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Switching-based Regulation of Uncertain Stable Linear Systems Affected by an Unknown Harmonic Disturbance

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Abstract: The problem of rejection of a sinusoidal disturbance with *unknown* frequency acting on an *unknown* single-input-single-output stable linear system is addressed in this paper. We present a new hybrid approach that does not require knowledge of the frequency response of the transfer function over the range of frequencies of interest. The proposed controller reposes upon a switching strategy within a family of linear controllers based on the adaptive feedforward / internal model control methodology. A dead-beat frequency estimation method is integrated in the controller. The method also accounts for the presence of bounded sensor noise as well as imprecise frequency estimation. It is shown that, within a finite number of switchings, the regulation error is ultimately bounded by a function of the norm of the noise that depends on the choice of the controller and the estimator gains.

Keywords: output regulation, disturbance rejection, switching algorithms, active noise control, feedforward control

1. PROBLEM FORMULATION

The problem of rejecting periodic disturbances of unknown frequency, in the presence of plant uncertainty, is a fundamental issue in control theory and applications (see Patt et al. (2005); Esbrook et al. (2013); Basturk and Krstic (2013) and references therein, to cite but a few recent contributions.) In this work, we consider SISO LTI plant models of the form

$$\begin{aligned} \dot{x} &= A(\mu)x + B(\mu)[u(t) - d(t)]\,, \quad x(0) = x_0 \in \mathbb{R}^n \\ y &= C(\mu)x\,, \quad y_d = y + \nu \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state of an internally stable uncertain plant model, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is the regulated output, $y_d \in \mathbb{R}$ is the measured output and $\nu \in \mathbb{R}$ is a bounded additive measurement noise, satisfying $\|\nu(\cdot)\|_\infty \leq \bar{\nu} < \infty$ for some given constant $\bar{\nu} > 0$. System (1) is affected by a sinusoidal disturbance (of *unknown* frequency, amplitude and phase)

$$d(t) = a\sin(\omega^* t + \phi_0) \tag{2}$$

The vectors $\mu \in \mathbb{R}^p$ and $\mu_d := \mathrm{col}(a, \omega^*, \phi_0) \in \mathbb{R}^3$ collect the uncertain parameters of the plant model and the disturbance, respectively. It is assumed that μ ranges on a given known compact set, $\mathcal{P} \subset \mathbb{R}^p$. Moreover, the disturbance $d(t)$ satisfies the following assumption:

Assumption 1. The unknown amplitude a and frequency ω^* of the disturbance $d(t)$ are bounded respectively by

$$0 \leq a \leq \bar{a}, \quad \underline{\omega} < \omega^* \leq \bar{\omega}$$

for some known positive constants $\bar{a}, \bar{\omega}$ and $\underline{\omega}$.

The transfer function from u to y of system (1) is denoted by $W_\mu(s) := C(\mu)(sI - A(\mu))^{-1}B(\mu)$, where I denotes the identity matrix. System (1) is assumed to be internally stable, robustly with respect to $\mu \in \mathcal{P}$:

Assumption 2. There exist positive constants a_1, a_2 and a_0 such that the solution $P_x : \mathbb{R}^p \to \mathbb{R}^{n\times n}$ of the Lyapunov equation $P_x(\mu)A(\mu) + A^T(\mu)P_x(\mu) = -I$ satisfies $a_1 I \leq P_x(\mu) \leq a_2 I$. Moreover, $-\mathrm{Re}\{\lambda\} \geq a_0$ for all $\lambda \in \mathrm{spec}\, A(\mu)$, $\forall \mu \in \mathcal{P}$. ◁

As the disturbance is generated by the LTI exosystem

$$\begin{aligned} \dot{w} &= \omega^* T w, \quad w(0) = w_0 \in \mathbb{R}^2 \\ d &= \Gamma w \end{aligned} \tag{3}$$

where

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

the problem can be formally stated as follows:

Problem 1. (Output Regulation Problem) For system (1), design a dynamic output-feedback controller of the form

$$\begin{aligned} \dot{\xi} &= f_a(\xi, y_d)\,, \quad \xi(0) = \xi_0 \in \mathbb{R}^m \\ u &= h_a(\xi, y_d) \end{aligned} \tag{4}$$

such that, for all $\mu \in \mathcal{P}$ and $\omega^* \in [\underline{\omega}, \bar{\omega}]$, the trajectories of the closed-loop system (1), (3) and (4) are bounded, and the output of the plant satisfies $\limsup_{t\to\infty} |y(t)| \leq$

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$r(\|\nu\|_\infty)$, where $r(\cdot)$ is a class-$\mathcal{N}$ function [1] that depends on the controller (4). ◁

Remark 1.1. Assumption 1 implies that the initial condition w_0 is taken over a compact set $\mathcal{W} \subset \mathbb{R}^2$. Without loss of generality, we take $\mathcal{W}$ to be an arbitrary, but fixed, compact set that is invariant under the flow of (3).

The majority of techniques proposed in the literature to solve the given problem assume prior knowledge and persistence of the sign of either the real part or the imaginary part of the frequency response (i.e. the sign of $W_\mu(j\omega)$ does not change) over the range of the frequencies of interest. This assumption is known as SPR-like conditions in the literature (Bodson et al. (1994); Marino and Tomei (2015).) Under the assumption that ω^* is known, the recent work Wang et al. (2016, 2018) have proposed a multiple-model adaptive control scheme and a switched-based strategy, respectively, that remove the necessity of SPR-like conditions. In the presence of uncertainties on the frequency of the disturbance, Battistelli et al. (2017) proposes a robust switching solution for discrete-time linear system given a nominal plant model. Liu et al. (2009) and Marino and Tomei (2011) address similar problems under the assumption that the system is minimum-phase with a known relative degree, which is relaxed in Marino and Tomei (2017), but an SPR-like condition is still required.

