbb{R}_{\mathrm{an}}$) we have the natural morphism of sites $\nu : X_{sa} \to X_{\mathrm{def}}$ induced by the inclusion $\mathrm{Op}(X_{\mathrm{def}}) \subset \mathrm{Op}(X_{sa})$.

Since a compact sub-analytic subset is a compact globally sub-analytic subset (hence a definably compact definable subset in the o-minimal structure $\mathbb{R}_{\mathrm{an}}$), we have $\Gamma(U; \nu_* \circ f_{!!}F) \subseteq \Gamma(U; f_{\underset{\cdot}{\wr}} \circ \nu_* F)$ for $U \in \mathrm{Op}(X_{\mathrm{def}})$. However, in general, we have $\nu_* \circ f_{!!} \neq f_{\underset{\cdot}{\wr}} \circ \nu_*$.

For example, suppose that $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $f : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the first coordinate and $U = (0, 1)$. Let $\phi : (0, 1) \to (0, +\infty)$ be given by $\phi(t) = \frac{1}{x(1-x)}$. Set $S = \{(x, y) \in (0, 1) \times \mathbb{R} : y = \phi(x))\}$ and consider the sheaf A_S. Let $s \in \Gamma(f^{-1}(U); A_S)$. Then $f_| : \mathrm{supp}(s) \to U$ is definably proper (and so proper in Def), so $s \in \Gamma(U; f_{\underset{\cdot}{\wr}} \circ \nu_* A_S)$, but $s \notin \Gamma(U; \nu_* \circ f_{!!}A_S) = \Gamma(U; f_{!!}A_S)$ since there is no compact (globally) sub-analytic subset K of X such that $\mathrm{supp}(s) \subseteq K$.

Remark 3.15 The o-minimal proper direct image functor $f_{\underset{\cdot}{\wr}}$ in comparison with the sub-analytic proper direct image functor $f_{!!}$ seems to be more natural since it commutes with restrictions in the sense of Lemma 3.7, as the classical Verdier proper direct image functor $f_!$ does in topological spaces. The property of that lemma is not satisfied by the functor $f_{!!}$ since there are less compact subsets on $f^{-1}(W)$ than there are intersections of $f^{-1}(W)$ with compact subsets of X.

3.3 *f*-soft sheaves

Here we introduce the f-soft sheaves where $f : X \to Y$ is a morphism in $\widetilde{\mathrm{Def}}$. We show that the full additive subcategory of f-soft sheaves is $f_{\underset{\cdot}{\wr}}$-injective and is stable under $\varinjlim$ and its flat objects are also stable under $\bullet \otimes F$ for all $F \in \mathrm{Mod}(A_X)$.

Consider a fiber $f^{-1}(\alpha)$ of a morphism $f : X \to Y$ in $\widetilde{\mathrm{Def}}$ and let c be the family of complete supports on $f^{-1}(\alpha)$. Recall that a sheaf F on $f^{-1}(\alpha)$ is *c-soft* if and only if the restriction $\Gamma(f^{-1}(\alpha); F) \to \Gamma(K; F)$ is surjective for every $K \in c$.

Definition 3.16 Let $f : X \to Y$ be a morphism in $\widetilde{\mathrm{Def}}$ and let F be a sheaf in $\mathrm{Mod}(A_X)$. We say that F is *f-soft* if for any $\alpha \in Y$ the sheaf $F_{|f^{-1}(\alpha)}$ in $\mathrm{Mod}(A_{f^{-1}(\alpha)})$ is c-soft.

By Remark 3.3 we have:

Remark 3.17 Let $a_X : X \to \mathrm{pt}$ be the morphism to a point in $\widetilde{\mathrm{Def}}$ and let F be a sheaf in $\mathrm{Mod}(A_X)$. Then F is a_X-soft if and only if it is c-soft.

Remark 3.18 Let $f : X \to Y$ be a morphism in $\widetilde{\mathbf{A}}$. To prove a property of f-soft sheaves in $\mathrm{Mod}(A_X)$ we have to take an arbitrary $\alpha \in Y$ and prove the corresponding property for c-soft sheaves in $\mathrm{Mod}(A_{f^{-1}(\alpha)})$. We will use Proposition 2.33 to be able to replace $f : X \to Y$

by a suitable morphism $f' : X' \to Y'$ in $\widetilde{\mathbf{A}}$ such that $\alpha \in Y'$ and is closed in Y'. After that we take $a \models \alpha$ a realization of α, $\mathbb{S}$ a prime model of the first-order theory of $\mathbb{M}$ over $\{a\} \cup M$ and $r_{|} : (f'^{\mathbb{S}})^{-1}(a) \to f'^{-1}(\alpha)$ the homeomorphism of Lemma 2.27. It follows that $(f'^{\mathbb{S}})^{-1}(a)$ is an object of $\widetilde{\mathrm{Def}(\mathbb{S})}$ and c is a normal and constructible family of supports on $(f'^{\mathbb{S}})^{-1}(a)$ because $f' : X' \to Y'$ is a morphism in $\widetilde{\mathbf{A}}$ and by Proposition 2.36, c is a normal and constructible family of supports on $f'^{-1}(\alpha)$. Therefore, we will be able to transfer results for c-soft sheaves on o-minimal spectral spaces [18, Section 3] to c-soft sheaves on the fiber $f^{-1}(\alpha)$ since all we need is that c is a family of normal and constructible supports.

Remark 3.19 In the paper [18] we assumed that A is a field, but that was only used to ensure that $\bullet \otimes G \simeq G \otimes \bullet$, for $G \in \mathrm{Mod}(A_X)$, is exact. For this reason, our results here about $\bullet \otimes G \simeq G \otimes \bullet$ will come with the flatness assumption.

Remark 3.20 Unfortunately in the statement of [18, Lemmas 3.2 and 3.3] and also in [18, Proposition 3.7], the assumption that Y (resp. $D \cap Y$) has a fundamental system of normal and constructible locally closed neighborhoods in X is missing. Note however that this does not affect the main results of [18, Section 3] that we will use here since in those results $Y = X$ and each D satisfies the hypothesis as Φ is normal and constructible. For Lemma 3.25 below (which uses directly [18, Lemma 3.2]) one gets the fundamental system of normal and constructible locally closed neighborhoods of K in X first assuming α is closed in Y (Proposition 2.33 as explained in Remark 3.18) and using Proposition 2.37.

The first application of the above method is:

Proposition 3.21 *Let $f : X \to Y$ be a morphism in $\widetilde{\mathbf{A}}$. Then filtrant inductive limits of f-soft sheaves in* $\mathrm{Mod}(A_X)$ *are f-soft.*

Proof Let $(F_i)_{i \in I}$ be a filtrant inductive family of f-soft sheaves in $\mathrm{Mod}(A_X)$. We have to show that $\varinjlim F_i$ is an f-soft sheaf in $\mathrm{Mod}(A_X)$, i.e., for every $\alpha \in Y$ the sheaf $(\varinjlim F_i)_{|f^{-1}(\alpha)} = \varinjlim(F_{i|f^{-1}(\alpha)})$ in $\mathrm{Mod}(A_{f^{-1}(\alpha)})$ is c-soft.

So fix $\alpha \in Y$ and by Proposition 2.33, let $Y' \in \mathrm{Op}^{\mathrm{cons}}(Y)$ be such that $\alpha \in Y'$ and α is closed in Y'. Let $X' = f^{-1}(Y')$ and $f' = f_{|Y'}$. Then $f'^{-1}(\alpha) = f^{-1}(\alpha)$ and so $F_{i|f^{-1}(\alpha)} = F_{i|f'^{-1}(\alpha)}$. Hence the sheaf $\varinjlim(F_{i|f^{-1}(\alpha)})$ in $\mathrm{Mod}(A_{f^{-1}(\alpha)})$ is c-soft if and only if the sheaf $\varinjlim(F_{i|f'^{-1}(\alpha)})$ in $\mathrm{Mod}(A_{f'^{-1}(\alpha)})$ is c-soft.

Let $a \models \alpha$ a realization of α, $\mathbb{S}$ a prime model of the first-order theory of $\mathbb{M}$ over $\{a\} \cup M$ and $r_{|} : (f'^{\mathbb{S}})^{-1}(a) \to f'^{-1}(\alpha)$ the homeomorphism of Lemma 2.27. Then the sheaf $\varinjlim(F_{i|f'^{-1}(\alpha)})$ in $\mathrm{Mod}(A_{f'^{-1}(\alpha)})$ is c-soft if and only if the sheaf $\varinjlim(r^{-1}F_{i|(f'^{\mathbb{S}})^{-1}(a)})$ in $\mathrm{Mod}(A_{(f'^{\mathbb{S}})^{-1}(a)})$ is c-soft.

