REMARKS ON SOBOLEV-MORREY-CAMPANATO SPACES DEFINED ON $C^{0,\gamma}$ DOMAINS

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Abstract. We discuss a few old results concerning embedding theorems for Campanato and Sobolev-Morrey spaces adapting the formulations to the case of domains of class $C^{0,\gamma}$, and we present more recent results concerning the extension of functions from Sobolev-Morrey spaces defined on those domains. As a corollary of the extension theorem we obtain an embedding theorem for Sobolev-Morrey spaces on arbitrary $C^{0,\gamma}$ domains.

1. Introduction

In the seminal papers [9, 10] Sergio Campanato introduced the spaces that nowadays are named after him, and used them to prove embedding theorems for Sobolev-Morrey spaces defined on bounded open sets Ω in $\mathbb{R}^n$. In particular, it was proved that if f is a function belonging to the Campanato space $\mathcal{L}_p^\lambda(\Omega)$ with $n<\lambda$ (and $\lambda \le n+p$) then f is Hölder continuous with exponent $(\lambda-n)/p$ that is for some $c>0$

$$|f(x)-f(y)| \le c|x-y|^{\frac{\lambda-n}{p}}, \tag{1.1}$$

for all $x, y \in \Omega$, and it was also proved that if f is a function in the Sobolev-Morrey space $W_p^{l,\lambda}(\Omega)$ with $0 \le \lambda < n$, $n-\lambda < pl$ (and $pl < n-\lambda+p$) then f is Hölder continuous with exponent $l+\frac{\lambda-n}{p}$ that is for some $c>0$

$$|f(x)-f(y)| \le c|x-y|^{l+\frac{\lambda-n}{p}}, \tag{1.2}$$

for all $x, y \in \Omega$. Here, for simplicty, $l \in \mathbb{N}$ and $W_p^{l,\lambda}(\Omega)$ is the space of functions with weak derivatives up to order l in the classical Morrey space $L_p^\lambda(\Omega)$, but we note that the focus of [9, 10] was mainly on the case of fractional order of smoothness l since the case of integer exponents was already discussed in [25, 35]. See Section 2 for precise definitions.

The importance of these spaces is evident in regularity theory and harmonic analysis. The classical regularity approach was based on the singular integrals theory approach introduced by A.P. Calderón and A. Zygmund [7]. Using this approach based on the heat kernel, J. Nash [33] was able to solve the XIX Hilbert problem about the analyticity of the solutions to regular problems in the calculus of variations. One year before, E. De Giorgi [19] proved the same result with a different approach. He introduced a suitable function space, the so-called De Giorgi class, proved that any solution to regular problems in the calculus of variations belongs to this class and showed the embeddings of the De Giorgi classes in the space of Hölder continuous functions. It was a natural question to ask if this approach could recover the classical Calderón and Zygmund theory for equations with regular coefficients (i.e., continuous or Hölder continuous coefficients). This question was proposed by Ennio De Giorgi and Guido Stampacchia and solved by S. Campanato with the introduction of the Campanato spaces. The regularity in L^p spaces was proved by S. Campanato and G. Stampacchia [11] with the supplementary hypotheses of the Hölder continuity of the coefficients (for a proof of such a result, with only the assumption of the continuity of the coefficients, see [13]). These spaces were used for proving regularity of

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2010 *Mathematics Subject Classification.* 46E35, 46E30, 42B35.

Key words and phrases. Sobolev, Morrey, Campanato spaces, irregular domains.

solutions to elliptic/parabolic systems/equations in variational/nonvariational form of the second (and higher) order (see, for instance [12] and [23]).

The other important field of application of these function spaces is harmonic analysis. T. Walsh [42] proved that the dual space of the Hardy space $H^p(R^N)$ is exactly the Campanato space. The theory of Hardy spaces $H^p(R^N)$ has important applications in harmonic analysis and partial differential equations (for instance, see [22, 37, 38, 39]). We recall that when $p \in (1, \infty)$, $L^p(R^N)$ and $H^p(R^N)$ are isomorphic; but when $p \in (0, 1]$, some of singular integrals (for example, Riesz transforms) are bounded on $H^p(R^N)$ but not on $L^p(R^N)$ and this fact makes the space $H^p(R^N)$ the right space where to study the theory of the boundedness of operators. In [22] C. Fefferman and E.M. Stein characterised the Hardy space $H^1(R^N)$ as the predual of the space $\mathrm{BMO}(R^N)$. The atomic and the molecular characterizations of $H^p(R^N)$ and their applications were studied by many authors; see, for example, [14, 15, 31, 32, 39, 40]. These characterizations (atomic and molecular) are necessary to extend the theory of Hardy spaces to spaces of homogeneous type in the sense of R.R. Coifman and G. Weiss [16, 17], which is, by far, one of the most general setting for singular integrals.

Going back to the initial work of S. Campanato, we note that inequality (1.1) was proved under the assumption that Ω satisfies the so-called property (A) which requires the existence of a constant $M > 0$ such that

$$|\Omega \cap B(x, r)| \geq M r^n \,, \tag{1.3}$$

for all $x \in \Omega$ and all $r > 0$ smaller than the diameter of Ω.

Inequality (1.2) was obtained under the stronger assumption that Ω is of class $C^{0,1}$, which means that, locally at the boundary, Ω can be represented as the subgraph of a Lipschitz continuous function (possibly after a rotation of coordinates).

Note that for $\lambda = 0$ we have $W^{l,\lambda}_p(\Omega) = W^l_p(\Omega)$ and inequality (1.2) is the celebrated Sobolev-Morrey inequality.

In this paper, we consider the case of open sets Ω of class $C^{0,\gamma}$ with $0 < \gamma \leq 1$ which means that the functions describing the boundary of Ω are Hölder continuous of exponent γ. It is a matter of folklore that passing from Lipschitz to Hölder continuity assumptions at the boundary of an open set is highly nontrivial (see e.g., the recent paper [29]), and it is interesting to note that also S. Campanato himself devoted his paper [8] to the study of embeddings for Sobolev spaces on open sets with power-type cusps at the boundary. We refer to the extensive monograph [34] for a recent introduction to the analysis of function spaces on irregular domains. We also refer to the classical monograph [28] for an introduction to Morrey-Campanato spaces on regular domains.