In this paper, we seek a novel switching-based solution to eliminate the need for SPR-like conditions in presence of model uncertainty *for both the plant and the exosystem.* A family of four certainty-equivalence candidate controllers is designed off-line, ensuring that at least one of the candidate controllers provides stability of the closed-loop system. Given the availability of frequency estimate (possibly not equal to ω^*), a switching logic is devised, which is capable of removing from the family of admissible controllers those that display destabilizing performance. It is shown that the switching sequence terminates in finite time, and the resulting controller is able to reject the disturbance completely if the frequency estimate is accurate. If not, the frequency estimation error will be corrected in finite time by employing the deadbeat method presented in Pin et al. (2017). A distinctive feature of this solution is that a reliable disturbance rejection (or attenuation in the presence of sensor noise) is achieved within a finite number of switchings. Another advantage of the presented approach is its modularity: the design of the frequency estimator can be completely decoupled from the design of the switching logic. In this paper, we employ a non-asymptotic estimator that provides finite time estimation, but any of many existing robust frequency estimation techniques can in principle be employed in the proposed architecture.

This paper is organized as follows: In Section 2, we introduce the overall architecture of the supervisory switching control systems and the candidate controller family. Our main contribution, a state-norm-estimator-based switching control strategy, is presented and analyzed in Section 3. An illustrative example is provided in Section 4.

[1] A class-$\mathcal{N}$ function $r(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is non-negative, continuous and strictly increasing, but does not necessarily satisfy $r(0) = 0$.

Notation Throughout the paper, $\lambda_i(M)$, $i = 1, \ldots, n$, denotes eigenvalues of the matrix $M \in \mathbb{R}^{n\times n}$, whereas $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ denote the maximum and minimum eigenvalues of M, respectively. We denote with $\|\cdot\|$ both the Euclidean vector norm and the corresponding induced matrix norm.

2. STATE-SHARING MULTI-CONTROLLER

The overall control architecture follows the general paradigm of supervisory control architecture proposed in Morse (1995). The measured output y_d of the plant drives a bank of controllers, each one generating a candidate feedback signal $\hat{d}^i$, which is an estimate of the disturbance. The control signal applied to the plant is

$$u(t) = \hat{d}(t) := \hat{d}^{\sigma(t)}(t) \tag{5}$$

where $\sigma : [0, \infty) \mapsto \mathcal{I}$ is a piecewise-constant switching signal taking values in the index set of the family of the candidate controllers $\mathcal{I} := \{1, 2, 3, 4\}$. The system that generates the switching signal is referred to as the *supervisory system.* The candidate controllers C^i, $i \in \mathcal{I}$, are designed as

$$\begin{aligned} \dot{\hat{w}}^i &= \hat{\omega} T \hat{w}^i - k\phi^i y_d\,, \quad \hat{w}^i(0) = \hat{w}^i_0 \in \mathbb{R}^2 \\ \hat{d}^i &= \Gamma \hat{w}^i\,, \quad i \in \mathcal{I} \end{aligned} \tag{6}$$

where $\hat{\omega}$ is a *constant estimate* of ω^* and $\phi^i \in \mathbb{R}^2$, $i \in \mathcal{I}$, are *constant* vectors given by

$$\phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \phi^2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix},\ \phi^3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix},\ \phi^4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{7}$$

In Marino and Tomei (2015), it has been shown that one of the dynamic feedback controllers in (6) solves Problem 1 given a suitable controller gain $k > 0$ and an accurate frequency estimate, i.e. the estimation error $\tilde{\omega} := \hat{\omega} - \omega^* = 0$. In this paper, we will first show that at least one of candidate controllers in (6) is capable of stabilizing the closed-loop system even with a non-zero frequency estimation error.

The dimension of the overall controller can be reduced by employing a shared-state parametrization. Consider a coordinate change $\hat{w}^i_c := M^i_c \hat{w}^i$, $i \in \mathcal{I}$, where

$$M^i_c := \phi^i_1 I - \phi^i_2 T$$

and $\phi^i_j, j = 1, 2$ represents the j-th element of ϕ^i. This yields the set of controllers (equivalent to (6))

$$\begin{aligned} \dot{\hat{w}}_c &= \hat{\omega} T \hat{w}_c - k G_c y_d\,, \quad \hat{w}_c(0) = \hat{w}_{c0} \in \mathbb{R}^2 \\ \hat{d}^i &= \varphi^{iT} \hat{w}_c\,, \qquad i \in \mathcal{I} \end{aligned} \tag{8}$$

Remark 2.1. Although the state-sharing form (8) is used for the implementation of the algorithm, in the forthcoming analysis for the sake of simplicity we will use its equivalent form (6).

