

# Relaxation Functions in Thermo-electro-viscoelasticity

Adriano Montanaro<sup>1,a)</sup>

<sup>1</sup>*Dep. of Mathematics ‘Tullio Levi-Civita’, University of Padua, Italy.*

<sup>a)</sup>adriano.montanaro@unipd.it

**Abstract.** In paper [2] the theory [1] for a thermo-electro-mechanical simple body  $\mathcal{B}$  with fading memory is used to set up a thermo-electro viscoelastic theory, which is obtained by a linearization procedure and the Riesz representation theorem. This is done within the Green-Naghdy scheme of thermodynamics. Hence several restrictions on the various relaxation functions are found that extend the ones in [3] for finitely thermoviscoelastic materials. The restrictions are obtained from an internal dissipation inequality that is a consequence of a dissipation inequality.

## INTRODUCTION

Green-Naghdy thermodynamics [4] is based on the notion of *thermal displacement*  $\alpha = \alpha(\mathbf{X}, t)$  at the material point  $\mathbf{X}$  and time  $t$ . Then

$$T = \dot{\alpha}, \quad \beta = \nabla_{\mathbf{X}} \alpha, \quad \gamma = \nabla_{\mathbf{x}} T \quad (1)$$

are the *empirical temperature* (‘thermal displacement rate’), *thermal displacement gradient* and *empirical-temperature gradient*, respectively. Note that we have

$$\dot{\beta} = \nabla_{\mathbf{X}} \dot{\alpha} = \frac{\partial \dot{\alpha}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{F}^T \gamma. \quad (2)$$

The invariance of the free-energy  $\Psi$  in a rigid rotation is assured when  $\Psi$  is an arbitrary functional of the referential quantities

$$\Phi := (T, \beta, \mathbf{E}, \mathbf{W}), \quad \dot{\beta}, \quad \Phi^t := (T^t, \beta^t, \mathbf{E}^t, \mathbf{W}^t), \quad (3)$$

where eq. (2) holds,

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I}) \quad (4)$$

is the Green-Lagrange strain tensor, and

$$\mathbf{W} = -\frac{\partial \phi}{\partial \mathbf{X}} = -\frac{\partial \phi}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{F}^T \mathbf{E}^M. \quad (5)$$

is the Lagrangian counterpart of the Maxwellian electric vector  $\mathbf{E}^M$ . Hence we assume a constitutive equation of the form

$$\Psi = \tilde{\Psi}(\Phi, \dot{\beta}, \Phi^t) \quad (6)$$

that of course is invariant under rigid rotations of the deformed and polarized body.

## Linear thermo-electro-viscoelasticity

In order to define a viscoelastic material we extend here the assumptions in [3, p.213] and so we assume that  $\delta^2 \tilde{\Psi}$  is independent of  $\Phi$  and is a completely continuous bilinear functional of its remaining arguments. By the Riesz

representation theorem, we have the following incremental expression for the free-energy functional

$$\Psi = \tilde{\Psi}(\Phi, \Phi^t(s)) = \tilde{\Sigma}(\Phi, \dot{\beta}) + \int_0^{+\infty} \mathbf{z}(\Phi; s) \cdot_h [\Phi^t(s) - \Phi] ds + \quad (7)$$

$$\begin{aligned} & + \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} [\Phi^t(s) - \Phi] \cdot_h \frac{\partial^2 \mathbf{M}(s, u)}{\partial s \partial u} [\Phi^t(s) - \Phi] ds du + \\ & + \int_0^{+\infty} \int_0^{+\infty} [T^t(s) - T] \frac{\partial^2 \mathbf{B}_1(s, u)}{\partial s \partial u} \cdot [\beta^t(u) - \beta] ds du + \int_0^{+\infty} \int_0^{+\infty} [T^t(s) - T] \frac{\partial^2 \mathbf{B}_2(s, u)}{\partial s \partial u} \cdot [\mathbf{E}^t(u) - \mathbf{E}] ds du + \\ & + \int_0^{+\infty} \int_0^{+\infty} [T^t(s) - T] \frac{\partial^2 \mathbf{B}_3(s, u)}{\partial s \partial u} \cdot [\mathbf{W}^t(u) - \mathbf{W}] ds du + \int_0^{+\infty} \int_0^{+\infty} [\beta^t(s) - \beta] \cdot \frac{\partial^2 \mathbf{M}_1(s, u)}{\partial s \partial u} [\mathbf{E}^t(u) - \mathbf{E}] ds du + \quad (8) \\ & + \int_0^{+\infty} \int_0^{+\infty} [\beta^t(s) - \beta] \cdot \frac{\partial^2 \mathbf{M}_2(s, u)}{\partial s \partial u} [\mathbf{W}^t(u) - \mathbf{W}] ds du + \int_0^{+\infty} \int_0^{+\infty} [\mathbf{W}^t(s) - \mathbf{W}] \cdot \frac{\partial^2 \mathbf{M}_3(s, u)}{\partial s \partial u} [\mathbf{E}^t(u) - \mathbf{E}] ds du, \end{aligned}$$

where the second integral on the right of the above equality componentwise writes as