But $(f'^{\mathbb{S}})^{-1}(a)$ is an object of $\widetilde{\mathrm{Def}(\mathbb{S})}$ and c is a normal and constructible family of supports on $(f'^{\mathbb{S}})^{-1}(a)$ because $f' : X' \to Y'$ is a morphism in $\widetilde{\mathbf{A}}$ and by Proposition 2.36, c is a normal and constructible family of supports on $f'^{-1}(\alpha)$. Since the sheaves $F_{i|f^{-1}(\alpha)}$ in $\mathrm{Mod}(A_{f^{-1}(\alpha)})$ are c-soft if and only if the sheaves $F_{i|f'^{-1}(\alpha)}$ in $\mathrm{Mod}(A_{f'^{-1}(\alpha)})$ are c-soft if and only if the sheaves $r^{-1}F_{i|(f'^{\mathbb{S}})^{-1}(a)}$ in $\mathrm{Mod}(A_{(f'^{\mathbb{S}})^{-1}(a)})$ are c-soft, by [18, Corollary 3.5] in $\widetilde{\mathrm{Def}(\mathbb{S})}$, we conclude that the sheaf $\varinjlim(r^{-1}F_{i|(f'^{\mathbb{S}})^{-1}(a)})$ in $\mathrm{Mod}(A_{(f'^{\mathbb{S}})^{-1}(a)})$ is c-soft as required. ☐

Since restriction is exact and commutes with $\otimes$, by the method of Remark 3.18 in combination with [18, Proposition 3.13] and Remark 3.19 we have:

Proposition 3.22 *Let $f : X \to Y$ be a morphism in $\widetilde{\mathbf{A}}$. If $G \in$* $\mathrm{Mod}(A_X)$ *is f-soft and flat, then for every $F \in$* $\mathrm{Mod}(A_X)$ *we have that $G \otimes F$ is f-soft.*

By Relative fiber formula (Proposition 3.8) we have:

Proposition 3.23 *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\widetilde{\mathbf{A}}$ and let F be a sheaf in $\mathrm{Mod}(A_X)$. If F is $(g \circ f)$-soft, then $f_{!!}F$ is g-soft.*

Proof Let $\beta \in Z$, we shall prove that the restriction $(f_{!!}F)_{|g^{-1}(\beta)}$ is c-soft. Let $K \subset K'$ be elements of the family of complete supports c on $g^{-1}(\beta)$. By Relative fiber formula (Proposition 3.8), there is a commutative diagram

$$\begin{tikzcd}
\Gamma(K'; f_{!!}F) \arrow[r, "\psi_{K'}"] \arrow[d] & \Gamma_c(f^{-1}(K'); F) \arrow[d] \\
\Gamma(K; f_{!!}F) \arrow[r, "\psi_K"] & \Gamma_c(f^{-1}(K); F)
\end{tikzcd}$$

induced by the restrictions, with $\psi_{K'}$ and ψ_K isomorphisms. Since F is $(g \circ f)$-soft, the arrow on the right is surjective by [18, Proposition 3.4 (3)]. Therefore, the arrow on the left is also surjective. This implies that, if K' is a constructible element of the family of complete supports c on $g^{-1}(\beta)$, then $(f_{!!}F)_{|K'}$ is soft, which implies $(f_{!!}F)_{|g^{-1}(\beta)}$ c-soft by [18, Proposition 3.4 (4)]. □

A special case which follows by Remark 3.5 and [18, Proposition 3.4] is the following. Compare also with [30, Chapter II, Proposition 2.5.7 (ii)].

Remark 3.24 Let $f : X \to Y$ be a morphism in $\widetilde{\mathrm{Def}}$ and let F be a sheaf in $\mathrm{Mod}(A_X)$. Suppose that the family c of supports on X (resp. on Y) is such that every $C \in c$ has a neighborhood D in X (resp. in Y) such that $D \in c$. If F is c-soft, then $f_{!!}F$ is c-soft.

Lemma 3.25 *Let $f : X \to Y$ be a morphism in $\widetilde{\mathrm{Def}}$ with X an open subspace of a normal space in $\widetilde{\mathrm{Def}}$ and let F be a sheaf in $\mathrm{Mod}(A_X)$. If F is flabby, then F is f-soft. In particular, the full additive subcategory of $\mathrm{Mod}(A_X)$ of f-soft sheaves is cogenerating, i.e. for every $F' \in \mathrm{Mod}(A_X)$ there exists an f-soft $F \in \mathrm{Mod}(A_X)$ and an exact sequence $0 \to F' \to F$.*

Proof Let $\alpha \in Y$ and let K be an element of the family c of complete supports on $f^{-1}(\alpha)$. Then K is quasi-compact in X (Remark 2.10). Since X is an open subspace of a normal space in $\widetilde{\mathrm{Def}}$, by [18, Lemma 3.2], the canonical morphism

$$\varinjlim_{K \subseteq U} \Gamma(U; F) \to \Gamma(K; (F_{|f^{-1}(\alpha)})_{|K})$$

where U ranges through the family of open constructible subsets of X, is an isomorphism. Since F is flabby $\Gamma(X; F) \to \Gamma(U; F)$ is surjective. The exactness of filtrant $\varinjlim$ implies that $\Gamma(X; F) \to \Gamma(K; F_{|f^{-1}(\alpha)})_{|K})$ is surjective. This morphism factors through $\Gamma(f^{-1}(\alpha); F_{|f^{-1}(\alpha)})$ and the result follows. □

Proposition 3.26 *Let $f : X \to Y$ be a morphism in $\widetilde{\mathbf{A}}$. Then the full additive subcategory of $\mathrm{Mod}(A_X)$ of f-soft sheaves is $f_{!!}$-injective, i.e.:*

(1) *For every $F \in \mathrm{Mod}(A_X)$ there exists an f-soft $F' \in \mathrm{Mod}(A_X)$ and an exact sequence $0 \to F \to F'$.*

(2) *If* $0 \to F' \to F \to F'' \to 0$ *is an exact sequence in* $\mathrm{Mod}(A_X)$ *and* F' *is* f*-soft, then* $0 \to f_! F' \to f_! F \to f_! F'' \to 0$ *is an exact sequence.*

(3) *If* $0 \to F' \to F \to F'' \to 0$ *is an exact sequence in* $\mathrm{Mod}(A_X)$ *and* F' *and* F *are* f*-soft, then* F'' *is* f*-soft.*

Proof (1) By (A2) X is homeomorphic to an open subspace of a normal space in $\widetilde{\mathrm{Def}}$, therefore, by Lemma 3.25 the full additive subcategory of $\mathrm{Mod}(A_X)$ of f-soft sheaves is cogenerating.

(2) Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Mod}(A_X)$ with F' is f-soft. Then for every $\alpha \in Y$, the sequence $0 \to F'_{|f^{-1}(\alpha)} \to F_{|f^{-1}(\alpha)} \to F''_{|f^{-1}(\alpha)} \to 0$ is exact and $F'_{|f^{-1}(\alpha)}$ is a c-soft sheaf in $\mathrm{Mod}(A_{f^{-1}(\alpha)})$. By the method of Remark 3.18 in combination with [18, Proposition 3.7 (2)], $0 \to \Gamma_c(f^{-1}(\alpha); F'_{|f^{-1}(\alpha)}) \to \Gamma_c(f^{-1}(\alpha); F_{|f^{-1}(\alpha)}) \to \Gamma_c(f^{-1}(\alpha); F''_{|f^{-1}(\alpha)}) \to 0$ is an exact sequence for every $\alpha \in Y$. By the Fiber formula (Corollary 3.12), the sequence $0 \to f_! F' \to f_! F \to f_! F'' \to 0$ is exact.

(3) Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{Mod}(A_X)$ with F' and F both f-soft. Then for every $\alpha \in Y$, the sequence $0 \to F'_{|f^{-1}(\alpha)} \to F_{|f^{-1}(\alpha)} \to F''_{|f^{-1}(\alpha)} \to 0$ is exact and $F'_{|f^{-1}(\alpha)}$ and $F_{|f^{-1}(\alpha)}$ are c-soft sheaves in $\mathrm{Mod}(A_{f^{-1}(\alpha)})$. By the method of Remark 3.18 in combination with [18, Proposition 3.7 (3)], $F''_{|f^{-1}(\alpha)}$ is c-soft. Since α was arbitrary, F'' is f-soft.

□

3.4 Projection and base change formulas

Here we prove the projection and base change formulas.

For our next proposition we need the following topological lemma which is an adaptation of [30, Lemma 2.5.12]:

Lemma 3.27 *Let* X *be a quasi-compact topological space with a basis of open quasi-compact neighborhoods and let* Φ *be a family of supports on* X*. Let* $F \in \mathrm{Mod}(A_X)$ *and let* M *be a flat* A*-module . Then there is a natural isomorphism.*

$$\Gamma_\Phi(X; F) \otimes M \simeq \Gamma_\Phi(X; F \otimes M_X).$$

In particular, if F *is* Φ*-soft, then* $F \otimes M_X$ *is* Φ*-soft.*

Proof We may assume that $X \in \Phi$. If $X = \cup_j U_j$ is a finite cover of X by open quasi-compact neighborhoods, then

$$0 \to \Gamma(X; F) \overset{\lambda}{\to} \oplus_j \Gamma(U_j; F) \overset{\mu}{\to} \oplus_{j,i} \Gamma(U_j \cap U_i; F)$$

is exact. Applying the exact functor $\bullet \otimes M$, recall M is flat, we obtain the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X; F) \otimes M \arrow[r, "\lambda"] \arrow[d, "\varphi"] & \oplus_j \Gamma(U_j; F) \otimes M \arrow[r, "\mu"] \arrow[d, "\psi"] & \oplus_{j,i} \Gamma(U_j \cap U_i; F) \otimes M \arrow[d, "\vartheta"] \\
0 \arrow[r] & \Gamma(X; F \otimes M_X) \arrow[r, "\lambda'"] & \oplus_j \Gamma(U_j; F \otimes M_X) \arrow[r, "\mu'"] & \oplus_{j,i} \Gamma(U_j \cap U_i; F \otimes M_X)
\end{tikzcd}$$

Observe also that for every $x \in X$ we have

$$\varinjlim_{U}(\Gamma(U; F) \otimes M) \simeq \varinjlim_{U}\Gamma(U; F \otimes M_X),$$

where U ranges through the family of open quasi-compact neighborhoods of x. In fact, both sides of this auxiliary isomorphism are isomorphic to $F_x \otimes M$. Thus, if $s \in \Gamma(X; F) \otimes M$ is such that $\varphi(s) = 0$, then we can find, by the auxiliary isomorphism and quasi-compactness of X, a finite covering $X = \cup_j U_j$ by open quasi-compact neighborhoods such that $\lambda(s) = 0$. Therefore, $s = 0$ and φ is injective. If we apply the same argument to U_j and $U_j \cap U_i$ instead of X we see that ψ and ϑ are also injective.

To show that φ is surjective, take $t \in \Gamma(X; F \otimes M_X)$. By the auxiliary isomorphism above, there exists a finite covering $X = \cup_j U_j$ by open quasi-compact neighborhoods such that $\lambda'(t)$ is in the image of ψ. But by injectivity of ϑ it follows that t is in the image of φ. ☐

By the Fiber formula (Corollary 3.12) and Lemma 3.27 we have:

Proposition 3.28 (Projection formula) *Let $f : X \to Y$ be a morphism in $\widetilde{\mathbf{A}}$, $F \in \mathrm{Mod}(A_X)$ and $G \in \mathrm{Mod}(A_Y)$. If G is flat, then the natural morphism*

$$f_{\underset{\cdot}{\wr}}F \otimes G \to f_{\underset{\cdot}{\wr}}(F \otimes f^{-1}G)$$

is an isomorphism.