3. STATE NORM ESTIMATOR-BASED SWITCHING

We first present the development of a norm estimator-based supervisor system that solves the following problem:

Problem 2. (Stabilizing Problem) Given compact sets $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{W} \subset \mathbb{R}^2$, find a selection for the gain $k > 0$ and a design for the *supervisory system* such that for all $\hat{\omega} \in [\underline{\omega}, \bar{\omega}]$ trajectories of the closed-loop system(1), (3), (5) and

(6), originating from any $x_0 \in \mathcal{X}$ and $\hat{w}_0^i \in \mathcal{W}$ are bounded and satisfy

(a) $\lim_{t\to\infty} |y(t)| = 0$ if $\|\nu(\cdot)\|_\infty = 0$ and $|\tilde{\omega}| = 0$;

(b) $\limsup_{t\to\infty} |y(t)| \le r'(\max\{\|\nu\|_\infty, |\tilde{\omega}|\})$ if $\|\nu(\cdot)\|_\infty \neq 0$ and/or $|\tilde{\omega}| \neq 0$.

where $r'(\cdot)$ is a class-$\mathcal{N}$ function that depends on the tuning parameters of the controller and the switching mechanism. ◁

In the light of the setup of Problem 2, the results will be valid in a semi-global sense, that is, on the basis of an arbitrary (but fixed) choice of compact sets for the initial conditions of the plant, the controller and the exoystem.

3.1 Switching Logic

The proposed supervisor is a cascade connection of two subsystems: a scheduling logic Σ_m and a routing function $\beta(\cdot) : \{1, 2, 3, \cdots\} \mapsto \mathcal{I}$. The output of the Σ_m subsystem $m(\cdot) : [0, +\infty) \mapsto \mathbb{Z}^+$, termed *the switching sequence*, is a piecewise-constant signal to be determined. The routing function $\beta(\cdot)$ employed here is constructed to satisfy the *revisitation property* (Morse (1995))

$$\beta(\sigma) := \mathrm{mod}(m + \sigma(0) - 1, 4) + 1,$$

where $\sigma(0) \in \mathcal{I}$ is the initial selection for the active controller. The idea behind the switching logic is that the supervisory system keeps adjusting σ through the index set $\mathcal{I}$ along a pre-specified path $\beta(\sigma)$ until the output y_d is small in a suitable sense.

The scheduling logic Σ_m exploits three auxiliary signals to make decisions: a state-norm estimator that generates the piecewise-continuous switching threshold $\bar{J}$, a performance index signal J obtained through an auxiliary filter and a monotonic decreasing signal J_ε that represents the norm-bound of the transient term.

The flow chart of the switching logic Σ_m is given in Figure 1, where $N > 1$ and $\varepsilon_y, \varepsilon_d, \varepsilon_0$ are constants will be determined later. After the algorithm is initiated, the performance index J evolves according to

$$\dot{J} = -\delta J + \delta y_d^2(t), \qquad J(0) = 0. \tag{9}$$

where $\delta > 0$ is a forgetting factor.

3.2 Generation of the switching threshold $\bar{J}$

Let the sequence $\{T_m\}_{m=1}^{\infty}$ denote the set of time instants at which switching takes place, where it is assumed that the same controller is kept in the interval $[T_m, T_{m+1})$. Consider the changes of coordinates $\zeta := \hat{w}^\sigma - w$ and $z := x - \Pi(\mu)\zeta$, where $\Pi(\mu) \in \mathbb{R}^{n\times 2}$ is the unique solution of the Sylvester equation

$$\hat{\omega}\Pi(\mu)T = A(\mu)\Pi(\mu) + B(\mu)\Gamma.$$

Then, the dynamics of the interconnection of the plant (1), the exosystem (3) and the active candidate controller (6) can be written as

$$\begin{aligned}
\dot{\zeta} &= \hat{\omega}T\zeta - k\phi^\sigma y_d + \tilde{\omega}Tw\,, \quad \zeta(T_m) = \zeta_0 \in \mathbb{R}^2 \\
\dot{z} &= A(\mu)z + k\Pi(\mu)\phi^\sigma y_d + \tilde{\omega}\Pi(\mu)Tw\,, \quad z(T_m) = z_0 \in \mathbb{R}^n \\
y_d &= C(\mu)z + \vartheta^T(\mu,\hat{\omega})\zeta + \nu
\end{aligned} \tag{10}$$

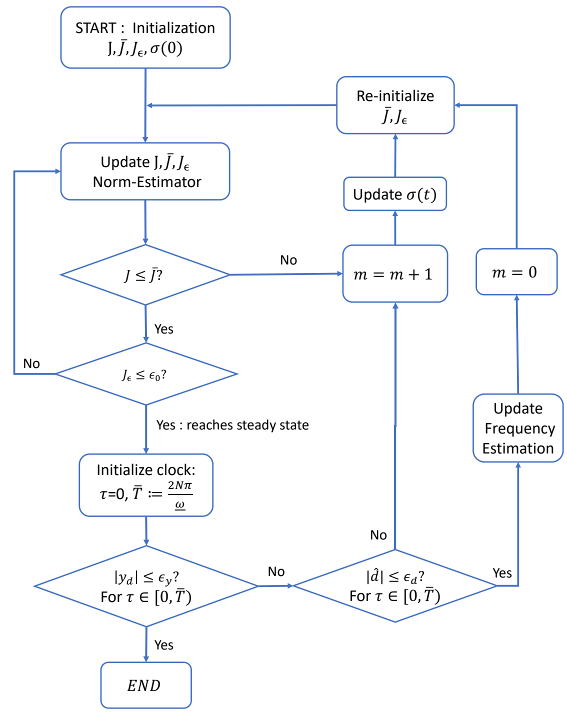

Fig. 1. Flowchart of the switching logic.

where

$$\vartheta^T(\mu,\hat{\omega}) := C(\mu)\Pi(\mu) = \Big(\mathrm{Re}\{W_\mu(j\hat{\omega})\},\ \mathrm{Im}\{W_\mu(j\hat{\omega})\}\Big)\,.$$