Broadly speaking, one may say that classical embedding theorems for Sobolev-Morrey-Campanato spaces hold on $C^{0,\gamma}$ domains provided one replaces (in the inequalities involved) the dimension n of the underlying space by $n_\gamma = (n-1)/\gamma + 1$, a fact which also appeared in [8]. It is important to note that $n_\gamma > n$ if $\gamma < 1$, and this typically leads to a deterioration in the estimates. For instance, if Ω is a domain with outer power-type cusps with exponent γ, property (A) above holds provided n is replaced by n_γ in the right-hand side of (1.3). On the other hand, we observe that if one wishes to control $|\Omega \cap B(x, r)|$ from above, the best one can do is to write $|\Omega \cap B(x, r)| \leq c r^n$, since it is impossible here to use n_γ. This discrepancy between the upper and lower bounds for $|\Omega \cap B(x, r)|$, indicates that the standard Euclidean metric is not suitable to deal with cusps and suggests to adapt the balls $B(x, r)$ to the type of domain under consideration. For example, if Ω is given by the cusp

$$\{(\bar{x}, x_n) \in \mathbb{R}^n : \ \bar{x} \in \mathbb{R}^{n-1}, \ x_n > |\bar{x}|^\gamma\} \tag{1.4}$$

with $\gamma < 1$, then one should replace the Euclidean ball $B(x,r)$ by the anisotropic ball

$$B_\gamma(x,r) = \left\{ y \in R^n : \ |\bar{x} - \bar{y}| < r^{\frac{1}{\gamma}}, \ |x_n - y_n| < r \right\} \tag{1.5}$$

in which case $|\Omega \cap B_\gamma(x,r)|$ is asymptotic to r^{n_γ} as $r \to 0$, and the discrepancy above disappears. Accordingly, in the right-hand side of inequalities (1.1), (1.2) one has to replace the Euclidean distance $|x-y|$ by the anisotropic one $|\bar{x}-\bar{y}|^\gamma + |x_n - y_n|$. This idea was already used by G.C. Barozzi [1] where some results of S. Campanato are extended to the case of domains with power-type cusps, and was further extended by Giuseppe Da Prato in the fundamental paper [18] where more general metrics were considered. We also note that final results for domains satisfying horn-type conditions are contained in the classical monograph [2, 3].

Although the existing literature seems to provide a complete picture of this subject, we have found it quite surprising that some results contained in the above mentioned papers, incorporate quite restrictive assumptions. In particular, in the analysis of inequality (1.2) for anisotropic metrics, [1, Theorem 3] eventually assumes for simplicity that Ω is a parallelepiped, [18, Theorem 4.1] assumes that Ω is convex and the estimate in [3, Theorem 27.4.2] is proved for all $x, y \in \Omega$ such that the segment $[x,y]$ is contained in Ω.

A different approach to the analysis of function spaces in domains of class $C^{0,\gamma}$ was suggested by Victor I. Burenkov in [4, 5] where he defined a new extension operator which, contrary to other classical extension operators, allows to deal not only with Lipschitz domains but also with $C^{0,\gamma}$ domains (as well as with anisotropic Sobolev spaces and extensions from manifolds of dimension $m < n$). Note that the flexibility of Burenkov's Extension Operator has been recently exploited in [21] where it is proved that this operator preserves general Sobolev-Morrey spaces, including the case of the classical Sobolev-Morrey spaces $W_p^{l,\lambda}(\Omega)$.

If $\gamma < 1$ then deterioration in the smoothness of the extended functions is expected and, in fact, Burenkov's Extension Operator maps the Sobolev space $W_p^l(\Omega)$ to the Sobolev space $W_p^{[\gamma l]}(\mathbb{R}^n)$ where $[\gamma l]$ is the integer part of γl. The exponent $[\gamma l]$ is sharp (in terms of Sobolev spaces). Thus, having a function extended to the whole of $\mathbb{R}^n$ allows to apply embedding theorems in $\mathbb{R}^n$ and eventually to return to Ω by mere restriction. Although the target space $W_p^{[\gamma l]}(\mathbb{R}^n)$ is sharp, it is observed already in [5] that in general the embedding theorems proved via this procedure are not sharp since the deterioration given by $[\gamma l]$ is too much for this purpose. However, this procedure has the advantage of giving at least some information even in most difficult cases.

The goal of the present paper is twofold. First, we revise the above mentioned old results by adapting their formulation to the case of elementary domains of class $C^{0,\gamma}$. In passing, we also indicate how it is possible to replace the convexity assumption in [18, Theorem 4.1] by the assumption that a Poincaré inequality for balls holds, see Theorem 2.5. Secondly, we indicate how to adapt the proofs of [21] to the case of domains of class $C^{0,\gamma}$, in order to prove that Burenkov's Extension Operator maps the Sobolev-Morrey space $W_p^{l,\lambda}(\Omega)$ to the Sobolev-Morrey space $W_p^{[\gamma l],\gamma\lambda}(\mathbb{R}^n)$, analysing also the case of Morrey norms defined by even more general weights, see Theorem 3.2. Note the extra deterioration in the Morrey exponent which passes from λ to $\gamma\lambda$. Moreover, we apply this extension result to recover an estimate of type (1.2) in domains of class $C^{0,\gamma}$, see Corollary 3.1. We observe that, although the new estimate is not sharp, it is obtained without any extra geometric assumptions on Ω or on the points $x, y \in \Omega$ as done by other authors.

It is important to observe that our extension result is obtained by using Morrey norms involving Euclidean balls both in Ω and in $\mathbb{R}^n$, even though for elementary domains of form (1.4) it would be natural to use anisotropic balls of type (1.5) in Ω. This is due to technical reasons involved in our proofs, which prevents us from controlling reflected balls in an anistropic way, see Lemma 3.3. On the other hand, since our final goal is to deal with general open sets Ω of class $C^{0,\gamma}$ (where cusps may have a different orientation depending on the part of the boundary under consideration), in principle there is no special reason why one should use the balls of type (1.5) in the whole of Ω. Thus, either one changes the definition of the Morrey spaces, adapting balls to the orientation of each local chart or uses, for uniformity, Euclidean balls in the whole of Ω. Our approach eventually leads us to choose the second option.

With reference to the problem of the extension of Sobolev-Morrey spaces, besides [21], we would also like to quote the papers [27], [30], [41].

2. EMBEDDING THEOREMS ON ELEMENTARY $C^{0,\gamma}$ DOMAINS

In this paper the elements of $\mathbb{R}^n$, $n \geq 2$, are denoted by $x = (\overline{x}, x_n)$ with $\overline{x} = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. For any $\gamma \in]0,1]$, we consider the metric δ_γ in $\mathbb{R}^n$ defined by

$$\delta_\gamma(x,y) = \max\{|\bar{x}-\bar{y}|^\gamma, |x_n - y_n|\}\,,$$

for all $x, y \in \mathbb{R}^N$ and we denote by $B_\gamma(x,r)$ the corresponding open balls of centre x and radius r, that is

$$\begin{aligned} B_\gamma(x,r) &= \{y \in \mathbb{R}^n : \ \delta_\gamma(x,y) < r\} \\ &= \big\{y \in R^n : \ |\bar{x}-\bar{y}| < r^{\frac{1}{\gamma}}, \ |x_n - y_n| < r\big\}. \end{aligned}$$

Note that the Lebesgue measure of $B_\gamma(x,y)$ is given by

$$|B_\gamma(x,r)| = 2\omega_{n-1} r^{n_\gamma}$$

where

$$n_\gamma = \frac{n-1}{\gamma} + 1\,,$$

and ω_{n-1} is the measure of the unit ball in $\mathbb{R}^{n-1}$. Note also that $n_\gamma = n + (n-1)(\frac{1}{\gamma} - 1)$, hence $n_\gamma \geq n$ and equality occurs if and only if either $n = 1$ or $\gamma = 1$.