Proof The morphism $f^{-1} \circ f_{\underset{\cdot}{\wr}} \to f^{-1} \circ f_* \to \mathrm{id}$ induces the morphism $f^{-1}(f_{\underset{\cdot}{\wr}}F \otimes G) \simeq f^{-1} \circ f_{\underset{\cdot}{\wr}}F \otimes f^{-1}G \to F \otimes f^{-1}G$ and we obtain the morphism $f_{\underset{\cdot}{\wr}}F \otimes G \to f_{\underset{\cdot}{\wr}}(F \otimes f^{-1}G)$ by adjunction.

To prove that it is an isomorphism, let $\alpha \in Y$. Then

$$\begin{aligned}(f_{\underset{\cdot}{\wr}}(F \otimes f^{-1}G))_\alpha &\simeq \Gamma_c(f^{-1}(\alpha); (F \otimes f^{-1}G)_{|f^{-1}(\alpha)})\\ &\simeq \Gamma_c(f^{-1}(\alpha); F_{|f^{-1}(\alpha)} \otimes (f^{-1}G)_{|f^{-1}(\alpha)})\\ &\simeq \Gamma_c(f^{-1}(\alpha); F_{|f^{-1}(\alpha)} \otimes G_\alpha)\\ &\simeq \Gamma_c(f^{-1}(\alpha); F_{|f^{-1}(\alpha)}) \otimes G_\alpha\\ &\simeq (f_{\underset{\cdot}{\wr}}F)_\alpha \otimes G_\alpha\\ &\simeq (f_{\underset{\cdot}{\wr}}F \otimes G)_\alpha,\end{aligned}$$

by the Fiber formula (Corollary 3.12), Lemma 3.27 and using also the fact that $(f^{-1}G)_{|f^{-1}(\alpha)} \simeq G_\alpha$. ☐

We now proceed to the proof of the base change formula. By Lemma 2.27, we have:

Remark 3.29 *Let $f : X \to Y$ be a morphism in $\widetilde{\mathrm{Def}}$ and $\alpha \in Y$. Let $a \models \alpha$ a realization of α and $\mathbb{S}$ a prime model of the theory of $\mathbb{M}$ over $\{a\} \cup M$. Then we have a commutative diagram*

$$\begin{tikzcd}
(f^{\mathbb{S}})^{-1}(a) \arrow[r, hook] \arrow[d, "r_|"] & X(\mathbb{S}) \arrow[r, "f^{\mathbb{S}}"] \arrow[d, "r"] & Y(\mathbb{S}) \arrow[d, "r"] \\
f^{-1}(\alpha) \arrow[r, hook] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

with the restriction $r_|$ a homeomorphism (Lemma 2.27).

If $F \in \mathrm{Mod}(A_X)$ *then*

$$F(\mathbb{S})_{|(f^{\mathbb{S}})^{-1}(a)} = (r_{|})^{-1} F_{|f^{-1}(\alpha)}$$

where here and below we use the notation $F(\mathbb{S}) = r^{-1}F$.

Lemma 3.30 *Consider a cartesian square in* $\widetilde{\mathrm{Def}}$

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Let $\gamma \in Y'$, $v \models \gamma$ *a realization of* γ *and* $\mathbb{S}$ *a prime model of the first-order theory of* $\mathbb{M}$ *over* $\{v\} \cup M$. *Let* $u = g^{\mathbb{S}}(v)$ *(so* $u \models g(\gamma)$ *is a realization of* $g(\gamma)$*). Then we have a commutative diagram*

$$\begin{tikzcd} (f'^{\mathbb{S}})^{-1}(v) \arrow[r, hook] \arrow[d, "g_{|}'^{\mathbb{S}}"] & X'(\mathbb{S}) \arrow[r, "f'^{\mathbb{S}}"] \arrow[d, "g'^{\mathbb{S}}"] & Y'(\mathbb{S}) \arrow[d, "g^{\mathbb{S}}"] \\ (f^{\mathbb{S}})^{-1}(u) \arrow[r, hook] & X(\mathbb{S}) \arrow[r, "f^{\mathbb{S}}"] & Y(\mathbb{S}) \end{tikzcd}$$

in $\widetilde{\mathrm{Def}}(\mathbb{S})$ *with* $g'^{\mathbb{S}} : (f'^{\mathbb{S}})^{-1}(v) \to (f^{\mathbb{S}})^{-1}(u)$ *an homeomorphism.*

Proof Indeed, after removing the tilde, we have a similar commutative diagram of continuous $\mathbb{S}$-definable maps between $\mathbb{S}$-definable spaces with the restriction to the $\mathbb{S}$-definable fibers an $\mathbb{S}$-definable homeomorphism. □

By Lemmas 2.27 and 3.30 we have:

Lemma 3.31 *Consider a cartesian square in* $\widetilde{\mathrm{Def}}$

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Let $\gamma \in Y'$, $v \models \gamma$ *a realization of* γ *and* $\mathbb{S}$ *a prime model of the first-order theory of* $\mathbb{M}$ *over* $\{v\} \cup M$. *Let* $u = g^{\mathbb{S}}(v)$ *(so* $u \models g(\gamma)$ *is a realization of* $g(\gamma)$*) and let* $\mathbb{K}$ *is a prime model of the first-order theory of* $\mathbb{M}$ *over* $\{u\} \cup M$. *If* $\mathbb{K} = \mathbb{S}$, *then we have a homeomorphism*

$$g'_{|} : f'^{-1}(\gamma) \to f^{-1}(g(\gamma))$$

which induces an isomorphism

$$\Gamma_c(f'^{-1}(\gamma); (g'^{-1}F)_{|f'^{-1}(\gamma)}) \simeq \Gamma_c(f^{-1}(g(\gamma)); F_{|f^{-1}(g(\gamma))})$$

for every $F \in \mathrm{Mod}(A_X)$.

Proof We have the following commutative diagram

$$\begin{tikzcd}
(f'^{\mathbb{S}})^{-1}(v) \arrow[r, hook] \arrow[d, "g'^{\mathbb{S}}_{|}"] & X'(\mathbb{S}) \arrow[r, "r"] \arrow[d, "g'^{\mathbb{S}}"] & X' \arrow[d, "g'"] & f'^{-1}(\gamma) \arrow[l, hook'] \arrow[d, "g'_{|}"] \\
(f^{\mathbb{S}})^{-1}(u) \arrow[r, hook] & X(\mathbb{S}) \arrow[r, "r"] & X & f^{-1}(g(\gamma)). \arrow[l, hook']
\end{tikzcd}$$

Since $g'^{\mathbb{S}}_{|} : (f'^{\mathbb{S}})^{-1}(v) \to (f^{\mathbb{S}})^{-1}(u)$ is a homeomorphism by Lemma 3.30 and the restrictions $(r_{|})^{-1} : (f'^{\mathbb{S}})^{-1}(v) \to f'^{-1}(\gamma)$ and $(r_{|})^{-1} : (f^{\mathbb{S}})^{-1}(u) \to f^{-1}(g(\gamma))$ are also homeomorphisms (Lemma 2.27 and $\mathbb{K} = \mathbb{S}$) the result follows. □

From now on until the end of the section we need to assume also (A3).

Lemma 3.32 *Consider a cartesian square in* $\widetilde{\mathrm{Def}}$

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

Suppose that $f : X \to Y$ *satisfies (A3).*

If $\gamma \in Y'$, *then there exists an isomorphism*

$$\Gamma_c(f'^{-1}(\gamma); (g'^{-1}F)_{|f'^{-1}(\gamma)}) \simeq \Gamma_c(f^{-1}(g(\gamma)); F_{|f^{-1}(g(\gamma))})$$

for every $F \in \mathrm{Mod}(A_X)$.

Proof Let $v \models \gamma$ be a realization of γ and $\mathbb{S}$ a prime model of the first-order theory of $\mathbb{M}$ over $\{v\} \cup M$. Set $u = g^{\mathbb{S}}(v)$ and note that $u \models g(\gamma)$ is a realization of $g(\gamma)$. Let $\mathbb{K}$ is a prime model of the first-order theory of $\mathbb{M}$ over $\{u\} \cup M$. Thus since $u \in S$, by Fact 2.26, we have either $\mathbb{K} = \mathbb{M}$ and $\mathbb{M} \neq \mathbb{S}$ or $\mathbb{K} = \mathbb{S}$. So we proceed with the proof by considering the two cases.
Case $\mathbb{K} = \mathbb{M}$ and $\mathbb{M} \neq \mathbb{S}$: We have $u = g(\gamma)$. Then we have

$$\begin{aligned}
\Gamma_c(f^{-1}(g(\gamma)); F_{|f^{-1}(g(\gamma))}) &= \Gamma_c(f^{-1}(u); F_{|f^{-1}(u)}) \\
&\simeq \Gamma_c((f^{\mathbb{S}})^{-1}(u); F(\mathbb{S})_{|(f^{\mathbb{S}})^{-1}(u)}) \\
&\simeq \Gamma_c((f'^{\mathbb{S}})^{-1}(v); (g'^{\mathbb{S}})^{-1}F(\mathbb{S})_{|(f'^{\mathbb{S}})^{-1}(v)}) \\
&\simeq \Gamma_c(f'^{-1}(\gamma); (g'^{-1}F)_{|f'^{-1}(\gamma)})
\end{aligned}$$

where the first isomorphism follows by (A3), the second follows from Lemma 3.30 and the third follows from Lemma 2.27 together with Remark 3.29.