Since the plant model is unknown, the parameter $\vartheta \in \mathbb{R}^2$ is unknown, but assumed to range in a known set. Specifically, let the set $\Theta \subset \mathbb{R}^2$ be the annular region defined, for given real numbers $0 < \delta_1 < \delta_2$, as

$$\Theta := \left\{\vartheta \in \mathbb{R}^2 \mid \delta_1^2 \le \vartheta_1^2 + \vartheta_2^2 \le \delta_2^2\right\} \tag{11}$$

and consider the following assumption that replaces the typical SPR-like conditions :

Assumption 3. The unknown parameter vector ϑ satisfies $\vartheta(\mu,\hat{\omega}) \in \mathrm{int}\,\Theta$ for all $\mu \in \mathcal{P}$ and all $\hat{\omega} \in [\underline{\omega}, \bar{\omega}]$. ◁

Define $\eta := \mathrm{col}(\zeta, z) \in \mathbb{R}^{n+2}$, and rewrite (10) as

$$\begin{aligned}
\dot{\eta} &= E_\sigma\eta + \tilde{\omega}Hw + kF\nu \\
y_d &= L\eta + \nu
\end{aligned} \tag{12}$$

with

$$E_\sigma = \begin{pmatrix} \hat{\omega}T - k\phi^\sigma\vartheta^T & -k\phi^\sigma C \\ k\Pi\phi^\sigma\vartheta^T & A + k\Pi\phi^\sigma C \end{pmatrix},$$

$$H := \begin{pmatrix} T \\ \Pi T \end{pmatrix}, \quad F := \begin{pmatrix} -\phi^\sigma \\ \Pi\phi^\sigma \end{pmatrix}$$

and $L := \begin{pmatrix} \vartheta^T & C \end{pmatrix}$. Note that the dependence of matrices on μ has been dropped for the sake of notational simplicity. For future use, let ϱ_X denotes the maximum value of the matrix norm of matrix $X(\mu)$, defined by $\varrho_X := \max_{\mu\in\mathcal{P}} ||X(\mu)||$. Two subsets $\mathcal{I}_k$ and $\mathcal{I}_k^*$, which depends on the choice of the gain $k > 0$ are defined as follows:

$$\mathcal{I}_k := \{\sigma \in \mathcal{I} : \mathrm{Re}\{\lambda_{\max}(E_\sigma)\} < 0\} \tag{13}$$

$$\mathcal{I}_k^* := \{\sigma \in \mathcal{I} : \mathrm{Re}\{\lambda_{\max}(E_\sigma)\} \le -\alpha(k)\} \tag{14}$$

where $\alpha(k)$ is a class-$\mathcal{K}$ function to be determined. Note that $\mathcal{I}_k^* \subseteq \mathcal{I}_k \subseteq \mathcal{I}$ for all $k > 0$.

Fix, arbitrarily, a positive constant $r_0 > 0$, such that

$$\mathcal{X} := \{x_0 \in \mathbb{R}^n : |x_0| \leq r_0\}.$$

The following property of Lyapunov function candidates is instrumental in the ensuing analysis:

Property 1. There exist a scalar $k_1 > 0$ and constants $c_3 \geq c_2 > c_1 > 0$ such that the solution $P_o : (\vartheta, \varepsilon, \hat{\omega}) \mapsto \mathbb{R}^{2\times 2}$ of the parametrized family of Lyapunov equations

$$P_o\left[\hat{w}T - \varepsilon\,\vartheta\vartheta^T\right] + \left[\hat{w}T - \varepsilon\,\vartheta\vartheta^T\right]^T P_o = -\varepsilon\,\vartheta^T\vartheta I \quad (15)$$

satisfies $c_1 I \leq P_o \leq c_2 I$ and $\|P_o\| \leq c_3$ for all $(\vartheta, \varepsilon, \hat{\omega}) \in \Theta \times (0, k_1] \times [\underline{\omega}, \bar{\omega}]$. ◁

The next lemma establishes the fact that at least one candidate controller defined in (6) solves Problem 2 for the closed-loop system (12) with a proper selection of k.

Lemma 3. If Assumptions 1-3 hold, there exists a constant $k^* > 0$, such that for any $k \in (0, k^*]$ and $\hat{\omega} \in [\underline{\omega}, \bar{\omega}]$, $\mathcal{I}_k$ and $\mathcal{I}_k^*$ are a non-empty sets with $\alpha(k) := \min\{\frac{k\delta_1^2}{8\sqrt{2}\delta_2 c_2}, \frac{1}{16a_2}\}$.

The proofs of Property 1 and Lemma 3 are omitted.

For any $k \in (0, k^*]$ and for all $\hat{\omega} \in [\underline{\omega}, \bar{\omega}]$, if the active controller satisfies $\sigma \in \mathcal{I}_k$, then by virtue of Lemma 3, the output of the plant satisfies $y_d(t) = y_{ss}(t) + y_{tr}(t) + \nu(t)$ for all $t \in [T_m, T_{m+1})$, where $y_{ss}(t)$ is the steady-state response and $y_{tr}(t)$ is an exponentially decaying term representing the transient response.

Remark 3.1. The strategy behind the proposed switching mechanism is to develop a threshold based on an upper bound of $y_d(t)$ under the assumption that $\sigma \in \mathcal{I}_k^*$. Then, the fact that the performance index $J(t)$ violates the threshold $\bar{J}$ implies that $\sigma(t) \notin \mathcal{I}_k^*$. On the other hand, boundedness of the output suggests a stabilizing property of the active controller. However, a pathological case may still exist where the active controller introduces a pair of eigenvalues on the imaginary axis, while boundedness is preserved. In that case, the output may still be bounded by the threshold, but the active controller is not a stabilizing one. We will show that this case can be ruled out by suitable checking on the input signal $\hat{d}$.