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \dot{\Phi}^t(s) \cdot_h \frac{\partial^2 \mathbf{M}(s, u)}{\partial s} \dot{\Phi}^t(u) ds du = \quad (9) \\ & = \int_0^{+\infty} \int_0^{+\infty} [T^t(s) - T] \frac{\partial^2 m_1(s, u)}{\partial s} [T^t(u) - T] ds du + \int_0^{+\infty} \int_0^{+\infty} [\beta^t(s) - \beta] \cdot \frac{\partial^2 \mathbf{m}_2(s, u)}{\partial s} [\beta^t(u) - \beta] ds du + \\ & + \int_0^{+\infty} \int_0^{+\infty} [\mathbf{E}^t(s) - \mathbf{E}] \cdot \frac{\partial^2 \mathbf{m}_3(s, u)}{\partial s} [\mathbf{E}^t(u) - \mathbf{E}] ds du + \int_0^{+\infty} \int_0^{+\infty} [\mathbf{W}^t(s) - \mathbf{W}] \cdot \frac{\partial^2 \mathbf{m}_4(s, u)}{\partial s} [\mathbf{W}^t(u) - \mathbf{W}] ds du. \end{aligned}$$

In the above constitutive relation  $\Phi = \Phi^t(0)$ ,  $\beta = \beta^t(0)$ , etc.;  $\tilde{\Sigma}(\Phi, \dot{\beta})$  is an arbitrary function of  $\Phi$ ,  $\dot{\beta}$ , and the material relaxation functions  $\mathbf{z}(\Phi; s)$ ,  $\mathbf{M}(s, u)$ ,  $\mathbf{B}_i(s, u)$ ,  $\mathbf{M}_i(s, u)$  fulfill the following conditions, where  $\text{Lin}\mathcal{V}$  denotes the linear space of endomorphisms on any given linear space  $\mathcal{V}$ :

$\mathbf{z}(\Phi; s)$  is  $\mathcal{V}$ -valued,  $\mathcal{V} = \mathbb{R} \times \mathbb{R}^3 \times \text{Lin} \times \mathbb{R}^3$ ;

$\mathbf{M}(s, u)$  is  $\text{Lin}\mathcal{H}$ -valued, where  $\mathcal{H}$  is a suitable Hilbert space of past histories, whence  $\mathbf{M} = (m_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4) \in (\text{Lin}\mathbb{R} \times \text{Lin}\mathbb{R}^3 \times \text{Lin}(\text{Lin}) \times \text{Lin}\mathbb{R}^3)$

$\mathbf{B}_2(s, u)$  is symmetric tensor-valued;

$\mathbf{B}_1(s, u)$  and  $\mathbf{B}_3(s, u)$  are  $\mathbb{R}^3$ -valued;

$\mathbf{M}_2(s, u)$  is second-order tensor-valued;

$\mathbf{M}_1(s, u)$  and  $\mathbf{M}_3(s, u)$  are third-order tensor-valued.

From the Riesz representation theorem we also have that

$\mathbf{z}(\Phi; s)$ ,  $\mathbf{M}(s, u)$ ,  $\mathbf{B}_i(s, u)$ ,  $\mathbf{M}_i(s, u) \rightarrow 0$  as either  $s$  or  $u \rightarrow 0$ .

## Restrictions on the material relaxation functions

The following restrictions on the material relaxation functions are deduced by extending the procedure in [3]. Remind that for a second- or fourth-order tensor  $\mathbf{A}$  by  $\mathbf{A} \leq \mathbf{0}$  ( $\mathbf{A} \geq \mathbf{0}$ ) we mean  $\mathbf{A}$  is negative semi-definite (positive semi-definite). Accordingly, by  $\mathbf{A} \leq \mathbf{B}$  we mean  $\mathbf{A} - \mathbf{B}$  is negative semi-definite.