Given $p \in [1, \infty[$, a function $\phi :]0,\infty[\to]0,\infty[$ and an open set Ω in $\mathbb{R}^n$, for all $f \in L^p(\Omega)$ we set

$$\|f\|_{L^\phi_{p,\gamma}(\Omega)} := \sup_{x\in\Omega} \sup_{r>0} \left(\frac{1}{\phi(r)} \int_{B_\gamma(x,r)\cap\Omega} |f(y)|^p dy \right)^{\frac{1}{p}}$$

and

$$|f|_{\mathcal{L}^\phi_{p,\gamma}(\Omega)} := \sup_{x\in\Omega} \sup_{r>0} \left(\frac{1}{\phi(r)} \int_{B_\gamma(x,r)\cap\Omega} |f(y) - \fint_{B_\gamma(x,r)\cap\Omega} f(z)dz|^p dy \right)^{\frac{1}{p}}\,.$$

The generalised Morrey spaces are defined by

$$L^\phi_{p,\gamma}(\Omega) = \{f \in L^p(\Omega) : \ \|f\|_{L^\phi_{p,\gamma}(\Omega)} < \infty\},$$

and the generalised Campanato spaces are defined by

$$\mathcal{L}^\phi_{p,\gamma}(\Omega) = \{f \in L^p(\Omega) : \ |f|_{\mathcal{L}^\phi_{p,\gamma}(\Omega)} < \infty\}.$$

For any $l \in \mathbb{N}$, we consider also the Sobolev-Morrey spaces

$$W^{l,\phi}_{p,\gamma}(\Omega) = \{f \in L^p(\Omega) : \ D^\alpha f \in L^\phi_{p,\gamma}(\Omega), \ \forall |\alpha| \leq l\}$$

endowed with the norm

$$\|f\|_{W^{l,\phi}_{p,\gamma}(\Omega)} = \sum_{|\alpha|\leq l} \|D^\alpha f\|_{L^\phi_{p,\gamma}(\Omega)}\,.$$

If $\lambda \geq 0$ and $\phi(r) = \min\{r^\lambda, 1\}$ for all $r > 0$ then the corresponding spaces will be denoted by $L^\lambda_{p,\gamma}(\Omega)$, $\mathcal{L}^\lambda_{p,\gamma}(\Omega)$, $W^{l,\lambda}_{p,\gamma}(\Omega)$. Since $|\cdot|_{\mathcal{L}^\lambda_{p,\gamma}(\Omega)}$ is a semi-norm, it is customary to endow the Campanato space $\mathcal{L}^\lambda_{p,\gamma}(\Omega)$ with the norm defined by

$$\|f\|_{\mathcal{L}^\lambda_{p,\gamma}(\Omega)} := \|f\|_{L^p(\Omega)} + |f|_{\mathcal{L}^\lambda_{p,\gamma}(\Omega)},$$

for all $f \in \mathcal{L}^\lambda_{p,\gamma}(\Omega)$.

Note that $L^\lambda_{p,1}(\Omega)$, $\mathcal{L}^\lambda_{p,1}(\Omega)$ are the classical Morrey and Campanato spaces respectively (recall that $L^\lambda_{p,1}(\Omega)$ contains only the zero function for $\lambda > n$ and it coincides with $L^\infty(\Omega)$ for $\lambda = n$ by the Lebesgue differentiation theorem, see [28] for more details concerning the limiting cases).

We consider elementary Hölder continuous domains Ω in $\mathbb{R}^n$ with exponent $\gamma \in]0,1]$ of the form

$$\Omega = \{x = (\overline{x}, x_n) \in \mathbb{R}^n : \ \bar{x} \in W, \ a < x_n < \varphi(\overline{x})\}, \tag{2.1}$$

where $-\infty \leq a < \infty$, W is a smooth or convex open set in $\mathbb{R}^{n-1}$, and $\varphi : W \to \mathbb{R}$ is a Hölder continuous function with exponent γ satisfying the condition $\varphi(\bar{x}) > a + \delta$ for some $\delta > 0$. In particular, there exists a positive constant M such that

$$|\varphi(\overline{x}) - \varphi(\overline{y})| \leq M|\overline{x} - \overline{y}|^\gamma, \ \forall \ \overline{x}, \overline{y} \in \mathbb{R}^{n-1}. \tag{2.2}$$

The best constant M in inequality (2.2) is denoted by $\mathrm{Lip}_\gamma\varphi$. For $\gamma = 1$ we obtain Lipschitz continuous domains. It is well known that Lipschitz continuous domains satisfy the usual cone condition. Similarly, Hölder continuous domains satisfy a generalisation of that condition which we call the cusp condition. Namely, for any $x \in \mathbb{R}^n$ and $h > 0$, we set

$$C_\gamma(x, h, M) = \{y \in \mathbb{R}^N : \ x_n - h < y_n < x_n - M|\bar{y} - \bar{x}|^\gamma\} \tag{2.3}$$

and we call it a cusp with exponent γ, vertex x, height h and opening M. Then we can prove the following simple lemma which, by the way, is essential in order to apply the general results of [1, 3, 18].

Lemma 2.1. *Let $\gamma \in]0,1]$ and Ω be an elementary Hölder continuous domain in $\mathbb{R}^n$ as in (2.1) with $W = \mathbb{R}^{n-1}$ and $a = -\infty$. Then for all $x \in \bar{\Omega}$ and $h > 0$, we have*

$$C_\gamma(x, h, \mathrm{Lip}_\gamma\varphi) \subset \Omega. \tag{2.4}$$

Moreover, there exists $c > 0$ depending only on n, γ and $\mathrm{Lip}_\gamma\varphi$ such that

$$|B_\gamma(x, r) \cap \Omega| \geq cr^{n_\gamma}, \tag{2.5}$$

for all $x \in \bar{\Omega}$ and $r > 0$.

Proof. Given a cusp $C_\gamma(x, h, \mathrm{Lip}_\gamma\varphi)$ as in the statement, for any point $y \in C_\gamma(x, h, \mathrm{Lip}_\gamma\varphi)$ we have

$$y_n < x_n - \mathrm{Lip}_\gamma\varphi\,|\bar{x} - \bar{y}|^\gamma \leq \varphi(\bar{x}) - \mathrm{Lip}_\gamma\varphi\,|\bar{x} - \bar{y}|^\gamma \leq \varphi(\bar{y}),$$

where the third inequality follows from the Hölder continuity of φ. Thus, $C_\gamma(x, h, \mathrm{Lip}_\gamma\varphi) \subset \Omega$. Inequality (2.5), easily follows from (2.4), the inclusion $C_\gamma(x, r, 1) \subset B_\gamma(x, r)$ and the fact that $|C_\gamma(x, h, M)| = ch^{n_\gamma}$ where c is a positive constant depending only on n, γ, M. □

Given two function spaces $X(\Omega)$ $Y(\Omega)$, we write $X(\Omega) \simeq Y(\Omega)$ to indicate that any function $f \in X(\Omega)$ equals almost everywhere in Ω a function $g \in Y(\Omega)$ and viceversa, and that the two norms $\|\cdot\|_{X(\Omega)}$, $\|\cdot\|_{Y(\Omega)}$ are equivalent. Note that, for the sake of simplicity, two functions f, g as above will be denoted by the same symbol (being aware of this identification is particularly important when stating Hölder continuity estimates).

