Non-compact subsets of the Zariski space of an integral domain
NON-COMPACT SUBSETS OF THE ZARISKI SPACE OF AN INTEGRAL DOMAIN

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Abstract. Let $V$ be a minimal valuation overring of an integral domain $D$ and let $\text{Zar}(D)$ be the Zariski space of the valuation overrings of $D$. Starting from a result in the theory of semistar operations, we prove a criterion under which the set $\text{Zar}(D) \setminus \{V\}$ is not compact. We then use it to prove that, in many cases, $\text{Zar}(D)$ is not a Noetherian space, and apply it to the study of the spaces of Kronecker function rings and of Noetherian overrings.

1. Introduction

The Zariski space $\text{Zar}(K|D)$ of the valuation rings of a field $K$ containing a domain $D$ was introduced (under the name abstract Riemann surface) by O. Zariski, who used it to show that resolution of singularities holds for varieties of dimension 2 or 3 over fields of characteristic 0 [32, 33]. In particular, Zariski showed that $\text{Zar}(K|D)$, endowed with a natural topology, is always a compact space [34, Chapter VI, Theorem 40]; this result has been subsequently improved by showing that $\text{Zar}(K|D)$ is a spectral space (in the sense of Hochster [18]), first in the case where $K$ is the quotient field of $D$ [4, 5], and then in the general case [8, Corollary 3.6(3)]. The topological aspects of the Zariski space has subsequently been used, for example, in real and rigid algebraic geometry [19, 31] and in the study of representation of integral domains as intersections of valuation overrings [26, 27, 28]. In the latter context, i.e., when $K$ is the quotient field of $D$, two important properties for subspaces of $\text{Zar}(K|D)$ to investigate are the properties of compactness and of Noetherianess.

In this paper, we concentrate on the case where $K$ is the quotient field of $D$, studying subspaces of $\text{Zar}(K|D) = \text{Zar}(D)$ that are not compact. The starting point is a criterion based on semistar operations, proved in [8, Theorems 4.9 and 4.13] (see also [11, Proposition 4.5] for a slightly stronger version) and integrated, as in [9, Example 3.7], with

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the use of the two-faced definition of the integral closure/b-operation, either through valuation overrings or through equations of integral dependence (see e.g. [20, Chapter 6]). In particular, we analyze sets of the form Zar\((D) \setminus \{V\}\), where \(V\) is a minimal valuation overring of \(D\): we show in Section 3 that such a space is compact only if \(V\) can be obtained from \(D\) in a very specific way (more precisely, as the integral closure of a localization of a finitely generated algebra over \(D\)), and we follow up in Sections 4 and 5 by showing that this condition implies a bound on the dimension of \(V\) in relation with the dimension of \(D\) (Proposition 4.3) and a quite strict condition on the intersection of sets of prime ideals of \(D\) (Theorem 5.1). Section 6 is dedicated to a brief application of these criteria to the study of Kronecker function rings (the definition will be recalled later).

In Section 7, we consider the set Over\((D)\) of overrings of \(D\) (which is known to be itself a spectral space [7, Proposition 3.5]). Using the result proved in the previous sections, we show that, when \(D\) is a Noetherian domain, some distinguished subspaces of Over\((D)\) (for example, the subspace of overrings of \(D\) that are Noetherian) are not spectral.

2. Preliminaries and notation

2.1. Spectral spaces. A topological space \(X\) is a spectral space if there is a ring \(R\) such that \(X\) is homeomorphic to the prime spectrum Spec\((R)\), endowed with the Zariski topology. Spectral spaces can be characterized in a purely topological way as those spaces that are \(T_0\), compact, with a basis of open and compact subset that is closed by finite intersections and such that every irreducible closed subset has a generic point (i.e., it is the closure of a single point) [18, Proposition 4].

On a spectral space \(X\) it is possible to define two new topologies: the inverse and the constructible topology.

The inverse topology is the topology on \(X\) having, as a basis of closed sets, the family of open and compact subspaces of \(X\). Endowed with the inverse topology, \(X\) is again a spectral space [18, Proposition 8]; moreover, a subspace \(Y \subseteq X\) is closed in the inverse topology if and only if \(Y\) is compact (in the original topology) and \(Y = Y^{\text{gen}}\) [8, Remark 2.2 and Proposition 2.6], where

\[
Y^{\text{gen}} := \{z \in X \mid z \leq y \text{ for some } y \in Y\} = \{z \in X \mid y \in \text{Cl}(z) \text{ for some } y \in Y\},
\]

with Cl\((z)\) denoting the closure of the singleton \(\{z\}\) (again, in the original topology) and \(\leq\) is the order induced by the original topology [17, d-1], which coincides on Spec\((R)\) with the set-theoretic inclusion.

The constructible topology on \(X\) (also called patch topology) is the coarsest topology such that the open and compact subsets of \(X\) are both open and closed. Endowed with the constructible topology, \(X\)
is a spectral space that is also Hausdorff (see [30, Propositions 3 and 5], [29] or [14, Proposition 5]), and the constructible topology is finer than both the original and the inverse topology. A subset of $X$ closed in the constructible topology is said to be a proconstructible subset of $X$; if $Y$ is proconstructible, then it is a spectral space when endowed with the topology induced by the original spectral topology of $X$, and the constructible topology on $Y$ is exactly the topology induced by the constructible topology on $X$ (this follows from [3, 1.9.5(vi-vii)]).

2.2. **Noetherian spaces.** A topological space $X$ is Noetherian if $X$ verifies the ascending chain condition on the open subsets, or equivalently if every subspace of $X$ is compact. Examples of Noetherian spaces are finite spaces and the prime spectra of Noetherian rings. If $\text{Spec}(R)$ is a Noetherian space, then every proper ideal of $R$ has only finitely many minimal primes (see e.g. the proof of [2, Chapter 4, Corollary 3, p.102] or [1, Chapter 6, Exercises 5 and 7]).

2.3. **Overrings and the Zariski space.** Let $D \subseteq K$ be an extension of integral domains. We denote the set of all rings contained between $D$ and $K$ by $\text{Over}(K|D)$; if $K$ is a field (not necessarily the quotient field of $D$), the set of all valuation rings containing $D$ with quotient field $K$ is denoted by $\text{Zar}(K|D)$, and it is called the Zariski space (or the Zariski-Riemann space) of $D$.

The Zariski topology on $\text{Over}(K|D)$ is the topology having, as a subbasis, the sets of the form

$$B(x_1, \ldots, x_n) := \{ T \in \text{Over}(K|D) \mid x_1, \ldots, x_n \in T \},$$

as $\{x_1, \ldots, x_n\}$ ranges among the finite subsets of $K$. Under this topology, both $\text{Over}(K|D)$ [7, Proposition 3.5] and its subspace $\text{Zar}(K|D)$ [5, 4] are spectral spaces, and the order induced by this topology is the inverse of the set-theoretic inclusion. In particular, every $Y \subseteq \text{Over}(K|D)$ with a minimum element is compact, and, if $Z$ is an arbitrary subset of $\text{Over}(K|D)$, then $Z^{\text{gen}} = \{ T \in \text{Over}(K|D) \mid T \supseteq A \text{ for some } A \in Z \}$.