Case $\mathbb{K} = \mathbb{S}$: Then by Lemma 3.31 we have

$$\Gamma_c(f^{-1}(g(\gamma)); F_{|f^{-1}(g(\gamma))}) \simeq \Gamma_c(f'^{-1}(\gamma); (g'^{-1}F)_{|f'^{-1}(\gamma)}).$$

□

We are now ready to prove the base change formula:

Proposition 3.33 (Base change formula) *Consider a cartesian square in* $\widetilde{\mathbf{A}}$

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

and let $F \in \mathrm{Mod}(A_X)$. *Suppose that* $f : X \to Y$ *satisfies (A3). Then*

$$g^{-1} \circ f_{\underset{\cdot}{\wr}} F \xrightarrow{\sim} f'_{\underset{\cdot}{\wr}} \circ g'^{-1} F.$$

Proof Let us construct the morphism $g^{-1} \circ f_{\underset{\cdot}{\wr}} \to f'_{\underset{\cdot}{\wr}} \circ g'^{-1}$. We shall construct first the morphism

$$f_{\underset{\cdot}{\wr}} \circ g'_* \to g_* \circ f'_{\underset{\cdot}{\wr}}.$$

Let $U \in \mathrm{Op}^{\mathrm{cons}}(Y)$ and $G \in \mathrm{Mod}(A_{X'})$. A section $t \in \Gamma(U;\, f_{\underset{\cdot}{\wr}} \circ g'_* G)$ is defined by a section $s \in \Gamma((f \circ g')^{-1}(U); G)$ such that $\mathrm{supp}(s) \subset g'^{-1}(Z)$ for a closed constructible subset Z of $f^{-1}(U)$ such that $f_{|Z} : Z \to U$ is proper. Since

$$\begin{tikzcd}
g'^{-1}(Z) \arrow[r, "f'_{|}"] \arrow[d, "g'_{|}"] & g^{-1}(U) \arrow[d, "g_{|}"] \\
Z \arrow[r, "f_{|}"] & U
\end{tikzcd}$$

is a cartesian square in $\widetilde{\mathrm{Def}}$, $(f \circ g')^{-1}(U) = (g \circ f')^{-1}(U)$ and by Proposition 2.9 (5), the restriction $f'_{|g'^{-1}(Z)} : g'^{-1}(Z) \to g^{-1}(U)$ is proper. Therefore, $s \in \Gamma((g \circ f')^{-1}(U); G) = \Gamma(f'^{-1}(g^{-1}(U)); G)$ such that $\mathrm{supp}(s) \subset g'^{-1}(Z)$ for a closed constructible subset $g'^{-1}(Z)$ of $f'^{-1}(g^{-1}(U))$ such that $f'_{|g'^{-1}(Z)} : g'^{-1}(Z) \to g'^{-1}(U)$ is proper. Then s defines a section of $\Gamma(U;\, g_* \circ f'_{\underset{\cdot}{\wr}} G)$ and we obtain $f_{\underset{\cdot}{\wr}} \circ g'_* \to g_* \circ f'_{\underset{\cdot}{\wr}}$.

To construct $g^{-1} \circ f_{\underset{\cdot}{\wr}} \to f'_{\underset{\cdot}{\wr}} \circ g'^{-1}$ consider the morphism $f_{\underset{\cdot}{\wr}} \to f_{\underset{\cdot}{\wr}} \circ g'_* \circ g'^{-1} \to g_* \circ f'_{\underset{\cdot}{\wr}} \circ g'^{-1}$, where the second arrow is induced by $f_{\underset{\cdot}{\wr}} \circ g'_* \to g_* \circ f'_{\underset{\cdot}{\wr}}$. We define $g^{-1} \circ f_{\underset{\cdot}{\wr}} \to f'_{\underset{\cdot}{\wr}} \circ g'^{-1}$ by adjunction.

To prove that $g^{-1} \circ f_{\underset{\cdot}{\wr}} F \to f'_{\underset{\cdot}{\wr}} \circ g'^{-1} F$ is an isomorphism, let us take $\gamma \in Y'$. Then by the Fiber formula (Corollary 3.12)

$$\begin{aligned}
(g^{-1} \circ f_{\underset{\cdot}{\wr}} F)_\gamma &\simeq (f_{\underset{\cdot}{\wr}} F)_{g(\gamma)} \\
&\simeq \Gamma_c(f^{-1}(g(\gamma));\, F_{|f^{-1}(g(\gamma))})
\end{aligned}$$

and

$$(f'_{\underset{\cdot}{\wr}} \circ g'^{-1} F)_\gamma \simeq \Gamma_c(f'^{-1}(\gamma);\, (g'^{-1} F)_{|f'^{-1}(\gamma)}).$$

Therefore, we have to show that

$$\Gamma_c(f^{-1}(g(\gamma));\, F_{|f^{-1}(g(\gamma))}) \simeq \Gamma_c(f'^{-1}(\gamma);\, (g'^{-1} F)_{|f'^{-1}(\gamma)}).$$

But this is proved in Lemma 3.32. □

3.5 Derived proper direct image

Here we derive the proper direct image and prove the derived projection and base change formulas. As corollaries we obtain the universal coefficients formula and the Künneth formula.

Recall that A is a commutative ring with unit and if X is a topological space, in particular an object of $\widetilde{\mathrm{Def}}$, we denote by $\mathrm{Mod}(A_X)$ the category of sheaves of A-modules on X (called also A-sheaves on X).

Since $\mathrm{Mod}(A_X)$ is an abelian category we may consider its derived category

$$\mathrm{D}(A_X) := \mathrm{D}(\mathrm{Mod}(A_X))$$

and its full triangulated subcategories

$$\mathrm{D}^*(A_X) := \mathrm{D}^*(\mathrm{Mod}(A_X))$$

where $* = -, +, b$.

Since $\mathrm{Mod}(A_X)$ has enough injectives we may right derive (resp. derive) the classical left exact (resp. exact) functors

$$R\mathcal{H}om_{A_X}(\bullet, \bullet),\ \ Rf_*,\ \ f^{-1},\ \ (\bullet)_Z,\ \ R\Gamma_Z(X; \bullet),\ \ R\Gamma(X; \bullet)$$

on A-sheaves on X, where $Z \subseteq X$ is a locally closed subset.

Remark 3.34 In order to left derive the functor

$$\bullet \otimes_{A_X} \bullet$$

we will need to assume that the ring A has finite weak global dimension, $\mathrm{wgld}(A) < \infty$. The weak global dimension of A is the smallest n such that every A-module has a flat resolution of length n, equivalently, it is the smallest n such that $\mathrm{Tor}_j^A(M, N) = 0$ for any $j > n$ and any A-modules M and N. Alternatively we can assume that A has finite global homological dimension, $\mathrm{gld}(A) < \infty$, since $\mathrm{wgld}(A) \leq \mathrm{gld}(A)$. The global dimension of A is the smallest n such that every A-module has a projective resolution of length n. See [30, Exercises I. 28 and I. 29].

If $\mathrm{wgld}(A) < \infty$, then by the observations on page 110 in [30], if $F \in \mathrm{D}^b(A_X)$ (resp. $F \in \mathrm{D}^+(A_X)$), then F is quasi-isomorphic to a bounded complex (resp. a complex bounded from below) of flat A-sheaves. Therefore, we may define the left derived functor

$$\bullet \otimes \bullet : \mathrm{D}^*(A_X) \times \mathrm{D}^*(A_X) \to \mathrm{D}^*(A_X)$$

with $* = -, +, b$.

Below we will use freely the properties relating the above derived operations and we refer to reader to [30, Chapter II, Section 2.6] for details.

Let $f : X \to Y$ be a morphism in $\widetilde{\mathbf{A}}$. We are going to consider the right derived functor of proper direct image

$$Rf_{\underset{\cdot}{\wr}} : \mathrm{D}^+(A_X) \to \mathrm{D}^+(A_Y).$$

If $F \in \mathrm{D}^+(A_X)$ then since the f-soft sheaves are $f_{\underset{\cdot}{\wr}}$-injective (Proposition 3.26), there is a complex F' of f-soft sheaves quasi-isomorphic to F and

$$Rf_{\underset{\cdot}{\wr}}F \simeq f_{\underset{\cdot}{\wr}}F'.$$

Furthermore, if $g : Y \to Z$ is another morphism in $\widetilde{\mathbf{A}}$, then by Proposition 3.23,

$$R(g \circ f)_{\underset{\cdot}{\wr}} \simeq Rg_{\underset{\cdot}{\wr}} \circ Rf_{\underset{\cdot}{\wr}}.$$

Note also that by Theorem 3.39, the functor $Rf_{\underset{\cdot}{\wr}}$ induces a functor:

$$Rf_{\underset{\cdot}{\wr}} : \mathrm{D}^b(A_X) \to \mathrm{D}^b(A_Y).$$

Deriving the projection formula (Proposition 3.28) we have:

Theorem 3.35 (Derived projection formula) *Let $f : X \to Y$ be a morphism in $\widetilde{\mathbf{A}}$. Let $F \in \mathrm{D}^+(A_X)$ and $G \in \mathrm{D}^+(A_Y)$. Suppose that* $\mathrm{wgld}(A) < \infty$. *Then there is a natural isomorphism*

$$Rf_{\underset{\cdot}{\wr}} F \otimes G \simeq Rf_{\underset{\cdot}{\wr}}(F \otimes f^{-1}G).$$

Proof First note that, if $G \in \mathrm{Mod}(A_Y)$ is flat, then by Lemma 3.27, $\bullet \otimes f^{-1}G$ sends f-soft sheaves to f-soft sheaves. (Indeed, for every $\alpha \in Y$, the restriction $(\bullet)_{|f^{-1}(\alpha)}$ commutes with $\otimes$ and $(f^{-1}G)_{|f^{-1}(\alpha)} \simeq G_\alpha$ which is also flat.)

Now let $F \in \mathrm{D}^+(A_X)$ and $G \in \mathrm{D}^+(A_Y)$. Let F' be a complex of f-soft sheaves quasi-isomorphic to F (Lemma 3.25). By Remark 3.34, there exists a complex G' bounded from below of flat sheaves quasi-isomorphic to G. Then $F' \otimes f^{-1}G'$ is a complex of f-soft sheaves quasi-isomorphic to $F \otimes f^{-1}G$. Therefore, by Proposition 3.26,

$$Rf_{\underset{\cdot}{\wr}} F \otimes G \simeq f_{\underset{\cdot}{\wr}} F' \otimes G' \simeq f_{\underset{\cdot}{\wr}}(F' \otimes f^{-1}G') \simeq Rf_{\underset{\cdot}{\wr}}(F \otimes f^{-1}G),$$

where the second isomorphism follows from Proposition 3.28. □

Recall that F is *φ-acyclic* where φ is a left exact functor if $R^k\varphi F = 0$ for all $k \neq 0$. In such a situation, the full subcategory of φ-acyclic objects is φ-injective [30, Exercises I.19].