The following theorem can be deduced by the general result [18, Theorem 3.1] combined with inequality (2.5) which guarantees that Ω is of type (A) as required in [18, Theorem 3.1].

Here, $C^{0,\alpha}(\bar{\Omega},\delta_\gamma)$ denotes the space of Hölder continuous functions with exponent α with respect to the metric δ_γ.

Theorem 2.1 (Campanato-Da Prato)**.** *Let Ω be a bounded elementary Hölder continuous domain with exponent $\gamma \in]0,1]$, $\lambda > 0$. The following statements hold:*

(i) *If $\lambda < n_\gamma$ then $\mathcal{L}^{\lambda}_{p,\gamma}(\Omega) \simeq L^{\lambda}_{p,\gamma}(\Omega)$.*

(ii) *If $\lambda > n_\gamma$ then $\mathcal{L}^{\lambda}_{p,\gamma}(\Omega) \simeq C^{0,\alpha}(\bar{\Omega},\delta_\gamma)$ where*

$$\alpha = \frac{\lambda - n_\gamma}{p}\,;$$

in particular, there exists $c > 0$ such that for all $f \in \mathcal{L}^{\lambda}_{p,\gamma}(\Omega)$ and for all $x, y \in \Omega$ we have

$$|f(x) - f(y)| \le c|f|_{\mathcal{L}^{\lambda}_{p,\gamma}(\Omega)}(|\bar{x} - \bar{y}|^\gamma + |x_n - y_n|)^{\frac{\lambda - n_\gamma}{p}}\,. \tag{2.6}$$

The following result is direct application of a general result in [3, Theorem 27.4.2, Remark 27.4.3] combined with inclusion (2.4) which guarantees that Ω satisfies the γ-horn condition described in [2, p. 153]. As customary, we denote by $[x, y]$ the segment connecting two points x and y in $\mathbb{R}^n$.

Theorem 2.2 (Sobolev-Morrey Embedding for elementary $C^{0,\gamma}$ domains)**.** *Let Ω be an elementary Hölder continuous domain with exponent $\gamma \in]0,1]$. Let $l \in \mathbb{N}$, $\lambda > 0$, $p \in [1,\infty[$ be such that*

$$pl > n_\gamma - \lambda$$

and[1] *$\gamma(l + \frac{\lambda - n_\gamma}{p}) < 1$. Then there exists $c > 0$ such that for all $f \in W^{l,\lambda}_{p,\gamma}(\Omega)$ and for all $x, y \in \Omega$ such that $[x, y] \subset \Omega$ we have*

$$|f(x) - f(y)| \le c\|f\|_{W^{l,\lambda}_{p,\gamma}(\Omega)}|x - y|^{\gamma\left(l + \frac{\lambda - n_\gamma}{p}\right)}\,. \tag{2.7}$$

Note that by setting formally $l = 0$ in (2.7), one essentially obtains estimate (2.6). It is interesting to observe that the previous result (with minor modifications) was proved in [1] in the case of a parallelepiped.

Theorem 2.3 (Barozzi)**.** *Let Ω be a parallelepiped in $\mathbb{R}^n$ of the form $\Omega = \Pi_{i=1}^{n}]a_i, b_i[$ with $-\infty < a_i < b_i < \infty$ for all $i = 1, \dots, n$. Let $\gamma \in]0,1]$, $l \in \mathbb{N}$, $\lambda > 0$, $p \in [1,\infty[$ be such that*

$$pl > n_\gamma - \lambda$$

and such that $l + \frac{\lambda - n_\gamma}{p} \le 1$. Then for any $\epsilon > 0$ there exists $c > 0$ such that for all $f \in W^{l,\lambda}_{p,\gamma}(\Omega)$ and for all $x, y \in \Omega$ we have

$$|f(x) - f(y)| \le c\|f\|_{W^{l,\lambda}_{p,\gamma}(\Omega)}(|\bar{x} - \bar{y}|^\gamma + |x_n - y_n|)^{l + \frac{\lambda - n_\gamma}{p} - \epsilon}\,.$$

Moreover, the following theorem can be deduced by a more general result obtained by G. Da Prato in [18, Theorem 4.1] for $l = 1$ in the case of a convex set.

Theorem 2.4. *Let Ω be a bounded convex domain in $\mathbb{R}^n$. Let $\gamma \in]0,1]$, and $\eta = \frac{n_\gamma}{n} + n - n_\gamma$. Let $\lambda > 0$, $p \in [1,\infty[$ be such that*

$$p\eta > n_\gamma - \lambda\,.$$

Then there exists $c > 0$ such that for all $f \in W^{1,\lambda}_{p,\gamma}(\Omega)$ and for all $x, y \in \Omega$ we have

$$|f(x) - f(y)| \le c\|f\|_{W^{1,\lambda}_{p,\gamma}(\Omega)}(|\bar{x} - \bar{y}|^\gamma + |x_n - y_n|)^{\eta + \frac{\lambda - n_\gamma}{p}}\,.$$

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[1] If viceversa $\gamma(l + \frac{\lambda - n_\gamma}{p}) > 1$ then one has Lipschitz continuity; in the case $\gamma(l + \frac{\lambda - n_\gamma}{p}) = 1$ one gets Hölder continuity with any exponent less than 1.

Remark 1. *We note that the constant η in Theorem 2.4 replaces the constant $l = 1$ in the previous theorems. Since $\eta < 1$ for $\gamma < 1$, we have a deterioration in the estimates. This seems to be due to the fact that the result in* [18, Theorem 4.1] *is quite general and is stated in order to embrace more general types of metrics.*

We now explain where the exponent η in Theorem 2.4 comes from. The main ingredient is a quantitative Poincaré-Wirtinger inequality for bounded convex domains B in $\mathbb{R}^n$, namely the inequality

$$\|f - f_B\|_{L^p(B)} \le \left(\frac{\omega_n}{|B|}\right)^{1-\frac{1}{n}} d^n \|\nabla f\|_{L^p(B)}, \quad \forall f \in W^1_p(B), \tag{2.8}$$

where ω_n denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$, d denotes the Euclidean diameter of B and $f_B = ⨍_B f(x)dx$ (see, e.g., [24, p.164]).

It follows from (2.8) and Hölder's inequality that if Ω is a convex domain in $\mathbb{R}^n$ and $f \in W^1_p(\Omega)$ then for all $x \in \Omega$ and $r > 0$ we have

$$\begin{aligned} &\|f - f_{\Omega\cap B_\gamma(x,r)}\|_{L^1(\Omega\cap B_\gamma(x,r))} \\ &\qquad \le \omega_n^{1-\frac{1}{n}} |\Omega \cap B_\gamma(x,r)|^{\frac{1}{n}-\frac{1}{p}} d_r^n \|\nabla f\|_{L^p(\Omega\cap B_\gamma(x,r))}, \end{aligned} \tag{2.9}$$

where d_r denotes the Euclidean diameter of $\Omega \cap B_\gamma(x,r)$. If in addition we have that $\nabla f \in L^\lambda_{p,\gamma}(\Omega)$, we obtain

$$\|f - f_{\Omega\cap B_\gamma(x,r)}\|_{L^1(\Omega\cap B_\gamma(x,r))} \le c|\Omega \cap B_\gamma(x,r)|^{\frac{1}{n}-\frac{1}{p}} d_r^n r^{\frac{\lambda}{p}}$$

hence

$$\|f - f_{\Omega\cap B_\gamma(x,r)}\|_{L^1(\Omega\cap B_\gamma(x,r))} \le c r^{n_\gamma(\frac{1}{n}-\frac{1}{p})} r^{\frac{\lambda}{p}} d_r^n \tag{2.10}$$

since the measure of $|\Omega \cap B_\gamma(x,r)|$ is controlled from above by a multiple of r^{n_γ}.