We denote by $\text{Zar}_{\text{min}}(D)$ the set of minimal elements of $\text{Zar}(D)$; since $\text{Zar}(D)$ is a spectral space, every $V \in \text{Zar}(D)$ contains an element $W \in \text{Zar}_{\text{min}}(D)$.

If $K$ is the quotient field of $D$, then we set $\text{Over}(K|D) =: \text{Over}(D)$ and $\text{Zar}(K|D) =: \text{Zar}(D)$. Elements of $\text{Over}(D)$ are called overrings of $D$, elements of $\text{Zar}(D)$ are the valuation overrings of $D$ and elements of $\text{Zar}_{\text{min}}(D)$ are the minimal valuation overrings of $D$.

The center map is the application

$$\gamma : \text{Zar}(K|D) \longrightarrow \text{Spec}(D)$$

$$V \longmapsto m_V \cap D,$$
where $m_V$ is the maximal ideal of $V$. When Zar$(K|D)$ and Spec$(D)$ are endowed with the respective Zariski topologies, the map $\gamma$ is continuous ([34, Chapter VI, §17, Lemma 1] or [4, Lemma 2.1]), surjective (this follows, for example, from [1, Theorem 5.21] or [15, Theorem 19.6]) and closed [4, Theorem 2.5].

2.4. Semistar operations. Let $D$ be a domain with quotient field $K$. Let $F(D)$ be the set of $D$-submodules of $K$, $\mathcal{F}(D)$ be the set of fractional ideals of $D$, and $\mathcal{F}_f(D)$ be the set of finitely generated fractional ideals of $D$.

A semistar operation on $D$ is a map $\star : F(D) \rightarrow F(D)$, $I \mapsto I^\star$, such that, for every $I, J \in F(D)$ and every $x \in K$,

1. $I \subseteq I^\star$;
2. if $I \subseteq J$, then $I^\star \subseteq J^\star$;
3. $(I^\star)^\star = I^\star$;
4. $x \cdot I^\star = (xI)^\star$.

Given a semistar operation $\star$, the map $\star_f$ is defined on every $E \in F_f(D)$ by

$$E^\star_f = \bigcup \{F^\star | F \in \mathcal{F}_f(D), F \subseteq E\}.$$  

The map $\star_f$ is always a semistar operation; if $\star = \star_f$, then $\star$ is said to be of finite type. Two semistar operations of finite type $\star_1, \star_2$ are equal if and only if $I^{\star_1} = I^{\star_2}$ for every $I \in \mathcal{F}_f(D)$. See [25] for general informations about semistar operations.

If $\Delta \subseteq \text{Zar}(D)$, then $\wedge_\Delta$ is defined as the semistar operation on $D$ such that

$$I^{\wedge_\Delta} := \bigcap \{IV | V \in \Delta\}$$

for every $D$-submodule $I$ of $K$; a semistar operation of type $\wedge_\Delta$ is said to be a valuative semistar operation. By [11, Proposition 4.5], $\wedge_\Delta$ is of finite type if and only if $\Delta$ is compact (in the Zariski topology of Zar$(D)$). If $\Delta, \Lambda \subseteq \text{Zar}(D)$, then $\wedge_\Delta = \wedge_\Lambda$ if and only if $\Delta^{\text{gen}} = \Lambda^{\text{gen}}$ [10, Lemma 5.8(1)], while $(\wedge_\Delta)_f = (\wedge_\Lambda)_f$ if and only if $\Delta$ and $\Lambda$ have the same closure with respect to the inverse topology [8, Theorem 4.9]. The semistar operation $\wedge_{\text{Zar}(D)}$ is usually denoted by $b$ and called the $b$-operation.

3. THE USE OF MINIMAL VALUATION DOMAINS

The starting point of this paper is the following well-known result.

Proposition 3.1 (see e.g. [20, Proposition 6.8.2]). Let $I$ be an ideal of an integral domain $D$; let $x \in D$. Then, $x \in IV$ for every $V \in \text{Zar}(D)$ if and only if there are $n \geq 1$ and $a_1, \ldots, a_n \in D$ such that $a_i \in I^i$ and

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$
An inspection of the proof of the previous proposition given in [20] shows that this result does not really rely on the fact that $I$ is an ideal of $D$, or on the fact that $x \in D$; indeed, it applies to every $D$-submodule $I$ of the quotient field $K$, and to every $x \in K$. In the terminology of semistar operations, this means that, for each $I \in F(D)$, $I^b = I^{\Lambda_{\z(D)}}$ is exactly the set of $x \in K$ that verifies an equation like (1), with $a_i \in I^i$. We are interested in generalizing that proof in a different way; we need the following definitions.

**Definition 3.2.** Let $D$ be an integral domain and let $\Delta, \Lambda \subseteq Over(D)$. We say that $\Lambda$ dominates $\Delta$ if, for every $T \in \Delta$ and every $M \in \text{Max}(T)$, there is a $A \in \Lambda$ such that $T \subseteq A$ and $1 \notin MA$.

For example, $\z(D)$ dominates every subset of $\text{Over}(D)$, while the set of localizations of $D$ dominates $\{D\}$.

**Definition 3.3.** Let $D$ be an integral domain domain. We denote by $D[F_f]$ the set of finitely generated $D$-algebras of $\text{Over}(D)$, or equivalently

$$D[F_f] := \{D[I] : I \in F_f(R)\}.$$ 

Even if the proof of the following result essentially repeats the proof of [20, Proposition 6.8.2], we restate it here for clarity.

**Proposition 3.4.** Let $D$ be an integral domain, and suppose that $\Delta \subseteq \z(D)$ dominates $D[F_f]$. Then, for every finitely generated ideal $I$ of $D$, $I^{\Lambda_{\Delta}} = I^b$.

**Proof.** Clearly, $I^b \subseteq I^{\Lambda_{\Delta}}$. Suppose thus that $x \in I^{\Delta_{\Delta}}$, $x \neq 0$, and let $I = (i_1, \ldots, i_k)D$. Define $J := x^{-1}I \in F_f(D)$, and let $A := D[J] = D[x^{-1}i_1, \ldots, x^{-1}i_k]$; by definition, $J \subseteq A$.

If $JA \neq A$, then there is a maximal ideal $M$ of $A$ containing $J$, and thus, by domination, there is a valuation domain $V \in \Delta$ containing $A$ whose maximal ideal $m_V$ is such that $JV \subseteq m_V$, and thus $IV \subseteq x m_V$.

However, $x \in I^b \subseteq IV$, which implies $x \in x m_V$, a contradiction.