In order to prove the derived base change formula, we need the following lemma in which we assume (A3):

Lemma 3.36 *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\widetilde{\mathrm{Def}}$. Suppose that $f : X \to Y$ satisfies (A3). Then $g'^{-1}(\bullet)$ sends f-soft sheaves to $f'_{\underset{\cdot}{\wr}}$-acyclic sheaves.[3]

Proof Let $F \in \mathrm{Mod}(A_X)$ be f-soft. Then $F_{|f^{-1}(\alpha)}$ is $\Gamma_c(f^{-1}(\alpha); \bullet)$-acyclic for every $\alpha \in Y$. We must show that the restriction $(g'^{-1}F)_{|f'^{-1}(\gamma)}$ is $\Gamma_c(f'^{-1}(\gamma); \bullet)$-acyclic for every $\gamma \in Y'$.

So take $\gamma \in Y'$. Let $v \models \gamma$ be a realization of γ and $\mathbb{S}$ a prime model of the first-order theory of $\mathbb{M}$ over $\{v\} \cup M$. Set $u = g^{\mathbb{S}}(v)$ and note that $u \models g(\gamma)$ is a realization of $g(\gamma)$. Let $\mathbb{K}$ is a prime model of the first-order theory of $\mathbb{M}$ over $\{u\} \cup M$. Thus since $u \in S$, by Fact 2.26, we have either $\mathbb{K} = \mathbb{M}$ and $\mathbb{M} \neq \mathbb{S}$ or $\mathbb{K} = \mathbb{S}$. So we proceed with the proof by considering the two cases.

---

[3] In topology it is trivial that $g'^{-1}(\bullet)$ sends f-soft sheaves to f'-soft sheaves. Here this is not evident.

Case $\mathbb{K}=\mathbb{M}$ and $\mathbb{M}\neq\mathbb{S}$: We have $u=g(\gamma)$ and $F_{|f^{-1}(g(\gamma))}=F_{|f^{-1}(u)}$. Since $F_{|f^{-1}(u)}$ is $\Gamma_c(f^{-1}(u);\bullet)$-acyclic, by (A3), $F(\mathbb{S})_{|(f^{\mathbb{S}})^{-1}(u)}$ is $\Gamma_c((f^{\mathbb{S}})^{-1}(u);\bullet)$-acyclic. Since $g'^{\mathbb{S}}:(f'^{\mathbb{S}})^{-1}(v)\to(f^{\mathbb{S}})^{-1}(u)$ is a homeomorphism (Lemma 3.30) and on the other hand, $((g'^{\mathbb{S}})^{-1}F(\mathbb{S}))_{|(f'^{\mathbb{S}})^{-1}(v)}=(g'^{\mathbb{S}}_{|})^{-1}(F(\mathbb{S})_{|(f^{\mathbb{S}})^{-1}(u)})$, we conclude that $((g'^{\mathbb{S}})^{-1}F(\mathbb{S}))_{|(f'^{\mathbb{S}})^{-1}(v)}$ is $\Gamma_c((f'^{\mathbb{S}})^{-1}(v);\bullet)$-acyclic. As $(r_{|})^{-1}:f'^{-1}(\gamma)\to(f'^{\mathbb{S}})^{-1}(v)$ is a homeomorphism (Lemma 2.27), by Remark 3.29, we have that $(g'^{-1}F)_{|f'^{-1}(\gamma)}$ is $\Gamma_c(f'^{-1}(\gamma);\bullet)$-acyclic.
Case $\mathbb{K}=\mathbb{S}$: Since $g'_{|}:f'^{-1}(\gamma)\to f^{-1}(g(\gamma))$ is a homeomorphism (Lemma 3.31), $(g'^{-1}F)_{|f'^{-1}(\gamma)}=(g'_{|})^{-1}(F_{|f^{-1}(g(\gamma))})$ and $F_{|f^{-1}(g(\gamma))}$ is $\Gamma_c(f^{-1}(g(\gamma));\bullet)$-acyclic, we have that $(g'^{-1}F)_{|f'^{-1}(\gamma)}$ is $\Gamma_c(f'^{-1}(\gamma);\bullet)$-acyclic. □

Let $f:X\to Y$ be a morphism in $\widetilde{\mathbf{A}}$. The full additive subcategory of $\mathrm{Mod}(A_X)$ of $f_{\underset{\cdot}{\wr}}$-acyclic sheaves is $f_{\underset{\cdot}{\wr}}$-injective. Therefore, if $F\in\mathrm{D}^+(A_X)$ and F' is a complex of $f_{\underset{\cdot}{\wr}}$-acyclic sheaves quasi-isomorphic to F, then

$$Rf_{\underset{\cdot}{\wr}}F\simeq f_{\underset{\cdot}{\wr}}F'.$$

Deriving the base change formula (Proposition 3.33) we have:

Theorem 3.37 (Derived base change formula) *Consider a cartesian square*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

in $\widetilde{\mathbf{A}}$. *Suppose that* $f:X\to Y$ *satisfies (A3). Then there is an isomorphism in* $\mathrm{D}^+(A_{Y'})$, *functorial in* $F\in\mathrm{D}^+(A_X)$:

$$g^{-1}\circ Rf_{\underset{\cdot}{\wr}}F\simeq Rf'_{\underset{\cdot}{\wr}}\circ g'^{-1}F.$$

Proof Let $F\in\mathrm{D}^+(A_X)$ and let F' be a complex of f-soft sheaves quasi-isomorphic to F (Lemma 3.25). By Lemma 3.36, g'^{-1} sends f-soft sheaves to $f'_{\underset{\cdot}{\wr}}$-acyclic sheaves. So $g'^{-1}F'$ is a complex of $f'_{\underset{\cdot}{\wr}}$-acyclic sheaves quasi-isomorphic to $g'^{-1}F$. Therefore, since the full subcategory of f-soft sheaves is $f_{\underset{\cdot}{\wr}}$-injective (Proposition 3.26) and the full additive subcategory of $f'_{\underset{\cdot}{\wr}}$-acyclic sheaves is $f'_{\underset{\cdot}{\wr}}$-injective, by Proposition 3.33 we have

$$\begin{aligned}
g^{-1}\circ Rf_{\underset{\cdot}{\wr}}F &\simeq g^{-1}\circ f_{\underset{\cdot}{\wr}}F'\\
&\simeq f'_{\underset{\cdot}{\wr}}\circ g'^{-1}F'\\
&\simeq Rf'_{\underset{\cdot}{\wr}}\circ g'^{-1}F.
\end{aligned}$$

□

Combining the the derived projection and base change formulas we obtain:

Theorem 3.38 (Künneth formula) *Consider a cartesian square*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"] \arrow[dr, dashed, "\delta"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

in $\widetilde{\mathbf{A}}$ *where* $\delta = f \circ g' = g \circ f'$. *Suppose that* $f : X \to Y$ *satisfies (A3). Suppose that* $\mathrm{wgld}(A) < \infty$. *There is a natural isomorphism*

$$R\delta_{!!}(g'^{-1}F \otimes f'^{-1}G) \simeq Rf_{!!}F \otimes Rg_{!!}G$$

for $F \in \mathrm{D}^+(A_X)$ *and* $G \in \mathrm{D}^+(A_{Y'})$.

Proof Using the derived projection formula and the derived base change formula we deduce

$$Rf'_{!!}(g'^{-1}F \otimes f'^{-1}G) \simeq (Rf'_{!!} \circ g'^{-1}F) \otimes G \simeq (g^{-1} \circ Rf_{!!}F) \otimes G.$$

Using the derived projection formula once again we find

$$Rg_{!!} \circ Rf'_{!!}(g'^{-1}F \otimes f'^{-1}G) \simeq Rg_{!!}((g^{-1} \circ Rf_{!!}F) \otimes G) \simeq Rf_{!!}F \otimes Rg_{!!}G$$

and the result follows since $R\delta_{!!} \simeq Rg_{!!} \circ Rf'_{!!}$. □

3.6 A bound for the cohomology of proper direct image

Here we find a bound for the cohomology of the proper direct image.

The *cohomological dimension of* $f_{!!}$ is the smallest n such that $R^k f_{!!}F = 0$ for all $k > n$ and all sheaves F in $\mathrm{Mod}(A_X)$.

Theorem 3.39 *Let* $f : X \to Y$ *be a morphism in* $\widetilde{\mathbf{A}}$. *Then the cohomological dimension of* $f_{!!}$ *is bounded by* $\dim X$.

Proof Let $F \in \mathrm{Mod}(A_X)$ and let $\alpha \in Y$. Taking a f-soft resolution of F (Proposition 3.26) one checks easily that $(Rf_{!!}F)_\alpha \simeq R\Gamma_c(f^{-1}(\alpha); F_{|f^{-1}(\alpha)})$ (the Fiber formula—Corollary 3.12).

By the method of Remark 3.18 in combination with [18, Theorem 3.12], we see that $H^k(Rf_{!!}F)_\alpha \simeq (R^k f_{!!}F)_\alpha \simeq R^k\Gamma_c(f^{-1}(\alpha); F_{|f^{-1}(\alpha)}) = 0$ if $k > \dim X$. Since α was arbitrary the result follows. □

3.7 Universal coefficients and Künneth formulas

Here we prove the universal coefficients formula and the Künneth formula.