In the general framework of [18], it is then assumed that $d_r \le cr^\beta$ for some constant $\beta \ge 1$ which in our case is $\beta = 1$ and cannot be better. Keeping track of β, we obtain from (2.10) that

$$\|f - f_{\Omega\cap B_\gamma(x,r)}\|_{L^1(\Omega\cap B_\gamma(x,r))} \le c r^{n_\gamma(\frac{1}{n}-\frac{1}{p})} r^{\frac{\lambda}{p}} r^{n\beta} \tag{2.11}$$

which means that $f \in \mathcal{L}^\theta_{1,\gamma}(\Omega)$ with

$$\theta = \frac{n_\gamma}{n} + n\beta + \frac{\lambda - n_\gamma}{p}.$$

If $\theta > n_\gamma$, that is $(n_\gamma/n + n\beta - n_\gamma)p > n_\gamma - \lambda$, we deduce by Theorem 2.1 (ii) that $u \in C^{0,\alpha}(\bar{\Omega}, \delta_\gamma)$ with

$$\alpha = \frac{n_\gamma}{n} + n\beta + \frac{\lambda - n_\gamma}{p} - n_\gamma,$$

which for $\beta = 1$ yields

$$\alpha = \eta + \frac{\lambda - n_\gamma}{p}.$$

This explains the appearance of η in Theorem 2.4.

We now reformulate the statement of [18, Theorem 4.1] in order to relax a bit the convexity assumption on Ω. Namely, assume that Ω is a bounded domain in $\mathbb{R}^n$ such that condition (2.5) is satisfied and such that the following p-Poincaré inequality holds

$$⨍_{\Omega\cap B_\gamma(x,r)} |f - f_{\Omega\cap B_\gamma(x,r)}|dx \le c_p r^{\tilde{\eta}} \left(⨍_{\Omega\cap B_\gamma(x,\tau r)} |\nabla f|^p dx\right)^{\frac{1}{p}}, \tag{2.12}$$

for all $f \in W^1_p(\Omega)$ and $r > 0$, where $\tau \geq 1$ and $\tilde{\eta} > 0$ are a fixed constants. In particular, if $f \in W^{1,\lambda}_{p,\gamma}(\Omega)$ we have

$$\begin{aligned}
&\|f - f_{\Omega\cap B_\gamma(x,r)}\|_{L^1(\Omega\cap B_\gamma(x,r))} \\
&\qquad \leq c_p r^{\tilde{\eta}} |\Omega \cap B_\gamma(x, \tau r)|^{1-\frac{1}{p}} \|\nabla f\|_{L^p(\Omega\cap B_\gamma(x,r))} \\
&\qquad \leq c r^{\tilde{\eta}} (\tau r)^{n_\gamma(1-\frac{1}{p})} \|\nabla f\|_{L^p(\Omega\cap B_\gamma(x,r))} \leq c r^{n_\gamma(1-\frac{1}{p})+\frac{\lambda}{p}+\tilde{\eta}},
\end{aligned}$$

for some $c > 0$ independent of r.

This implies that $f \in \mathcal{L}^{\theta}_{1,\gamma}(\Omega)$ with

$$\theta = n_\gamma + \tilde{\eta} + \frac{\lambda - n_\gamma}{p}.$$

If $\theta > n_\gamma$, that is $p\tilde{\eta} > n_\gamma - \lambda$, by the original result of [18, Theorem 3.I] we deduce that $u \in C^{0,\alpha}(\overline{\Omega}, \delta_\gamma)$ with

$$\alpha = \tilde{\eta} + \frac{\lambda - n_\gamma}{p}.$$

Note that for applying [18, Theorem 3.I] we need only condition (2.5).

In conclusion, the following variant of Theorem 2.4 holds.

Theorem 2.5. *Let Ω be a bounded domain in $\mathbb{R}^n$ such that condition* (2.5) *holds, and let $p \in [1, \infty[$. Assume that the p-Poincaré inequality* (2.12) *holds. Let $\gamma \in]0, 1]$ and $\lambda > 0$ be such that*

$$p\tilde{\eta} > n_\gamma - \lambda\,.$$

Then there exists $c > 0$ such that for all $f \in W^{1,\lambda}_{p,\gamma}(\Omega)$ and for all $x, y \in \Omega$ we have

$$|f(x) - f(y)| \leq c\|f\|_{W^{1,\lambda}_{p,\gamma}(\Omega)} (|\bar{x} - \bar{y}|^\gamma + |x_n - y_n|)^{\tilde{\eta}+\frac{\lambda-n_\gamma}{p}}\,.$$

We observe that inequality (2.9) implies the validity of inequality (2.12) with $\tilde{\eta} = \frac{n_\gamma}{n} + n - n_\gamma$, which is the constant η used in Theorem 2.4. We also note that assuming the validity of p-Poincaré inequalities of type (2.12) is nowadays standard in Analysis on Metric Spaces. For instance, we refer to the celebrated paper [26] where general p-Poincaré inequalities of the form

$$\fint_B |f - f_B| d\mu \leq c_p r \left(\fint_{\tau B} g^p d\mu\right)^{\frac{1}{p}} \tag{2.13}$$

are considered. Here g is the upper gradient of f, B is an arbitrary ball of radius r in a metric space space X, τB is the concentric ball of radius τr for a fixed $\tau \geq 1$ and μ is a suitable measure in X. Sufficient conditions ensuring the validity of (2.13) are known in the literature and are discussed e.g., in [26, § 10]. See also [20] for a more recent work on this subject. We note that the study of inequalities of the type (2.12) in domains with cusps or domains of class $C^{0,\gamma}$ is very delicate and in general one does not expect their validity, in particular for outer cusps. Conditions for the validity of global (p, p)-Poincaré inequalities (which means that the power p appears also in the left-hand side of (2.13)) in domains with inner cusps and more generally John domains or L^p-averaging domains are given in [36] where, besides an interesting counterexample, a class of domains admitting moderately sharp outer 'spires' is also analyzed.