Hence, $JA = A$, i.e., $1 = j_1a_1 + \cdots + j_n a_n$ for some $j_i \in J$, $a_i \in A$; expliciting the elements of $A$ as elements of $D[J]$ and using $J = x^{-1}I$, we find that there must be an $N \in \mathbb{N}$ and elements $i_t \in I^t$ such that $x^N = i_1 x^{N-1} + \cdots + i_{N-1} x + i_N$, which gives an equation of integral dependence of $x$ over $I$. Therefore, $x \in I^b$, as requested.

We can now use the properties of valuative semistar operations to study compactness.

**Proposition 3.5.** Let $D$ be an integral domain, and let $\Delta \subseteq \z(D)$ be a set that dominates $D[F_f]$. Then, $\Delta$ is compact if and only if it contains $\z_{\text{min}}(D)$.

**Proof.** If $\Delta$ contains $\z_{\text{min}}(D)$, then $\mathcal{U}$ is an open cover of $\Delta$ if and only if it is an open cover of $\z(D)$; thus, $\Delta$ is compact since $\z(D)$ is.
Conversely, suppose $\Delta$ is compact. By Proposition 3.4, $I^\Delta = I^b$ for every finitely generated ideal $I$; hence, $(\wedge\Delta)_f = b_f = b$. By [10, Lemma 5.8(1)], it follows that the closure of $\Delta$ with respect to the inverse topology of $\text{Zar}(D)$ is the whole $\text{Zar}(D)$; however, since $\Delta$ is compact, its closure in the inverse topology is exactly $\Delta^{\text{gen}} = \Delta^\uparrow = \{W \in \text{Zar}(D) \mid W \supseteq V \text{ for some } V \in \Delta\}$. Hence, $\Delta$ must contain $\text{Zar}_{\text{min}}(D)$.

Thus, to find a subset of $\text{Zar}(D)$ that is not compact, it is enough to find a $\Delta$ that dominates $D[F_f]$ but that does not contain $\text{Zar}_{\text{min}}(D)$. The easiest case where this criterion can be applied is when $\Delta = \text{Zar}(D) \setminus \{V\}$ for some $V \in \text{Zar}_{\text{min}}(D)$.

**Theorem 3.6.** Let $D$ be an integral domain and let $V \in \text{Zar}_{\text{min}}(D)$. If $\text{Zar}(D) \setminus \{V\}$ is compact, then $V$ is the integral closure of $D[x_1, \ldots, x_n]_M$ for some $x_1, \ldots, x_n \in K$ and some $M \in \text{Max}(D[x_1, \ldots, x_n])$.

**Proof.** If $\Delta := \text{Zar}(D) \setminus \{V\}$ is compact, then by Proposition 3.5 it cannot dominate $D[F_f]$. Hence, there is a finitely generated fractional ideal $I$ such that $\Delta$ does not dominate $A := D[I]$, and so a maximal ideal $M$ of $A$ such that $1 \in MW$ for every $W \in \Delta$. In particular, $A \neq K$ (otherwise $M$ would be $(0)$).

However, there must be a valuation ring containing $A_M$ whose center (on $A_M$) is $MA_M$, and the unique possibility for this valuation ring is $V$: it follows that $V$ is the unique valuation ring centered on $MA_M$. However, the integral closure of $A_M$ is the intersection of the valuation rings with center $MA_M$ (since every valuation ring containing $A_M$ contains a valuation ring centered on $MA_M$ [15, Corollary 19.7]); thus, $V$ is the integral closure of $A_M$. $\square$

4. THE DIMENSION OF $V$

Before embarking on using Theorem 3.6, we prove a simple yet general result.

**Proposition 4.1.** Let $D$ be an integral domain. If $\text{Zar}(D)$ is a Noetherian space, so is $\text{Spec}(D)$.

**Proof.** The claim follows from the fact that $\text{Spec}(D)$ is the continuous image of $\text{Zar}(D)$ through the center map $\gamma$, and that the image of a Noetherian space is still Noetherian. $\square$

Note that the converse of this proposition is far from being true (this is, for example, a consequence of Proposition 5.4 or of Proposition 7.1).

The problem in using Theorem 3.6 is that it is usually difficult to control the behaviour of finitely generated algebras over $D$. We can, however, control the behaviour of the prime spectrum of $D$. 

Proof. Suppose every $PD \hookrightarrow be the canonical spectral map associated to the inclusion $D[M]$. Let $D \hookrightarrow A = D[a_1, \ldots, a_n] \hookrightarrow A_M \simeq D[X_1, \ldots, X_n]/b \hookrightarrow V$. 

Figure 1. Rings involved in the proof of Proposition 4.3.

Lemma 4.2. Let $D$ be an integral domain, and let $V \in Zar(D)$ be the integral closure of $D_M$, for some $M \in Spec(D)$. Then, the set of prime ideals of $D$ contained in $M$ is linearly ordered.

Proof. Let $P, Q$ be two prime ideals of $D$ contained in $M$; then, $PD_M, QD_M \in Spec(D_M)$. Since $D_M \subseteq V$ is an integral extension, $PD_M = P' \cap D_M$ and $QD_M = Q' \cap D_M$ for some $P', Q' \in Spec(V)$; however, $V$ is a valuation domain, and thus (without loss of generality) $P' \subseteq Q'$. Hence, $PD_M \subseteq QD_M$ and $P \subseteq Q$, as requested. \qed

Proposition 4.3. Let $D$ be an integral domain, let $V \in Zar_{min}(D)$ and suppose that $Zar(D) \setminus \{V\}$ is compact. Let $\iota_V : Spec(V) \rightarrow Spec(D)$ be the canonical spectral map associated to the inclusion $D \hookrightarrow V$. For every $P \in Spec(D)$, $|\iota_V^{-1}(P)| \leq 2$; in particular, dim($V$) \leq 2 dim($D$).

Proof. Suppose $|\iota_V^{-1}(P)| > 2$: then, there are prime ideals $Q_1 \subseteq Q_2 \subseteq Q_3$ of $V$ such that $\iota_V(Q_1) = \iota_V(Q_2) = \iota_V(Q_3) =: P$. If $Zar(D) \setminus \{V\}$ is compact, by Theorem 3.6 there is a finitely generated $D$-algebra $A := D[a_1, \ldots, a_n]$ such that $V$ is the integral closure of $A_M$, for some maximal ideal $M$ of $A$. We can write $A_M$ as a quotient $D[X_1, \ldots, X_n]/b$, where $X_1, \ldots, X_n$ are independent indeterminates and $a, b \in Spec(D[X_1, \ldots, X_n])$. Since $A_M \subseteq V$ is an integral extension, $Q_i \cap A \neq Q_j \cap A$ if $i \neq j$.