Restrictions on each  $m \in \{m_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4\}$ :

$$\frac{\partial m}{\partial u}(s, s) \leq 0 \quad \forall s, \quad (10)$$

$$\frac{\partial m(s, s)}{\partial u} \pm 2 \frac{\partial m(s, u)}{\partial u} + \frac{\partial m(u, u)}{\partial u} \leq \mathbf{0} \quad \forall s, u. \quad (11)$$

Restrictions on  $\mathbf{B}_1$ :

$$\left| \frac{\partial B_{1ij}}{\partial s}(s, u) + \frac{\partial B_{1ij}}{\partial u}(s, u) \right| \leq 4 \frac{\partial m_{2ij}}{\partial u}(s, s) \frac{\partial m}{\partial u}(u, u) \quad (i, j \text{ not summed}) \quad \forall s, u. \quad (12)$$

Restrictions on  $\mathbf{B}_2$ :

$$\left| \frac{\partial B_{2ij}}{\partial s}(s, u) + \frac{\partial B_{2ij}}{\partial u}(s, u) \right| \leq 4 \frac{\partial m_{3ijij}}{\partial u}(s, s) \frac{\partial m}{\partial u}(u, u) \quad (i, j \text{ not summed}) \quad \forall s, u. \quad (13)$$

Restrictions on  $\mathbf{B}_3$ :

$$\left| \frac{\partial B_{3pq}}{\partial s}(s, u) + \frac{\partial B_{3pq}}{\partial u}(s, u) \right| \leq 4 \frac{\partial m_{4pq}}{\partial u}(s, s) \frac{\partial m}{\partial u}(u, u) \quad (p, q \text{ not summed}) \quad \forall s, u. \quad (14)$$

Restrictions on  $\mathbf{M}_1$ :

$$\left| \frac{\partial M_1^{pqr}}{\partial s}(s, u) + \frac{\partial M_1^{pqr}}{\partial u}(s, u) \right|^2 \leq 4 \frac{\partial m_3^{pqrs}}{\partial u}(s, s) \frac{\partial m_2^{pp}}{\partial u}(u, u) \quad (p, q, r \text{ not summed}) \quad \forall s, u. \quad (15)$$

Restrictions on  $\mathbf{M}_2$ :

$$\left| \frac{\partial M_2^{pq}}{\partial s}(s, u) + \frac{\partial M_2^{pq}}{\partial u}(s, u) \right|^2 \leq 4 \frac{\partial m_4^{pq}}{\partial u}(s, s) \frac{\partial m_2^{pp}}{\partial u}(u, u) \quad (p, q \text{ not summed}) \quad \forall s, u. \quad (16)$$

Restrictions on  $\mathbf{M}_3$ :

$$\left| \frac{\partial M_3^{pqr}}{\partial s}(s, u) + \frac{\partial M_3^{pqr}}{\partial u}(s, u) \right|^2 \leq 4 \frac{\partial m_3^{pqrs}}{\partial u}(s, s) \frac{\partial m_4^{pp}}{\partial u}(u, u) \quad (p, q, r \text{ not summed}) \quad \forall s, u. \quad (17)$$

### Restrictions from the minimality of the free energy in equilibrium

There are further restrictions on the relaxation functions that can be deduced from the minimality of the free energy in equilibrium. For  $i = 1, \dots, 4$  they write as

$$\begin{aligned} \mathbf{m}_i(s, s) &\geq \mathbf{0} \quad \forall s, u, \\ \mathbf{m}_i(s, s) + \mathbf{m}_i(u, u) \pm 2\mathbf{m}_i(s, u) &\geq \mathbf{0} \quad \forall s, u, \\ \mathbf{m}_i(s, s) &\geq \mathbf{m}_i(u, u) \quad \forall s \leq u, \end{aligned}$$

and

$$\mathbf{m}_i(\hat{s}, \hat{s}) \geq \pm \mathbf{m}_i(s, u) \quad \forall s, u$$

where  $\hat{s} = \min(s, u)$ .

Then by a limit procedure on letting  $\epsilon \rightarrow 0$  from all the above restrictions on the relaxation functions we obtain:  
for  $h = 2, 3, 4$

$$\begin{aligned} \mathbf{m}_h(0, 0) &\text{ is symmetric,} \\ \mathbf{m}_h(0, 0) &\geq \mathbf{0}, \\ \frac{\partial \mathbf{m}_h}{\partial u}(0, 0) &\leq \mathbf{0}, \\ \mathbf{m}_h(0, 0) &\geq \pm \mathbf{m}_h(0, u). \end{aligned}$$

Moreover,

$$\mathbf{B}_2(0, 0) \text{ is symmetric,} \quad (18)$$

and for  $i = 1, 2, 3$ ,

$$\begin{aligned} \mathbf{B}_i(0, 0) &\geq \mathbf{0}, \\ \frac{\partial \mathbf{B}_i}{\partial u}(0, 0) &\leq \mathbf{0}, \\ \mathbf{B}_i(0, 0) &\geq \pm \mathbf{B}_i(0, u). \end{aligned}$$

## Conclusions

We have used the theory in [1] for thermo-electro-mechanical simple materials with fading memory to set up a thermo-electro viscoelastic theory, which is obtained by a linearization procedure with the Riesz representation theorem. The usefulness of this future work lies in its possible applications in material theories, e.g. for modeling bone tissues.

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