3. EXTENSION OF SOBOLEV-MORREY SPACES FOR $C^{0,\gamma}$ DOMAINS

3.1. The case of elementary domains of class $C^{0,\gamma}$.

Let Ω be an elementary Hölder continuous domain in $\mathbb{R}^n$ with exponent $\gamma \in]0, 1]$ as in (2.1), with $W = \mathbb{R}^{n-1}$ and $a = -\infty$. Following [5, 6], we set $G = \mathbb{R}^n \setminus \overline{\Omega}$ and

$$G_k = \{x \in G : 2^{-k-1} < \rho_n(x) \leq 2^{-k}\}$$

for all $k \in \mathbb{Z}$, where $\rho_n(x) = x_n - \varphi(\overline{x})$ is the signed distance from $x \in \mathbb{R}^n$ to ∂G in the x_n direction and we consider a partition of unity associated with the covering $\{G_k\}_{k\in\mathbb{Z}}$ of G satisfying a number of properties. Namely, it is proved in [5] that for every $k \in \mathbb{Z}$ there exists $\psi_k \in C^\infty(\mathbb{R}^n)$ such that

(i) $\displaystyle\sum_{k=-\infty}^{\infty} \psi_k = \begin{cases} 1, & \text{if } x \in G, \\ 0, & \text{if } x \notin G; \end{cases}$

(ii) $G = \cup_{k=-\infty}^{\infty} \mathrm{supp}\psi_k$ and the covering $\{\mathrm{supp}\psi_k\}_{k\in\mathbb{Z}}$ has multiplicity equal to 2;

(iii) $G_k \subset \mathrm{supp}\psi_k \subset G_{k-1} \cup G_k \cup G_{k+1}$, for all $k \in \mathbb{Z}$;

(iv) $|D^\alpha \psi_k(x)| \le c(\alpha) 2^{k\left(\frac{|\overline{\alpha}|}{\gamma} + \alpha_n\right)}$, for all $x \in \mathbb{R}^n, k \in \mathbb{Z}, \alpha \in \mathbb{N}_0^n$.

Note the appearance of γ in the exponent in item (iv) above.

Burenkov's Extension Operator was defined in [5] as follows. Let $l \in \mathbb{N}$ and $1 \le p \le \infty$. For every $f \in W^{l,p}(\Omega)$, we set

$$(Tf)(x) = \begin{cases} f(x), & \text{if } x \in \Omega, \\ \displaystyle\sum_{k=-\infty}^{\infty} \psi_k(x) f_k(x), & \text{if } x \in G, \end{cases} \tag{3.1}$$

where

$$f_k(x) = \int_{\mathbb{R}^n} f(\overline{x} - 2^{-\frac{k}{\gamma}}\overline{z}, x_n - A2^{-k}z_n)\omega(z)dz =$$

$$= A^{-1} 2^{\frac{k}{\gamma}(n-1)+k} \int_{\mathbb{R}^n} \omega(2^{\frac{k}{\gamma}}(\overline{x} - \overline{y}), A^{-1}2^k(x_n - y_n)) f(y) dy\,,$$

A is a sufficiently large constant depending only on n and M in (2.2) (in [5] it is chosen for example $A = 200(1 + Mn)$) and $\omega \in C_c^\infty(\mathbb{R}^n)$ is a kernel of mollification defined by

$$\omega(x) = \omega_1(x_1)\cdots\omega_n(x_n),\ \omega_i \in C_c^\infty(1/2, 1),\ \int_{-\infty}^{+\infty} \omega(x_i)dx_i = 1,\ \int_{-\infty}^{+\infty} \omega_i(x_i)x_i^k dx_i = 0$$

for all $i = 1, \dots, n$, $k = 1, \dots, l$.

Among other results (in particular, concerning anisotropic Sobolev spaces), it is proved in [5] that the operator T is a linear continuous operator from $W_p^l(\Omega)$ to $W_p^{[\gamma l]}(\mathbb{R}^n)$ where $[\gamma l]$ is the integer part of γl.

The following theorem is a generalisation of the extension theorem proved in [21] in the case of Lispchitz domains, that is for $\gamma = 1$. Considering a number of technical issues appearing in the case $\gamma < 1$, we assume for simplicity that the function ϕ defining the Morrey norm satisfies the condition $\phi(r) = 1$ for all $r > 1$.

Theorem 3.1. *Let Ω be an elementary Hölder continuous domain in $\mathbb{R}^n$ with exponent $\gamma \in]0,1]$, with $W = \mathbb{R}^{n-1}$ and $a = -\infty$. Let $l \in \mathbb{N}$, $p \in [1,\infty[$, and $\phi :]0,\infty[\to]0,\infty[$ satisfy the condition $\phi(r) = 1$ for all $r > 1$. Then the operator T maps $W_{p,1}^{l,\phi}(\Omega)$ continuously to $W_{p,1}^{[\gamma l],\phi_\gamma}(\mathbb{R}^n)$, where ϕ_γ is defined by $\phi_\gamma(r) = \phi(r^\gamma)$ for all $r \ge 0$. In particular, T maps the space $W_{p,1}^{l,\lambda}(\Omega)$ to the space $W_{p,1}^{[\gamma l],\gamma\lambda}(\mathbb{R}^n)$, for any $\lambda \ge 0$.*

The proof of Theorem 3.1 can be carried out by adapting the corresponding proof of [21] in a suitable way. Since the adaptation is quite technical and touches a number of delicate points, we indicate here the main steps starting from the first but crucial lemmas which we combine in the following statement. Here $\tilde{G}_k = G_{k-1} \cup G_k \cup G_{k+1} = \{x \in G : 2^{-k-2} < \rho_n(x) \le 2^{-k+1}\}$ for all $k \in \mathbb{Z}$ and $\mathrm{diam}\, C$ denotes the Euclidean diameter of a set C.

Lemma 3.1. *Assume that* $B_1(x,r)\cap G\neq\emptyset$ *for some* $x\in\mathbb{R}^n$ *and* $r>0$. *Let* $h\in\mathbb{Z}$ *be the minimal integer such that* $B_1(x,r)\cap G_h\neq\emptyset$. *Let* $k\in\mathbb{Z}$ *be such that* $k\geq h+3$ *and* $B_1(x,r)\cap\tilde{G}_k\neq\emptyset$. *Then*

$$|2^{-(h+3)}-2^{-k}|\leq c(r+r^\gamma)\,, \tag{3.2}$$

where c *depends only on* γ *and* $\mathrm{Lip}_\gamma\varphi$.

Moreover, given $E>0$ *there exists* $S>0$ *depending only on* γ, $\mathrm{Lip}_\gamma\varphi$, E, *and a lower bound for* h *such that for every* $\eta\in\mathbb{R}^n$, *with* $|\eta|<E$,

$$\mathrm{diam}\left(\bigcup_{k=h+3}^{\infty}\left(B_1(x,r)\cap\tilde{G}_k-(2^{-\frac{k}{\gamma}}\bar\eta,2^{-k}\eta_n)\right)\right)\leq S(r+r^\gamma)\,. \tag{3.3}$$

Proof. By our assumptions we deduce that $\{x\in B_1(x,r):\ \rho_n(x)=2^{-h-2}\},\{x\in B_1(x,r):\ \rho_n(x)=2^{-k+1}\}\neq\emptyset$ hence there exist $y,w\in B_1(x,r)$ with $y_n-\varphi(\bar y)=2^{-h-2}$ and $w_n-\varphi(\bar w)=2^{-k+1}$. Since $|y_n-x_n|,|\bar y-\bar w|<2r$, by the Hölder continuity of φ we get

$$\begin{aligned}|2^{-(h+3)}-2^{-k}|&=\frac{1}{2}|2^{-h-2}-2^{-k+1}|=\frac{1}{2}|y_n-\varphi(\bar y)-w_n+\varphi(\bar w)|\\&\leq\frac{1}{2}(|y_n-w_n|+\mathrm{Lip}_\gamma\varphi|\bar y-\bar w|^\gamma)\leq\frac{1}{2}\left(2r+\mathrm{Lip}_\gamma\varphi(2r)^\gamma\right)\end{aligned}$$

and (3.2) follows.