For $i \in \{1, 2, 3\}$, let $q_i$ be the prime ideal of $D[X_1, \ldots, X_n]$ whose image in $A$ is $Q_i$; then, $q_1, q_2$ and $q_3$ are distinct, $q_i \cap D = P$ for each $i$, and the set of ideals between $q_1$ and $q_3$ is linearly ordered (by Lemma 4.2). However, the prime ideals of $D[X_1, \ldots, X_n]$ contracting to $P$ are in a bijective and order-preserving correspondence with the prime ideals of $F[X_1, \ldots, X_n]$, where $F$ is the quotient field of $D/P$; since $F[X_1, \ldots, X_n]$ is a Noetherian ring, there are an infinite number of prime ideals between the ideals corresponding to $q_1$ and $q_3$. This is a contradiction, and $|\iota_V^{-1}(P)| \leq 2$.

For the “in particular” statement, take a chain $(0) \subseteq Q_1 \subseteq \cdots \subseteq Q_k$ in $Spec(V)$. Then, the corresponding chain of the $P_i := Q_i \cap D$ has length at most $\dim(D)$, and moreover $\iota^{-1}((0)) = \{(0)\}$. Hence, $k + 1 \leq 2 \dim(D) + 1$ and $\dim(V) \leq 2 \dim(D)$. \qed

The valuative dimension of $D$, indicated by $dim_v(D)$, is defined as the supremum of the dimensions of the valuation overrings of $D$; we have always $\dim(D) \leq \dim_v(D)$, and $dim_v(D)$ can be arbitrarily large.
with respect to \( \dim(D) \) [15, Section 30, Exercises 16 and 17]. In particular, with the notation of the previous proposition, the cardinality of \( \iota_{V}^{-1}(P) \) can be arbitrarily large: for example, if \((D, \mathfrak{m})\) is local and one-dimensional, then \( |\iota_{V}^{-1}(\mathfrak{m})| = \dim_{v}(D) \).

**Corollary 4.4.** Let \( D \) be an integral domain such that \( \text{Zar}(D) \) is Noetherian. Then, \( \dim_{v}(D) \leq 2 \dim(D) \).

**Proof.** If \( \text{Zar}(D) \) is Noetherian, then in particular \( \text{Zar}(D) \setminus \{V\} \) is compact for every \( V \in \text{Zar}_{\text{min}}(D) \). Hence, \( \dim(V) \leq 2 \dim(D) \) for every \( V \in \text{Zar}_{\text{min}}(D) \), by Proposition 4.3; since, if \( W \supseteq V \) are valuation domain, \( \dim(W) \leq \dim(V) \), the claim follows. \( \square \)

**Proposition 4.5.** Let \( D \) be an integral domain, and let \( V \in \text{Zar}_{\text{min}}(D) \) be such that \( \text{Zar}(D) \setminus \{V\} \) is compact; let \( (0) \subsetneq P_1 \subsetneq \cdots \subsetneq P_k \) be the chain of prime ideals of \( V \) and let \( Q_i := P_i \cap D \). Denote by \( \text{ht}(P) \) the height of the prime ideal \( P \). Then:

(a) for every \( 0 \leq t \leq \dim(D) \), we have
\[
\dim(V) \leq \dim_{v}(D_{Q_t}) + 2(\dim(D) - \text{ht}(Q_t));
\]
(b) if \( D_{Q_t} \) is a valuation domain, then
\[
\dim(V) \leq 2 \dim(D) - \text{ht}(Q_t).
\]

**Proof.** (a) Let \( (0) \subsetneq Q^{(1)} \subsetneq Q^{(2)} \subsetneq \cdots \subsetneq Q^{(s)} \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_k \) be the chain \((0) \subsetneq P_1 \subsetneq \cdots \subsetneq P_k \) without the repetitions, and let \( a \) be the index such that \( Q^{(a)} = Q_t \). For every \( b > a \), by the proof of Proposition 4.3 there can be at most two prime ideals of \( V \) over \( Q^{(b)} \); on the other hand, \( V_{P_t} \) is a valuation overring of \( D_{Q_t} \), and thus \( t = \dim(V_{P_t}) \leq \dim_{v}(D_{Q_t}) \).

Therefore,
\[
\dim(V) \leq t + 2(s - a) \leq \dim_{v}(D_{Q_t}) + 2(\dim(D) - \text{ht}(Q_t))
\]
since each ascending chain of prime ideals starting from \( Q_t \) has length at most \( \dim(D) - \text{ht}(Q_t) \).

Point (b) follows, since \( \dim(V) = \dim_{v}(V) \) for every valuation domain \( V \). \( \square \)

**Example 4.6.** A class of integral domain whose Zariski space is Noetherian is constituted by the class of Prüfer domains with Noetherian spectrum. Indeed, if \( D \) is a Prüfer domain then the valuation overrings of \( D \) are exactly the localizations of \( D \) at prime ideals; thus, the center map \( \gamma \) establishes a homeomorphism between \( \text{Zar}(D) \) and \( \text{Spec}(D) \). Thus, if the latter is Noetherian also the former is Noetherian.

In this case, \( \dim(D) = \dim_{v}(D) \).

**Example 4.7.** It is also possible to construct domains whose Zariski space is Noetherian but with \( \dim(D) \neq \dim_{v}(D) \). For example, let \( L \) be a field, and consider the ring \( A := L + Y L(X)[[Y]] \), where \( X \) and \( Y \) are independent indeterminates. Then, the valuation overrings
of $A$ different from $F := L(X)((Y))$ are the rings in the form $V + YL(X)[[Y]]$, as $V$ ranges among the valuation rings containing $L$ and having quotient field $L(X)$; that is, $\text{Zar}(A) \setminus \{F\} \simeq \text{Zar}(L(X)|L)$. By the following Corollary 5.5, $\text{Zar}(A)$ is a Noetherian space.

From this, we can construct analogous examples of arbitrarily large dimension. Indeed, if $R$ is an integral domain with quotient field $K$, and $T := R + XK[[X]]$, then as above $\text{Zar}(T)$ is composed by $K((X))$ and by rings of the form $V + XK[[X]]$, as $V$ ranges in $\text{Zar}(R)$; in particular, $\text{Zar}(T) = \{K((X))\} \cup \mathcal{A}$, where $\mathcal{A} \simeq \text{Zar}(R)$. Thus, $\text{Zar}(T)$ is Noetherian if $\text{Zar}(R)$ is. Moreover, $\dim(T) = \dim(R) + 1$ and $\dim_v(T) = \dim_v(R) + 1$.

Consider now the sequence of rings $R_1 := L + YL(X)[[Y]]$, $R_2 := R_1 + YQ(R_1)[[Y_2]]$, \ldots, $R_n := R_{n-1} + YQ(R_{n-1})[[Y_n]]$, where $Q(R)$ indicates the quotient field of $R$ and each $Y_i$ is an indeterminate over $Q(R_{i-1})((Y_{i-1}))$. Recursively, we see that each $\text{Zar}(R_n)$ is Noetherian, while $\dim(R_n) = n \neq n + 1 = \dim_v(R_n)$.