Theorem 3.40 (Universal coefficients formula) *Let* X *be an object of* $\widetilde{\mathrm{Def}}$ *such that* c *is a normal and constructible family of supports on* X. *Let* M *be a flat* A*-module. Then there is an isomorphism*

$$H^*_c(X; M_X) \simeq H^*_c(X; A_X) \otimes M.$$

Proof Let F' be a complex of c-soft sheaves quasi-isomorphic to A_X [18, Proposition 3.7]. By Lemma 3.27, $F' \otimes M_X$ is a complex of c-soft sheaves quasi-isomorphic to $A_X \otimes M_X \simeq M_X$. Therefore, by Lemma 3.27,

$$R\Gamma_c(X; A_X \otimes M_X) \simeq \Gamma_c(X; F' \otimes M_X) \simeq \Gamma_c(X; F') \otimes M \simeq R\Gamma_c(X; A_X) \otimes M.$$

Hence, by the purely homological algebra result in [30, Exercise I.24], applied to the exact bifunctor $\bullet \otimes \bullet : \mathrm{Mod}(A) \times \mathrm{Mod}(A) \to \mathrm{Mod}(A)$, we have

$$\begin{aligned} H^k_c(X; M_X) &= H^k(R\Gamma_c(X; M_X)) \\ &\simeq H^k(R\Gamma_c(X; A_X)) \otimes M. \\ &= H^k_c(X; A_X) \otimes M. \end{aligned}$$

□

Theorem 3.41 (Künneth formula) *Consider the cartesian square*

$$\begin{tikzcd} X \times Y \arrow[r, "p_Y"] \arrow[d, "p_X"'] \arrow[rd, dashed, "a_{X\times Y}"] & Y \arrow[d, "a_Y"] \\ X \arrow[r, "a_X"'] & \mathrm{pt} \end{tikzcd}$$

in $\widetilde{\mathbf{A}}$. *Suppose that* $a_X : X \to \mathrm{pt}$ *satisfies (A3) and* A *is a field. Then for any* $k \in \mathbb{Z}$ *there is a natural isomorphism*

$$H^k_c(X \times Y; A_{X\times Y}) \simeq \bigoplus_{p+q=k} (H^q_c(X; A_X) \otimes H^p_c(Y; A_Y)).$$

Proof We have $A_{X\times Y} \simeq {p_X}^{-1}A_X \simeq {p_Y}^{-1}A_Y$ and by Theorem 3.38 we obtain

$$Ra_{X\times Y\text{¡}}A_{X\times Y} \simeq Ra_{X\text{¡}}A_X \otimes Ra_{Y\text{¡}}A_Y.$$

Therefore, by the purely homological algebra result in [30, Exercise I.24], applied to the exact bifunctor $\bullet \otimes \bullet : \mathrm{Mod}(A) \times \mathrm{Mod}(A) \to \mathrm{Mod}(A)$, we have

$$\begin{aligned} H^k_c(X \times Y; A_{X\times Y}) &= H^k(Ra_{X\times Y\text{¡}}A_{X\times Y}) \\ &\simeq \bigoplus_{p+q=k} H^p(Ra_{X\text{¡}}A_X) \otimes H^q(Ra_{Y\text{¡}}A_Y) \\ &= \bigoplus_{p+q=k} (H^q_c(X; A_X) \otimes H^p_c(Y; A_Y)). \end{aligned}$$

□

4 Poincaré–Verdier Duality

In this section we prove the local and the global Verdier duality, we introduce the A-orientation sheaf and prove the Poincaré and the Alexander duality.

The proofs here follow in a standard way from the results already obtained in the previous section. Compare with the topological case in [30, Chapter III, Sections 3.1 and 3.3].

4.1 Poincaré–Verdier duality

Here we show that the derived proper direct image functor $Rf_{\underset{\cdot}{\wr}}$ extends to $\mathrm{D}(A_X)$ and both $Rf_{\underset{\cdot}{\wr}}$ and its extension have a right adjoint $f^{\underset{\cdot}{\wr}}$. We then deduce the basic properties of the right adjoint $f^{\underset{\cdot}{\wr}}$ and the local and the global Verdier duality.

Let $f : X \to Y$ be a morphism in $\widetilde{\mathbf{A}}$. Let $\mathcal{J}$ be the full additive subcategory of $\mathrm{Mod}(A_X)$ of f-soft sheaves. As a consequence of the results we proved for $f_{\underset{\cdot}{\wr}}$ and for $f_{\underset{\cdot}{\wr}}$-acyclic sheaves we have the following properties:[4]

$$\begin{cases} \mathcal{J} \text{ is cogenerating;} \\ f_{\underset{\cdot}{\wr}} \text{ has finite cohomological dimension;} \\ \mathcal{J} \text{ is } f_{\underset{\cdot}{\wr}}\text{-injective;} \\ \mathcal{J} \text{ is stable under small } \oplus; \\ f_{\underset{\cdot}{\wr}} \text{ commutes with small } \oplus\,. \end{cases}$$

Therefore, by [32, Proposition 14.3.4] the functor $Rf_{\underset{\cdot}{\wr}} : \mathrm{D}^+(A_X) \to \mathrm{D}^+(A_Y)$ extends to a functor

$$Rf_{\underset{\cdot}{\wr}} : \mathrm{D}(A_X) \to \mathrm{D}(A_Y)$$

such that:

(i) for every $F \in \mathrm{D}(A_X)$ we have

$$Rf_{\underset{\cdot}{\wr}}F \simeq f_{\underset{\cdot}{\wr}}F'$$

where F' is a complex of $f_{\underset{\cdot}{\wr}}$-acyclic sheaves quasi-isomorphic to F;

(ii) $Rf_{\underset{\cdot}{\wr}}$ commutes with small $\oplus$.

Theorem 4.1 *Let $f : X \to Y$ be a morphism in $\widetilde{\mathbf{A}}$. Then the functor $Rf_{\underset{\cdot}{\wr}} : \mathrm{D}(A_X) \to \mathrm{D}(A_Y)$ admits a right adjoint*

$$f^{\underset{\cdot}{\wr}} : \mathrm{D}(A_Y) \to \mathrm{D}(A_X).$$

The functor $f^{\underset{\cdot}{\wr}}$ will thus satisfy an isomorphism

$$\mathrm{Hom}_{\mathrm{D}(A_Y)}(Rf_{\underset{\cdot}{\wr}}F; G) \simeq \mathrm{Hom}_{\mathrm{D}(A_X)}(F; f^{\underset{\cdot}{\wr}}G)$$

functorial in $F \in \mathrm{D}(A_X)$ and $G \in \mathrm{D}(A_Y)$. Moreover, the restriction

$$f^{\underset{\cdot}{\wr}} : \mathrm{D}^+(A_Y) \to \mathrm{D}^+(A_X)$$

is well defined and it is the right adjoint to the restriction $Rf_{\underset{\cdot}{\wr}} : \mathrm{D}^+(A_X) \to \mathrm{D}^+(A_Y)$.

Proof The existence of right adjoint $f^{\underset{\cdot}{\wr}} : \mathrm{D}(A_Y) \to \mathrm{D}(A_X)$ is a consequence of the Brown representability theorem (see [32, Corollary 14.3.7] for details).

For the second part we have to show that if $G \in \mathrm{D}^+(A_Y)$, then $f^{\underset{\cdot}{\wr}}G \in \mathrm{D}^+(A_X)$. We may assume that $G \in \mathrm{D}^{\geq 0}(A_Y)$. Let N_0 be the dimension of X. Then the cohomological dimension

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[4] As usual in category theory [32], to avoid set theoretic issues, we assumed throughout the paper that we are working in a (very big) fixed universe, so below by small sum we mean a sum of a family indexed by a set in this universe.

of $f_{\underset{\cdot}{\wr}}$ is bounded by N_0 (Theorem 3.39). Set $a = -N_0 - 1$ for short. If $F \in \mathrm{D}^{\leq a}(A_X)$, then $Rf_{\underset{\cdot}{\wr}}F \in \mathrm{D}^{\leq -1}(A_Y)$ and

$$0 = \mathrm{Hom}_{\mathrm{D}(A_Y)}(Rf_{\underset{\cdot}{\wr}}F, G) \simeq \mathrm{Hom}_{\mathrm{D}(A_X)}(F, f^{\underset{\cdot}{\wr}}G).$$

Hence for each $F \in \mathrm{D}^{\leq a}(A_X)$ we have $\mathrm{Hom}_{\mathrm{D}(A_X)}(F, f^{\underset{\cdot}{\wr}}G) = 0$. In particular, if $F = \tau^{\leq a} f^{\underset{\cdot}{\wr}}G$, then

$$\mathrm{Hom}_{\mathrm{D}(A_X)}(\tau^{\leq a} f^{\underset{\cdot}{\wr}}G, f^{\underset{\cdot}{\wr}}G) \simeq \mathrm{Hom}_{\mathrm{D}^{\leq a}(A_X)}(\tau^{\leq a} f^{\underset{\cdot}{\wr}}G, \tau^{\leq a} f^{\underset{\cdot}{\wr}}G) = 0$$

and so $\tau^{\leq a} f^{\underset{\cdot}{\wr}}G = 0$. This implies, by definition, that $f^{\underset{\cdot}{\wr}}G \in \mathrm{D}^{+}(A_X)$. □

The bound of the f-soft dimension of $\mathrm{Mod}(A_X)$ implies the following result:

Proposition 4.2 *Let $f : X \to Y$ be a morphism in $\widetilde{\mathbf{A}}$. Let $F \in \mathrm{Mod}(A_X)$, $G \in \mathrm{Mod}(A_Y)$. Let N_0 be the dimension of X. Then*

$$\mathrm{Hom}(F, H^{-N_0} f^{\underset{\cdot}{\wr}}G) \simeq \mathrm{Hom}(R^{N_0} f_{\underset{\cdot}{\wr}}F, G).$$

Proof Let $I^\bullet$ be a complex of injective objects quasi-isomorphic to $f^{\underset{\cdot}{\wr}}G$. As in the proof of Theorem 4.1 we have $f^{\underset{\cdot}{\wr}}G \simeq \tau^{\geq -N_0} I^\bullet$ and hence we have the exact sequence

$$0 \to H^{-N_0} f^{\underset{\cdot}{\wr}}G \to I^{-N_0} \to I^{-N_0+1}$$

that implies the isomorphism

$$\mathrm{Hom}(F, H^{-N_0} f^{\underset{\cdot}{\wr}}G) \simeq \mathrm{Hom}_{\mathrm{D}^{+}(A_X)}(F, f^{\underset{\cdot}{\wr}}G[-N_0]).$$

On the other hand the complex $Rf_{\underset{\cdot}{\wr}}F[N_0]$ is concentrated in negative degree and hence

$$\mathrm{Hom}_{\mathrm{D}^{+}(A_Y)}(Rf_{\underset{\cdot}{\wr}}F[N_0], G) \simeq \mathrm{Hom}(R^0 f_{\underset{\cdot}{\wr}}F[N_0], G) \simeq \mathrm{Hom}(R^{N_0} f_{\underset{\cdot}{\wr}}F, G).$$

Then the result follows from Theorem 4.1. □

If a morphism $f : X \to Y$ in $\widetilde{\mathbf{A}}$ is a homeomorphism onto a locally closed subset of Y, then $f_{\underset{\cdot}{\wr}}$ is (isomorphic to) the extension by zero functor. What about $f^{\underset{\cdot}{\wr}}$?