We now prove (3.3). Let $k\geq h+3$ be such that $B_1(x,r)\cap\tilde{G}_k\neq\emptyset$. Let $a\in B_1(x,r)\cap\tilde{G}_{h+3}$ and $b\in B_1(x,r)\cap\tilde{G}_k$. By (3.2), for all $\eta\in\mathbb{R}^n$, with $|\eta|<E$, we have

$$\begin{aligned}|b_n-2^{-k}\eta_n-(a_n-2^{-(h+3)}\eta_n)|&\leq|b_n-a_n|+|2^{-k}-2^{-(h+3)}||\eta_n|\\&\leq 2r+cE(r+r^\gamma)\end{aligned}$$

and

$$\begin{aligned}|\bar b-2^{-\frac{k}{\gamma}}\bar\eta-(\bar a-2^{-\frac{h+3}{\gamma}}\bar\eta)|&\leq|\bar b-\bar a|+|2^{-\frac{k}{\gamma}}-2^{-\frac{h+3}{\gamma}}||\bar\eta|\\&\leq c\max\{1,2^{-h(1-\gamma)/\gamma}\}(r+r^\gamma)\end{aligned}$$

which proves (3.3). □

Another crucial step in the proof is [21, Lemma 2.4, (ii)] which has to be modified as follows. As in [6, Chap. 6], for every $k\in\mathbb{Z}$ we set

$$\tilde\Omega_k=\{x\in\Omega:2^{-k-2}<|\rho_n(x)|\leq b2^{-k+1}\}\,,$$

where $b=10A$.

Lemma 3.2. *Assume that* $B_1(x,r)\cap G\neq\emptyset$ *for some* $x\in\mathbb{R}^n$ *and* $r>0$. *Let* $f\in W^{l,p}(\Omega)$ *and* $\mathcal{U}\subset\mathbb{R}^n$ *be a fixed measurable set with* $d:=\sup\{\rho_n(x):\ x\in B_1(x,r)\cap\mathcal{U}\}<\infty$. *Then there exists* $c>0$ *and* $m\in\mathbb{N}$ *depending only on* n, l, p, M, ω, d, *and for every* $\alpha\in\mathbb{N}_0^n$ *with* $|\alpha|\leq l$ *there exists a function* g_α *independent of* $r,\mathcal{U}$, *such that for every* $z\in\mathbb{R}^n$ *with* $|z|\leq c$ *there exist* m *balls* $B_1(x_z^{(i)},r^\gamma)$, $i=1,\dots,m$, *such that*

$$\begin{aligned}&\|D^\alpha f_k-g_\alpha\|^p_{L^p(B_1(x,r)\cap\mathcal{U}\cap\tilde{G}_k)}\\&\qquad\leq c2^{pk(\frac{|\bar\alpha|}{\gamma}+\alpha_n-l)}\int_{|z|\leq c}\sum_{|\beta|=l}\|D^\beta f\|^p_{L^p(\cup_{i=1}^m B_1(x_z^{(i)},r^\gamma)\cap\tilde\Omega_k)}dz,\end{aligned} \tag{3.4}$$

for all $k\in\mathbb{N}$.

The proof of the previous lemma follows the lines of [21, Lemma 2.4, (ii)]. We omit the lengthy details but we explain how this lemma is used and how the modified exponent $pk(\frac{|\bar{\alpha}|}{\gamma}+\alpha_n-l)$ affects the final result. Namely, in order to prove Theorem 3.1, one has to estimate the derivatives $D^{\alpha}Tf$ of the extension Tf of a function f. By applying the Leibnitz rule one ends up with estimating $D^{\alpha-\beta}\psi_k D^{\beta}f_k$ for all $\beta\le\alpha$. The difficult part of the work concerns the case $\beta<\alpha$ and $k>0$. One observes that $\sum_{k\in\mathbb{Z}}D^{\alpha-\beta}\psi_k D^{\beta}f_k = \sum_{k\in\mathbb{Z}}D^{\alpha-\beta}\psi_k(D^{\beta}f_k-g_{\beta})$ for $\beta<\alpha$ since g_{β} does not depend on k. Thus, one has to estimate $D^{\beta}f_k-g_{\beta}$. By combining the previous lemma with property (iv) of the partition of unity, we have

$$\begin{aligned}(3.5)\quad \|D^{\alpha-\beta}\psi_k(D^{\beta}f_k-g_{\beta})\|^p_{L^p(B_1(x,r)\cap\mathcal{U}\cap\tilde{G}_k)} &\le c2^{pk(\frac{|\bar{\alpha}-\bar{\beta}|}{\gamma}+\alpha_n-\beta_n)}\|D^{\beta}f_k-g_{\beta}\|^p_{L^p(B_1(x,r)\cap\mathcal{U}\cap\tilde{G}_k)}\\ &\le c2^{pk(\frac{|\bar{\alpha}-\bar{\beta}|}{\gamma}+\alpha_n-\beta_n)}2^{pk(\frac{|\bar{\beta}|}{\gamma}+\beta_n-l)}\int_{|z|\le c}\sum_{|\beta|=l}\|D^{\beta}f\|^p_{L^p(\cup_{i=1}^m B_1(x_z^{(i)},r^{\gamma})\cap\tilde{\Omega}_k)}dz\,.\end{aligned}$$

We note that the exponent of the power of 2 in the right-hand side of (3.5) equals

$$pk\left(\frac{|\bar{\alpha}-\bar{\beta}|}{\gamma}+\alpha_n-\beta_n\right)+pk\left(\frac{|\bar{\beta}|}{\gamma}+\beta_n-l\right)=pk\left(\frac{|\bar{\alpha}|}{\gamma}+\alpha_n-l\right)$$

hence one can control the right-hand side of (3.5), provided that exponent is non-positive, that is

$$(3.6)\qquad |\bar{\alpha}|+\gamma\alpha_n\le\gamma l\,.$$

Inequality (3.6) explains why one gets $[\gamma l]$ as index of smoothness in the target Sobolev space $W^{[\gamma l],\phi}_{\lambda,\gamma}(\mathbb{R}^n)$ in Theorem 3.1.

Moreover, in estimate (3.4) we have the quantity

$$\|D^{\beta}f\|^p_{L^p(\cup_{i=1}^m B_1(x_z^{(i)},r^{\gamma})\cap\tilde{\Omega}_k)}$$

and, since the balls have radius r^{γ}, one eventually controls that quantity via

$$\phi(r^{\gamma})\|D^{\beta}f\|^p_{L^{\phi}_{p,1}(\Omega)}$$

which explains the appearance of the new weight ϕ_{λ} in Theorem 3.1. For further details, we refer to the proof of [21, Theorem 2.5].