5. Intersections of prime ideals

The results of the previous sections, while very general, are often difficult to apply, because it is usually not easy to determine the valuative dimension of a domain $D$. More applicable criteria, based on the prime spectrum of $D$, are the ones that we will prove next.

**Theorem 5.1.** Let $D$ be a local integral domain, and suppose there is a set $\Delta \subseteq \text{Spec}(D)$ and a prime ideal $Q$ such that:

1. $Q \notin \Delta$;
2. no two members of $\Delta$ are comparable;
3. $\bigcap \{P \mid P \in \Delta\} = Q$;
4. $D_Q$ is a valuation domain.

Then, for any minimal valuation overring $V$ of $D$ contained in $D_Q$, $\text{Zar}(D) \setminus \{V\}$ is not compact; in particular, $\text{Zar}(D)$ is not Noetherian.

**Proof.** Note first that, since $V$ is a minimal valuation overring, its center $M$ on $D$ must be the maximal ideal of $D$ [15, Corollary 19.7]. Suppose that $\text{Zar}(D) \setminus \{V\}$ is compact: by Theorem 3.6, there is a finitely generated $D$-algebra $A := D[x_1, \ldots, x_n]$ such that $V$ is the integral closure of $A_M$ for some $M \in \text{Max}(A)$.

Let $I := x_1^{-1}D \cap \cdots \cap x_n^{-1}D \cap D = (D :_D x_1) \cap \cdots \cap (D :_D x_n)$. If $I \subseteq Q$, then $(D :_D x) \subseteq Q$ for some $x_i := x$; then, since $D_Q$ is flat over $D$,

$$(D_Q :_{D_Q} x) = (D :_D x)D_Q \subseteq QD_Q,$$

and in particular $x \notin D_Q$. However, $V \subseteq D_Q$, and thus $x \notin V$, a contradiction. Hence, we must have $I \not\subseteq Q$.

In this case, there must be a prime ideal $P_1 \in \Delta$ not containing $I$. Moreover, $I \cap P_1 \not\subseteq Q$ too, and thus there is another prime $P_2 \in \Delta$. 


$P_1 \neq P_2$, not containing $I$. By Lemma 4.2, the prime ideals of $A$ inside $M$ are linearly ordered; in particular, we can suppose without loss of generality that $\text{rad}(P_2A) \subseteq \text{rad}(P_1A)$.

Let now $t \in P_2 \setminus P_1$; then, $t \in \text{rad}(P_1A)$, and thus there are $p_1, \ldots, p_k \in P_1$, $a_1, \ldots, a_n \in A$ such that $t^e = p_1a_1 + \cdots + p_k a_k$ for some positive integer $e$. For each $i$, $a_i = B_i(x_1, \ldots, x_n)$, where $B_i$ is a polynomial over $D$ of total degree $d_i$; let $d := \sup\{d_1, \ldots, d_k\}$, and take an $r \in I \setminus P_1$ (recall that $I \not\subseteq P_1$). Then, $r^d B_i(x_1, \ldots, x_n) \in D$ for each $i$; therefore,

$$r^d t^e = p_1 r^d a_1 + \cdots + p_k r^d a_k \in p_1 D + \cdots + p_k D \subseteq P_1.$$

However, by construction, both $r$ and $t$ are out of $P_1$; since $P_1$ is prime, this is impossible. Hence, $\text{Zar}(D) \setminus \{V\}$ is not compact, and $\text{Zar}(D)$ is not Noetherian.

The first corollaries of this result can be obtained simply by putting $Q = (0)$. Recall that a $G$-domain (or Goldman domain) is an integral domain such that the intersection of all nonzero prime ideals is nonzero. They were introduced by Kaplansky for giving a new proof of Hilbert’s Nullstellensatz (see for example [22, Section 1.3]).

**Corollary 5.2.** Let $D$ be a local domain of finite dimension, and suppose that $D$ is not a $G$-domain. Then, $\text{Zar}(D) \setminus \{V\}$ is not compact for every $V \in \text{Zar}_{\text{min}}(D)$.

**Proof.** Since $D$ is finite-dimensional, every prime ideal of $D$ contains a prime ideal of height 1; since $D$ is not a $G$-domain, it follows that the intersection of the set $\text{Spec}^1(D)$ of the height-1 prime ideals of $D$ is $(0)$. The localization $D_{(0)}$ is the quotient field of $D$, and thus a valuation domain; therefore, we can apply Theorem 5.1 to $\Delta := \text{Spec}^1(D)$.

**Corollary 5.3.** Let $D$ be a local domain. If $D$ has infinitely many height-1 primes, then $\text{Zar}(D)$ is not Noetherian.

**Proof.** Let $I$ be the intersection of all height-1 prime ideals. If $I \neq (0)$, every height-one prime of $D$ would be minimal over $I$; since there is an infinite number of them, $\text{Spec}(D)$ would not be Noetherian, and by Proposition 4.1 neither $\text{Zar}(D)$ would be Noetherian. Hence, $I = (0)$. But then we can apply Theorem 5.1 (for $Q = I$).

Note that the hypothesis that $D$ is local is needed in Theorem 5.1 and in Corollary 5.3: for example, $\mathbb{Z}$ has infinitely many height-1 primes, and $\bigcap\{P \mid P \in \text{Spec}^1(D)\} = (0)$, but $\text{Zar}(\mathbb{Z}) \simeq \text{Spec}(\mathbb{Z})$ is a Noetherian space.

**Proposition 5.4.** Let $D$ be an integral domain. If $D$ is not a field, then $\text{Zar}(D[X])$ is a not a Noetherian space.

**Proof.** Since $D$ is not a field, there exist a nonzero prime ideal $P$ of $D$. For any $a \in P$, let $p_a$ be the ideal of $D[X]$ generated by $X - a$;
then, each \( p_a \) is a prime ideal of height 1, \( p_a \neq p_b \) if \( a \neq b \), and \( \bigcap \{ p_a \mid a \in P \} = (0) \).

The prime ideal \( m := PD[X] + XD[X] \) contains every \( p_a \); by Corollary 5.3, \( \text{Zar}(D[X]_m) \) is not Noetherian. Therefore, neither \( \text{Zar}(D[X]) \) is Noetherian. \( \square \)

**Corollary 5.5.** Let \( F \subseteq L \) be a transcendental field extension.

(a) If \( \text{trdeg}_F(L) = 1 \) and \( L \) is finitely generated over \( F \) then \( \text{Zar}(L|F) \) is Noetherian.

(b) If \( \text{trdeg}_F(L) > 1 \) then \( \text{Zar}(L|F) \) is not Noetherian.

**Proof.** (a) Let \( L = F(\alpha_1, \ldots, \alpha_n) \); without loss of generality we can suppose that \( \alpha_1 \) is transcendental over \( F \). Then, the extension \( F(\alpha_1) \subseteq L \) is algebraic and finitely generated, and thus finite.