Recast (12) as a parallel interconnection of two subsystems

$$\Sigma_1 : \begin{cases} \dot{\eta}_1 = E_\sigma \eta_1 + \tilde{\omega} H w, \quad \eta_1(0) = 0 \\ y_1 = L\eta_1 \end{cases} \quad (16)$$

$$\Sigma_2 : \begin{cases} \dot{\eta}_2 = E_\sigma \eta_2 + kF\nu, \quad \eta_2(0) = \eta(T_m) \\ y_2 = L\eta_2 + \nu \end{cases} \quad (17)$$

with $\eta(t, \eta(T_m)) = \eta_1(t, 0) + \eta_2(t, \eta(T_m))$ and $y_d(t) = y_1(t) + y_2(t)$. Subsystem Σ_1 generates the response of the plant with respect to $w(t)$. Using integration by parts, one can easily obtain the output of Σ_1 as

$$y_1(t) = y_{1,ss}(t) + y_{1,tr}(t) \quad (18)$$

with

$$y_{1,ss}(t) := -\tilde{\omega} L[E_\sigma^2 + \omega^{*2} I]^{-1}\big\{E_\sigma H + \omega^* HT)\big\}w(t)$$

and

$$y_{1,tr}(t) := \tilde{\omega} L[E_\sigma^2 + \omega^{*2} I]^{-1}\big\{E_\sigma \mathrm{e}^{E_\sigma(t-T_m)} H + \omega^* \mathrm{e}^{E_\sigma(t-T_m)} HT\big\}w(t),$$

where I is an identity matrix of suitable dimension. Thanks to Assumption 1 and Lemma 3, the terms $y_{1,ss}(t)$ and $y_{1,tr}(t)$ are norm-bounded by

$$|y_{1,ss}(t)| \leq \kappa_1 |\tilde{\omega}|, \quad |y_{1,tr}(t)| \leq \kappa_1 \mathrm{e}^{-\alpha(k)(t-T_m)} |\tilde{\omega}| \quad (19)$$

respectively, with $\kappa_1 = \varrho_L \kappa_0(\alpha)(\varrho_{E_\sigma}\varrho_H + \varrho_H \bar{\omega})\bar{a}$ and

$$\kappa_0(\alpha) := \frac{2^{n/2}\sqrt{n+2}(\bar{\omega}^2 + \varrho_{E_\sigma}^2)}{\alpha^{2n+4}}.$$

The constant κ_0 is the norm bound of the matrix $||[E_\sigma^2 + \omega^{*2} I]^{-1}||$, which can be determined using the results in Piazza and Politi (2002).

The output $y_2(t)$ is the forced response $y_{2,fo}(t)$ to the measurement noise ν, and it is bounded by

$$\begin{aligned} |y_{2,fo}(t)| &= |L\int_0^t \mathrm{e}^{E_\sigma(t-\tau)} kF\nu(\tau)d\tau + \nu| \\ &\leq k||L||\,||F||\,||-E_\sigma^{-1}||\,\mathrm{e}^{E_\sigma t}(\mathrm{e}^{-E_\sigma t} - 1)|\nu| \\ &\leq k\varrho_L\,\varrho_F\,||-E_\sigma^{-1}||\,(1 - \mathrm{e}^{E_\sigma t})\bar{\nu} + \bar{\nu} \leq (\kappa_2 + 1)\bar{\nu} \end{aligned} \quad (20)$$

for all $t \geq 0$, where $\kappa_2 = k\varrho_L\varrho_F/\alpha(k)$. The computation of κ_2 makes use of the identity $\lambda_{\max}(-E_\sigma^{-1}) = 1/\lambda_{\min}(-E_\sigma) \leq 1/\alpha(k)$. The free response $y_{2,fr}(t)$ from the initial state at time T_m satisfies

$$\begin{aligned} |y_{2,fr}(t)| &= |L\mathrm{e}^{E_\sigma(t-T_m)}\eta(T_m)| \\ &\leq \varrho_L \mathrm{e}^{-\alpha(k)(t-T_m)}||\eta(T_m)||. \end{aligned} \quad (21)$$

Recalling the coordinate change $\eta = \mathrm{col}(\zeta, z)$, $z = x - \Pi\zeta$ and $\zeta = \hat{w}^\sigma - w$, one obtains

$$\begin{aligned} ||\eta(T_m)|| &\leq ||\zeta(T_m)|| + ||\varrho_\Pi \zeta(T_m)|| + ||x(T_m)|| \\ &\leq (\varrho_\Pi + 1)(\bar{a} + ||\hat{\omega}^{\sigma(T_m)}(T_m)||) + ||x(T_m)||. \end{aligned} \quad (22)$$

All variables in (22) are available except for the last term $||x(T_m)||$, which is estimated by the norm-estimator

$$\Sigma_{norm}: \quad \dot{\xi}(t) = -\frac{1}{2a_2}\xi(t) + 2a_2^2\varrho_B^2(\hat{d}(t)^2 + \bar{a}^2) \quad (23)$$

where $\xi(0) = 0 \in \mathbb{R}$ and $\bar{a}, a_2$ are defined in Assumption 1 and 2, respectively. Here, we restrict the initial condition to be zero just for simplicity of the analysis (Krichman et al. (2001).)