3.2. **The case of general domains of class $C^{0,\gamma}$.**

We recall the definition of open sets with $C^{0,\gamma}$ boundary. Here and in the sequel, given a set C in $\mathbb{R}^n$ and $d>0$ we denote by C_d the set $\{x\in C:\mathrm{dist}(x,\partial C)>d\}$.

Definition 1. *Let* $\gamma\in]0,1]$, $d>0$, $M\ge0$, $s\in\mathbb{N}\cup\{\infty\}$. *Let* $\{V_j\}_{j=1}^s$ *be a family of cuboids, i.e. for every* $j=\overline{1,s}$ *there exists an isometry* λ_j *in* $\mathbb{R}^n$ *such that*

$$\lambda_j(V_j)=\Pi_{i=1}^n]a_{i,j},b_{i,j}[$$

where $0<a_{i,j}<a_{i,j}+d<b_{i,j}$. *Assume that* $D:=\sup_{j=\overline{1,s}}\mathrm{diam}V_j<\infty$, $(V_j)_d\ne\emptyset$ *for all* $j=\overline{1,s}$, *and that the multiplicity of the covering* $\{V_j\}_{j=1}^s$ *is finite. We then say that* $\mathcal{A}=(s,d,\{V_j\}_{j=1}^s,\{\lambda_j\}_{j=1}^s)$ *is an atlas.*

Let $M\ge0$. *We say that an open set* Ω *in* $\mathbb{R}^n$ *is of class* $C^{0,\gamma}_M(\mathcal{A})$ *if the following conditions are satisfied:*

(i) For every $j=\overline{1,s}$, *we have* $\Omega\cap (V_j)_d\neq\emptyset$.

(ii) $\Omega\subset\cup_{j=1}^s (V_j)_d$.

(iii) For every $j=\overline{1,s}$, *the set* $\mathcal{H}_j:=\lambda_j(\Omega\cap V_j)$ *satisfies the following condition: either* $\mathcal{H}_j=\Pi_{i=1}^n]a_{i,j},b_{i,j}[$ *(in which case* $V_j\subset\Omega$*), or* $\mathcal{H}_j$ *is a bounded elementary Hölder continuous domain of the form*

$$\mathcal{H}_j=\{x\in\mathbb{R}^n:\ \bar{x}\in W_j,\ a_{n,j}<x_n<\varphi_j(\bar{x})\}$$

where φ_j *is a real-valued Hölder continuous function with exponent* γ, *defined on* $W_j=\Pi_{i=1}^{n-1}]a_{i,j},b_{i,j}[$ *such that*

$$a_{n,j}+d<\varphi_j \text{ and } \mathrm{Lip}_\gamma\varphi_j\le M$$

(in which case $V_j\cap\partial\Omega\neq\emptyset$*).*

Finally, we say that an open set Ω *in* $\mathbb{R}^n$ *is of class* $C^{0,\gamma}$ *if it is of class* $C^{0,\gamma}_M(\mathcal{A})$ *for some* M *and* $\mathcal{A}$.

The definition of Burenkov's Extension Operator for a general domain of class $C^{0,\gamma}$ is given by pasting together the extension operators defined on each chart of the atlas as follows. Following [6, p.265], given an open set Ω of class $C^{0,\gamma}_M(\mathcal{A})$, we consider a family of functions $\{\psi\}_{j=1}^s$ such that $\psi_j\in C^\infty_c(\mathbb{R}^n)$, supp$\psi_j\subset (V_j)_d$, $0\le\psi_j\le 1$, $\sum_{j=1}^s\psi_j^2(x)=1$ for all $x\in\Omega$ and such that $\|D^\alpha\psi_j\|_{L^\infty(\mathbb{R}^n)}\le M$ for all $j=\overline{1,s}$ and $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\le l$, where M depends only on n,l,d.

Burenkov's Extension Operator T is defined from $W^l_p(\Omega)$ to $W^{[\gamma l]}_p(\mathbb{R}^n)$ by

$$Tf=\sum_{j=1}^s\psi_jT_j(f\psi_j), \tag{3.7}$$

for all $f\in W^{l,p}(\Omega)$, where T_j are the extension operators defined on each domain $\Omega\cap V_j$. See [21] for details.

Then, we have he following. Recall that ϕ_γ is defined by $\phi_\gamma(r)=\phi(r^\gamma)$ for all $r\ge 0$.

Theorem 3.2. *Let* Ω *be an open set in* $\mathbb{R}^n$ *of class* $C^{0,\gamma}$ *with* $\gamma\in]0,1]$. *Let* $l\in\mathbb{N}$, $p\in[1,\infty[$, *and* $\phi:]0,\infty[\to]0,\infty[$ *satisfying the condition* $\phi(r)=1$ *for all* $r>1$. *Then the operator* T *maps* $W^{l,\phi}_{p,1}(\Omega)$ *continuously to* $W^{[\gamma l],\phi_\gamma}_{p,1}(\mathbb{R}^n)$. *In particular,* T *maps the space* $W^{l,\lambda}_{p,1}(\Omega)$ *to the space* $W^{[\gamma l],\gamma\lambda}_{p,1}(\mathbb{R}^n)$, *for any* $\lambda\ge 0$.

The proof of Theorem 3.2 can be carried out by pasting together local extensions operators provided by Theorem 3.1 in each cuboid of the covering of Ω. This argument is described in detail in the proof of [21, Theorem 3.3]. Finally, we can deduce the following

Corollary 3.1. *Let* Ω *be an open set in* $\mathbb{R}^n$ *of class* $C^{0,\gamma}$ *with* $\gamma\in]0,1]$. *Let* $l\in\mathbb{N}$, $p\in[1,\infty[$, *and* $\lambda>0$. *If*

$$p[\gamma l]>n-\gamma\lambda$$

and $[\gamma l]+\frac{\gamma\lambda-n}{p}<1$ *then there exists* $c>0$ *such that for all* $f\in W^{l,\lambda}_{p,1}(\Omega)$ *and for all* $x,y\in\Omega$ *we have*

$$|f(x)-f(y)|\le c\|f\|_{W^{l,\lambda}_{p,1}(\Omega)}|x-y|^{[\gamma l]+\frac{\gamma\lambda-n}{p}}. \tag{3.8}$$

The proof of the previous corollary follows immediately by Theorem 3.2 and estimate (2.7) applied with $\gamma=1$ and l replaced by $[\gamma l]$. Indeed, by Theorem 3.2, any functions $f\in W^{l,\lambda}_{p,1}(\Omega)$ is extended to the whole of $\mathbb{R}^n$ as a function of $W^{[\gamma l],\gamma\lambda}_{p,1}(\mathbb{R}^n)$ to which the classical Sobolev-Morrey Theorem applies.

ACKNOWLEDGMENTS

The authors are very thankful to Prof. Victor I. Burenkov for useful discussions and for suggesting the study of the action of his extension operator on Sobolev-Morrey spaces defined on domains with Hölder continuous boundaries. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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