Each \( V \in \text{Zar}(L|F) \) must contain either \( \alpha_1 \) or \( \alpha_1^{-1} \); therefore, \( \text{Zar}(L|F) = \text{Zar}(L[F(\alpha_1)]) \cup \text{Zar}(L[F(\alpha_1^{-1})]) \). However, \( \text{Zar}(L[A]) = \text{Zar}(A') \) for every domain \( A \), where we denote by \( A' \) the integral closure of \( A \) in \( L \); since \( F[\alpha_1] \) (respectively, \( F[\alpha_1^{-1}] \)) is a principal ideal domain and \( F(\alpha_1) \subseteq L \) is finite, the integral closure of \( F[\alpha_1] \) (resp., \( F[\alpha_1^{-1}] \)) is a Dedekind domain, and thus \( \text{Zar}(L[F(\alpha_1)]) = \text{Zar}(F[\alpha_1]) \simeq \text{Spec}(F[\alpha_1]) \) is Noetherian. Being the union of two Noetherian spaces, \( \text{Zar}(L|F) \) is itself Noetherian.

(b) Suppose \( \text{trdeg}_F(L) > 1 \). Then, there are \( X,Y \in L \) such that \( \{X,Y\} \) is an algebraically independent set over \( F \); in particular, we have a continuous surjective map \( \text{Zar}(L|F) \rightarrow \text{Zar}(F(X,Y)|F) \) given by \( V \mapsto V \cap F(X,Y) \). However, \( \text{Zar}(F(X,Y)|F) \) contains \( \text{Zar}(F[X,Y]) \); by Proposition 5.4, the latter is not Noetherian, since \( F[X,Y] \) is the polynomial ring over \( F[X] \), a domain of dimension 1. Thus, \( \text{Zar}(L|F) \) is not Noetherian. \( \square \)

The condition that \( \bigcap \{P \mid P \in \Delta \} = Q \) of Theorem 5.1 can be slightly generalized, requiring only that the intersection is contained in \( Q \). However, doing so we can only prove that \( \text{Zar}(D) \) is not Noetherian, without always finding a specific \( V \) such that \( \text{Zar}(D) \setminus \{V\} \) is not compact.

**Proposition 5.6.** Let \( D \) be a local integral domain, and suppose there is a set \( \Delta \subseteq \text{Spec}(D) \) and a prime ideal \( Q \) such that:

1. \( Q \notin \Delta \);
2. no two members of \( \Delta \) are comparable;
3. \( \bigcap \{P \mid P \in \Delta \} \subseteq Q \);
4. \( D_Q \) is a valuation domain.

Then, \( \text{Zar}(D) \) is not Noetherian.

**Proof.** If \( \text{Spec}(D) \) is not Noetherian, by Proposition 4.1 neither is \( \text{Zar}(D) \); suppose that \( \text{Spec}(D) \) is Noetherian.

Let \( I := \bigcap \{P \mid P \in \Delta \} \); since an overring of a valuation domain is still a valuation domain, we can suppose that \( Q \) is a minimal prime
of $I$. Since $D$ has Noetherian spectrum, the radical ideal $I$ has only a finite number of minimal primes, say $Q =: Q_1, Q_2, \ldots, Q_n$; let $\Delta_i := \{ p \in \Delta \mid Q_i \subseteq p \}$ and $I_i := \bigcap \{ p \mid p \in \Delta_i \}$. By standard properties of minimal primes, $\Delta = \Delta_1 \cup \cdots \cup \Delta_n$ and $I = I_1 \cap \cdots \cap I_n$.

In particular, $I_1 \cap \cdots \cap I_n \subseteq Q$; hence, $I_k \subseteq Q$ for some $k$. However, $Q_k \subseteq I_k$, and thus $Q_k \subseteq Q$; since different minimal primes of the same ideal are not comparable, $k = 1$ and $Q \subseteq I_1 \subseteq Q$, i.e., $I_1 = Q$. Then, $\Delta_1$ is a family of primes satisfying the hypothesis of Theorem 5.1; in particular, $\text{Zar}(D)$ is not Noetherian.

An essential prime of a domain $D$ is a $P \in \text{Spec}(D)$ such that $D_P$ is a valuation domain. $D$ is an essential domain if it is equal to the intersection of the localizations of $D$ at the essential primes. If, moreover, the family of the essential primes is compact, then $D$ can be called a Prüfer $\nu$-multiplication domain (PrVM $D$ for short) [12, Corollary 2.7]; note that the original definition of PrVMs was given through star operations (more precisely, $D$ is a PrVM if and only if $D_P$ is a valuation ring for every $t$-maximal ideal $P$ [16, 21]).

**Proposition 5.7.** Let $D$ be an essential domain. Then, $\text{Zar}(D)$ is Noetherian if and only if $D$ is a Prüfer domain with Noetherian spectrum.

**Proof.** If $D$ is a Prüfer domain with Noetherian spectrum, then $\text{Zar}(D) \simeq \text{Spec}(D)$ is Noetherian (see Example 4.6). Conversely, suppose $\text{Zar}(D)$ is Noetherian: by Proposition 4.1, $\text{Spec}(D)$ is Noetherian. Let $E$ be the set of essential prime ideals of $D$: since $\text{Spec}(D)$ is Noetherian, $E$ is compact, and thus $D$ is a PrVM.

Suppose by contradiction that $D$ is not a Prüfer domain. Then, there is a maximal ideal $M$ of $D$ such that $D_M$ is not a valuation domain; since the localization of a PrVM is a PrVM [21, Theorem 3.11], and $\text{Zar}(D_M)$ is a subspace of $\text{Zar}(D)$, without loss of generality we can suppose $D = D_M$, i.e., we can suppose that $D$ is local.

Since $E$ is compact, every $P \in E$ is contained in a maximal element of $E$; let $\Delta$ be the set of such maximal elements. Clearly, $D = \bigcap \{ D_P \mid P \in \Delta \}$. If $\Delta$ were finite, $D$ would be an intersection of finitely many valuation domains, and thus it would be a Prüfer domain [15, Theorem 22.8]; hence, we can suppose that $\Delta$ is infinite. Let $I := \bigcap \{ P \mid P \in \Delta \}$.

Each $P \in \Delta$ contains a minimal prime of $I$; however, since $\text{Spec}(D)$ is Noetherian, $I$ has only finitely many minimal primes. It follows that there is a minimal prime $Q$ of $I$ that is not contained in $\Delta$; in particular, $\bigcap \{ P \mid P \in \Delta \} \subseteq Q$, and thus we can apply Proposition 5.6. Hence, $\text{Zar}(D)$ is not Noetherian, which is a contradiction.