Lemma 4. If Assumptions 1 and 2 hold and $x_0 \in \mathcal{X}$ then the state of the plant is norm-bounded by

$$||x(t)||^2 \leq \frac{\xi(t)}{a_1} + \frac{a_2 r_0}{a_1}\mathrm{e}^{-\frac{1}{2a_2}t}, \quad t \geq 0 \quad (24)$$

where $\xi(t)$ is given by (23).

Proof Consider the Lyapunov candidate $V(x) = x^T P_x x$, whose derivative along trajectories of (1) reads as

$$\dot{V}(x) = -x^T x + 2x^T P_x B(\hat{d}(t) - d(t)).$$

Applying Young's inequality, it follows that

$$\begin{aligned} \dot{V}(x) &\leq -\frac{1}{2}x^T x + 2||P_x||^2||B||^2(\hat{d}(t)^2 + d(t)^2) \\ &\leq -\frac{1}{2a_2}V(x) + 2a_2^2\varrho_B^2(\hat{d}(t)^2 + \bar{a}^2). \end{aligned} \quad (25)$$

Substituting the identity $2a_2^2\varrho_B^2(\hat{d}(t)^2 + \bar{a}^2) = \dot{\xi} + \frac{1}{2a_2}\xi$ from (23) into (25), one obtains

$$\dot{V}(x) - \dot{\xi} \leq -\frac{1}{2a_2}(V(x) - \xi)$$

which implies

$$V(x) \leq \xi(t) + \mathrm{e}^{-\frac{1}{2a_2}t}(V(x_0) - \xi(0)) \leq \xi(t) + e^{-\frac{1}{2a_2}t}a_2 r_0$$

where the assumption $x_0 \in \mathcal{X}$ has been used. Due to the fact that $||x(t)||^2 \leq V(t)/a_1$, (24) follows. □

Substituting the value of $||x(T_m)||$ given by (24) into (22), one obtains the norm-bound of $y_{2,fr}(t)$ as

$$|y_{2,fr}(t)| \leq \kappa_3 \mathrm{e}^{-\alpha(k)(t-T_m)} \tag{26}$$

with $\kappa_3 := \varrho_L[(\varrho_\Pi + 1)(\bar{a} + ||\hat{\omega}^{\sigma(T_{m-1})}(T_m)||) + (\frac{\xi(t)}{a_1} + \frac{a_2 r_0}{a_1}\mathrm{e}^{-\frac{1}{2a_2}T_m})^{\frac{1}{2}}\ ]$. In summary, by virtue of (19), (20) and (26), if $\sigma \in \mathcal{I}_k^*$ then the output y_d is norm-bounded as

$$\begin{aligned}|y_d(t)| &\leq |y_1(t)| + |y_2(t)| \\ &\leq \kappa_1\bar{\omega} + (\kappa_2+1)\bar{\nu} + J_\varepsilon(t) := \bar{y}\end{aligned} \tag{27}$$

where $J_\varepsilon(t)$ is the norm bound of the transient terms

$$J_\varepsilon(t) := (\kappa_1\bar{\omega} + \kappa_3)\mathrm{e}^{-\alpha(k)(t-T_m)} \geq |y_{1,tr}| + |y_{2,fr}| \tag{28}$$

Applying (27) to (9), one obtains the dynamics of $\bar{J}$ as

$$\dot{\bar{J}}(t) = -\delta\bar{J}(t) + \delta\bar{y}^2(t), \quad \bar{J}(T_m) = J_0 \tag{29}$$

for $t \in [T_m, T_{m+1})$, where $J_0 > 0$ can be chosen arbitrarily.

3.3 Development of thresholds $\varepsilon_0, \varepsilon_y$ and ε_d

Thanks to Lemma 3 and (9), the threshold (29) is valid for all the candidate controllers $\sigma \in \mathcal{I}_k^*$ with $k \in (0, k^*]$. Hence, there exists a positive integer $\bar{m} \leq 4$ such that the switching stops after $T_{\bar{m}}$. As $J_\varepsilon(t)$ is an exponentially decaying signal, for any fixed ε_0 there exists $T_{ss} > 0$ such that

$$J_\varepsilon(t) \leq \varepsilon_0, \qquad \forall t \geq T_{\bar{m}} + T_{ss}.$$

For practical implementation, we consider the system to have reached steady state when J_ε is smaller than or equal to a given tolerance $\varepsilon_0 > 0$.

Using (19) and (20), it follows that if $\sigma \in \mathcal{I}_k^*$ then the output of the plant reaches its steady state after $t \geq T_{\bar{m}} + T_{ss}$ and satisfies

$$|y_d(t)| \leq |y_{1,ss}(t)| + |y_{2,fo}(t)| + \varepsilon_0 \leq \kappa_1|\tilde{\omega}| + (\kappa_2+1)\bar{\nu} + \varepsilon_0.$$

Given ε_0, set

$$\varepsilon_y := \kappa_1\gamma_0(\bar{\nu}) + (\kappa_2+1)\bar{\nu} + \varepsilon_0, \tag{30}$$

where $\gamma_0(\cdot)$ is a class-$\mathcal{K}$ function that quantifies the inaccuracy of the frequency estimate provided by the estimator [2].

If $|y_d(t)| \leq \varepsilon_y$ does NOT hold for all $t \in [T_{\bar{m}} + T_{ss}, T_{\bar{m}} + T_{ss} + \bar{T}]$, [3], two cases for the closed-loop system may hold:

a) It is internally stable but $|\tilde{\omega}| > \gamma_0(\bar{\nu})$;

b) It has a neutrally stable mode.