Proposition 4.3 *Let $f : X \to Y$ be a morphism in $\widetilde{\mathbf{A}}$. If $f : X \to Y$ is a homeomorphism onto a locally closed subset of Y, then*

$$f^{\underset{\cdot}{\wr}}G \simeq f^{-1} \circ R\mathcal{H}om(A_{f(X)}, G) \simeq f^{-1} \circ R\Gamma_{f(X)}(G)$$

for every $G \in \mathrm{D}^{+}(A_Y)$. In particular:

- *if $f : X \to Y$ is a closed immersion, then* $\mathrm{id} \simeq f^{\underset{\cdot}{\wr}} \circ Rf_{\underset{\cdot}{\wr}}$*;*
- *if $f : X \to Y$ is an open immersion, then $f^{\underset{\cdot}{\wr}} \simeq f^{-1}$.*

Proof Let $G \in \mathrm{D}^{+}(A_Y)$ and $F \in \mathrm{D}^{+}(A_X)$. Then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}^{+}(A_Y)}(f_{\underset{\cdot}{\wr}}F, G) &\simeq \mathrm{Hom}_{\mathrm{D}^{+}(A_Y)}(f_{\underset{\cdot}{\wr}}F, R\mathcal{H}om(A_{f(X)}, G)) \\ &\simeq \mathrm{Hom}_{\mathrm{D}^{+}(A_X)}(f^{-1} \circ f_{\underset{\cdot}{\wr}}F, f^{-1} \circ R\mathcal{H}om(A_{f(X)}, G)) \\ &\simeq \mathrm{Hom}_{\mathrm{D}^{+}(A_X)}(F, f^{-1} \circ R\mathcal{H}om(A_{f(X)}, G)).\end{aligned}$$

Hence $f^{\underset{\cdot}{\wr}}G \simeq f^{-1} \circ R\mathcal{H}om(A_{f(X)}, G)$.

Suppose now that $f : X \to Y$ is a closed immersion. Then f is proper, $Rf_* \simeq Rf_{\underset{\cdot}{\wr}}$ and $f^{-1} \circ Rf_* \simeq \mathrm{id}$. On the other hand, we have the isomorphisms

$$\begin{aligned} f^{\underset{\cdot}{\wr}} \circ Rf_*F &\simeq f^{-1} \circ R\mathcal{H}om(A_{f(X)}, Rf_*F) \\ &\simeq f^{-1} \circ Rf_*R\mathcal{H}om(A_{f(X)}, F) \\ &\simeq f^{-1} \circ Rf_*F. \end{aligned}$$

Hence, $\mathrm{id} \simeq f^{\underset{\cdot}{\wr}} \circ Rf_{\underset{\cdot}{\wr}}$.

Suppose now that $f : X \to Y$ is an open immersion. Then $f^{-1} \circ Rf_* \simeq \mathrm{id}$ and $R\Gamma_{f(X)} \simeq Rf_* \circ f^{-1}$. Hence, $f^{\underset{\cdot}{\wr}} \simeq f^{-1}$. □

We now prove several useful properties of the dual $f^{\underset{\cdot}{\wr}}$ of the derived proper direct image functor $Rf_{\underset{\cdot}{\wr}}$.

Proposition 4.4 *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\widetilde{\mathbf{A}}$. Then $(g \circ f)^{\underset{\cdot}{\wr}} \simeq f^{\underset{\cdot}{\wr}} \circ g^{\underset{\cdot}{\wr}}$.*

Proof This follows from $R(g \circ f)_{\underset{\cdot}{\wr}} \simeq Rg_{\underset{\cdot}{\wr}} \circ Rf_{\underset{\cdot}{\wr}}$ and the adjunction in Theorem 4.1. □

Proposition 4.5 (Dual projection formula) *Let $f : X \to Y$ be a morphism in $\widetilde{\mathbf{A}}$. Let $F \in \mathrm{D}^b(A_Y)$ and $G \in \mathrm{D}^+(A_Y)$. Suppose that $\mathrm{wgld}(A) < \infty$. Then we have a natural isomorphism*

$$f^{\underset{\cdot}{\wr}} \circ R\mathcal{H}om(F, G) \simeq R\mathcal{H}om(f^{-1}F, f^{\underset{\cdot}{\wr}}G).$$

Proof This follows from the derived projection formula (Theorem 3.35), the adjunction in Theorem 4.1 and the adjunction

$$\mathrm{Hom}_{\mathrm{D}^+(A_Z)}(F \otimes H, G) \simeq \mathrm{Hom}_{\mathrm{D}^+(A_Z)}(H, R\mathcal{H}om(F, G)).$$

□

Proposition 4.6 (Dual base change formula) *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\widetilde{\mathrm{Def}}$. Suppose that $f : X \to Y$ and $f' : X' \to Y'$ are in $\widetilde{\mathbf{A}}$ and that $f : X \to Y$ satisfies (A3). Then there is an isomorphism in $\mathrm{D}^+(A_X)$, functorial in $F \in \mathrm{D}^+(A_{Y'})$:

$$f^{\underset{\cdot}{\wr}} \circ Rg_*F \simeq Rg'_* \circ f'^{\underset{\cdot}{\wr}}F.$$

Proof This follows from the derived base change formula (Theorem 3.37), the adjunction in Theorem 4.1 and the adjunction

$$\mathrm{Hom}_{\mathrm{D}^+(A_Z)}(H, Rh_*G) \simeq \mathrm{Hom}_{\mathrm{D}^+(A_X)}(h^{-1}H, G)$$

for every morphism $h : X \to Z$ in $\widetilde{\mathbf{A}}$. □

Theorem 4.7 (Local and global Verdier duality) *Let* $f : X \to Y$ *be a morphism in* $\widetilde{\mathbf{A}}$. *Then for* $F \in \mathrm{D}^b(A_X)$ *and* $G \in \mathrm{D}^+(A_Y)$, *we have the local Verdier duality*

$$Rf_* \circ R\mathcal{H}om(F, f^{\overset{\wr}{\cdot}}G) \simeq R\mathcal{H}om(Rf_{\overset{\wr}{\cdot}}F, G)$$

and the global Verdier duality

$$R\mathrm{Hom}(F, f^{\overset{\wr}{\cdot}}G) \simeq \mathrm{Hom}(Rf_{\overset{\wr}{\cdot}}F, G).$$

Proof We obtain the morphism

$$Rf_* \circ R\mathcal{H}om(F, f^{\overset{\wr}{\cdot}}G) \to R\mathcal{H}om(Rf_{\overset{\wr}{\cdot}}F, G)$$

by composing the canonical morphism

$$Rf_* \circ R\mathcal{H}om(F, f^{\overset{\wr}{\cdot}}G) \to R\mathcal{H}om(Rf_{\overset{\wr}{\cdot}}F, Rf_{\overset{\wr}{\cdot}} \circ f^{\overset{\wr}{\cdot}}G)$$

with the morphism $Rf_{\overset{\wr}{\cdot}} \circ f^{\overset{\wr}{\cdot}}G \to G$ obtained by adjunction (Theorem 4.1).

Let $V \in \mathrm{Op}(Y)$. Then we have

$$\begin{aligned} H^j(R\Gamma(V; Rf_* \circ R\mathcal{H}om(F, f^{\overset{\wr}{\cdot}}G))) &\simeq \mathrm{Hom}_{\mathrm{D}^+(A_{f^{-1}(V)})}(F_{|f^{-1}(V)}, f^{\overset{\wr}{\cdot}}G[j]_{|f^{-1}(V)}) \\ &\simeq \mathrm{Hom}_{\mathrm{D}^+(A_V)}(Rf_{\overset{\wr}{\cdot}}F_{|V}, G[j]_{|V}) \\ &\simeq H^j(R\Gamma(V; R\mathcal{H}om(Rf_{\overset{\wr}{\cdot}}F, G))) \end{aligned}$$

completing the proof of the first isomorphism. The second isomorphism is obtained from the first one by applying the functor $R\Gamma(Y; \bullet)$. ☐

4.2 Orientation and duality

Here we introduce the A-orientation sheaf and prove the Poincaré and the Alexander duality theorems.

As before, as the reader can easily verify, in all of our previous results for the proper direct image $f_{\overset{\wr}{\cdot}}$ of a morphism $f : X \to Y$ in $\widetilde{\mathbf{A}}$ the assumptions (A0), (A1) and (A2) were used only to show:

(i) if $\alpha \in Y$ is closed, then c is a normal and constructible family of supports on $f^{-1}(\alpha)$;
(ii) fiber formula;
(iii) the theory of f-soft sheaves.

For the morphism $a_X : X \to \mathrm{pt}$ we have that (i) holds if we assume that c is a normal and constructible family of supports on X, (ii) is Remark 3.3 and (iii) is the theory of c-soft sheaves developed already in [18, Section 3].