**Remark 5.8.** The previous proof can be interpreted using the terminology of the theory of star operations. Indeed, any essential prime $P$ is a $t$-ideal, i.e., $P = P^t$, where (for any ideal $J$ of $D$) $J^t := \bigcup \{ (D :
(D : I) | I ⊆ J is finitely generated] [21, Lemma 3.17] and if D is a PvMD then the set Δ of the maximal elements of E is exactly the set of t-maximal ideals, i.e., the set of the ideals I such that I = I^t and J ≠ J^t for every proper ideal I ⊆ J.

Corollary 5.9. Let D be a Krull domain. Then, Zar(D) is Noetherian if and only if dim(D) = 1, i.e., if and only if D is a Dedekind domain.

Proof. If dim(D) = 1 then D is Noetherian and so is Zar(D). If dim(D) > 1, then D is not a Prüfer domain; since each Krull domain is a PvMD, we can apply Proposition 5.7. □

Note that this corollary can also be proved directly from Corollary 5.3 since, if D is Krull, and P ∈ Spec(D) has height 2 or more, then D_P has infinitely many height-1 primes.

6. An application: Kronecker function rings

Let D be an integrally closed integral domain with quotient field K. For every V ∈ Zar(D), let V(X) := V[X]_{mV(X)} ⊆ K(X), where mV is the maximal ideal of V. If Δ ⊆ Zar(D), the Kronecker function ring of D with respect to Δ is

$$\text{Kr}(D, \Delta) := \bigcap \{V(X) \mid V \in \Delta\};$$

equivalently,

$$\text{Kr}(D, \Delta) = \{f/g \mid f, g \in D[X], g \neq 0, c(f) \subseteq (c(g))^\Delta\},$$

where c(f) is the content of f and △ is the semistar operation defined in Section 2.4. See [15, Chapter 32] or [13] for general properties of Kronecker function rings.

The set of Kronecker function rings it exactly the set of overrings of the basic Kronecker function ring Kr(D, Zar(D)); this set is in bijective correspondence with the set of finite-type valuative semistar operations [15, Remark 32.9], or equivalently with the set of nonempty subsets of Zar(D) that are closed in the inverse topology [8, Theorem 4.9].

Let K(D) be the set of Kronecker function rings T of D such that T ∩ K = D. Then, K(D) is in bijective correspondence with the set of finite-type valuative star operations, or equivalently with the set of inverse-closed representation of D through valuation rings, i.e., the sets Δ ⊆ Zar(D) that are closed in the inverse topology and such that \(\bigcap \{V \mid V \in \Delta\} = D\) [27, Proposition 5.10].

It has been conjectured [23] that K(D) is either a singleton (in which case D is said to be a vacant domain; see [6]) or infinite, and this has been proved to be the case for a wide class of pseudo-valuation domains [6, Theorem 4.10]. As a consequence of the following proposition, we will prove this conjecture for another class of domains.
Proposition 6.1. Let $D$ be an integrally closed integral domain such that $1 < |K(D)| < \infty$. Then, there is a minimal valuation overring $V$ of $D$ such that $\text{Zar}(D) \setminus \{V\}$ is compact.

Proof. Suppose $|K(D)| > 1$. Then, there is an inverse-closed representation $\Delta$ of $D$ different from $\text{Zar}(D)$; let $\Lambda := \text{Zar}(D) \setminus \Delta$. For each $W \in \Lambda$, let $\Delta(W) := \Delta \cup \{W\}^\uparrow$; then, every $\Delta(W)$ is an inverse-closed representation of $D$, and $\Delta(W) \neq \Delta(W')$ if $W \neq W'$ (since, without loss of generality, $W \not\subseteq W'$, and thus $W \notin \Delta(W')$). Hence, each $W \in \Lambda$ give rise to a different member of $K(D)$; since $|K(D)| < \infty$, it follows that $\Lambda$ is finite.

If now $V$ is minimal in $\Lambda$, then $\text{Zar}(D) \setminus \{V\} = \Delta \cup (\Lambda \setminus \{V\})$ is closed by generizations; since $\Lambda$ is finite, it follows that $\text{Zar}(D) \setminus \{V\}$ is the union of two compact subspaces, and thus it is itself compact. □

Corollary 6.2. Let $D$ be an integrally closed local integral domain, and suppose there exist a set $\Delta \subseteq \text{Spec}(D)$ of incomparable nonzero prime ideals such that $\bigcap \{P \mid P \in \Delta\} = (0)$. Then, $|K(D)| \in \{1, \infty\}$.

Proof. By Theorem 5.1, each $\text{Zar}(D) \setminus \{V\}$ is noncompact. The claim now follows from Proposition 6.1. □

7. Overrings of Noetherian Domains

If $D$ is a Noetherian domain, Theorem 3.6 admits a direct application, without using any of the results proved in Sections 4 and 5. Indeed, if $D$ is Noetherian with quotient field $K$, then it is the same for any localization of $D[x_1, \ldots, x_n]$, for arbitrary $x_1, \ldots, x_n \in K$; thus, the integral closure of $D[x_1, \ldots, x_n]_M$ is a Krull domain for each maximal ideal $M$ of $D[x_1, \ldots, x_n]$ ([24, (33.10)] or [20, Theorem 4.10.5]). Since a domain that is both Krull and a valuation ring must be a field or a discrete valuation ring, Theorem 3.6 implies that $\text{Zar}(D) \setminus \{V\}$ is not compact as soon as $V$ is a minimal valuation overring of dimension 2 or more.

We can actually say more than this; the following is a proof through Proposition 3.5 of an observation already appeared in [9, Example 3.7].

Proposition 7.1. Let $D$ be a Noetherian domain with quotient field $K$, and let $\Delta$ be the set of valuation overrings of $D$ that are Noetherian (i.e., $\Delta$ is the union of $\{K\}$ with the set of discrete valuation overrings of $D$). Then, $\Delta$ is compact if and only if $\dim(D) = 1$.

Proof. If $\dim(D) = 1$, then $\Delta = \text{Zar}(D)$, and thus it is compact.

On the other hand, for every ideal $I$ of $D$, $I^{\Delta} = I^b$ [20, Proposition 6.8.4]; however, if $\dim(D) > 1$, then $\text{Zar}(D)$ contains elements of dimension 2, and thus $\Delta$ cannot contain $\text{Zar}_{\min}(D)$. The claim now follows from Proposition 3.5. □

Remark 7.2.
(1) The equality $I^\Delta = I^b$ holds also if we restrict $\Delta$ to be the set of discrete valuation overrings of $D$ whose center is a maximal ideal of $D$ [20, Proposition 6.8.4]. For each prime ideal of height 2 or more, by passing to $D_P$, we can thus prove that the set of discrete valuation overrings of $D$ with center $P$ is not compact (and in particular it is infinite).

(2) The previous proposition also allows a proof of the second part of Corollary 5.5 without using Theorem 5.1, since $F[X,Y]$ is a Noetherian domain of dimension 2.