As shown in Fig. 1, we distinguish case a) and b) by checking the value of $\hat{d}(t)$ on a time interval of length $\bar{T} > 0$ seconds. This is possible due to the band-pass-filtering feature of the controllers in (6) stated in the following lemma (proof omitted due to space limitation):

Lemma 5. There exist a class-$\mathcal{KL}$ function $\gamma_k(k, |\tilde{\omega}|)$, a class-$\mathcal{K}$ function $\gamma_v(\bar{\nu})$ and constant $k^{**} \in (0, k^*]$ such that

$$|\hat{d}| \leq \gamma_k(k, |\tilde{\omega}|) + \gamma_v(\bar{\nu}), \quad \forall t \geq T_{\bar{m}} + T_{ss}. \tag{31}$$

for all $k \in (0, k^{**}]$ and all $\sigma \in \mathcal{I}_k$.

Set $\varepsilon_d := \gamma_k(k, \gamma_0(\bar{\nu})) + \gamma_v(\bar{\nu})$. If the condition $|\hat{d}(t| \leq \varepsilon_d$ for all $t \geq T_{\bar{m}} + T_{ss}$ is verified for a sufficiently small ε_d, then $y_{ss}(t)$ is guaranteed to contain only one harmonic component with frequency equal to ω^*. Otherwise, case b) occurs. In that case (which is highly unlikely in practice), the controller that is currently active needs to be deselected and switched to the next one in the family of candidate controllers.

Problem 2 is readily solved by the proposed switching mechanism with the selection $k \in (0, k^{**}]$. However, if $|y_d(t)| > \varepsilon_y$ for $t \geq T_{\bar{m}} + T_{ss}$, the output regulation problem (Problem 1) has not been solved yet. The next step is to obtain a new $\hat{\omega}$ via the deadbeat estimator and repeat the stabilizing controller selection procedure. In this regard, it follows that if the estimator provides a bound on the estimation error of the form $|\tilde{\omega}| \leq \gamma_0(\bar{\nu})$, then Problem 1 is solved with $r(\bar{v}) := \varepsilon_y = \kappa_1\gamma_0(\bar{\nu}) + (\kappa_2+1)\bar{\nu} + \varepsilon_0$.

Remark 3.2. It worth mentioning that, in absence of sensor noise, the ultimate bound of the output signal only depends on the value of ε_0, which, in theory, can be chosen arbitrarily small.

Remark 3.3. The state-norm-estimator-based switching logic presented in this section is a pseudo pre-routed switching. The feature that distinguishes the proposed one from the pre-routed switching logic presented in Morse (1995) is that the proposed switching mechanism guarantees that the stabilizing controller will be selected within at most 4 switchings. According to Morse (1995), if the cardinality of $\mathcal{I}$ is large, the performance of the pre-routed switching will be significantly degrades. This is completely avoided in the proposed switching mechanism, thanks to the small cardinality [4] of $\mathcal{I}$ and the guaranteed termination of the switching sequence.

4. ILLUSTRATIVE EXAMPLE

In this section, a numerical example is presented to demonstrate the effectiveness and robustness of the proposed method. Consider the stable non-minimum phase plant model described by the transfer function

$$W_1(s) = \frac{2(s-1)}{s^2 + 2s + 5} \tag{32}$$

and disturbance signal given by $d(t) = 5\sin(\omega^* t)$ with $\omega^* = 3$. The same setting of the controller gain, $k = 0.25$, has been used in all simulations. The Runge-Kutta integration method has been employed for all simulations with fixed sampling interval $T_s = 10^{-4}s$. The two frequency estimates used in the simulation are listed in Table 4, together with the corresponding frequency response parameter and set of stabilizing controllers. The initial frequency estimation error is $|\tilde{\omega}| = 2$ and the initial controller C^2 is not a stabilizing one. The additive noise $\nu(t)$ is selected as a random noise with uniform distribution

[2] In this study, we employ the deadbeat estimator of Pin et al. (2017). This type of estimator yields in finite time an accurate estimation of the frequency of a sinusoidal signal. In the presence of additive noise, the estimated frequency $\hat{\omega}$ provided by the deadbeat estimator enters into a neighborhood of the true value ω^* in finite time and the frequency estimation error $\tilde{\omega}$ satisfies an asymptotic bound with respect to the magnitude of the noise.

[3] $\bar{T} := 2N\pi/\underline{\omega}$, $N \geq 1$, is a multiple of the largest period of any sinusoidal signal with frequency satisfying Assumption 1.

[4] For supervisory systems with more than four candidate controllers, the same thresholds can be used but the transient performance may not necessarily improve.

Frequency	ϑ^T	$\mathcal{I}_k^*$
$\omega_1 = 1$ [rad/s]	$(-0.2,\ 0.6)$	{4}
$\omega_2 = 3$ [rad/s]	$(0.85,\ -0.23)$	{1,2}

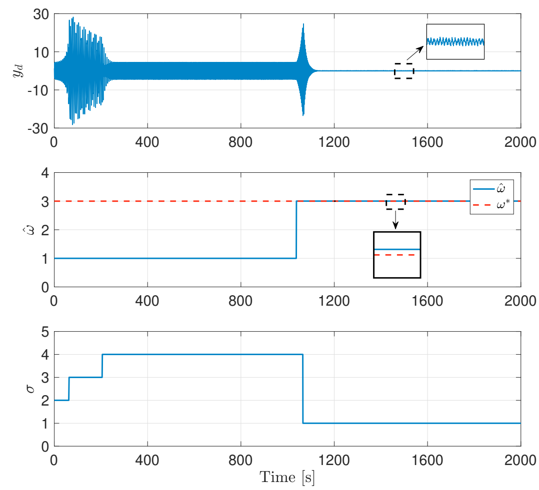

Fig. 2. Regulated output $y_d(t)$ (top), frequency estimate $\hat{\omega}(t)$(center) and switching signal $\sigma(t)$ (bottom).