Let X be an object of $\widetilde{\mathrm{Def}}$ such that c is a normal and constructible family of supports on X. Then the functor $Ra_{X\overset{\wr}{\cdot}} : \mathrm{D}^+(A_X) \to \mathrm{D}^+(\mathrm{Mod}(A))$ admits a right adjoint

$$a_X^{\overset{\wr}{\cdot}} : \mathrm{D}(\mathrm{Mod}(A)) \to \mathrm{D}(A_X)$$

(Theorem 4.1) and we have the dual projection formula (Proposition 4.5), the local and the global Verdier duality (Theorem 4.7) for $a_{X\overset{\wr}{\cdot}}$. In particular, we obtain the form of the global Verdier duality proved already in [18] (assuming A is a field), where $a_X^{\overset{\wr}{\cdot}}A_X$ is the *dualizing complex*:

Theorem 4.8 (Absolute Poincaré duality) *Let X be an object of $\widetilde{\mathrm{Def}}$ such that c is a normal and constructible family of supports on X. Then we have a natural isomorphism*

$$R\mathrm{Hom}(F, a_X^! A) \simeq R\mathrm{Hom}(R\Gamma_c(X; F), A)$$

as F varies through $\mathrm{D}^b(A_X)$.

Let X be an object of $\widetilde{\mathrm{Def}}$ such that c is a normal and constructible family of supports on X. We want to define the notion of orientation on X. We follow the definition for topological manifolds of dimension n [28, page 194], however we have to impose the following condition which in the topological case is true. We say that X *has an A-orientation sheaf* if for every $U \in \mathrm{Op}^{\mathrm{cons}}(X)$ there exists an admissible (finite) cover $\{U_1, \ldots, U_\ell\}$ of U such that for each i we have

$$H_c^p(U_i; A_X) = \begin{cases} A & \text{if} \quad p = \dim X \\ \\ 0 & \text{if} \quad p \neq \dim X. \end{cases} \tag{1}$$

Theorem 4.9 *Let X be an object of $\widetilde{\mathrm{Def}}$ of dimension n such that c is a normal and constructible family of supports on X. Suppose that X has an A-orientation sheaf. Then the A-pre-sheaf $\mathcal{O}r_X$ on X with sections*

$$\Gamma(U; \mathcal{O}r_X) \simeq \mathrm{Hom}(H_c^n(U; A_X), A)$$

is an A-sheaf, called the A-orientation sheaf on X, such that $\mathcal{O}r_X \simeq H^{-n} a_X^! A$. Moreover $\mathcal{O}r_X$ is locally constant and there is a quasi-isomorphism

$$\mathcal{O}r_X[n] \simeq a_X^! A.$$

Proof Setting $f = a_X$, $F = A_U$, $G = A$ in Proposition 4.2 we obtain

$$\Gamma(U; H^{-n} a_X^! A) \simeq \mathrm{Hom}(H_c^n(X; A_U), A) \simeq \mathrm{Hom}(H_c^n(U; A_X), A),$$

where the second isomorphism follows from [18, Corollary 3.9].

On the other hand, (1), implies $R\Gamma(U_i; a_X^! A) \simeq \mathrm{RHom}(R\Gamma_c(U_i; A_X), A) \simeq A[-n]$, i.e. $a_X^! A$ is concentrated in degree $-n$ and the sheaf $H^{-n} a_X^! A$ is locally isomorphic to A_X. □

If $A = \mathbb{Z}$ we call $\mathcal{O}r_X$ the *orientation sheaf on X*.

In particular we recover [18, Theorem 4.11]:

Remark 4.10 *(Poincaré duality in cohomology)* When A is a field, setting $(\bullet)^\vee = \mathrm{Hom}(\bullet, A)$ we obtain:

$$H^p(X; \mathcal{O}r_X) \simeq H_c^{n-p}(X; A_X)^\vee.$$

Using the pure homological algebra result [28, Proposition VI.4.6] we also have:

Corollary 4.11 *Let X be an object of $\widetilde{\mathrm{Def}}$ of dimension n such that c is a normal and constructible family of supports on X. Suppose that X has an orientation sheaf $\mathcal{O}r_X$. Then there is a short exact sequence of abelian groups:*

$$0 \to \mathrm{Ext}^1(H_c^{k+1}(X; \mathbb{Z}_X), \mathbb{Z}) \to H^{n-k}(X; \mathcal{O}r_X) \to \mathrm{Hom}(H_c^k(X, \mathbb{Z}_X), \mathbb{Z}) \to 0.$$

In particular $H^{n-k}(X; \mathcal{O}r_X) \simeq \mathrm{Hom}(H_c^k(X, \mathbb{Z}_X), \mathbb{Z})$ when $H_c^{k+1}(X, \mathbb{Z}_X)$ has no torsion.

Proof By the pure homological algebra result [28, Proposition VI.4.6] we have

$$0 \to \mathrm{Ext}^1(H^{k+1}C^\bullet, \mathbb{Z}) \to H^{-k}\mathrm{RHom}(C^\bullet, \mathbb{Z}) \to \mathrm{Hom}(H^k C^\bullet, \mathbb{Z}) \to 0$$

for any bounded complex $C^\bullet$ of abelian groups. Applying this to $C^\bullet = R\Gamma_c(X; \mathbb{Z}_X)$ and using $R\mathrm{Hom}(R\Gamma_c(X; \mathbb{Z}_X), \mathbb{Z}) \simeq R\mathrm{Hom}(\mathbb{Z}_X, a_X^{\wr}\mathbb{Z}) \simeq R\Gamma(X; \mathcal{O}r_X[n])$ (by Theorems 4.8 and 4.9) the result follows. ◻

Corollary 4.12 *Let X be an object of $\widetilde{\mathrm{Def}}$ of dimension n such that c is a normal and constructible family of supports on X. Suppose that X has an orientation sheaf $\mathcal{O}r_X$. Then there exists an isomorphism*

$$H^n(X; \mathcal{O}r_X) \simeq \mathrm{Hom}(H^0_c(X; \mathbb{Z}_X), \mathbb{Z}) \simeq \mathbb{Z}^l$$

where l is the number of complete connected components of X.

Proof By Corollary 4.11 (with $k = 0$) and since $H^0_c(X; \mathbb{Z}_X) = \mathbb{Z}^l$ where l is the number of complete connected components of X, the result follows once we show that $H^1_c(X; \mathbb{Z}_X)$ is torsion free. But this is [7, Chapter I, Exercise 11 and Chapter II, Exercise 28]. ◻

By an *A-orientation* we understand an isomorphism $A_X \simeq \mathcal{O}r_X$. We shall say that X is *A-orientable* if an A-orientation exists and *A-unorientable* in the opposite case. For $A = \mathbb{Z}$ we simply say *orientation*, *orientable* or *unorientable*.

From [18, Theorem 3.12] (dim X is a bound on the cohomological c-dimension of X) and Corollary 4.11, arguing as in [18, Proposition 4.13] we have:

Proposition 4.13 *Let X be an object of $\widetilde{\mathrm{Def}}$ of dimension n such that c is a normal and constructible family of supports on X. Suppose that X has an orientation sheaf $\mathcal{O}r_X$. Then*

(1) *$H^n_c(X; \mathbb{Z}_X) \simeq \mathbb{Z}$ if X is orientable.*
(2) *$H^n_c(X; \mathbb{Z}_X) \simeq 0$ if X is unorientable.*

If is Z a closed constructible subset of X, then setting $F = A_Z$ in Theorem 4.8 we obtain:

Theorem 4.14 (Alexander duality) *Let X be an object of $\widetilde{\mathrm{Def}}$ of dimension n such that c is a normal and constructible family of supports on X. Suppose that X is A-orientable. If Z a closed constructible subset of X, then there exists a quasi-isomorphism*

$$R\Gamma_Z(X; A_X) \simeq R\mathrm{Hom}(R\Gamma_c(Z; A_X), A)[n].$$

In particular we recover [18, Theorem 4.14]:

Remark 4.15 *(Alexander duality in cohomology)* When A is a field, setting $(\bullet)^\vee = \mathrm{Hom}(\bullet, A)$ we obtain:

$$H^p_Z(X; A_X) \simeq H^{n-p}_c(Z; A_X)^\vee.$$

Using the pure homological algebra result [28, Proposition VI.4.6] we also have:

Corollary 4.16 *Let X be an object of $\widetilde{\mathrm{Def}}$ of dimension n such that c is a normal and constructible family of supports on X. Suppose that X is orientatable. Then there is a short exact sequence of abelian groups:*

$$0 \to \mathrm{Ext}^1(H^{k+1}_c(Z; \mathbb{Z}_X), \mathbb{Z}) \to H^{n-k}_Z(X; \mathbb{Z}_X) \to \mathrm{Hom}(H^k_c(Z, \mathbb{Z}_X), \mathbb{Z}) \to 0.$$

In particular $H^{n-k}_Z(X; \mathbb{Z}_X) \simeq \mathrm{Hom}(H^k_c(Z; \mathbb{Z}_X), \mathbb{Z})$ when $H^{k+1}_c(Z; \mathbb{Z}_X)$ has no torsion.

Proof By the pure homological algebra result [28, Proposition VI.4.6] we have

$$0 \to \mathrm{Ext}^1(H^{k+1}C^\bullet, \mathbb{Z}) \to H^{-k}\mathrm{RHom}(C^\bullet, \mathbb{Z}) \to \mathrm{Hom}(H^k C^\bullet, \mathbb{Z}) \to 0$$

for any bounded complex $C^\bullet$ of abelian groups. Applying this to $C^\bullet = R\Gamma_c(Z; \mathbb{Z}_X)$ and using $R\mathrm{Hom}(R\Gamma_c(Z; \mathbb{Z}_X), \mathbb{Z})[n] \simeq R\Gamma_Z(X; \mathbb{Z}_X)$ (by Theorem 4.14) the result follows. □

Corollary 4.17 *Let X be an object of $\widetilde{\mathrm{Def}}$ of dimension n such that c is a normal and constructible family of supports on X. Suppose that X is orientatable. Then there exists an isomorphism*

$$H^n_Z(X; \mathbb{Z}_X) \simeq \mathrm{Hom}(H^0_c(Z; \mathbb{Z}_X), \mathbb{Z}) \simeq \mathbb{Z}^l$$

induced by the given orientation, where l is the number of complete connected components of Z.

Proof By Corollary 4.16 (with $k = 0$) and since $H^0_c(Z; \mathbb{Z}_X) = \mathbb{Z}^l$ where l is the number of complete connected components of Z, the result follows once we show that $H^1_c(Z; \mathbb{Z}_X)$ is torsion free. But this is [7, Chapter I, Exercise 11 and Chapter II, Exercise 28]. □

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