By Proposition 7.1, in particular, the space $\Delta$ of Noetherian valuation overrings of $D$ (where $D$ is Noetherian and $\text{dim}(D) \geq 2$) is not a spectral space, since it is not compact. Our next purpose is to see $\Delta$ as an intersection $X \cap \text{Zar}(D)$, for some subset $X$ of $\text{Over}(D)$, and use this representation to prove facts about $X$. We start with using the inverse topology.

**Proposition 7.3.** Let $D$ be a Noetherian domain with quotient field $K$, and let:
- $X_1$ be the set of all overrings of $D$ that are Noetherian and of dimension at most 1;
- $X_2$ be the set of all overrings of $D$ that are Dedekind domains ($K$ included).

For $i \in \{1, 2\}$, the following are equivalent:

(i) $X_i$ is compact;
(ii) $X_i$ is spectral;
(iii) $X_i$ is proconstructible in $\text{Over}(D)$;
(iv) $\text{dim}(D) = 1$.

**Proof.** (i) $\implies$ (iii). In both cases, $X = X^\text{gen}$: for $X_1$ see [22, Theorem 93], while for $X_2$ see e.g. [15, Theorem 40.1] (or use the previous result and [15, Corollary 36.3]). (iii) $\implies$ (ii) $\implies$ (i) always holds.

(iv) $\implies$ (i). If $\text{dim}(D) = 1$, then $X_1 = \text{Over}(D)$, while $X_2 = \text{Over}(D')$, where $D'$ is the integral closure of $D$, and both are compact since they have a minimum.

(iii) $\implies$ (iv). If $X_i$ is proconstructible, so is $X_i \cap \text{Zar}(D)$ (since $\text{Zar}(D)$ is also proconstructible), and in particular $X_i \cap \text{Zar}(D)$ is compact. However, in both cases, $X_i \cap \text{Zar}(D)$ is exactly the set of Noetherian valuation overrings of $D$; by Proposition 7.1, $\text{dim}(D) = 1$. □

**Remark 7.4.** The equivalence between the first three conditions of Proposition 7.3 holds for every subset $X \subseteq \text{Over}(D)$ such that $X = X^\text{gen}$ (and every domain $D$). In particular, it holds if $X$ is the set of overrings of $D$ that are principal ideal domains, and, with the same proof of the other cases, we can show that if $D$ is Noetherian and these conditions hold, then $\text{dim}(D) = 1$. However, it is not clear if, when $D$ is Noetherian and $\text{dim}(D) = 1$, this set is actually compact.
Another immediate consequence of Proposition 7.1 is that the set NoethOver(D) of Noetherian overrings of D is not proconstructible as soon as D is Noetherian and dim(D) \geq 2: indeed, if it were, then NoethOver(D) \cap Zar(D) = \Delta would be proconstructible, against the fact that \Delta is not compact. However, this is also a consequence of a more general result. We need a topological lemma.

Lemma 7.5. Let Y \subseteq X be spectral spaces. Suppose that there is a subbasis \mathcal{B} of X such that, for every B \in \mathcal{B}, both B and B \cap Y are compact. Then, Y is a proconstructible subset of X.

Proof. The hypothesis on \mathcal{B} implies that the inclusion map Y \hookrightarrow X is a spectral map; by [3, 1.9.5(vii)], it follows that Y is a proconstructible subset of X. \qed

Proposition 7.6. Let D be an integral domain with quotient field K, and let D[\mathcal{F}_f] be the set of finitely generated D-algebras contained in K.

(a) D[\mathcal{F}_f] is dense in Over(D), with respect to the constructible topology.

(b) Let X such that D[\mathcal{F}_f] \subseteq X \subseteq Over(D). Then, X is spectral in the Zariski topology if and only if X = Over(D).

Proof. (a) A basis of the constructible topology is given by the sets of type U \cap (X \setminus V), as U and V ranges in the open and compact subsets of Over(D). Such an U can be written as B_1 \cup \cdots \cup B_n, where each B_i = B(x_1^{(i)}, \ldots, x_n^{(i)}) is a basic open set of Over(D); thus, we can suppose that U = B(x_1, \ldots, x_n). Suppose \Omega := U \cap (X \setminus V) is nonempty; we claim that A := D[x_1, \ldots, x_n] \in \Omega \cap D[\mathcal{F}_f]. Clearly A \in D[\mathcal{F}_f] and A \in U; let T \in \Omega. Then, T \in U, and thus A \subseteq T; therefore, A is in the closure Cl(T) of T, with respect to the Zariski topology. But X \setminus V is closed, and thus Cl(T) \subseteq X \setminus V; i.e., A \in X \setminus V. Hence, A \in \Omega \cap D[\mathcal{F}_f], which in particular is nonempty, and D[\mathcal{F}_f] is dense.

(b) Suppose X is spectral. For every x_1, \ldots, x_n, the set X \cap B(x_1, \ldots, x_n) has a minimum (i.e., D[x_1, \ldots, x_n]), so it is compact. Since the family of all B(x_1, \ldots, x_n) is a basis, by Lemma 7.5 it follows that X is proconstructible. By the previous point, we must have X = Over(D). \qed

Corollary 7.7. Let D be a Noetherian domain. The spaces

- NoethOver(D) := \{T \in Over(D) \mid T is Noetherian\}, and
- KrullOver(D) := \{T \in Over(D) \mid T is a Krull domain\}

are spectral if and only if \text{dim}(D) = 1.

Proof. If \text{dim}(D) = 1, then the claim follows by Proposition 7.3.
If \( \dim(D) \geq 2 \), then \( \text{NoethOver}(D) \) is not spectral by Proposition 7.6(b) and the Hilbert Basis Theorem; the case of \( \text{KrullOver}(D) \) follows in the same way, since \( \text{KrullOver}(D) \cap B(x_1, \ldots, x_n) \) has always a minimum (i.e., the integral closure of \( D[x_1, \ldots, x_n] \)). □

More generally, consider a property \( P \) of Noetherian domains such that every field and every discrete valuation ring satisfies \( P \); for example, \( P \) may be the property of being regular, Gorenstein or Cohen-Macaulay. Let \( X_P(D) \) be the set of overrings of \( D \) satisfying \( P \); then, \( X_P(D) \cap \text{Zar}(D) \) is not compact, and thus \( X_P(D) \) is not proconstructible. On the other hand, if \( X_P(T) \) is compact for every overring of \( D \) that is finitely generated as a \( D \)-algebra, then by Lemma 7.5 it follows that \( X_P(D) \) cannot be a spectral space. Thus, the assignment \( D \mapsto X_P(D) \) cannot be “too good”: either some \( X_P(T) \) is not compact, or \( X_P(D) \) is not spectral.

**Question.** Let \( P \) be the property of being regular, the property of being Gorenstein or the property of being Cohen-Macaulay. Is it possible to characterize for which Noetherian domains \( D \) there is a \( T \in \text{Over}(D) \) such that \( X_P(T) \) is not compact and for which \( X_P(D) \) is not spectral?

### Section 8. Acknowledgments

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### References


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