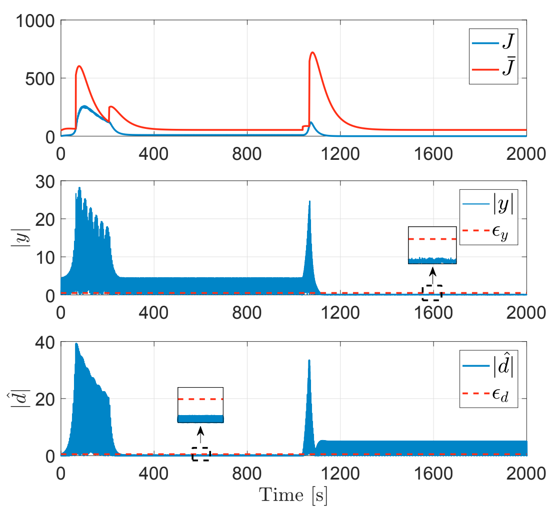

Fig. 3. Filtered norm-bound of output (top), norm-bound of output (center) and norm-bound of input $\hat{d}(t)$ (bottom).

within the interval $[-0.05, 0.05]$. The thresholds are chosen as $\varepsilon_y = \varepsilon_d = 0.5$ and $\varepsilon_0 = 1 \times 10^{-2}$.

The results of the simulations are shown in Fig. 2 and Fig. 3. After 2 switches the supervisor selects the stabilizing controller C^4, and the closed-loop system reaches steady state in about 300 seconds. Then, the deadbeat estimator provides a frequency estimate with small mismatch, $\hat{\omega} \approxeq 3$ at about $t = 1000$ seconds. For this new estimate, the family of stabilizing controller comprises C^1 and C^2. The supervisory system is automatically reactivated, and asymptotic regulation of the plant output to zero is achieved with the new stabilizing controller C^1. Under the effect of the estimation error, the proposed approach is capable of attenuating the disturbance and confining the output of the plant to a small residual set, all with a finite number of switches.

REFERENCES

Basturk, H.I. and Krstic, M. (2013). Adaptive wave cancelation by acceleration feedback for ramp-connected air cushion-actuated surface effect ships. *Automatica*, 49(9), 2591–2602.

Battistelli, G., Selvi, D., and Tesi, A. (2017). Robust switching control: Stability analysis and application to active disturbance attenuation. *IEEE Transactions on Automatic Control*, 62(12), 6369–6376.

Bodson, M., Sacks, A., and Khosla, P. (1994). Harmonic generation in adaptive feedforward cancellation schemes. *IEEE Transactions on Automatic Control*, 39(9), 1939–1944.

Esbrook, A., Tan, X., and Khalil, H.K. (2013). An indirect adaptive servocompensator for signals of unknown frequencies with application to nanopositioning. *Automatica*, 49(7), 2006–2016.

Krichman, M., Sontag, E.D., and Wang, Y. (2001). Input-output-to-state stability. *SIAM Journal on Control and Optimization*, 39(6), 1874–1928.

Liu, L., Chen, Z., and Huang, J. (2009). Parameter convergence and minimal internal model with an adaptive output regulation problem. *Automatica*, 45(5), 1306–1311.

Marino, R. and Tomei, P. (2011). An adaptive learning regulator for uncertain minimum phase systems with undermodeled unknown exosystems. *Automatica*, 47(4), 739–747.

Marino, R. and Tomei, P. (2015). Output regulation for unknown stable linear systems. *IEEE Transactions on Automatic Control*, 60(8), 2213–2218.

Marino, R. and Tomei, P. (2017). Hybrid adaptive multi-sinusoidal disturbance cancellation. *IEEE Transactions on Automatic Control*, 62(8), 4023–4030.

Morse, A.S. (1995). Control using logic-based switching. In *Trends in Control. A European Perspective*, 69–113. Springer Verlag, London, UK.

Patt, D., Liu, L., Chandrasekar, J., Bernstein, D.S., and Friedmann, P.P. (2005). Higher-harmonic-control algorithm for helicopter vibration reduction revisited. *Journal of Guidance, Control, and Dynamics*, 28(5), 918–930.

Piazza, G. and Politi, T. (2002). An upper bound for the condition number of a matrix in spectral norm. *Journal of Computational and Applied Mathematics*, 143(1), 141–144.

Pin, G., Chen, B., and Parisini, T. (2017). Robust finite-time estimation of biased sinusoidal signals: A Volterra operators approach. *Automatica*, 77, 120 – 132.

Wang, Y., Pin, G., Serrani, A., and Parisini, T. (2018). Switching-based sinusoidal disturbance rejection for uncertain stable linear systems. In *2018 Annual American Control Conference (ACC)*, 4502–4507.

Wang, Y., Pin, G., Serrani, A., and Parisini, T. (2016). Removing SPR-like conditions in adaptive feedforward control of uncertain systems. In *Proceedings of the 55th IEEE Conference on Decision and Control CDC 2016*, 4728–4733. Las Vegas, NV.